Vote Buying I: General Elections

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Abstract

We examine the consequences of vote buying, assuming this practice were allowed and free of stigma. Two parties compete in a binary election and may purchase votes in a sequential bidding game via up-front binding payments and/or campaign promises (platforms) that are contingent upon the outcome of the election. We analyze the role of the parties' and voters' preferences in determining the winner and the payments to voters.

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1 Introduction

The practice of vote buying appears in many societies and organizations, and in different forms. Obvious examples include direct payments to voters, donations to a legislator's

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campaign by special interest groups, the buying of the voting shares of a stock, and the promise of specific programs or payments to voters conditional on the election of a candidate. Our purpose here is to explore the consequences of vote buying. The aim is both to enhance the understanding of those forms of vote buying that are widely practiced, such as making campaign promises, and to shed light on the hypothetical question of what might happen if vote buying were allowed where it is currently prohibited. The latter question can of course help us think about the rationale behind current social conventions. To do so we study how vote buying would function in an environment in which it is allowed and free of stigma.

We inquire how voters' preferences over outcomes and parties' valuations of winning affect the outcome of the election; how the institutional environment—whether parties can purchase votes with up-front payments or can only make campaign promises—affects the outcome; and how vote buying affects the efficiency of the outcome.

We address these questions using the following model. We initially focus on a complete-information environment, but later allow for some incomplete information. There is a finite population of voters choosing between two competing parties. Each of the parties has a value for winning and is interested in obtaining a majority of the votes while spending as little as possible. We examine two scenarios: one in which the parties only compete in campaign promises (that are contingent upon the outcome of the election, but not upon the actual vote); and the other where parties compete in up-front vote buying (where the payment is contingent on the vote, but not on the outcome). In both scenarios the parties make offers in a sequential and alternating bidding process. Although voters are not formally modeled as players, their assumed behavior is motivated by considerations of utility maximization.

The answers to the first two questions raised above are intertwined. The identity of the winning party and the distribution of payments to voters depend not just on voter preferences and party valuations, but also critically on whether up-front vote buying is permitted or only campaign promises are allowed. When parties compete only through campaign promises, the total payments received by voters tend to be substantially higher than under up-front vote buying. Moreover, when parties compete only through campaign promises, the voters whose preferences matter are a specific subset of the voters near the median voter.

Both these features are broadly consistent with the analysis of Anderson and Tollison (1990) who claim that vote buying was wide spread (though never fully legal) in Britain and the USA prior to the introduction of secret ballots towards the end of the nineteenth and beginning of twentieth centuries. They claim that when vote buying occurred the sums involved were quite small. Moreover, they argue that the elimination of vote buying contributed to the historical rise in government expenditures on redistributive policies. The low payments in up-front vote buying also seem consistent with the observation that the price of stocks with voting rights is generally similar to that of non-voting stocks (Lamont and Thaler (2001)).

The answer to the efficiency question is that with no vote buying, with campaign promises or with up-front vote buying, the outcome could be Pareto efficient or inefficient. This independence of efficiency from the trading environment follows since in all three situations voters' preferences are not fully accounted for in determining the winner of the election.

There are several lines of related literature: the study of Colonel Blotto games (e.g., Laslier and Picard (2002)); the political science literature on lobbying (e.g., Groseclose and Snyder (1996)), common agency (Bernheim and Whinston (1986)), campaign promises (e.g., Lindbeck and Weibull (1987), Myerson (1993)), and vote buying (e.g., Buchanan and Tullock (1962), Anderson and Tollison (1990), Piketty (1994)); and the finance literature on corporate control and takeover battles (e.g., Grossman and Hart (1988), Harris and Raviv (1988)). We also have a companion paper (Dekel, Jackson, and Wolinsky (2006a)), with a related but distinct model. Discussing how our conclusions relate to those in the literature will be easier after the presentation of our model and results, so we defer this discussion to Section 5.6.

2 A Model of Vote Buying

Two "parties," X and Y, compete in an election with an odd number, N, of voters. We may think of the parties as candidates in the election or supporters of two alternatives. A party needs $m = (N+1)/2$ votes to win the election. Prior to the election the parties try to influence the voting by offering money payments to voters. Each voter i is characterized by parameters U_i^X and U_i^Y that are interpreted as the utility she obtains from a victory of X and Y respectively. Let $U_i = U_i^X - U_i^Y$ and label voters so that U_i is non-increasing in i. Under this labeling, we refer to voter m as the median voter and, without loss of generality, suppose that voter m is a supporter of party X ($U_m > 0$). There is a smallest money unit $\varepsilon > 0$, so offers can only be made in multiples of ε. To avoid dealing with ties, it is assumed that U_i is not an integer multiple of ε.

2.1 The Vote Buying Games

We consider two types of offers that parties can make to voters.

(1) Up-front payments: a binding agreement that gives the party full control of the vote in exchange for an up-front payment to the voter.

(2) Campaign promises: a promise that has to be honored by the party if it is elected; the voter maintains control of the vote.[1]

The parties alternate in making offers. In the up-front buying game, party k announces in its turn an offer to buy up to m votes at price p^k; in the campaign promises game party k announces campaign promises $c_i^k \geq 0$ to be given to voter i if k is elected.

