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OPTIMAL ALLOCATION OF PUBLIC GOODS: A SOLUTION TO THE "FREE RIDER PROBLEM"

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Abstract

This paper presents a general equilibrium model in which private commodities are allocated through competitive markets and public commodities according to government allocation and taxing rules that depend on information communicated to the government by consumers regarding their preferences. A wide range of strategic behavior for consumers in their communication with the government is allowed; in particular, consumers may understate their preferences and be "free riders" if they choose.

Although several examples of allocation - taxation schemes falling within the general model are discussed, the major contribution of the paper is the formulation of a particular government allocation - taxation scheme for which the behavioral equilibria are Pareto-optimal. That is, given the government rules, consumers find it in their self-interest to reveal their true preferences for public goods.

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I. INTRODUCTION

It is widely believed that the achievement of a Pareto-optimal allocation of resources via decentralized methods in the presence of public goods is fundamentally incompatible with individual incentives. Samuelson [23], in particular, has argued this point most forcefully in showing the difficulties of extending the competitive market system to cover the allocation of public goods. This belief is so firmly embedded in conventional wisdom that the problem has acquired a name - the Free Rider Problem - and a considerable amount of work has been devoted to attempts at mitigating or circumventing the difficulties it poses.

In this paper we present a decentralized method for determining optimal levels of public goods even when consumers are allowed extensive opportunities to pursue their own self-interest and be "free riders" if they so choose. Basically our method consists of appending to the traditional general equilibrium competitive private ownership economy (as formulated, for example, by Debreu [5]), an explicit procedure for determining consumers' demands for public goods and their tax burdens. Even though consumers are completely free to misrepresent their demands for public goods, the tax and allocation rules we specify are structured in such a way that in equilibrium it is in each consumer's individual self-interest to reveal his true demand or valuation of the public goods. Thus, we have not assumed away the Free Rider Problem, but have provided a possible solution to it.1/

In Section II we formulate a class of mechanisms for allocating public goods by adding a special agent - the government - to the standard Arrow-Debreu model of a private ownership economy. The government (which could be thought of as a computer) chooses according to fixed rules the level of public goods to be provided and the taxes to be levied on consumers based on market prices for all goods and the information ("messages") communicated by consumers.

Consumers are assumed to know (or to be able to discover) the government rules and are free to communicate any message they desire. The government has no way of verifying the "correctness" or "truth" of the information communicated by consumers since it has no basis on which to compare alternative messages from a consumer. In addition to choosing what message to send the government, consumers also choose (purchase) private goods bundles on competitive markets. In making their decisions, consumers are assumed to maximize their preferences over consumption bundles (containing both private and public goods) subject to their budget constraints (which include their tax burdens). We assume consumers behave competitively; that is, they treat as parameters the market prices for goods, their shares of firms' profits, and the messages sent the government by other consumers.

This assumption of competitive behavior is, we feel, the natural extension to a general equilibrium model with public goods of the competitive behavioral assumption made for the Arrow-Debreu private goods only model. It also is in the spirit of Samuelson's remarks to the effect that under a mechanism for determining public goods allocations that relies on consumers to communicate their demands or valuations, "any one person can hope to snatch some selfish benefit in a way not possible under the self-policing competitive pricing of private goods." [23, p. 38]. Thus under our competitive behavioral assumption, a rational consumer, in maximizing his preferences (i.e. attempting "to snatch some selfish benefit") will consider how his message affects the government's determination of the quantity of public goods to be provided and the taxes he must pay. But, just as a competitive consumer on private goods markets takes prices as given, he takes the aggregate effect of the other consumers' messages on the public goods quantity and on his taxes as given also.

Producers are also assumed to behave competitively; that is, as profit maximizers treating prices as parameters.

A member of this class of mechanisms is thus specified by any set of government allocation and taxing rules. Two examples of well known rules are presented at the end of Section II, both to illustrate the broad coverage of our general model and to emphasize the fact that these particular schemes do not lead to Pareto-optimal equilibrium allocations.

In Section III the basic mechanism we propose to solve the Free Rider Problem is presented. The mechanism is described in terms of neo-classical economic concepts that enable us to prove using standard calculus methods the First Fundamental Welfare Theorem - the efficiency or Pareto-optimality of a competitive equilibrium. As a Corollary to this result we prove that under our mechanism a consumer always has an incentive to communicate his true marginal willingness to pay for public goods. Thus, he has nothing to gain (and in fact will lose) by being a free rider and concealing his true marginal willingness to pay.

However, the form of the mechanism presented in Section III is restrictive and unsatisfactory from several standpoints. It involves communicating entire inverse demand functions (willingness to pay functions) and may lead to unbalanced government budgets. In Section IV we present a new form of the mechanism that avoids these difficulties. We then prove both Fundamental Welfare Theorems (optimality of a competitive equilibrium and unbiasedness) under conditions very similar to those used by Debreu in [5] or Arrow and Hahn in [3]. Also, in order to show that these theorems are non-vacuous we discuss, in Section IV.5, the existence question.

Finally, in Section V we discuss some of the literature that is related to this paper.

II. COMPETITIVE PRIVATE OWNERSHIP ECONOMIES WITH GOVERNMENT

II.1 The Economy

The model we consider is an Arrow-Debreu private ownership economy [1/] with public goods and a government. There are L private goods (indexed $\ell = 1,\ldots,L$) and K public goods (indexed $k = 1,\ldots,K$). A bundle of private goods is denoted by x and is an element of the private goods commodity space $\mathbb{R}^L$ (the L-dimensional Euclidean space). A bundle of public goods is denoted by y and is an element of the public goods commodity space $\mathbb{R}^K$. Prices for private and public goods are denoted by the vectors $p \in \mathbb{R}^L$ and $q \in \mathbb{R}^K$ respectively, and the price vector $(p,q) \in \mathbb{R}^{L+K}$ of all goods is denoted by s.

The model has two types of ordinary economic agents - consumers and producers - plus a special agent - the government. There are I consumers (indexed $i = 1,\ldots,I$); each is characterized by a) a consumption set $\mathcal{X}^i \subseteq \mathbb{R}^{L+K}$, b) a preference relation $\precsim_i$ on $\mathcal{X}^i$, and c) an initial endowment of private [3/] goods, $\omega^i \in \mathbb{R}^L$.

There are J producers (indexed $j = 1,\ldots,J$); each is characterized by a production set, $Z^j \subseteq \mathbb{R}^{L+K}$. Each element $z^j = (z_1^j, z_2^j)$ in the set Z^j is a technologically feasible input-output vector whose negative components denote inputs and whose positive components denote outputs. Associated with each producer j is a profit share distribution $\langle\theta^{ij}\rangle_i$ such that $0 \leq \theta^{ij} \leq 1$ and $\Sigma_i \theta^{ij} = 1$, where θ^{ij} is the i^{th} consumer's share of producer (firm) j's profits.

Thus far no distinction has been made between private and public goods except for their labeling. The distinction results from specifying that the entire net production of public goods, $\Sigma_j z_2^j = z_2$ is consumed by each consumer, whereas the net production of private goods, $\Sigma_j z_1^j = z_1$, must be divided among the consumers. This distinction is formalized by the definition of an attainable allocation:

Definition 2.1: (i) An allocation is an $(I + 1 + J)$ - tuple $\{\langle x^i \rangle, y, \langle z^j \rangle\}$,[4/] where $x^i \in \mathbb{R}^L$, $y \in \mathbb{R}^K$, and $z^j \in \mathbb{R}^{L+K}$.

(ii) An attainable allocation is any allocation such that:

a) $(x^i, y) \in \mathcal{X}^i$ for $i = 1,\ldots,I$,

b) $z^j \in Z^j$ for $j = 1,\ldots,J$, and

c) $(\Sigma_i (x^i - \omega^i), y) = \Sigma_j z^j$.

A private ownership economy will be denoted by

$\mathcal{E} \equiv \{\langle \mathcal{X}^i, \succsim_i, \omega^i \rangle, \langle Z^j \rangle, \langle \theta^{ij} \rangle\}$.

II.2 The Government

In a private ownership economy, private goods are purchased by consumers in private markets; public goods are purchased in private markets and provided to the consumers by the special economic agent - the government. This agent has, therefore, two basic tasks to perform. First, it must choose the quantity of each of the K public goods it will purchase and provide the consumers. Second, it must raise, through taxes, the necessary funds to finance its purchases of the public goods. In order to carry out these tasks in a socially desirable or non-arbitrary manner, the government will have to communicate with the consumers. To make precise the concept of communication, we specify an abstract set M to be the language or message space. Each consumer, i, selects an element $m^i \in M$ where m^i is interpreted to be the consumer's message to the government.

In addition to the language, M, the government is characterized by rules that specify a) what public goods bundle to purchase, the allocation rule, and b) what taxes to levy on consumers, the tax rules. Given the language, the rules define specific quantities of public goods and taxes for every I-tuple of messages $m \equiv (m^1,\ldots,m^I)$ received from consumers and every price vector $s = (p,q)$ prevailing in the private markets for private and public goods.

Formally, the allocation rule is a function $y: M^I \times \mathbb{R}^{L+K} \to \mathbb{R}^K$. Thus $y(m,s)$ is the vector of public goods purchased by the government and supplied to consumers if it receives the messages $m = (m^1,\ldots,m^I)$ from consumers and the prices prevailing in the market place are s. The consumers' tax rules are formally specified as (real-valued) functions $C^i: M^I \times \mathbb{R}^{L+K} \to \mathbb{R}$, $i = 1,\ldots,I$. Thus, $C^i(m,s)$ is the lump-sum tax levied on consumer i when the government receives the messages m and the market prices are s. 5/

A government, G, is then completely specified by a language M, an allocation rule $y(\cdot)$, and consumer tax rules, $\langle C^i(\cdot)\rangle$. We write $G = \{M,y,\langle C^i\rangle\}$

II.3 Producer Behavior

Producers are assumed to behave as price - taking profit maximizers. That is, given prices $s = (p,q)$, producer j chooses an input-output vector in his production set Z^j so as to maximize $s \cdot z^j$.

Definition 2.2: (i) The supply correspondence of the $j^{\underline{th}}$ firm, $\varphi^j : \mathbb{R}^{L+K} \to \mathbb{R}^{L+K}$, is defined by:

$$\varphi^j(s) \equiv \{z^j \in Z^j \mid s \cdot z^j \text{ is maximal over } Z^j\}.$$

(ii) The profit function of the $j^{\underline{th}}$ firm $\pi^j : \mathbb{R}^{L+K} \to \mathbb{R}$ is defined by: $\pi^j(s) \equiv s \cdot \varphi^j(s)$

II.4 Consumer Behavior

Each consumer must make two decisions; he must choose a private goods consumption bundle, $x^i \in \mathbb{R}^L$, and a message, $m^i \in M$, to send the government. Consumers are assumed to take as given the prices of all goods, their shares of the firms' profits, and the messages of all other consumers. Consumers do consider the fact that the message they send may affect the quantity, y, of public goods provided and the tax, C^i, levied by the government. Thus they will choose a decision pair (x^i, m^i) to maximize preferences over consumption bundles (x^i, y) subject to a budget constraint.

Definition 2.3: (i) The budget correspondence of the $i^{\underline{th}}$ consumer, $\beta^i : M^{I-1} \times \mathbb{R}^{L+K} \to \mathbb{R}^L \times M$, is defined by[6]

$$\beta^i(m)^i(,s) \equiv \{(\bar{x}^i, \bar{m}^i) \in \mathbb{R}^L \times M \mid (\bar{x}^i, y(m/\bar{m}^i, s)) \in \mathcal{X}^i, p \cdot \bar{x}^i + C^i(m/\bar{m}^i, s) \leq w^i(s)\}$$

where $w^i(s) \equiv p\,\omega^i + \Sigma_j \theta^{ij} \pi^j(s)$ is his wealth.