These games share the following common features.

- A fresh price offer (or a promise) made to a voter cannot be lower than those previously made by the same party to the same voter.
- When a party moves, it observes all past offers and promises by each party to each voter.
- The bidding process ends when two consecutive offers (one by X and one by Y) go by without any change in who would win if the game ended in those rounds.
- Once the bidding process ends, voters tender their votes to the parties and the party that collects more than half the votes wins.

Voters are not modeled as players. In the description of each of the games below we make direct assumptions on how voters tender their votes given their preferences and the final bids they face.

Party k has a utility W^k for winning, so party k's (net) payoff is the probability of k winning times W^k less the total payments by k to voters. To avoid dealing with ties, it is assumed that W^k is not an integer multiple of ε.

Thus, the parties' payoffs are modeled as are bidders' payoffs in an auction. This corresponds to a view that control of the government is an economic asset and that political competition is a contest of profit maximizers to obtain this asset at minimal

[1] Thus in case (1) payments are contingent on the individual's vote but not on the outcome of the election, and under (2) the opposite holds. There are other possibilities, like having the payments be contingent on both, which we do not analyze.

cost. This is a somewhat stark view of political competition, similar in flavor to Downsian models, but provides a useful path through the problem.

We focus on the complete information version of the games where the parties' and the voters' preferences are known to the parties when they bid. In order to identify robust conclusions we also consider the case where budgets are imperfectly known. Strategies for the parties are defined in the obvious way in each case. In the complete-information game we study subgame-perfect equilibrium; we discuss the solution concept for the incomplete-information case when we apply it below.

3 Campaign Promises

We begin by studying the game where only campaign promises are permitted. Here party k's net payoff is $W^k - \sum_i c_i^k$ if k wins having made promises $(c_1^k, \ldots, c_N^k)$ to the voters, 0 if k loses, and $-\infty$ if the game never ends.

We assume that voter i will vote for X if and only if $c_i^X + U_i^X > c_i^Y + U_i^Y$, where (c_i^X, c_i^Y) are the final promises of the parties to voter i. Recall that $U_i = U_i^X - U_i^Y$ and let $n = |\{i : U_i > 0\}|$ be the number of a priori supporters of X. The analogous number for Y is simply $N - n$. Since $U_m > 0$ and U_i is nonincreasing in i, it follows that $m \leq n$. Given a number z, let $\lceil z \rceil^{\varepsilon}$ be the smallest multiple of ε greater than z, and let $\lfloor z \rfloor_{\varepsilon}$ be the largest multiple of ε smaller than z.

Let $\overline{U} = \sum_{i=m}^{n} \lceil U_i \rceil^{\varepsilon} > 0$ be the minimal sum that Y has to promise to voters in order to secure the support of a minimal majority, in case X does not promise anything. Thus $\overline{U}$ is one possible measure of the preference advantage that X enjoys over Y. In Figure 1 the solid line is $\lceil U_i \rceil^{\varepsilon}$, the line crosses the axis at n, the long vertical segment is at m, and the marked (red) area is $\overline{U}$.

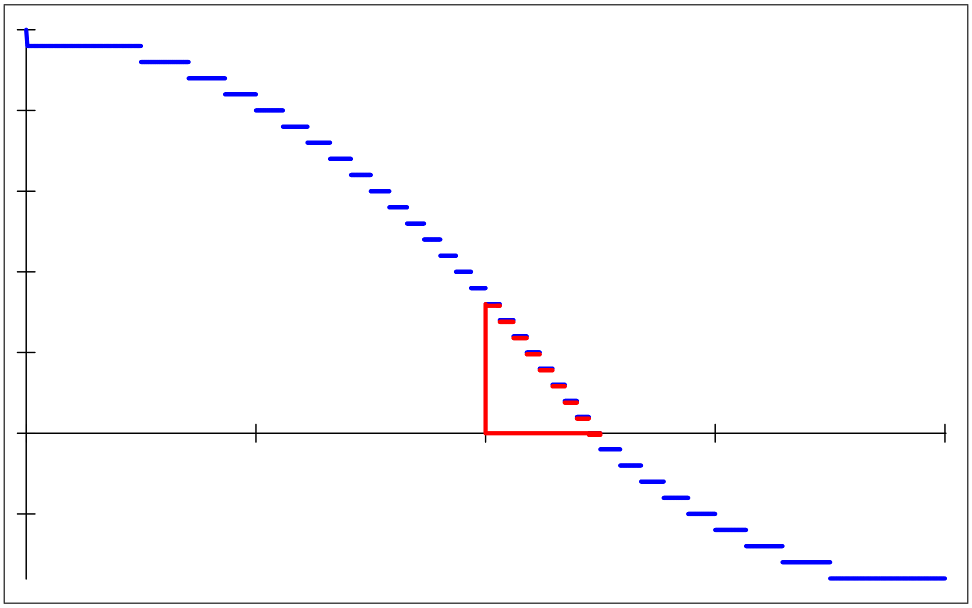

Figure 1: X's advantage in the campaign-promises game

Proposition 1 *There exists an equilibrium in the campaign-promises game. In any equilibrium Y wins if and only if $\lfloor W^Y \rfloor_\varepsilon \geq \lfloor W^X \rfloor_\varepsilon + \overline{U}$.*

The idea behind Proposition 1 is easily explained. Party Y must spend at least $\overline{U}$ in order to secure a majority. If the two parties were to compete, they would compete over the minimum-cost voters. The competition back and forth will lead to the winner being the party with the largest value once an expense of $\overline{U}$ has been incurred by Y.[2]

In this game there are many equilibria because the loser's behavior is not pinned down, as it is certain to lose and will not have to honor the promises it makes. Note however that strategies that prescribe quitting below, or bidding above, one's value only make sense if one is certain of the other party's value and behavior. Hence we introduce uncertainty over the parties' values and consider a refinement that selects what seems to be the natural outcome. The outcome on which we focus arises when parties use Least Expensive Majority (LEM) strategies, in which each party purchases the least expensive majority in turn, provided that their total commitment does not exceed their value. The identity of the winner would still be the same as above, but the total payment of the winner would be the loser's value adjusted by the magnitude $\overline{U}$, as spelled out in the proposition.

[2]Note that this characterization is easily extended to any voting rule, including ones that might be nonanonymous and/or nonneutral, and might include weights, veto players, or other special considerations. The critical calculation is the minimum expenditure that Y has to purchase in order to secure a winning vote, and so one can calculate a corresponding $\overline{U}$ for any voting rule.

The refinement we consider is "ex post perfect equilibrium:" a profile of strategies for each player (specifying a behavioral strategy for each realization of type) that form a subgame perfect equilibrium relative to any profile of realized types.[3] Given our use of subgame perfection in the complete-information game, this seems to be a natural refinement for the incomplete-information game. While it is clear that such equilibria might not always exist in general environments, they are very robust and compelling equilibria when they do exist, which they do in our setting.

The values of each party are distributed on a finite set $\mathcal{W}$. The difference between any two adjacent values in $\mathcal{W}$ is no more than ε, and $\mathcal{W}$ does not include integer multiples of ε.

Proposition **2** *Consider the campaign promises game with any full-support distribution over $\mathcal{W}$.*

1. *LEM strategies constitute an ex post perfect equilibrium.*
2. *In any ex post perfect equilibrium Y wins if $\lfloor W^Y \rfloor_\varepsilon \geq \lfloor W^X \rfloor_\varepsilon + \overline{U}$ and X wins otherwise.*
3. *In any ex post perfect equilibrium if Y wins then Y promises exactly $\lfloor W^X \rfloor_\varepsilon + \overline{U}$ and if X wins then X promises exactly $\max\{\lfloor W^Y \rfloor_\varepsilon - \overline{U} + \varepsilon, 0\}$.*
4. *In any ex post perfect equilibrium only voters between $\hat{m} = \{\min i : \lceil U_i \rceil^\varepsilon = \lceil U_m \rceil^\varepsilon\}$ and $\hat{n} = \{\min i : U_i > -\varepsilon\}$ can receive positive payments.*

Thus, in ex post perfect equilibria, the loser promises an amount equal to its value to a subset of the "near median" voters between $\hat{m}$, the first voter with the median preferences, and $\hat{n}$, the last voter whose preference for Y is marginal, i.e., less than ε. The winner commits—also to voters in this group—the minimal sum required to beat the loser. This sum amounts to the value of the loser plus or minus the magnitude $\overline{U}$ according to whether the winner is X or Y. (If $W^Y < \overline{U}$ then any strategy by Y that involves promises amounting to less than W^Y is an LEM strategy, and no payments are made, although Y might still make promises.)

[3] We believe the result holds under much weaker assumptions (that is, for more general solution concepts) but have not been able to prove such a conjecture. We have been able to show that the result also holds if we instead use ex post Nash equilibrium where players do not use weakly dominated strategies. That is neither a stronger nor weaker solution than ex post perfect equilibrium.

While payments are concentrated among the voters between $\hat{m}$ and $\hat{n}$, the particulars of which voters get how much can differ across equilibria. For example, in one equilibrium using LEM strategies in a case where $W^Y > W^X + \overline{U}$, the final outcome is that party X ends up offering $\lfloor W^X \rfloor_\varepsilon$ to a single voter, say voter m, and party Y ends up winning by offering $\lceil U_m \rceil^\varepsilon + \lfloor W^X \rfloor_\varepsilon$ to that voter and $\lceil U_i \rceil^\varepsilon$ to all voters $i \in [m+1, n]$. This happens by having the parties repeatedly outbid each other by a minimal amount for voter m. In another equilibrium with LEM strategies, X's budget is spread equally over voters $i \in [m, n]$, and Y matches all those bids and tops them off by $\lceil U_i \rceil^\varepsilon$ to compensate for these voters' initial preference for X.

One of the main objectives of this paper is to compare the equilibrium under campaign promises as described by Proposition 2 with the equilibrium under up-front vote buying to be derived below. But the analysis of the present section also serves to complement the literature on campaign promises. Myerson (1993) considered a simultaneous move model of redistributive promises assuming symmetry among voters and between parties. The model above allows heterogeneity in the preferences of the parties and the voters and uses this heterogeneity to identify the winner, the magnitude of the promises, and the identity of the voters who benefit from them. As discussed further in Section 5.6, the richer insights are made possible by the assumptions that the parties' promises are made sequentially and cannot be withdrawn. (This enables us to circumvent the technical difficulties encountered by Myerson and the earlier literature on "Colonel Blotto" games.)