(ii) The decision correspondence of the $i^{\underline{th}}$ consumer, $\delta^i : M^{I-1} \times \mathbb{R}^{L+K} \to \mathbb{R}^L \times M$ is defined by:

$$\delta^i(m^{)i(},s) \equiv \{(\bar{x}^i,\bar{m}^i) \in \beta^i(m^{)i(},s) \mid$$

$$(\bar{x}^i, y(m/\bar{m}^i,s)) \succcurlyeq_i (x^i, y(m/m^i,s)) \quad \text{for all } (x^i,m^i) \in \beta^i(m^{)i(},s)\}.$$

Loosely speaking, the consumer's choice maximizes the indirect utility of (x^i,m^i) given $(m^{)i(},s)$ subject to a budget constraint given $(m^{)i(},s)$.

II.5 Equilibrium

The definition of equilibrium for our model is a natural generalization of a competitive equilibrium for an Arrow-Debreu economy (without public goods):

Definition 2.4: A competitive equilibrium relative to the government $G = \{M, y(\cdot), \langle C^i(\cdot)\rangle\}$ in the private ownership economy $\mathcal{E} = \{\langle \mathcal{X}^i, \succcurlyeq_i, \omega^i\rangle, \langle Z^j\rangle, \langle \theta^{ij}\rangle\}$ is an $(I + J + 1)$-tuple $\{\langle x^i,m^i\rangle, \langle z^j\rangle, s\}$ of consumer decisions, producer decisions, and a price system such that:

a) $(x^i,m^i) \in \delta^i(m^{)i(},s)$ for all $i = 1,\ldots,I$ (preference maximization)

b) $z^j \in \varphi^j(s)$ for all $j = 1,\ldots,J$ (profit maximization),

c) $(\Sigma_i(x^i - \omega^i), y(m,s)) = \Sigma_j z^j$ (supply equals demand), and

d) $s \neq 0$

Remark 2.1: It can easily be seen that when there are no public goods and government, definition (2.4) reduces to the definition of a competitive equilibrium for a private ownership Arrow-Debreu economy.[7/] Let $\mathcal{X}^i \equiv X^i \times \{0\}$ for each i and $Z^j \equiv Y^j \times \{0\}$ for each j (where $0 \in R^K$). Also, let $y(m,s) \equiv 0, C^i(m,s) \equiv 0$ for all $(m,s) \in M^I \times \mathbb{R}^{L+K}$. Then $\{\langle x^i\rangle, \langle z^j\rangle, p\}$ is a Debreu equilibrium if and only if, for all I-tuples of messages $m = (m^1,\ldots,m^I)$, $\{\langle x^i,m^i\rangle, \langle z^j\rangle, (p,0)\}$ is an equilibrium relative to the government rules.

Remark 2.2: If consumers are never locally satiated, then at a competitive equilibrium relative to G it is necessary that the government's budget be in balance.

To see this, note that local non-satiation implies $p \cdot x^i + C^i(m,s)$ $p \cdot x^i + C^i(m,s) = \omega^i(s) \equiv p \cdot \omega^i + \Sigma_j \theta^{ij} \pi^j(s)$. Summary and substituting $s \cdot z_j = \pi^j(s)$ gives:

$$p \cdot \Sigma_i (x^i - \omega^i) + \Sigma_i C^i(m,s) = \Sigma_j s \cdot z_j = p \cdot \Sigma_j z_{1j} + q \cdot \Sigma_j z_{2j}$$

Since excess demand is zero at a competitive equilibrium,

$$\Sigma_i C^i(m,s) = q \cdot y(m,s).$$

II.6 Optimality

The two Fundamental Theorems of Welfare Economics assert for a private ownership economy (without public goods) that under suitable conditions (i) every competitive allocation is Pareto-optimal and (ii) every Pareto-optimal allocation is competitive for some initial distribution of endowments and profit shares. 8/ Competitive and Pareto-optimal allocations are defined for our economy $\mathcal{E}$ by:

Definition 2.5: (i) A competitive allocation relative to the government G in $\mathcal{E}$ is an allocation $\{\langle x^i \rangle, y, \langle z^j \rangle\}$ such that there exist messages $\langle m^i \rangle$ and a price system s such that $\{\langle x^i, m^i \rangle, \langle z^j \rangle, s\}$ is a competitive equilibrium relative to the government G in $\mathcal{E}$ and $y = y(m,s)$.

(ii) An allocation $\{\langle x^i \rangle, y, \langle z^j \rangle\}$ in $\mathcal{E}$ is Pareto-optimal if (a) it is attainable and (b) there does not exist another attainable allocation $\{\langle \hat{x}^i \rangle, \hat{y}, \langle \hat{z}^j \rangle\}$ such that $(\hat{x}^i, \hat{y}) \succsim_i (x^i, y)$ for $i = 1, \ldots, I$ and $(\hat{x}^{i_0}, \hat{y}) \succ_{i_0} (x^{i_0}, y)$ for some i_0.

Conventional wisdom suggests that it is not possible to find government rules such that the two fundamental theorems of welfare economics hold in private economies with public goods. In order to explain this pessimistic view and also to aid the understanding of the model detailed above, two examples of governments that have been discussed, more or less explicitly, in the literature may be considered. The examples are (i) a private market (or voluntary contributions) model and (ii) the Lindahl "pricing" procedure. We show that in neo-classical economies[9/] neither government optimally allocates resources in general.

Example 2.1: The Naive Government

In this example public goods are treated as if they were private goods. Each consumer reports to the government how much of each public good he wishes to buy. The government purchases the aggregate amount requested. Each consumer pays for the amount he requested; however, he is able to consume the total amount provided.

In terms of our model, the "naive government" $G^N \equiv \{M, y(\cdot), \langle C^i(\cdot)\rangle\}$ is specified by[10/]

(a) $M \equiv \mathbb{R}^K_+$

(b) $y(m,s) = \Sigma_i m^i$

(c) $C^i(m,s) = q \cdot m^i$ for $i = 1,\ldots,I$.

To see why this government is not optimal in neo-classical economies consider consumer i's problem: $\max_{x^i, m^i} U^i(x^i, \Sigma_h m^h)$ s.t. $p \cdot x^i + q \cdot m^i \leq w^i(s)$.

This consumer will choose (x^i, m^i) such that $U^i_\ell - \lambda^i p_\ell = 0$ for all ℓ and

$U^i_k - \lambda^i q_k = 0$ for all k. Hence, in equilibrium $\Sigma_i U^i_k / U^i_\ell = I(q_k/p_\ell) = I(F^j_k/F^j_\ell)$. Thus, for $I > 1$, an equilibrium allocation relative to G^N will not be optimal since each consumer will be a "free-rider" with respect to the public goods. In particular, in selecting his demand, m^i, a consumer will evaluate additional units in terms of their full marginal social costs q, but also only in terms of the marginal private benefit they will confer upon him. Thus, generally, too few resources will be devoted to the provision of public goods and too many to the provision of private goods for an equilibrium allocation to be Pareto-optimal.

Example 2.2 The Lindahl Government

In this example, the government rules are designed so that, if consumers report "truthfully" then an equilibrium allocation will be a Lindahl equilibrium allocation and thus, of course, Pareto-optimal. Each consumer is asked to report his marginal "willingness to pay" or his marginal rate of substitution between each public good and some numeraire private good. The amounts of the public goods provided are those such that the sum of the consumers' marginal willingness to pay equals the marginal costs, q, of providing the public goods. Each consumer is then taxed for the total quantity of each public good at a (per unit) rate equal to his reported marginal willingness to pay.

In terms of our model in a neo-classical economy the language, M, of the Lindahl Government is defined to be the space of all functions $m^i: \mathbb{R}^K_+ \to \mathbb{R}^K_+$ and $m^i(y)$ is interpreted as the K-dimensional vector of consumer i's marginal willingness to pay, [11/] in terms of some fixed numeraire private good, for an additional unit of each public good at the level y. To motivate this interpretation consider what it means for a consumer to report "truthfully" or to send the "true" message. We will say that $\mathring{m}^i \varepsilon M$ is the true message of consumer i if $\mathring{m}^i(\cdot)$ is his true vector of marginal rates of substitution; that is, if for each $y^* \varepsilon \mathbb{R}^K_+$, $(\bar{x}^{i*}, y^*)$ solves $\underset{(x^i,y)}{\text{Max}}\ U^i(x,y)$ subject to $p \cdot x^i + \mathring{m}^i(y^*)\ y \leq w^i$ then $\mathring{m}^i(\cdot)$ is a true message.[12]

Note that the true message $\mathring{m}^i(\cdot)$ is just the inverse (partial equilibrium) Marshallian demand surface and hence (see footnote 11) depends parametrically on the prices of private goods, p, and income, $w_i(p)$. It does not depend on the messages of the other consumers $m^{)i(}$.

With this language M the Lindahl Government, G^L, is completely defined by the rules:

a) $y(m,s)$ is any bundle of public goods such that for every k, $\Sigma_i m^i_k(y) = q_k$ unless $\Sigma_i m^i_k(y/y_k) < q_k$ for all $y_k \geq 0$ in which case $y_k = 0$,

b) $C^i(m,s) = m^i[y(m,s)] \cdot y(m,s)$, $i = 1,\ldots,I$,

The allocation rule selects that bundle of public goods such that the sum of all the <u>reported</u> marginal rates of substitutions for each public good equals its price (marginal cost). The tax rules assess each consumer i for the bundle y at the price $m^i(y)$ per unit.

It is easy to see that a competitive allocation relative to the Lindahl Government is Pareto-optimal <u>if all consumers report truthfully</u>, since a competitive equilibrium relative to the Lindahl Government is then a Lindahl equilibrium[13/] where consumer i's public goods prices are $t^i = \overset{\circ}{m}^i(y^*)$ and y^* is the Lindahl equilibrium level of public goods. And, as is well-known, a Lindahl equilibrium allocation is Pareto-optimal[14/].

However, in a neo-classical economy at a competitive equilibrium relative to the Lindahl Government each consumer will be falsely reporting his marginal willingness to pay and consequently too few resources will be allocated to public goods for Pareto-optimality to obtain.

To see this, notice that a consumer will choose (x^i, m^i) such that at (x^i, y) where $y = y(m/m^i, s)$ it will be true that $U^i_\ell - \lambda^i p_\ell = 0$ for all ℓ and $U^i_k - \lambda^i[m^i_k + (\partial m^i/\partial y_k)\cdot y] = 0$. Thus, in competitive equilibrium $\Sigma_i(U^i_k/U^i_\ell) = (q_k/p_\ell) + [\Sigma_i(\partial m^i/\partial y_k)]\cdot y/p_\ell$.
This last term will, in general, be non-zero and thus $\Sigma_i(U^i_k/U^i_\ell) \neq (F^j_k/F^j_\ell)$. Therefore in general, optimality will not obtain. Thus, although the rules of the Lindahl Government were designed to produce Lindahl equilibria if consumers are truthful, they create incentives for consumers to be untruthful. Hence, while Lindahl equilibrium allocations are, in general Pareto-optimal, competitive allocations relative to the Lindahl government are not.

III. A Class of Optimal Government Rules[15/]

In this section we present a class of governments for which the First Fundamental Welfare Theorem holds in neo-classical economies; that is, for such $\mathcal{E}$, if G is any government in the class, then every competitive allocation relative to G in $\mathcal{E}$ is Pareto-optimal. As a corollary to this result we show that under the government rules, a consumer will always communicate his true marginal willingness to pay for each public good at the level the government will provide given his message.