Finally, notice that, if there was only one voter, the campaign-promises game would be an English auction in which the seller has a known preference for one buyer over the other. With many voters, this analogy is not exact, but the model and analysis still resemble those of the English auction, where the competition is over the “marginal” voters (the least expensive voters whom the party that is initially losing would have to buy in order to win the election). The equilibrium in LEM strategies is the counterpart of the standard equilibrium in undominated strategies of the English auction.

4 Up-Front Vote Buying

We now consider the situation where up-front vote buying is permitted. In this game each firm in its turn offers a price p^k that constitutes a commitment to buy up to m votes at this price. Again, voters are not formally modeled as players in this game. Instead, it is assumed that, once the bidding ends, all voters try first to tender their votes to the highest bidder and those who are rationed by the winner tender their votes to the loser.

Thus, if $p^X > p^Y$ at the end of the bidding, X ends up getting the minimal majority of m voters that it needs at p^X per vote, while the remaining $N - m$ voters who are rationed out by X sell to Y at p^Y per vote. If when the bidding is over $p^X = p^Y$, the ties are broken using the voters' fundamental preferences captured by the parameters U_i^X and U_i^Y: if $U_i^X > U_i^Y$, voter i will try first to tender to X. Party k's net payoff is then $W^k - mp^k$ if k wins, $-(N - m)p^k$ if k loses, and $-\infty$ if the game never ends.

This is somewhat artificial. Besides assuming that the parties' offers are commitments that can only be increased—assumptions that are shared with the campaign promises model—the up-front buying model embodies a number of additional assumptions. First, the voters try to sell at the higher price ignoring their potential of being pivotal. Second, the parties make the same restricted offers to all voters. Third, voters wait to the end of the bidding process before tendering their votes.

The main purpose of adopting this model is to simplify the analysis. Consider first the decision to assume away pivot considerations. Since U_i^X and U_i^Y are the utility that i obtains from a victory of X and Y respectively, then a strategic voter i would compare

$$\begin{aligned} &p^X + \Pr(X \text{ wins} \mid \text{tender to } X)\, U_i^X + \Pr(Y \text{ wins} \mid \text{tender to } X)\, U_i^Y \\ \text{with } \; &p^Y + \Pr(X \text{ wins} \mid \text{tender to } Y)\, U_i^X + \Pr(Y \text{ wins} \mid \text{tender to } Y)\, U_i^Y \end{aligned} \tag{1}$$

and try to sell to X if the former expression is larger than the latter. Note that the probability of being pivotal is $\Pr(X \text{ wins} \mid \text{tender to } X) - \Pr(X \text{ wins} \mid \text{tender to } Y) = \Pr(Y \text{ wins} \mid \text{tender to } Y) - \Pr(Y \text{ wins} \mid \text{tender to } X)$. If this probability is negligible, then (1) reduces to a comparison between p^X and p^Y. Thus, the assumption that voters try to sell to the highest bidder is a simple way of encapsulating the assumption that endogenous pivot probabilities do not play an important role in the situations we would like to consider. We explain this further in Section 5.3 by arguing that in a more complete model pivot considerations are inconsequential in this setting even when voters are fully strategic.

The assumption that the parties make uniform restricted-price offers is not made in our companion paper, Dekel, Jackson and Wolinsky (2006a), where—among other differences in the modeling—the parties make direct offers to individuals rather than announce a uniform price. In that model the assumption that voters wait for the end is more compelling as it is weakly dominant for them to do so. The alternative model yields the same insights as the present one but is more complex to analyze and requires adding a (negligible) per-period bidding cost. If we incorporated such a cost throughout this paper it would complicate the results of the campaign-promises model. Nevertheless, it is worth noting that such a model (with the bidding costs) would yield the same conclusion as below.

Proposition 3 *In the uniform-offer up-front vote-buying game, if* $W^j > W^k + (m+1)\,\varepsilon$*,* $j \neq k$*, party* j *wins in (every) equilibrium and* j*'s total payments are bounded above by* $m\varepsilon$*.*

Proposition 3 says that, modulo some epsilons, the party with the higher value wins and makes negligible payments to voters. In contrast, when the competition between the parties is restricted to campaign promises, the voters' preferences have a direct effect on the outcome and some near-median voters might get substantial transfers. In a sense this confirms a popular view that vote buying would give more power to the powerful and not benefit the masses in comparison with competition via campaign promises.

Notice that the up-front buying model is closely related to an all-pay auction: any outstanding promises must be paid regardless of whether a party ends up winning. It is not exactly an all-pay auction since the winner pays m times the last price it offered, while the loser pays $(m-1)$ times the last price it offered. In contrast, campaign promises are not binding unless a candidate wins, and hence the interaction there resembles an English auction instead of an all-pay auction. Thus, when the competition is through up-front buying, it is not worthwhile for a party to make substantial offers if it is unlikely to win, but when the competition is through campaign promises it is worthwhile to bid even when the probability of winning is small.