To begin, we define the particular government $G^* = \{M, y(\cdot), \langle C^i(\cdot)\rangle\}$ by:[16/]

(3.1) a) $M \equiv \{m^i : \mathbb{R}^K_+ \to \mathbb{R} \mid m^i$ is strictly concave and twice differentiable$\}$

b) $y(m,s)$ maximizes $\Sigma_i m^i(y) - q \cdot y$ subject to $y \in \mathbb{R}^K_+$

c)
$$\begin{aligned} C^i(m,s) &\equiv \hat{C}^i[y(m,s); m^{)i(}, s] \\ &\equiv \alpha^i q \cdot y(m,s) - \Sigma_{h \neq i}[m^h(y(m,s)) - \alpha^h q \cdot y(m,s)] \\ &\quad + R^i(m^{)i(}, s) \end{aligned}$$

where $\Sigma_h \alpha^h = 1$ and $R^i : M^{I-1} \times \mathbb{R}^{L+K} \to \mathbb{R}$ is an arbitrary function, $i=1,\dots,I$.

This government is quite easy to interpret. A message m^i in M is interpreted as consumer i's <u>reported willingness to pay function</u>. Thus, if m^i is consumer i's message, $m^i(y)$ indicates the maximum i is reporting that he is willing to pay (in units of account) for the bundle y of public goods. This interpretation is justified by the result we show below (Corollary 3.2) that a consumer's best message always communicates his true <u>marginal</u> willingness to pay for each public good at the level provided by the government given his message.[17]

The allocation rule, $y(m,s)$, thus has the government provide the bundle of public goods that maximizes the <u>net</u> social reported willingness to pay or the <u>total reported consumer surplus</u>.

The consumer's tax rule $C^i(m,s)$ is interpreted as follows: the term $\alpha^i q \cdot y$ is called consumer i's <u>proportional cost share</u> of y and the term $m^h(y) - \alpha^h q \cdot y$ consumer h's <u>reported consumer surplus</u> (or reported net willingness to pay). Consumer i is thus simply assessed his proportional cost share of y minus the <u>reported consumer surplus</u> of the others[18] plus possibly some lump sum transfer $R^i(m^{)i(},s)$ that is independent of his message[19].

The following theorem establishes the First Fundamental Welfare Theorem for the government G^* in neo-classical economies.

<u>Theorem 3.1:</u> Let $\mathcal{E}$ be a neo-classical economy and suppose $\{\langle x^i,m^i\rangle,\langle z^j\rangle,s\}$ is a competitive equilibrium relative to the government G^* defined by (3.1 a-c) in $\mathcal{E}$. The competitive allocation $\{\langle x^i\rangle,y(m,s),\langle z^j\rangle\}$ is then Pareto-optimal in $\mathcal{E}$.

<u>Proof</u>: Observe that the decision rule δ^i of every consumer i can be derived in a two-stage process. The consumer may be thought to choose first a consumption bundle $(\bar{x}^i,\bar{y})$ to maximize $U^i(x^i,y)$ subject to $p \cdot x^i + \hat{C}^i(y:m^{)i(},s) \leq w^i(s)$. Next, $\bar{m}^i$ is chosen so that, given $m^{)i(}$ and s, $y(m/\bar{m}^i,s) = \bar{y}$. Since M contains all strictly concave, twice differentiable functions $m^i:\mathbb{R}^K_+ \to \mathbb{R}$ it clearly contains one $\bar{m}^i$ such that, for $m^{)i(}$ and s,

$$(3.2)\qquad \frac{\partial \overline{m}^i(\overline{y})}{\partial y_k} = q_k - \Sigma_{h\neq i}\frac{\partial m^h(\overline{y})}{\partial y_k} \quad \text{for all } k = 1,\ldots,K.$$

Since $y(m/\overline{m}^i,s)$ maximizes $\Sigma_{h\neq i}m^h(y) + \overline{m}^i(y) - q\cdot y$ in $\mathbb{R}^K_+$ and all $m^h(\cdot)$ are strictly concave and differentiable, it follows by (3.2) that $y(m/\overline{m}^i,s) = \overline{y}$. Thus if $(\overline{x}^i,\overline{m}^i) \in \delta^i(m)^i(,s)$, then $y(m/\overline{m}^i,s) = \overline{y}$ and $(\overline{x},\overline{y})$ maximizes $U^i(x^i,y)$ subject to the budget constraint.

Given this observation and the fact that $\mathcal{E}$ is neo-classical it is clear that at the competitive equilibrium relative to G^* in $\mathcal{E}$, there is a (Lagrange multiplier) $\lambda^i \in \mathbb{R}$ for each i such that $U^i_\ell = \lambda^i p_\ell$ for all ℓ, and $U^i_k = \lambda^i(\partial\hat{C}^i/\partial y_k)$ for all k. But $\partial\hat{C}/\partial y_k = q_k - \Sigma_{h\neq i}\partial m^h/\partial y_k$. Furthermore by the definition of $y(m,s)$ and M, $\Sigma_h \partial m^h/\partial y_k = q_k$. Thus $U^i_k = \lambda^i\partial m^i/\partial y_k$ and $\dfrac{U^i_k}{U^i_\ell} = \dfrac{\partial m^i/\partial y_k}{p_\ell}$. Summing over all i gives

$\Sigma_i \dfrac{U^i_k}{U^i_\ell} = \dfrac{\Sigma_i \partial m^i/\partial y_k}{p_\ell} = \dfrac{q_k}{p_\ell}$. Also $U^i_\ell/U^i_1 = p_\ell/p_1$. Finally, $z^j \in \varphi^j(s)$ implies $F^j_\ell/F^j_1 = p_\ell/p_1$ and $F^j_k/F^j_\ell = q_k/p_\ell$ for all ℓ and k. The conclusion follows since $\mathcal{E}$ is neo-classical.

Q.E.D.

Remark 3.1: Note that in view of Remark 2.2 and since consumers are locally non-satiated in neo-classical economies, the government's budget must be balanced at the competitive equilibrium. Thus, if for some $\mathcal{E}$ it is not possible for the government's budget to be balanced, then Theorem 3.1 is vacuous for that $\mathcal{E}$. We return to this point in Section IV.1.

Since the only properties of the language M that were used in the proof of Theorem 3.1 were the conditions of strict concavity and differentiability and the fact that M contains a message $\bar{m}^i$, given $m^{)i(}$ and s, such that (3.2) holds, any language M having these properties may be substituted for (3.1a) in the definition of the government G^* and the theorem will still hold. Thus, the First Fundamental Welfare Theorem holds for all such governments. The class of such governments is denoted by $\mathcal{G}^*$.

<u>Corollary 3.1:</u> Let $\mathcal{G}^*$ be the class of all governments $G = \{M, y(\cdot), \langle C^i(\cdot)\rangle\}$ defined by:

(3.3) a) M satisfies

i) $m^i \in M$ implies $m^i : \mathbb{R}_+^K \to \mathbb{R}$ and is strictly concave and twice differentiable

ii) for any $m^{)i(} \in M^{I-1}, s \in \mathbb{R}^{L+K}$, $y \in \mathbb{R}_+^K$, there exists some $\bar{m}^i \in M$ such that

$$\frac{\partial \bar{m}^i(y)}{\partial y_k} = q_k - \Sigma_{h \neq i} \frac{\partial m^h(y)}{\partial y_k} \quad \text{for all } k = 1,\dots,K$$

[Note that M may be a proper subset of the language space for G^* (3.1a).]

b) $y(m,s)$ maximizes $\Sigma_i m^i(y) - q \cdot y$ subject to $y \in \mathbb{R}_+^K$

c) $$C^i(m,s) \equiv \hat{C}^i[y(m,s); m^{)i(}, s]$$
$$\equiv \alpha^i q \cdot y(m,s) - \Sigma_{h \neq i}[m^h(y(m,s)) - \alpha^h q \cdot y(m,s)] + R^i(m^{)i(}, s)$$

where $R^i : M^{I-1} \times \mathbb{R}^{L+K} \to \mathbb{R}$ is an arbitrary function, $i = 1,\dots,I$.

Let $\mathcal{E}$ be a neo-classical economy and suppose $\{\langle x^i,m^i\rangle,\langle z^j\rangle,s\}$ is a competitive equilibrium in $\mathcal{E}$ relative to any government G in $\mathcal{G}^*$. The competitive allocation $\{\langle x^i\rangle,y(m,s),\langle z^j\rangle\}$ is then Pareto-optimal in $\mathcal{E}$.

<u>Proof:</u> Identical to the proof of Theorem 3.1

Q.E.D.

An interesting corollary to these two results is that for any government G in $\mathcal{G}^*$, given the prices s and messages of the other consumers $m^{)i(}$, the best message $\bar{m}^i$ of consumer i where $(\bar{x}^i,\bar{m}^i) \in \delta^i(m^{)i(},s)$ communicates his <u>true</u> marginal willingness to pay for the level of public goods provided, $y(m/\bar{m}^i,s)$. This result justifies the interpretation of the messages m^i in M as consumer's reported willingness to pay functions.

<u>Corollary 3.2:</u> Let $m^{)i(}$ be any messages of consumers other than i, s be any price vector, and $(\bar{x}^i,\bar{m}^i)$ be consumer i's best decision under a government G in $\mathcal{G}^*$; i.e. $(\bar{x}^i,\bar{m}^i) \in \delta^i(m^{)i(},s)$.

Define $v^i(y)$ to be the maximum $v^i \in \mathbb{R}$ such that
$U^i(\hat{x}^i(v^i),y) \geq U^i[\bar{x}^i,y(m/\bar{m}^i,s)]$ where
$\hat{x}^i(v^i)$ maximizes $U^i(x^i,y)$ subject to $p \cdot x^i \leq w^i(s) - v^i$.
[Thus, $v^i(y)$ is the maximum i would be willing to pay for the bundle y subject to remaining as well off as at his best decision $(\bar{x}^i,\bar{m}^i)$ and $v^i(y(m/\bar{m}^i,s))$ is his true willingness to pay for the bundle $y(m/\bar{m}^i,s)$.]

Then, for all $k = 1,\ldots,K$,

$$\left.\frac{\partial v^i(y)}{\partial y_k}\right|_{y\,=\,y(m/\bar{m}^i,s)} = \left.\frac{\partial \bar{m}^i(y)}{\partial y_k}\right|_{y\,=\,y(m/\bar{m}^i,s)}.$$

[Thus, i's reported marginal willingness to pay $\partial\bar{m}^i(y)/\partial y_k$ is equal to his true marginal willingness to pay $\partial v^i(y)/\partial y_k$ at the level $y(m/\bar{m}^i,s)$ provided by the government.]

<u>Proof</u>: Since $\mathcal{E}$ is neo-classical, at $v^i(y) = v^i$, $U^i[\hat{x}^i(v^i),y] = U^i[\bar{x}^i,y(m/\bar{m}^i,s)] = \bar{u}^i$. It then follows that

$$\left[\Sigma_\ell U^i_\ell \left(\frac{d\hat{x}^i_\ell}{dv^i}\right)\right]\frac{\partial v^i}{\partial y_k} + U^i_k = 0$$

By the definition of $\hat{x}^i(\cdot)$, $U^i_\ell[\hat{x}^i(v^i),y] = \lambda^i p_\ell$ for all ℓ and $\Sigma_\ell p_\ell d\hat{x}^i_\ell = -dv^i$. Thus,

$$\left[\Sigma_\ell U^i_\ell \left(\frac{d\hat{x}^i_\ell}{dv^i}\right)\right] \frac{\partial v^i}{\partial y_k} = -\lambda^i \frac{\partial v^i}{\partial y_k} = -\frac{U^i_\ell}{p_\ell}\frac{\partial v^i}{\partial y_k} \text{ and } \frac{\partial v^i}{\partial y_k} = p_\ell \frac{U^i_k}{U^i_\ell}$$

evaluated at all $(\hat{x}^i(v^i(y)),y)$.

But, $(\bar{x}^i,\bar{m}^i) \in \delta^i(m)^i(,s)$ implies

$$p_\ell \frac{U^i_k}{U^i_\ell} = \frac{\partial \hat{C}^i}{\partial y_k} = \frac{\partial \bar{m}^i}{\partial y_k}$$

evaluated at $(\bar{x}^i,y(m/\bar{m}^i,s))$.