4.1 Up-front buying with incomplete information about parties' values

In the campaign-promises game we identified a subset of equilibrium strategies that were robust. Here we show that the equilibrium outcome of Proposition 3 is robust to the introduction of some (small) uncertainty about the values. To see this consider the up-front buying game under the assumption that the parties are uncertain about the valuations of the other party. That is, W^k is now private information of party k. We show that, when there is sufficiently "little" incomplete information, there is a perfect Bayesian equilibrium (PBE) outcome that is close to the complete-information outcome.[4]

[4] The equilibrium we construct is not an ex-post perfect equilibrium. In the games with up-front vote buying, parties prefer not to make any payments if they lose; which leads to different strategic properties than in the campaign promises case where losers never have to make any payments.

PROPOSITION **4** *Assume that the W^k's are independent, that they have a common finite support, that* $\Pr\left(W^k = \tilde{W}^k\right) \geq 1-\eta$, *and that* $\tilde{W}^Y > \tilde{W}^X + (m+1)\varepsilon$. *For any* $\delta > 0$, *there exists* $\eta(\delta) > 0$ *such that if* $\eta < \eta(\delta)$, *then there is a PBE where players only use undominated strategies with an outcome that coincides with the complete information outcome (i.e.,* Y *wins paying no more than* $m\varepsilon$*) with probability at least* δ.

We believe this equilibrium is not unique. Since we are interested in robustness, rather than finding additional equilibria that disappear when there is complete information, we have not verified this conjecture.[5]

5 Discussion

5.1 Insights

The main insights of our analysis can be summarized as follows. First, with campaign promises, the party with the highest value, adjusted by the voters' preferences measure $\overline{U}$, wins and pays out the second highest value, subject to the same adjustment, to a group of "near median" voters. Second, with up-front vote-buying and no uncertainty, there will be only minimal spending in equilibrium. Third, our analysis highlights some important differences between competition through up-front vote buying and through campaign promises, both in terms of the expected cost of winning and in the determination of the winner. The outcome of competition in campaign promises is affected by the preferences of the voters and might involve substantial transfers to the voters, whereas the outcome of up-front vote buying is not affected by the voters' preferences and the voters receive only minimal transfers.[6]

As mentioned in the introduction, all of these features are broadly consistent with descriptive work on vote buying. Anderson and Tollison (1990) claim that during a period when vote buying was common the payments were small, and the elimination of vote buying led to an increase in redistributive policies. Low payments in up-front vote

[5] We do know that there are multiple equilibria in the context of a straight all-pay auction with jump bids permitted (see Dekel, Jackson and Wolinsky (2006b)). However, the two models differ sufficiently so that it is not trivial to check whether the multiplicity translates. (The winner here pays m times the price it offered, while the loser pays only $m-1$ times the price it offered, the bidding here is restricted to a grid, and the construction here is of PBE, rather than sequential equilibria.)

[6] These relatively sharp insights are facilitated by modeling assumptions that have been discussed in the paper, including the sequential bidding with offers that cannot be withdrawn and the (almost) complete information with regard to the parties' valuations.

buying also seem consistent with the observation that the price of stocks with voting rights is generally similar to that of non-voting stocks (Lamont and Thaler (2001)).

5.2 Efficiency

In the absence of any mechanism for buying and selling votes, the outcome of voting will in general be inefficient. There is simply nothing to make voters take into account the effect of their vote on others. A natural hypothesis then might be that the opening of trade will lead to efficient outcomes. Our analysis shows that the outcome of a vote-buying equilibrium is in general inefficient. In the up-front buying scenario only the parties' valuations matter: If voters strongly support X, but W^Y is larger than W^X, Y still wins. In the campaign-promise scenario only the preferences of voters near the median group affect the outcome, and hence, the outcome does not reflect the preferences of all voters.

Under what circumstances will vote buying result in efficiency? In the up-front vote buying game the equilibrium will be (approximately) efficient if the parties' valuations are proportional to the true surpluses. That is, if W^X stands in the same proportion to $\sum_i \lceil U_i \rceil^{\varepsilon}$ as W^Y is in proportion to $\sum_i \lceil -U_i \rceil^{\varepsilon}$. This would be the case if the party's valuation perfectly aggregated the values of its supporters.[7]

More fundamentally, the main source of inefficiency is that the voters are not pivotal with respect to the decision.[8] A non-pivotal voter will sell her vote regardless of how she values the parties. Hence, it is clear that vote buying cannot take such a person's preferences into account and thus would not be efficient.[9]

Does vote buying and selling entail greater welfare loss than would occur in its absence? It is easy to construct examples that generate higher or lower overall utility with

[7] For example, we could consider a stage taking place before the vote-buying game, where voters could contribute to the two parties. The vote-buying game would then be one where the parties can spend up to the budgets at their disposal (where budgets substitute for valuations, as discussed below). For certain specifications of such a contribution game, there exist some equilibria where the winning party would be the one whose supporters had a higher total valuation (following a logic similar to that behind the results of Bernheim and Whinston (1986)).

[8] Piketty (1994) presents an example illustrating a different sort of inefficiency that can emerge in vote-trading environments. His point is that when there are private signals about common values, voters may fail to account for the informational externalities concerning lost information when they sell their vote.