Hence, since $\hat{x}^i[v^i(y(m/\bar{m}^i,s))] = \bar{x}^i$,

$$\left.\frac{\partial v^i}{\partial y_k}\right|_{y(m/\bar{m}^i,s)} = \left. p_\ell \frac{U^i_k}{U^i_\ell}\right|_{(\bar{x}^i,y(m/\bar{m}^i,s))} = \left.\frac{\partial \bar{m}^i}{\partial y_k}\right|_{y(m/\bar{m}^i,s)}$$

Q.E.D.

IV. An Optimal, Unbiased Government

IV.1: Difficulties with the Abstract Government G^*

Although the government G^* defined in Section III is intuitively easy to understand it is not satisfactory for at least two reasons. First of all, the communication requirements of the government rules are undesirably complicated. Since the language M is a large space of functions, it contains some extremely complicated functions that would be difficult to imagine being communicated. Additionally, the allocation rule is defined only implicitly as a solution to a possibly complicated maximization problem. A more satisfactory government would be one for which the message space M is, say, a Euclidean space and the allocation and tax rules are easily computable functions of real vectors.

The second reason why G^* is unsatisfactory is related to the questions of unbiasedness and existence of a competitive equilibrium relative to a particular government. All proofs of unbiasedness and existence of which we are aware require establishing some form of Walras' Law; for example, the value of excess demand at any price vector is zero. For any government and any economy in which consumers are never locally satiated, in order for Walras' Law to hold it is necessary that, at any price vector s, the government's budget be balanced at the messages sent by consumers[20], i.e.

$$(4.1) \qquad \Sigma_i C^i(m,s) = q \cdot y(m,s) \quad \text{whenever} \quad (x^i,m^i) \in \delta^i(m^{)i(},s).$$

In particular, for the government G^* (or any government G in $\mathcal{G}^*$) it follows from the definition of the rules C^i that

$$(4.2) \qquad \Sigma_i R^i(m^{)i(},s) = (I-1)[\Sigma_i m^i(y,m,s)) - q \cdot y(m,s)]$$

for every s whenever $(x^i, m^i) \in \delta^i(m^{)i(}, s)$. Now, although the functions $R^i(m^{)i(}, s)$ are to a degree arbitrary, for each i, $R^i(m^{)i(}, s)$ cannot depend on m^i. Unfortunately, since the message space M for G^* is the space of all strictly convex twice differentiable functions from $\mathbb{R}^K_+$ to $\mathbb{R}$, there exist no functions R^i satisfying (4.2) for all $m \in M^I$ and s.[21]

Fortunately it is possible to exhibit a government that avoids these difficulties. In the remainder of this section we define this government and prove the two Fundamental Welfare Theorems for quite general economies with this government.

IV.2 An Optimal Government

The government we describe is called the Optimal (O) government and is defined, for $I \geq 3$, by:[22]

$G^O = \{M, y(\cdot), \langle C^i(\cdot)\rangle\}$ where

(4.3)

a) $M \equiv \mathbb{R}^K$

b) $y(m) \equiv \Sigma_i m^i$

c) $$C^i(m,s) \equiv \alpha^i q \cdot \Sigma_h m^h + \frac{\gamma}{2}\left[\frac{I-1}{I}\left(m^i - \frac{1}{I-1}\Sigma_{h\neq i} m^h\right)^2 - \frac{1}{2(I-1)(I-2)}\Sigma_{h\neq i}\Sigma_{h'\neq i}(m^h - m^{h'})^2\right]$$

$$= \alpha^i q \cdot \Sigma_h m^h + \frac{\gamma}{2}\left[\frac{I-1}{I}[m^i - \mu(m^{)i(})]^2 - \sigma(m^{)i(})^2\right]$$

where $\gamma > 0$, $\Sigma_i \alpha^i = 1$ and

(4.4)

a) $\mu^i \equiv \mu(m^{)i(}) \equiv \frac{1}{I-1}\Sigma_{h\neq i} m^h$

b) $$\sigma^{i^2} \equiv \sigma(m^{)i(})^2 \equiv \frac{1}{2(I-1)(I-2)}\Sigma_{h\neq i}\Sigma_{h'\neq i}(m^h - m^{h'})^2 =$$

$$= \frac{1}{I-2}\Sigma_{h\neq i}(m^h - \mu(m^{)i(}))^2$$

With the allocation rule (4.3b) one interpretation of a consumer's message $m^i \in \mathbb{R}^K$ is that it is the increment (or decrement) of each public good the consumer would like the government to add (or subtract) to the amounts requested by the others. Given the others' messages then, a rational consumer will communicate the message m^i such that the resulting bundle is the most desired one. 23/ Since every consumer can insure that the resulting allocation of public goods is his most desired bundle given the messages of the other consumers, in an equilibrium all consumers' most desired bundles must be equal. It is the role of the tax rules to ensure that this is possible. But, even though in equilibrium all consumers desire the same bundle, their messages and taxes will not generally be identical.

The tax rule (4.3c) specifies a consumer's tax as a proportional amount 24/ of the cost of the public goods, $\alpha_i q \cdot \Sigma_i m^i = \alpha_i q \cdot y(m)$, plus a positive multiple, $\frac{\gamma}{2}$, of the difference between the squared deviation (corrected for small samples) of the consumer's message m^i from the mean of the others' messages, $\frac{I-1}{I}(m^i - \mu^i)^2$, and the squared standard error (corrected for small samples) of the mean of the others' messages, σ^{i^2}. 25/ Thus, given the total amount of public goods requested by consumers, $\Sigma_h m^h$, consumer i's tax is larger as the amount he requests deviates from the average of the others' requests and smaller the greater the squared standard error of the mean of the others' messages.

It is interesting to note that according to these allocation and tax rules a consumer does not need to know the individual messages of all the other consumers. All a consumer needs to know in order to make his decisions are the prices of all goods $s = (p,q)$, the (scalar) parameters α_i and γ, the mean of all other consumers' messages μ^i (a K-dimensional vector) and the squared standard error of the mean of the others' messages σ^{i^2} (a scalar).

It can be shown that the Optimal government G^0 is a member of the class $\mathcal{G}^*$ defined in Corollary 3.1 (3.1 a-c) by defining the appropriate language. Define for every i a function $f^i : \mathbb{R}^K_+ \to R$ for each message $m^i \in M \equiv \mathbb{R}^K$ and public goods price vector q by:

$$(4.5)\quad f^i(y;m^i,q) \equiv (\gamma m^i + \alpha^i q)\cdot y - \frac{\gamma}{2I}\, y\cdot y \quad .$$

Since each message m^i defines such a function, a consumer's message m^i can be interpreted as communicating the function $f^i(\,\cdot\,;m^i,q)$. It is easy to verify that the space of such functions, $\mathcal{F}$, satisfies the conditions for a language of a member of $\mathcal{G}^*$; i.e.:

$$\mathcal{F} = \{f^i(\,\cdot\,;m^i,q)\,|\,f^i(y;m^i,q) \equiv (\gamma m^i + \alpha^i q)\cdot y - \frac{\gamma}{2I}\, y\cdot y,\ m^i \in \mathbb{R}^K, q \in \mathbb{R}^K\}$$

satisfies (3.3a). It can also be verified that the rules (3.3 b-c) are equivalent to (4.3 b-c) for this language. A message m^i from the language of the Optimal government communicates the parameters defining a quadratic willingness to pay function, and thus the Optimal government may be viewed as a parametric representation of a government in the class $\mathcal{G}^*$. By Corollary 3.1 a consumer's best message m^i is the vector of parameters of a quadratic approximation to his true willingness to pay function at the level $y(m) = \Sigma_h m^h$ of public goods.

With respect to the two difficulties of the abstract government G^* discussed at the beginning of this section, the Optimal government avoids them both. First of all, in the form defining G^0 [eq. (4.3 a-c)], the rules are extremely simple and easy to compute. Messages are just points in $\mathbb{R}^K$, the allocation rule $y(m)$ is just the sum of all consumers' messages and each consumer's tax is a simple quadratic function of all the messages.

Secondly, if the number of consumers I is greater than 2, then by summing all consumer taxes for any set of messages $\langle m^i \rangle$ and prices q, the government's budget will be balanced as required for Walras' Law to hold:[26/] i.e.

$$\Sigma_i C^i(m,s) = q \cdot y(m) \quad \text{for all} \quad m \in \mathbb{R}_+^K \ . \tag{4.6}$$

IV. 3 The Optimality of the Optimal Government

The theorem proved below establishes the First Fundamental Welfare Theorem for the Optimal government in quite general economies.

<u>Theorem 4.1:</u> (Optimality). Let $\mathcal{E}$ be an economy satisfying the following conditions for every $i = 1,\ldots,I$:

(a) (Continuity of Preferences) for every $(x^i,y) \in \mathcal{X}^i$, the sets $\{(\bar{x}^i,\bar{y}) \in \mathcal{X}^i \mid (\bar{x}^i,\bar{y}) \gtrsim_i (x^i,y)\}$ and $\{(\bar{x}^i,\bar{y}) \in \mathcal{X}^i \mid (x^i,y) \gtrsim_i (\bar{x}^i,\bar{y})\}$ are closed in $\mathcal{X}^i$,

(b) (Convexity of the Consumption Set and Preferences) $\mathcal{X}^i$ is convex and if $(\bar{x}^i,\bar{y})$ and (x^i,y) are in $\mathcal{X}^i$ with $(\bar{x}^i,\bar{y}) \succ_i (x^i,y)$, then $[\lambda\bar{x}^i + (1-\lambda)x^i, \lambda\bar{y} + (1-\lambda)y] \succ_i (x^i,y)$ for all $\lambda \in (0,1)$.

If $\{\langle x^i,m^i\rangle,\langle z^j\rangle,s\}$ is a competitive equilibrium relative to the Optimal government defined by (4.3 a-c) in $\mathcal{E}$ such that, for every $i=1,\ldots,I$,

(c) (Non-satiation) there exists $(x^{i*},y^*) \in \mathcal{X}^i$ such that $(x^{i*},y^*) \succ_i (x^i,y(m))$, and

(d) (No Minimum Wealth) there exists $(\bar{x}^i,\bar{m}^i)$ such that $(\bar{x}^i,y(m/\bar{m}^i)) \in X^i$ and $p\bar{x}^i + C^i(m/\bar{m}^i,s) < px^i + C^i(m,s)$, then the competitive allocation $\{\langle x^i\rangle, y(m), \langle z^j\rangle\}$ is a Pareto-optimal allocation for $\mathcal{E}$.

Remark 4.1: It is interesting to note that slightly stronger assumptions on preferences are needed for our theorem than are required for the analogous theorem in economies without public goods. Although Debreu also assumes convexity of preferences [5 ,p. 94], all that is required is local non-satiation at an equilibrium. Our proof requires convexity of preferences to ensure the existence of a hyperplane separating a consumer's budget and upper contour sets. In the Arrow-Debreu model the upper boundary of the budget set is itself the needed separating hyperplane. In our model, since C^i is not linear in y, the boundary of the budget set is not a hyperplane. 27/

Remark 4.2: The assumption that no consumer is in his minimum wealth condition at equilibrium is required for a reason fundamentally identical to the reason this condition is excluded in proving existence in an Arrow-Debreu economy. Consider an Arrow-Debreu economy in which the only possible relative price for commodity x that will not lead to an excess demand for some other commodity is zero. Suppose also that consumer i holds as initial endowment only commodity x and that his preferences are strictly monotone increasing in x. Then, at a relative price for x of zero, consumer i is in his minimum wealth condition and will demand unlimited quantities of x. Thus, no equilibrium will exist. 28/

In proving optimality for the Arrow-Debreu economy, since a true equilibrium is postulated, this circumstance is ruled out. The budget hyperplane separates the budget set from all strictly preferred points so that any preferred point lies strictly above the budget hyperplane. Although in an equilibrium of our public goods model any strictly preferred point must be outside the budget set, since the budget set is strictly convex along the boundary the separating hyperplane may contain strictly preferred points. This possibility is ruled out when the consumer is not in his minimum wealth condition.