[9] Buchanan and Tullock (1960) and Neeman (1999) make the point that, if decisions require unanimity, then vote trading could lead to efficiency, since then every voter is pivotal.

vote buying than with campaign promises or with neither. What we learn from our model is that vote buying may lead to parties' valuations rather than voter preferences being the driving force that determine the winner. Thus, if we think of a party's valuation as reflecting the profit that a certain narrow group will derive from taking over the government, then the opening of vote trading will elevate the relative importance of such groups, but of course nothing can be said in general on whether these biases are likely to produce lower total utility than simple voting.

While it is natural to ask how vote buying and campaign promises fare in terms of efficiency, our goal was not to find a mechanism that yields efficiency. That mechanism design question is trivial in the context studied here, where the parties have complete information. Rather we wanted to take the voting as given, and explore the implications of permitting trade.

5.3 Voter Behavior

Assuming, in the up-front buying model, that voters sell to the party that offers the higher p^k is a short-cut that embodies the assumption that pivot probabilities play a negligible role. If the voters were modeled as players, who at the end of the bidding decide simultaneously to which party to tender, then the behavior that we have assumed—that everybody tries first to tender to the party that offers the higher price—will still be an equilibrium behavior in the tendering subgame. But there might also be other equilibria that rely on pivot considerations. For example, there might be an equilibrium in the tendering subgame in which exactly $(N+1)/2$ voters tender to party j although $p^j < p^k$, since for each of these voters $U_i > p^k - p^j$. We think that pivot considerations of this sort are not truly important in the situations we would like to consider. In large elections there is inevitably sufficient noise to make the pivot probability of an individual voter insignificant. This can be modeled formally by introducing some "noise voters" into the model. The magnitude of such noise can be made small relative to the size of the electorate, hence leaving intact the essence of the analysis conducted above. At the same time, the noise can be significant enough to make the pivot probabilities negligible. To see this, suppose that, in addition to the N voters we consider, there are L noise voters each of whom votes randomly and independently with equal probability for each of the parties. Assume also that the L noise voters are not part of vote buying process. Let N and L be large, but L/N be very small. The large L implies a small pivot probability for each of the N voters who participate in the buying game. The small L/N implies that the analysis of the parties' competition over the N voters will be similar to the above analysis.

The bottom line is that we think that, for the purposes of our analysis, it is appropriate to abstract away from pivot considerations. We chose to do so in a straightforward way. As the preceding paragraph explains, this can be done in a more sophisticated way. However, if we were to adopt such an approach and carry it throughout, the complexity of the analysis would increase substantially without any gain in substance.

5.4 Budgets

Throughout the above analysis the parties were not subjected to budget constraints. We argue below that our main results have immediate analogs in the case in which the parties are constrained by budgets. Suppose that the parties have budgets B^X and B^Y, respectively. The constraint is that a party's offers at any point in the game are such that its liability if the game ended at that time would not exceed its respective budget. Let us retain the assumptions about parties' payoffs made above and assume that $B^X \leq W^X$ and $B^Y \leq W^Y$. That is, the parties are willing to spend up to their budgets in order to win but prefer spending less to more.

In the campaign promises case, the analog of Propositions 1 (and Proposition 2) are obtained by replacing W^k by B^k everywhere in the statements. That is, Y wins if and only if $B^Y \geq B^X + \overline{U}$ and with some uncertainty, the payments end up as in Proposition 2 and only the voters in the interval $[\hat{m}, \hat{n}]$ ever receive promises in equilibrium.

In the case of up-front buying, the analogs of Propositions 3 and 4 again hold with budgets replacing the valuations for winning, and thus the party with the larger budget (modulo some ε's) wins with a negligible total payment.

The proofs of these results are simpler than their counterparts without budget constraints, since the budget constraints together with the ε-grid bound the depth of the game tree.

Besides extending the basic model, the introduction of budget constraints also broadens the set of possible interpretations for the parties' motives. One may think of the parties as entities that seek to win at any cost up to the resource constraint that they face, irrespective of whether they are profit maximizers as we assumed earlier or are interested in winning for other reasons.

Notice however that if we introduce budget constraints, the meaning of the comparison between campaign promises and up-front buying is less clear than it was when the focus was on valuations alone as it was throughout the analysis. This is because it is not obvious that the same budget constraints should apply to these two scenarios, whereas it is natural to assume that parties' valuations are independent of the mode of competition.

5.5 Redistribution

We model the parties as self interested profit maximizers who seek to take control of the resources of the government at minimum cost. In questioning this view, one might argue that, even if the leaders of parties are out to appropriate resources for themselves and for a small group around them, the magnitude of the resources appropriated by such narrow groups is small relative to the total resources at stake and that redistribution of resources among broad groups of voters is a much more significant issue in electoral competition. Notice, however, that our approach does not conflict with this statement. First, the campaign promises and the up-front payments are instruments of distributing the resources of the government that the winner gets to control, independently of the actual motives of the parties. Second, our assumption that there are no negative transfers (taxes) and no minimum level of benefits that the government has to provide are just normalizations. It is possible to assume instead that there is a benchmark level of taxation that will be imposed on each voter and a benchmark level of benefits that the government must provide. The campaign promises and up-front payments of the model can then be viewed as modifications to those benchmark levels.