Proof of Theorem 4.1:

1. $p \cdot x^i + C^i(m,s) = w^i(s)$ for all i. Suppose not. Then, since $y(m) = \Sigma_i m^i$, assumptions (a-c) and the continuity of $C^i(\cdot)$ in m^i imply (via a standard argument) that there is a pair $(\bar{x}^i, \bar{m}^i)$ such that, $\bar{m}^i \in M$, $(\bar{x}^i, y(m/\bar{m}^i)) \in \mathcal{X}^i$, $p \cdot \bar{x}^i + C^i(m/\bar{m}^i, s) \leq w^i(s)$, and $(\bar{x}^i, y(m/\bar{m}^i)) \succ_i (x^i, y(m))$. This contradicts the fact that $(x^i, m^i) \in \delta^i(m^{)i(}, s)$.

2. For any $(\bar{x}^i, \bar{y}) \in \mathcal{X}^i$, there is an $\bar{m}^i \in M$ such that $\bar{y} = y(m/\bar{m}^i)$. Simply let $\bar{m}^i = \bar{y} - \Sigma_{h \neq i} m^h$.

3. $(\bar{x}^i, \bar{y}) \succsim_i (x^i, y(m))$, $\bar{y} = y(m/\bar{m}^i)$ implies $p \cdot \bar{x}^i + C^i(m/\bar{m}^i, s) \geq p \cdot x^i + C^i(m,s)$ If not, $p \cdot \bar{x}^i + C^i(m/\bar{m}^i, s) < w^i(s)$ (by 1). Then by the same argument of 1, $(x^i, m^i) \notin \delta^i(m^{)i(}, s)$ which is a contradiction.

4. $(\bar{x}^i, \bar{y}) \succ_i (x^i, y(m))$, $\bar{y} = y(m/\bar{m}^i)$ implies $p \cdot \bar{x}^i + C^i(m/\bar{m}^i, s) > p \cdot x^i + C^i(m,s)$. If not, $(x^i, m^i) \notin \delta^i(m^{)i(}, s)$ which is a contradiction.

5. Let $\hat{C}^i(y; m^{)i(}, s)$ be defined by:

$$\hat{C}^i(y; m^{)i(}, s) \equiv \alpha^i q \cdot y + \frac{\gamma}{2}\left[\frac{I-1}{I}(y - I\mu^i)^2 - \sigma^{i2}\right]$$

It is easily verified that $C^i(m,s) = \hat{C}^i[y(m);m^{)i(},s]$ for every (m,s). Define $\hat{C}^i_y \equiv \alpha_i q + \gamma \frac{I-1}{I}(y(m) - I\mu^i)$. If $(\hat{x}^i,\hat{y}) \underset{\sim i}{\succ} (x^i,y(m))$ then $p \cdot \hat{x}^i + \hat{C}^i_y \cdot \hat{y} \geq p \cdot x^i + \hat{C}^i_y \cdot y(m)$. To show this, let $A \equiv \{(\bar{x}^i,\bar{y}) \in \mathcal{X}^i \mid (\bar{x}^i,\bar{y}) \underset{\sim i}{\succ} (x^i,y(m))\}$. By (a and b), A is convex and $(x^i,y(m))$ is in the boundary of A. Let $B \equiv \{(\bar{x}^i,\bar{y}) \in \mathcal{X}^i \mid p \cdot \bar{x}^i + C^i(\bar{y};m^{)i(},s) \leq w^i(s)\}$. B is convex since $\hat{C}^i$ is a convex function of y. Also, since $(x^i,m^i) \in \beta^i(m^{)i(},s)$, $(x^i,y(m)) \in B$. By 1, $(x^i,y(m))$ is in the boundary of B.

Now, by 4, (relative interior A) $\cap$ (relative interior B) = $\emptyset$. Thus, there exists a hyperplane through $(x^i,y(m))$ separating A and B. It is easy to see that the vector $(p,\hat{C}^i_y)$ defines this hyperplane. The desired conclusion follows from this fact.

6. $(\bar{x}^i,\bar{y}) \succ_i (x^i,y(m))$ implies $p \cdot \bar{x}^i + \hat{C}^i_y \cdot \bar{y} > p \cdot x^i + \hat{C}^i_y \cdot y(m)$. Suppose not. By 5, $p \cdot \bar{x}^i + \hat{C}^i_y \cdot \bar{y} = p \cdot x^i + \hat{C}^i_y \cdot y(m)$. By (d), 2 and 5 there is $(\hat{x},\hat{y}) \in \mathcal{X}^i$ such that $p \cdot \hat{x}^i + \hat{C}^i_y \cdot \hat{y} < p \cdot x^i + \hat{C}^i_y \cdot y(m)$. Let $F \equiv \{(x^{i'},y') \in \mathcal{X}^i \mid (x^{i'},y') = (\lambda \hat{x}^i + (1-\lambda)\bar{x}^i, \lambda\hat{y} + (1-\lambda)\bar{y})$ for all $\lambda \in [0,1]\}$. By (a) there is a neighborhood N of $(\bar{x}^i,\bar{y})$ such that $(x^{i'},y') \in N \cap \mathcal{X}^i$ implies $(x^{i'},y') \succ_i (x^i,y(m))$. But $N \cap F \neq \emptyset$. This leads easily to a contradiction of 5.

7. Suppose that $\{\langle x^i\rangle,y(m),\langle z^j\rangle\}$ is not Pareto-optimal and let $\{\langle\hat{x}^i\rangle,\hat{y},\langle\hat{z}^j\rangle\}$ be a Pareto-superior feasible allocation. It then follows from 5 and 6 that $p \cdot \Sigma_i \hat{x}^i + \Sigma_i \hat{C}^i_y \cdot \hat{y} > p \cdot \Sigma_i x^i + \Sigma_i \hat{C}^i_y \cdot y(m)$. But

$\Sigma_i \hat{C}^i_y = q + \gamma\frac{I-1}{I}[Iy(m) - I\,\Sigma_i \mu^i]$ where $\Sigma_i\mu^i = \Sigma_i m^i = y(m)$. Thus

$\Sigma_i \hat{C}^i_y = q$. It follows then that

$$p \cdot \Sigma_i \hat{x}^i + q \cdot \hat{y} > p \cdot \Sigma_i x^i + q \cdot y(m).$$

But this contradicts the facts that $[\Sigma_i (x^i - \omega^i), y(m)] = \Sigma_j z^j$, $[\Sigma_i (\hat{x}^i - \omega^i), \hat{y}] = \Sigma_j \hat{z}^j$ and $z^j \in \varphi^j(s)$ for all j.

Q.E.D.

Remark 4.3: $\gamma > 0$ was not used in the proof of Theorem 4.1. Thus, the theorem holds for a government defined by the Optimal government rules when $\gamma = 0$:

$$M = \mathbb{R}^K$$

$$y(m) = \Sigma_i m^i$$

$$C^i(m,s) = \alpha_i q \cdot y(m) \quad \text{where} \quad \Sigma\alpha_i = 1$$

Although this government is even simpler than the Optimal government, it is essentially vacuous since competitive equilibrium relative to this government will rarely exist. [29/] However, $\gamma > 0$ is required to prove the second Fundamental Welfare Theorem of the next section.

IV.4 The Unbiasedness of the Optimal Government

The theorem proved below establishes the Second Fundamental Welfare Theorem for the Optimal government. The conditions the economy must satisfy to prove this theorem are identical to those in Debreu [5 , Theorem (1) of Section 6.4]. The exceptional cases which must be excluded are identical to those excluded in the private goods only model -- namely, Pareto-optimal allocations that would place the consumer in a minimum wealth condition at the prices which would support those optima.

In order to isolate the special assumptions necessary to exclude the exceptional cases and to expose the logic of the proof of the Theorem more openly, we prove it through a pair of Lemmata.

Lemma 4.1: Let $\{\langle x^i\rangle, y, \langle z^j\rangle\}$ be a Pareto-optimal allocation for the economy $\mathcal{E}$. If $\mathcal{E}$ satisfies, for every $i = 1,\ldots,I$, conditions (a) (continuity) and (b) (convexity) of Theorem 4.1,

(c) $Z \equiv \Sigma_j Z^j$ is convex, and

(d) for some $i_\circ$ and $(\hat{x}^{i_\circ}, \hat{y}) \in \mathcal{X}^{i_\circ}$, $(\hat{x}^{i_\circ}, \hat{y}) \succ_{i_\circ} (x^{i_\circ}, y)$

then there exists $m^{i*} \in M = \mathbb{R}^K$ for all i and $s^* \in \mathbb{R}^{L+K}$ such that

(1) $y(m^*) = y$

(2) $z^j \in \varphi^j(s^*)$ for $j = 1,\ldots,J$, and

(3) $(\bar{x}^i, y(m^*/\bar{m}^i)) \succsim_i (x^i, y)$ implies

$$p^* \cdot \bar{x}^i + C^i(m^*/\bar{m}^i, s^*) \geq p^* \cdot x^i + C^i(m^*, s^*) \quad \text{for all } i.$$

Proof: 1. There is a vector $\tau = (p^*, t^1, \ldots, t^I) \in \mathbb{R}^{L+IK}$, $\tau \neq 0$, such that, defining $q^* \equiv \Sigma_i t^i$ and $s^* \equiv (p^*, q^*)$,

(i) $z^j \in \varphi^j(s^*)$ for $j = 1,\ldots,J$, and

(ii) $(\bar{x}^i, \bar{y}) \succsim_i (x^i, y)$ implies $p^* \cdot \bar{x}^i + t^i \cdot \bar{y} \geq p^* \cdot x^i + t^i \cdot y$

for $i = 1,\ldots,I$.

[That is, there exist Lindahl prices supporting the Pareto-optimal allocation 30/].

To prove this, let

$$F^f \equiv \{(\bar{z}_1, \zeta^1, \ldots, \zeta^I) \in \mathbb{R}^{L+IK} \mid \zeta^i = \zeta^h = \bar{z}_2 \ \text{all } i,h, \text{ and } (\bar{z}_1, \bar{z}_2) \in Z\}$$

Let

$$F^c \equiv \{(\bar{z}_1,\zeta^1,\ldots,\zeta^I) \in \mathbb{R}^{L+IK} \mid \text{ there exists } \bar{x}^i \in \mathbb{R}^L, \text{ all } i, \text{ such that}$$
$$\bar{z}_1 = \Sigma_i\bar{x}^i, (\bar{x}^i,\zeta^i) \succsim_i (\bar{x}^i,y), (\bar{x}^i,\zeta^i) \in \mathcal{X}^i, (\bar{x}^{i_o},\zeta^{i_o}) \succ_{i_o} (x^{i_o},y) \text{ for some } i_o\}.$$

Now, F^f is convex and non-empty since Z is. F^c is convex by (a) and (b) and non-empty by (d). Thus, $G \equiv F^c - F^f$ is a non-empty convex subset of $\mathbb{R}^{L+IK}$.

Since $\{\langle x^i\rangle,y,\langle z^j\rangle\}$ is Pareto-optimal, $\bar{\omega} \notin G$ where $\bar{\omega} = (\Sigma_i\omega^i,0) \in \mathbb{R}^{L+IK}$. But by (a) and (b), $\bar{\omega} \in$ closure G. Thus, by Minkowski's theorem there is a hyperplane through $\bar{\omega}$ and bounding for G. That is, there exists a vector $\tau = (p^*,t^1,\ldots,t^I) \in \mathbb{R}^{L+IK}$, $\tau \neq 0$, such that if $g \in G$ then $\tau\cdot g > \tau\cdot\bar{\omega}$. Using (a), (b) and the fact that $\bar{\omega} = (\Sigma_i x^i - \Sigma_j z_1^j, y - \Sigma_j z_2^j,\ldots,y - \Sigma_j z_2^j)$ it can be shown that

(iii) $(\bar{z}_1,\zeta) \in F^f$ implies $\tau\cdot(\bar{z}_1,\zeta) \le \tau\cdot(z_1,z_2,\ldots,z_2)$,

and

(iv) $(\bar{z}_1,\zeta) \in$ closure F^c implies $\tau\cdot(\bar{z}_1,\zeta) \ge \tau\cdot(\Sigma_i x^i,y),\ldots,y)$.

The desired conclusions (i) and (ii) follow easily.

2. For each i, let $m^{i*} = \frac{1}{I}y + \frac{1}{\gamma}(t^i - \alpha^i q^*)$. m^{i*} is well-defined since $I > 0$ and $\gamma > 0$ [see Remark 4.3]. Then $y(m^*) = \Sigma_i m^{i*} =$ $= y + \frac{1}{\gamma}(\Sigma_i t^i - q^*) = y$ since $\Sigma_i t^i \equiv q^*$.

3. If $p^*\cdot\bar{x}^i + t^i\cdot y(m^*/\bar{m}^i) \geq p^*\cdot x^i + t^i\cdot y$, then $p^*\cdot\bar{x}^i + C^i(m^*/\bar{m}^i,s^*) \geq p^*\cdot x^i + C^i(m^*,s^*)$.

As in 5 of the proof of Theorem 4.1 we define $\hat{C}^i(\bar{y};m)^{i(*},s^*) =$ $\alpha^i q^*\cdot\bar{y} + \frac{\gamma}{2}\frac{I-1}{I}(\bar{y} - I\mu^i)^2 - \frac{\gamma}{2}\sigma^{i2}$. Substituting m^{h*} from 2, $I\mu^i =$ $\frac{I}{I-1}\Sigma_{h\neq i}[\frac{1}{I}y + \frac{1}{\gamma}(t^h - \alpha^h q) = y + \frac{I}{\gamma(I-1)}[(q - t^i) - (1 - \alpha^i)q] =$

$y + \frac{I}{\gamma(I-1)}(\alpha^i q - t^i)$. Hence

$\hat{C}^i(\bar{y};m)^i(^*,s^*) = \alpha_i q \cdot \bar{y} + \frac{\gamma}{2}\frac{(I-1)}{I}[(\bar{y}-y) - \frac{I}{\gamma(I-1)}(\alpha^i q - t^i)]^2 - \frac{\gamma}{2}\sigma^{i2}$

$= t^i \cdot \bar{y} + \frac{\gamma}{2}\frac{I-1}{I}(\bar{y}-y)^2 + T^i$ where T^i contains only terms which are independent of $\bar{y}$.

Suppose $p^* \cdot \bar{x}^i + t^i \cdot y(m^*/\bar{m}^i) \geq p^* \cdot x^i + t^i \cdot y$. Let $\bar{y} = y(m^*/\bar{m}^i)$. Then $p^* \cdot \bar{x}^i + \hat{C}^i(\bar{y};m)^i(,s^*) - \frac{\gamma}{2}\frac{(I-1)}{1}(\bar{y}-y)^2 \geq p^* \cdot x^i + \hat{C}^i(y;m)^i(^*,s^*)$ which implies that $p^* \cdot \bar{x}^i + \hat{C}^i(\bar{y};m)^i(^*,s^*) \geq p^* \cdot x^i + \hat{C}^i(y;m)^i(^*,s^*)$. But $\hat{C}^i(y(m^*/\bar{m}^i);m)^i(^*,s^*) = C^i(m^*/\bar{m}^i,s^*)$ and the desired conclusion follows.

4. Conclusion (1) of the Lemma is established in 2. Conclusion (2) is established in 1(i). Conclusion (3) follows from 1 (ii) and 3.

Q.E.D.

Remark 4.4: The proposition that the Optimal government is unbiased in economies satisfying (a-c) will follow from Lemma 4.1 if we show that conclusion (3) implies there exists a redistribution of initial endowments and profit shares such that $(x^i,m^{i*}) \in \delta^i(m)^i(^*,s^*)$ for all i. But, as is the case with the (private goods only) Arrow-Debreu model, an additional assumption is needed to establish this implication. In particular, no consumer should be in the minimum wealth condition given the messages and prices (m^*,s^*) required to support the Pareto-optimal allocation.

To establish the link between Lemma 4.1 and the unbiasedness of the Optimal government, we prove first that an appropriate reallocation exists (Lemma 4.2) and second that conclusion (3) of Lemma 4.1 implies $(x^i,m^{i*}) \in \delta^i(m)^i(^*,s^*)$ given this redistribution if no one is in the minimum wealth condition (Theorem 4.2).

Lemma 4.2: Let $\{\langle x^i\rangle, y, \langle z^j\rangle\}$ and (m^*, s^*) be as in Lemma 4.1. Then, if $p^* \neq 0$, there exist profit share distributions $\langle\theta^{ij*}\rangle$ and initial endowments $\langle\omega^{i*}\rangle$ such that $\Sigma_i\omega^{i*} = \Sigma_i\omega^i$, with the property that

$$p^*\cdot x^i + C^i(m^*,s^*) = p^*\cdot\omega^{i*} + \Sigma_j\theta^{ij*}\pi^j(s^*).$$

Proof: Let $\theta^{ij*} = \frac{1}{I}$ for all i,j. Let

$$\omega^{i*} = x^i - \frac{1}{I}(\Sigma_j z_1^j) + [C^i(m^*,s^*) - \frac{1}{I}q^*\cdot y]\cdot e \quad \text{where}$$

$e = (e_1,\ldots,e_L)$, $e_\ell = \frac{1}{p_\ell^* L_0}$ if $p_\ell^* \neq 0$, $e_\ell = 0$ if $p_\ell^* = 0$, and L_0 is the number of private goods with non-zero prices. Since $p^* \neq 0$, ω^{i*} is well-defined. Furthermore $\Sigma\omega^{i*} = \Sigma_i x^i - \Sigma_j z_1^j + [\Sigma_i C^i(m^*,s^*) - q^*\cdot y]e$. But $\Sigma_i C^i(m^*,s^*) = q^*\cdot y(m^*) = q^*\cdot y$ [See equation (4.9)].
Therefore, $\Sigma_i\omega^{i*} = \Sigma_i\omega^i$.
Finally, $p^*\cdot\omega^{i*} + \Sigma_j\theta^{ij*}\pi^j(s^*) =$

$$= p^*\cdot x^i - \frac{1}{I}p^*\cdot\Sigma_j z_1^j + C^i(m^*,s^*) - \frac{1}{I}q^*\cdot y + \frac{1}{I}p^*\cdot z_1 + \frac{1}{I}q^*\cdot z_2 =$$

$$= p^*\cdot x^i + C^i(m^*,s^*).$$

Q.E.D.

Remark 4.5: Although it is possible to redistribute such that profit shares are non-negative and sum to unity, it may be necessary to assign initial endowments such that $(\omega^{i*},0) \notin \mathcal{X}^i$ because of the terms $R^i(m)^{i}(^*,s^*)$ in the consumer tax rule. That is, the lump sum taxes may make $p^*\cdot x^{i*} + C^i(m^*,s^*)$ a large negative number. This can also occur in the Arrow-Debreu private goods only model under the usual redistribution, $\omega^{i*} = x^{i*} - \frac{1}{I}z_1^*$, if x^{i*} is small and z_1^* is large.

Remark 4.6: The requirement that $p^* \neq 0$ is needed since the fact that $\tau = (p^*,t^1,\ldots,t^I) \neq 0$ does not imply that $p^* \neq 0$ (or, even, that $s^* \neq 0$). We will ensure that $p^* \neq 0$ by assuming the existence of an always desired private commodity [see condition (d.1) of Theorem 4.2 below].

An alternative definition of the redistribution is possible in some cases when $p^* = 0$. No redistribution of initial endowments is made, i.e. $\omega^{i*} \equiv \omega^i$, but profit shares are redistributed by: $\theta^{ij*} \equiv C^i(m^*,s^*)/q^* \cdot y$. However, for this redistribution to be satisfactory both $C^i(m^*,s^*)$ and $q^* \cdot y$ must be strictly positive and to guarantee this additional assumptions would be required. It seems to us that guaranteeing $p^* \neq 0$ is a simpler approach.

We now prove the Second Fundamental Welfare Theorem for the Optimal government.

<u>Theorem 4.2:</u> (Unbiasedness) Let $\mathcal{E}$ be an economy satisfying the following conditions:

(a) (Continuity of Preferences) for all i, for every $(x^i,y) \in \mathcal{X}^i$, the sets $\{(\bar{x}^i,\bar{y}) \in X^i \mid (\bar{x}^i,\bar{y}) \succsim_i (x^i,y)\}$ and $\{(\bar{x}^i,\bar{y}) \in \mathcal{X}^i \mid (x^i,y) \succsim_i (\bar{x}^i,\bar{y})\}$ are closed in $\mathcal{X}^i$,

(b) (Convexity of the Consumption Set and Preferences) for all i, $\mathcal{X}^i$ is convex and if $(\bar{x}^i,\bar{y})$ and (x^i,y) are in $\mathcal{X}^i$ and

$$(\bar{x}^i,\bar{y}) \succ_i (x^i,y), \text{ then}$$

$$[\lambda\bar{x}^i + (1-\lambda)x^i,\ \lambda\bar{y} + (1-\lambda)y] \succ_i (x^i,y) \text{ for all } \lambda \in (0,1).$$

(c) (Convexity of Aggregate Production) $Z = \Sigma_j Z^j$ is convex in $\mathbb{R}^{L+K}$

Let $\{\langle x^i\rangle, y, \langle z^j\rangle\}$ be a Pareto-optimal allocation for $\mathcal{E}$. If there exists some private commodity, say $\ell = 1$, such that

(d.1) (Monotonicity in Commodity 1) for all i, $(\hat{x}^i,y) \in \mathcal{X}^i$, $\bar{x}^i_1 > \hat{x}^i_1$, $\bar{x}^i_\ell = \hat{x}^i_\ell$ for $\ell = 2,\ldots,L$ implies $(\bar{x}^i,y) \in X^i$ and $(\bar{x}^i,y) \succ_i (\hat{x}^i,y)$, and (d.2) (No Minimum Consumption) for all i, there is $(\hat{x}^i,\hat{y}) \in \mathcal{X}^i$ such that $\hat{x}^i_1 < x^i_1$, $\hat{x}^i_\ell = x^i_\ell$ for $\ell = 2,\ldots,L$ and $\hat{y} = y$,

then there exist messages $m^{i*} \in M$ for $i = 1,\ldots,I$ and prices s^* such that $\{\langle x^i, m^{i*}\rangle, \langle z^j\rangle, s^*\}$ is a competitive equilibrium relative to the Optimal government defined by (4.3 a-c) in $\mathcal{E}$ with $y = y(m^*)$, following if necessary a redistribution of initial endowments and profit shares.

Proof: 1. (a-c) and (d.1) imply (a-d) of Lemma 4.1. Hence, the conclusion of Lemma 4.1 holds.

2. (d.1) implies $p^*_1 > 0$. To see this, recall that $g \in G$ implies $\tau \cdot g > \tau \cdot \bar{\omega}$. [See 1 of Proof of Lemma 4.1)] Suppose then that $p^*_1 \leq 0$. Let $\{\langle \bar{x}^i\rangle, \bar{y}, \langle \bar{z}^j\rangle\}$ be such $\bar{y} = y$, $\bar{z}^j = z^j$ for all j and, for each i, $\bar{x}^i_\ell = x^i_\ell$ for $\ell > 1$ and $\bar{x}^i_1 > x^i_1$. Then

$$\bar{g} \equiv (\Sigma_i \bar{x}^i - \Sigma_j \bar{z}^j, \bar{y} - \Sigma_j \bar{z}^j, \ldots, \bar{y} - \Sigma_j \bar{z}^j) \in G \quad \text{by} \quad (d.1)$$

But

$$\tau \cdot \bar{g} = \tau \cdot \bar{\omega} + p^*_1(\Sigma_i(\bar{x}^i_1 - x^i_1)) \leq \tau \cdot \bar{\omega} \quad \text{since} \quad p^*_1 \leq 0. \quad \text{Contradiction}$$

3. Since $p^* \neq 0$, Lemma 4.2 applies. Let $w^{i*}(s^*) = p^* \cdot x^i + C^i(m^*, s^*)$ be the value of i's wealth after the redistribution.