Notice, however, that the campaign promises model cannot be simply extended to allow the parties to commit to new taxes in each round along with their other promises, since the possibility of offsetting promises by taxes would violate the assumption that the parties' promises cannot be withdrawn. Given our monotonicity constraints on offers, one way to incorporate taxes is as described above, with a benchmark tax level that will be levied on any voter unless it is reduced by other promises. However, the monotonicity constraint may seem even less appealing in such a context.

5.6 Related literature

We discuss below three literatures that dealt with vote buying and relate them to our analysis.

5.6.1 Colonel Blotto Games

In a "Colonel Blotto Game" two opposing armies simultaneously allocate forces among n fronts. Any given front is won by the army that committed a larger force to that front and the overall winner is the army that wins a majority of the fronts. This model has been also interpreted as a model of electoral competition, where each party wins the voters to whom it made the larger promise and the overall winner of the election is the party that managed to win a majority of the votes (Gross and Wagner's (1950) continuous version of a Colonel Blotto game is perhaps the earliest contribution adopting this interpretation). A simultaneous version of our campaign promises game with budget constraints (as explained earlier in this section) is also a Colonel Blotto game.

The problem is that Colonel Blotto games are notoriously difficult to solve, even in the simplest settings[10]. The existing analyses are of symmetric mixed strategy equilibria in which voters are treated identically (from an ex-ante point of view) and the parties are equally likely to win.

Myerson (1993) circumvents some of the technical difficulties of Colonel Blotto games by allowing candidates to meet the budget constraint on average, rather than exactly. In particular, Myerson considers a simultaneous move game that is similar to the campaign promises game we analyze, but where parties' can offer random payments to each voter and the payments need only meet the budget in expectation. As in the previous Colonel Blotto literature, Myerson assumes voters and parties are symmetric, and derives a symmetric mixed strategy equilibrium in which parties exhaust their budgets.

Our work circumvents the technical difficulties of this literature by making the bidding sequential and irreversible (past promises cannot be withdrawn or lowered). While the irreversibility may not always apply, these features permit a rich analysis. This enables us to consider heterogeneous voters and parties and examine how such heterogeneity affects the outcomes.

5.6.2 Other vote buying models

Groseclose and Snyder (1996) present a model of vote buying in a legislature. Their model can be thought of as a two-round version of our campaign promises or our alternative up-front vote buying model.[11] The restriction to two rounds gives the second mover a substantial advantage. The first mover has to purchase a supermajority of voters in order to successfully block the response of the second mover. Thus, for example, if all voters were indifferent between the parties, the first mover would need to make promises

[10] See Laslier and Picard (2002), Szentes and Rosenthal (2003), and Weinstein (2005) for some characterizations of equilibria.

[11] Given that each party only moves once in their model, it is irrelevant to the outcome as to whether the game is in terms of up-front vote buying or campaian promises.

totalling twice the value (or budget) of the second mover in order to win, since the second mover should not be able to purchase the least expensive 50%. As is evident from the above analysis, our more symmetric bidding process neutralizes the affect of the order of moves and consequently gets significantly different results both with respect to the identity of the winner, how much they pay, which voters they buy, and the mode of competition (up-front vote buying or campaign promises).[12]

Another feature of vote buying in legislative settings that differs from that of general elections is that legislators may care (substantially) about how they cast their vote independent of the outcome. In a companion paper, Dekel, Jackson and Wolinsky (2006a), we analyze similar alternating-move vote-buying games to the ones analyzed here, but in contexts where voters care about how they cast their vote and not just about the eventual outcome. For instance, a legislator might strongly prefer to vote against a certain bill even if the bill is sure to pass, given that his or her constituents might pay attention to the legislator's voting record in future campaigns. This changes the behavior of legislators (voters) significantly vis a vis the analysis in this paper, and hence also has a substantial impact on the strategic interaction of the vote buyers. For instance, the up front vote buying game with complete information can involve substantial payments by the winner and the identity of the winner depends in a subtle way on both the buyer's willingness to pay and the voters' preferences. That contrasts sharply with the analysis of general elections in this paper. The companion paper also has a different focus—it studies the impact of budget constraints on vote buyers. We refer the interested reader to the companion paper for more details.

5.6.3 Corporate control

The related literature on corporate control (Harris and Raviv(1988), Grossman and Hart (1988)) examines settings in which two alternative management teams—an incumbent and a rival—are competing to gain control of a corporation through acquisition of a majority of the shareholders' votes. The alternative teams are the counterparts of our parties and their private benefits from controlling the corporation are the counterparts of the parties' valuations for being elected. The shareholders are the counterparts of our voters with a special form of identical preferences based on the difference in share

[12] Other articles that address similar issues are sufficiently distant in terms of their focus and framework to be considered largely complementary to our discussion, and it does not seem useful to try to relate them to our analysis. These include Baron (2001), Buchanan and Tullock (1962), Kochin and Kochin (1998), Philipson and Snyder (1996), and Tobin (1970).