4. $(x^{i*}, y^*) \succ_i (x^i, y), y^* = y(m^*/\bar{m}^i)$ implies $p \cdot x^{i*} + C^i(m^*/\bar{m}^i, s^*) > w^{i*}(s^*)$.

Suppose not. Then (d.2) implies there is an $\bar{x}^i$ such that $(\bar{x}^i, y) \in \mathcal{X}^i$, $p \cdot \bar{x}^i + C^i(m^*, s^*) < w^{i*}(s^*)$, and $y = y(m^*)$. Conclusion (3) of Lemma 4.1 implies $p \cdot x^{i*} + C^i(m^*/\bar{m}^i, s^*) = w^{i*}(s^*)$. But Condition (d.1) implies that $(x^{i*}, y^*) \succ_i (\bar{x}^i, y)$. Now using Conditions (a),(b) and an argument identical to that in 6 of the proof of Theorem 4.1, a contradiction is reached. Thus, $(x^{i*}, m^{i*}) \varepsilon \delta^{i}(m)^{i}(^{*}, s^{*})$ following the redistribution.

5. Conclusions (1) and (2) of Lemma 4.1 and 4 above imply the desired conclusion.

Q.E.D.

<u>Remark 4.7:</u> Although it would be desirable to eliminate assumption (d) of Theorem 4.2, a complete elimination is not possible in general. In particular, our approach requires, given the supporting prices τ and messages $\langle m^{i*}\rangle$, that $p^{*} \neq 0$ and that

$$p^{*} \cdot x^{i} + C^{i}(m^{*}, s^{*}) > \underset{(x^{i'} m^{i})}{\mathrm{Min}} \; p^{*} \cdot x^{i'} + C^{i}(m^{*}/m^{i}, s) \quad \text{subject to}$$

$(x^{i'}, y(m^{*}/m^{i})) \varepsilon \mathcal{X}^{i}$. We have commented on the necessity for $p^{*} \neq 0$ in Remark 4.6. The second condition is simply that no consumer be in the minimum wealth condition [see condition (d) of Theorem 4.1] and is necessary to rule out "exceptional" cases. Such cases cause the same type of problems that occur in economies with private goods only. (See, e.g. Debreu [5 , p. 96, remarks following (1) of Section 6.4]). If $p^{*} \neq 0$ is guaranteed, these exceptional cases can be ruled out in the same way they are for private goods economies. Condition (d.2) suffices for this purpose.

IV.5. Some Remarks on Existence[31/]

Although Theorem 4.1 establishes that a competitive equilibrium relative to the Optimal government gives a Pareto-optimal resource allocation, unless it can be shown that competitive equilibria exist the theorem is potentially vacuous. Theorem 4.2 - the unbiasedness theorem - however establishes the non-vacuousness of Theorem 4.1. Specifically Theorem 4.2 shows that, given any preferences, technology, and aggregate endowments satisfying assumptions (a)-(c), if a Pareto-optimum exists and preferences also satisfy assumption (d) at this optimum, then there is at least one economy with those preferences and technology which has a competitive equilibrium relative to the Optimal government. Since it is an easy matter to exhibit preferences, a technology, and aggregate endowments satisfying these conditions, Theorem 4.1 is clearly not vacuous.

It is, however, not so easy to specify prior conditions on an economy sufficient to ensure that an equilibrium exists. For instance, there are economies which satisfy all the hypotheses (suitably adjusted for public goods) of Debreu's existence theorem [5, Theorem (i), Section 5.7], but which do not have an equilibrium. Now, all economies satisfying Debreu's assumptions have (a) a compact set of attainable allocations, (b) upper hemi-continuous, convex and non-empty valued supply correspondences in prices, and (c) upper hemi-continuous and convex valued consumer decision correspondences in prices and other consumers' messages. It is also true that for these economies Walras' Law will hold at all prices and messages for which all agents' decision correspondences are non-empty. Therefore, the only reason an equilibrium may not exist for such an economy is that some consumer's decision correspondence may be empty. This may occur in two ways.

First of all, although the set of attainable allocations is compact, the message space $M \equiv \mathbb{R}^K$ is not. Thus, the set of possible decisions is not compact and hence the decision $\delta^i(m^{)i(}, s)$ of a consumer may not be defined at some $(m^{)i(}, s)$ since the consumer may always prefer a larger (or smaller) message m^i to any given finite one. It is easy to show that an arbitrary large m^i can be afforded only if μ^i is arbitrarily small (negative). Thus, if no consumer ever desires to exhaust his budget totally to reduce the quantity of any public good then μ^i can be bounded below and hence the affordable m^i bounded above. A sufficient assumption to ensure this is that preferences are (locally) strongly monotonic in some private good.[32/] Under this assumption, the decision correspondence of a consumer will be defined although it may still be empty valued at some pair $(m^{)i(}, s)$.

The second reason for a possibly empty decision correspondence is that the budget set, $\beta^i(m^{)i(}, s)$, may be empty for some prices and messages of other consumers. Although it is not necessary to guarantee non-emptiness of the budget correspondence for <u>all</u> prices and messages of others,[33/] some condition is required to ensure that at market clearing prices a mutually consistent set of preference maximizing messages exist such that no consumer is bankrupt given the prices and other consumers' messages. The possibility of non-existence occurs in our model (under the Debreu assumptions) only when there is a diversity of tastes (preferences for public goods) <u>and</u> a sufficiently large number of consumers who prefer public goods levels that are high relative to the productive capacity of the economy.

An extreme case occurs if all consumers but one are identical. In This case, the exceptional consumer would receive no lump sum subsidy, σ^i

[c.f. (4.4)], while the others will receive a positive subsidy. If the identical consumers prefer to expend nearly all their income on public goods they may, unintentionally, bankrupt the exceptional consumer. If the exceptional consumer were identical to the other consumers this possibility of bankruptcy could not occur.

A sufficient condition to rule out this possibility of bankruptcy and still permit wide diversity of tastes is if every consumer is satiated in public goods at high levels of public goods.[34/] For **economies** with one public good and one private good it is sufficient to assume that satiation occurs, for all i, at a level $\hat{y} = \overline{y} - I/2\gamma(I-1)$ where $\overline{y} = \min_i \min_s y^i(s)$ and and $y^i(s) = \max y$ subject to $\alpha_i q \cdot y \leq w^i(s)$.

It can be shown (c.f. [16]) that all economies satisfying (1) the above satiation assumption (footnote 33), (2) the above monotonicity assumption (footnote 31), and (3) Debreu's hypotheses will have an equilibrium relative to the Optimal government. But it is clear that the satiation assumption, in particular, is much stronger than necessary. For example, in place of the satiation assumption it would be sufficient if the maximal aggregate marginal rate of substitution is less than the marginal rate of transformation for all public goods bundles y that are sufficiently large.[35/]

V. Some Remarks on the Literature

Although it is clearly impossible in this paper to discuss thoroughly all that has been written on the Free Rider Problem or on the incentive problem in public goods resource allocation models, there are three bodies of literature that are related to this paper and thus should be mentioned to place this paper in proper perspective. The first is the literature covering general equilibrium resource allocation models with public goods. The second concerns general optimal resource allocation mechanisms and individual incentives. The third relates to the development of the specific mechanism for determining the public goods allocation and tax shares presented in this paper.

Most contributions in the first body of literature as surveyed by Milleron [22] do not directly address the issue of finding mechanisms for optimally allocating public goods that take account of individual self-interested behavior. Rather, they have explored the existence of and the relationship between Lindahl equilibria on the one hand and Pareto-optimal and core allocations of resources on the other. However, as discussed above in Section II, Example 2.2, a natural mechanism formulated to achieve Lindahl equilibria (the Lindahl government) must rely on individual consumers to reveal truthfully their preferences for public goods and a competitive equilibrium relative to the Lindahl government is generally not Pareto-optimal.

Two papers discussed in Milleron's survey that do, however, formulate mechanisms for optimally allocating public goods and which also consider the incentive problem are those of Drèze and Vallée Poussin [7] and

Malinvaud [21]. Under the assumptions of these papers, these mechanisms provide incentives for consumers to correctly reveal their preferences for public goods and lead to Pareto-optimal allocations. However, the behavioral assumptions of these papers are more restrictive than those assumed in this paper. Essentially it is assumed that a consumer does not take other consumers' decisions as given (our competitive assumption), but rather believes they will choose decisions that are the least favorable ones for him. In game theoretic language, a consumer in these models is assumed to choose "minimax" decisions, whereas our competitive assumptions imply Nash equilibrium decisions are chosen. It can be shown that if consumers behave competitively in the Drèze-Vallée Poussin and Malinvaud models, then an equilibrium under their rules is not generally Pareto optimal (c.f. Groves and Ledyard [15, Section II. F, Example 2.4]).

In the second body of literature related to this paper are papers of Hurwicz [18] and Ledyard and Roberts [20]. These papers contain a theorem (proved by Hurwicz for pure exchange economies with private goods only and by Ledyard and Roberts for economies with public goods) stating that there exists in general no resource allocation mechanism that yields "individually rational" Pareto-optima which is also "individually incentive compatible" for all agents.[36/] Our results, of course, do not contradict this theorem; essentially their theorem implies that competitive behavior (which we assume is followed) is not optimal behavior. Just as in a finite agent Arrow-Debreu economy a sophisticated consumer can gain by considering how prices and his profit shares are affected by his own demand; in our public goods model, under our mechanism, a sophisticated consumer can gain by considering how equilibrium prices, his profit shares, and the other consumers' messages are affected by his own decisions.

We have made the competitive behavior assumption not only because it permits us to prove positive results, but also because it is consistent with the fundamental welfare theorems of economics and the implicit assumptions of the verbal literature on the Free Rider Problem.

The third body of related literature consists of papers developing and applying a class of incentive mechanisms for inducing agents to communicate truthfully to a central agent and thereby enabling him to take optimal or efficient decisions. The allocation and tax rules presented in section III, equations (3.1), are based on this class of mechanisms.

The first formulation of one of these incentive mechanisms was by Wm. Vickrey [26] in 1961, who developed his mechanism as a procedure for an exclusive public marketing agency to engage in counterspeculation in dealing with monopolistic suppliers and monopsonistic buyers. Nearly a decade later E. Clarke [4] in 1971 and T. Groves [12, 13] in 1969, independently rediscovered such mechanisms. Clarke developed a particular example of these mechanisms in a partial equilibrium model for determining the optimal quantity of a public good under the restrictive assumption that the income elasticity of demand for the public good is identically zero.[37/]

Groves developed and formulated analytically the entire class of these optimal incentive mechanisms in the context of general team decision models. His mechanisms were developed to provide a method of evaluating decentralized decision makers (e.g. divisional managers of a large firm) that would induce them to behave as team members. Groves [14] and Groves and Loeb [17] later applied these mechanisms to the problems of choosing optimal decisions in production models when externalities are present and choosing optimal levels of public inputs in production.

The models discussed in all of these papers are partial equilibrium models in which payoffs to the different decision makers can be directly compared and freely transferred. In the language of game theory, the models are n-person non-cooperative games with freely transferrable utility. For this group of models Green and Laffont [11] have shown that the class of incentive mechanisms formulated by Groves includes (up to an isomorphism) every possible deterministic incentive mechanism for inducing agents to report truthfully and thereby enabling optimal decisions to be made.

Our model and mechanism is to be distinguished from these others by (1) being a full general equilibrium model in which income effects are allowed (i.e. utility is not freely transferrable), and (2) by guaranteeing that the government's budget is always balanced, as is required by Walras Law for Pareto-optimality (c.f. Section IV.1). In the papers mentioned above, the government is unable to balance its budget, except in very special cases. Because of this difficulty, Vickrey viewed his mechanism as impractical although he did not recognize that the problem could be reduced in severity. For Groves's team models the agents payoffs are viewed as success indicators or evaluation measures and thus the budgetary imbalance is a purely accounting feature of the mechanism. For the production models of Groves and Loeb, a particular version of the mechanism was discussed which guaranteed the center (or government) a surplus. Clarke's version of the mechanism also guaranteed the government a surplus which he then assumed could be redistributed by "a truly lump-sum arrangement" [4, p. 29]. However, even under his restrictive assumptions of no income effects, no such deterministic lump-sum arrangement will exist for his mechanism.[38/] Thus his mechanism cannot yield Pareto-optimal resource allocations even when no income effects exist. Indeed, as is discussed in Section IV.1, we were forced to modify considerably the mechanism discussed in Section III just to ensure government budget balance without distorting individual incentives.[39/]

FOOTNOTES

1/ Our analysis is a static general equilibrium analysis in the same spirit as, for example, Debreu's Theory of Value [5]. Thus we do not suggest that it would be practical to implement directly the mechanism we propose. Indeed, the logical next step in the theoretical development of the mechanism would be to formulate an explicit adjustment process. Vernon Smith [24] has developed such an adjustment process for a simpler version of the mechanism discussed here which he has used in small group experiments. Under his adjustment process the incentives provided by the mechanism have led his experimental groups to the optimum rapidly.

2/ See Arrow and Debreu [2] or Debreu [5].

3/ We assume neither consumers nor the economy as a whole possess any initial endowments of public goods. This assumption could easily be relaxed.

4/ Throughout we use the notation $\langle x^i \rangle$ to denote the I-tuple $(x^1,\ldots,x^I)$ and similarly for $\langle z^j \rangle, \langle \theta^{ij} \rangle$, etc.

5/ It is possible to include under this formulation tax rules that depend on the level of public goods purchased. If $\hat{C}^i(y;m,s)$ is such a rule, simply let $C^i(m,s) = \hat{C}^i(y(m,s);m,s)$ where $y(\cdot)$ is the allocation rule.

6/ Throughout we use the notation:

$$m^{)i(} \equiv (m^1,\ldots,m^{i-1},m^{i+1},\ldots,m^I)$$

$$(m/\bar{m}^i) \equiv (m^1,\ldots,m^{i-1},\bar{m}^i,m^{i+1},\ldots,m^I).$$

7/ See, for example, Debreu [5]. Our notation is for Debreu's definition.

8/ See, for example, Debreu [5 , Theorems 6.3 and 6.4].

9/ A neo-classical economy is defined as an economy for which preferences (production transformations) are representable by twice differentiable strictly concave utility (convex production) functions $U^i(x^i,y)$ $(F^j(z^j) \leq 0)$ and for which a feasible allocation is Pareto-optimal if and only if:

(a) $U^i_\ell/U^i_1 = F^j_\ell/F^j_1$ for $\ell = 2,\ldots,L$ and all i and j,

and

(b) $\Sigma_i U^i_k/U^i_\ell = F^j_k/F^j_\ell$ for $k = 1,\ldots,K$; $\ell = 1,\ldots,L$ and all j

where $U^i_\ell \equiv (\partial U^i/\partial x^i_\ell)$ and $F^j_k = (\partial F^j/\partial z_{2k})$, etc.

10/ This is also a model of voluntary contributions since G^N may be alternatively defined by $M \equiv \mathbb{R}^K_+, y_k(m,s) = (\Sigma_i m^i_k)/q_k$, and $C^i(m,s) = \Sigma_k m^i_k$ without changing its properties. The message m^i_k, for example, would then be interpreted as the amount of unit of account which i contributes towards the purchase of k.

11/ An alternative, but equivalent, interpretation is that $m^i_k(y)$ is the "maximum price" i is willing to pay for an additional unit of k, given the public goods level y. Thus, $m^i_k(y)$ is simply the inverse (partial equilibrium) demand function of i for commodity k.

12/ An alternative definition results from letting $x^i(t^i, y)$ solve $\max_{x^i} U^i(x^i,y)$ subject to $p \cdot x^i \leq - t^i \cdot y + w^i$. Then $\overset{\circ}{m}{}^i(\cdot)$ is "true" if, for all $y \in \mathbb{R}^K_+$,

$MRS^i_{\ell k}[x^i(\overset{\circ}{m}{}^i(y),y),y] = \overset{\circ}{m}{}^i_k(y)/p_k$ for all ℓ,k.

13/ See Foley [8], for a definition of Lindahl equilibrium.

14/ See Foley [8 , Theorem, Section 6].

15/ This section is included to enable us to explain the economic content of the mechanism (government) we discuss in Section IV. By restricting the economies considered to be neo-classical, conventional marginal concepts can be used to explicate and interpret the mechanism's features.

16/ See Section V for a discussion of the literature in which these rules were developed and explored.

17/ Thus, the gradient of the message $m^i(\cdot)$, $\langle \partial m^i/\partial y_k \rangle_k$, is also interpretable as consumer i's inverse Marshallian demand surface. See footnote 11 and compare with the language of the Lindahl Government of Section II, Example 2.2.

18/ An alternative interpretation is based on the equivalent expression of the consumer tax rule:

$$C^i(m,s) = m^i[y(m,s)] - [\Sigma_h m^h(y(m,s)) - q \cdot y(m,s)] + R^i(m^{)i(},s).$$

Thus, the consumer is taxed exactly what he reports he is willing to pay minus a dividend equal to the total reported consumer surplus (plus a lump sum). Compare this to Drèze-Vallée Poussin [7] who use

$$C^i(m,s) = m^i[y(m,s)] - \alpha^i[\Sigma_h m^h(y(m,s)) - q \cdot y(m,s)]$$

where $\Sigma_i \alpha^i = 1$. It is shown in Groves and Ledyard [15] that a competitive equilibrium relative to these rules is generally non-optimal.

19/ For example $R^i(m^{)i(},s)$ might be a constant or be defined as

$$R^i(m^{)i(},s) = \Sigma_{h \neq i} m^h(\bar{y}) - q \cdot \bar{y}$$

where $\bar{y}$ is a fixed vector of public goods. Even though $m^h(\cdot)$ is function of y, $R^i(m^{)i(},s)$ is a scalar and does not depend on $m^i(\cdot)$.

20/ Since consumers are never locally satiated, at any price vector s, $p \cdot x^i + C^i(m,s) = w^i(s) \equiv p \cdot \omega^i + \Sigma_j \theta^{ij} \pi^j(s)$ where $\pi^j(s) = s \cdot z^j$, $(x^i, m^i) \in \delta^i(m)^i(,s)$ and $z^j \in \varphi^j(s)$. Thus,

$$\text{value of excess demand} = p \cdot [\Sigma_i (x^i - \omega^i) - \Sigma_j z_1^j] + q \cdot [y(m,s) - \Sigma_j z_2^j]$$

$$= q \cdot y(m,s) - \Sigma_i C^i(m,s).$$

Hence, for the value of excess demand to equal zero at s,

$$\Sigma_i C^i(m,s) = q \cdot y(m,s).$$

21/ This follows from a theorem of Hurwicz [19, Theorem 4, Part A]. Hurwicz's theorem proves that there exists no mechanism (Government) satisfying (4.1) such that if each consumer's preference maximizing message is independent of the other consumers' messages then the resulting allocation is Pareto-optimal. Now, under the government G^*, if consumers' preferences between after-tax income (which they spend on private goods) and public goods are quasi-linear in after-tax income (i.e. no "income effects" for public goods) then by a theorem of Groves [13, Theorem 1, or 14, Section 3] each consumer's true willingness to pay function $v^i(\cdot)$ (see Corollary 3.2) (which in this special case is independent of his after-tax income and hence all other consumers' messages) is his best message independent of the other consumers' messages. Since Hurwicz proves his theorem for this special class of preferences the result follows.

22/ The notation $(x-y)^2$ where x and y are vectors is the inner product and thus is a scalar:

$$(x-y)^2 \equiv \Sigma_j (x_j - y_j)^2$$

23/ Since the most desired bundle for consumer i may contain less of some public good than the aggregate amounts requested by the other consumers, we must permit negative messages. Hence, M was defined to be the entire space $\mathbb{R}^K$ rather than just the non-negative orthant.

24/ The proportionality factors α^i need not be constant. For each i, α^i may depend, for example, on the messages of others, $m^{)i(}$, and prices, s. The only restrictions are that (1) α^i cannot depend on m^i, and (2) $\Sigma_i \alpha^i = 1$.

25/ If the messages $\langle m^i \rangle$ were observations of I independently identically distributed random variables with mean μ and variance σ^2, then (a) $\mu^i = \mu(m^{)i(})$ is the uniform minimum variance unbiased estimator of μ given $m^{)i(}$, (b) both $\frac{I-1}{I}(m^i - \mu^i)^2$ and σ^{i^2} are unbiased estimators of σ^2, and (c) σ^{i^2} is the uniform minimum variance unbiased estimator of σ^2 given $m^{)i(}$. These observations suggest that a sampling approach to the determination of public goods allocation based on these allocation and tax rules might be worthwhile. For an example of research in this direction see Green and Laffont [10].

26/ This follows since $\Sigma_i \mu^i = \Sigma_i m^i$ which implies that

$$\frac{I-1}{I} \Sigma_i (m^i - \mu^i)^2 - \Sigma_i \sigma^{i^2} = 0.$$

27/ This phenomena has been encountered in general equilibrium models with transactions costs also; c.f. Foley [9].

28/ This is Arrow's "exceptional case" ; c.f. Arrow [1]

29/ At a competitive equilibrium, were one to exist, consumer i's budget constraint would have the normal vector $(p,\alpha^i q)$. Also in equilibrium each consumer must desire the same level of public goods. Thus, the consumers' marginal rates of substitution at the most desired level of public goods must be proportional to α^i, an unlikely occurrence for any prespecified constants α^i.

30/ The proof is similar to Foley's [8 , p. 68].

31/ A full analysis of existence of competitive equilibria relative to the Optimal government is contained in [16].

32/ Formally, for each i, for every $(x^i,y) \in \mathcal{X}^i$, there is some private good $\ell = 1,\ldots,L$ and $\varepsilon > 0$ such that

$$(x_1^i,\ldots,x_\ell^i+\varepsilon,\ldots,x_L^i,y) \succ_i (x^i,y)$$

33/ See, for example, Debreu [6].

34/ Formally, let $H = \{y \in \mathbb{R}_+^K \,|\, \alpha_i q\cdot(y + \frac{\alpha_i I}{2\gamma(I-1)} q) \leq w^i(s)$ for all s on the unit sphere S and all $i = 1,\ldots,I\}$ where $w^i(s) = p\cdot\omega^i + \Sigma_j \theta^{ij}\pi^j(s)$. Assume that if $(x^i,y) \in \mathcal{X}^i$ and $y \notin H$, then there exists a $\hat{y} \in H$, $\hat{y} < y$, such that $(x^i,\hat{y}) \succ_i (x^i,y)$.

35/ Specifically for all $y \notin H$, see footnote 33.

36/ These concepts are defined in Hurwicz [18].

37/ Tideman and Tullock [25] have recently explained and extended Clarke's work to a wider class of problems. However, their models are also partial equilibrium models and they also assume no income effects in the demand for public goods.

38/ See footnote 21 and discussion in the text.'

39/ The Hurwicz impossibility result, c.f. footnote 21, does not apply to our mechanism (4.3) since in the no income effects environments, a consumer's best replay message depends on the other consumers' messages. He is not allowed to send his "true" marginal valuation function, but is forced to approximate it.

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