value that will be generated under the two teams. The model of Harris and Raviv[13] (henceforth HR) resembles a two-round version of our up-front restricted price offers model. HR characterize an equilibrium where the efficient team wins; that is, the team that maximizes the total shareholder value plus its private benefit. This equilibrium relies critically on every voter believing that their tendering decision will be completely pivotal. In this sense the HR model takes an opposite view to ours. Whereas we assume away the pivot considerations on the grounds that they are marginal, these considerations are the central element of their model. Owing to this approach the HR equilibrium is very fragile in the sense that uncertainty about the number of shares, actions of other voters or offers, could destabilize it.[14] We believe the HR game has stable equilibria in which shareholders are not pivotal and the team with the larger private benefit wins.[15] These stable equilibria are the counterpart of the equilibrium we derive, except that the limitation to two rounds means that the price paid by the winner depends on whether it moves first or second (as in the analysis of Groseclose and Snyder (1996)).[16]

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[13] The related model of Grossman and Hart does not seem to have an explicit equilibrium model for the case that would be close to our model (what they call competition in restricted offers between parties with significant private benefits).

[14] Their model has a continuum of voters and so is not quite a closed game theoretic model. It appears that a large finite approximation to this equilibrium could be built, but the equilibrium would be unstable in that any shift in bidders' beliefs would lead to a change in their tendering strategies, and thus a movement to another equilibrium in the subgame (the one conjectured next in the text).

[15] These are equilibria that we conjecture, but are not mentioned by Harris and Raviv. We do not provide a formal analysis, as it would take a good deal of space to set up the model, for a relatively tangential point.

[16] Those equilibria do not exhibit the second mover advantage of the Groseclose and Snyder (1996) equilibria, since HR's model has restricted price offers while in the Groseclose and Snyder model the offers are targeted.

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7 Appendix

The appendix contains proofs of those results not proven in the main body of the paper.
PROPOSITION **1:** There exists an equilibrium in the campaign-promises game. In any equilibrium Y wins if and only if $\lfloor W^Y \rfloor_\varepsilon \geq \lfloor W^X \rfloor_\varepsilon + \overline{U}$.

Proof of Proposition 1: The proof is based on the following lemma in which we characterize the outcomes resulting when at least one player follows Least Expensive Majority (LEM) strategies. These are strategies such that each party in its turn acquires the least expensive majority so long as its total commitment does not exceed its value.

Let $C^{k\ell}$ denote the total promises made by party $k = X, Y$ up to some node ℓ of the game, and $\overline{U}^\ell > 0$ the minimal amount needed by Y to obtain a majority at that point. At the initial node $C^{j\ell} = 0$ and $\overline{U}^\ell = \overline{U}$.)

LEMMA **1** *:*

1. *If* $\lfloor W^Y \rfloor_\varepsilon - C^{Y\ell} \geq \lfloor W^X \rfloor_\varepsilon - C^{X\ell} + \overline{U}^\ell$ *and for* $k = X, Y$, $\lceil W^k \rceil_\varepsilon \geq C^{k\ell}$, *then*

   (a) *If X's strategy is LEM from* ℓ *onwards, then with an LEM strategy from* ℓ *onwards Y wins and spends* $\lfloor W^X \rfloor_\varepsilon - C^{X\ell} + C^{Y\ell} + \overline{U}^\ell$.

   (b) *If X's strategy is LEM from* ℓ *onwards, then to win Y must spend at least* $\lfloor W^X \rfloor_\varepsilon - C^{X\ell} + C^{Y\ell} + \overline{U}^\ell$.

   (c) *If Y's strategy is LEM from* ℓ *onwards then X cannot win without spending more than* W^X.

2. *If* $\lfloor W^Y \rfloor_\varepsilon - C^{Y\ell} < \lfloor W^X \rfloor_\varepsilon - C^{X\ell} + \overline{U}^\ell$ *and for* $k = X, Y$, $\lceil W^k \rceil_\varepsilon \geq C^{k\ell}$, *then*

(a) *If Y's strategy is LEM from ℓ onwards, then with an LEM strategy from ℓ onwards X wins and spends $\lfloor W^Y \rfloor_\varepsilon - C^{Y\ell} + C^{X\ell} + \overline{U}^\ell + \varepsilon$.*

(b) *If Y's strategy is LEM from ℓ onwards, then to win X must spend at least $\lfloor W^Y \rfloor_\varepsilon - C^{Y\ell} + C^{X\ell} - \overline{U}^\ell + \varepsilon$.*

(c) *If X's strategy is LEM from ℓ onwards, then Y cannot win without spending more than W^Y.*

Proof of Lemma 1: 1a and 2a follow immediately from the nature of the LEM strategies: Y initially must buy sufficiently many voters at cost $\overline{U}^\ell$ (the notion of "buying voters" stands here for making promises that would convince these voters to vote for the buying party if the bidding stops immediately after those promises were made); X then must buy one voter with an additional cost of ε; Y then must buy a voter back at additional cost ε; and so on. Iff $\lfloor W^Y \rfloor_\varepsilon - C^{Y\ell} \geq \lfloor W^X \rfloor_\varepsilon - C^{X\ell} + \overline{U}^\ell$ this process will reach a point where Y has promised not more than W^Y and in order stay in the game X has to increase its total outstanding promises to more than W^X and hence, by the hypothesis that X plays LEM, X stops.

1b is proved by induction on $\lfloor W^X \rfloor_\varepsilon$ as follows. By definition of $\overline{U}^\ell$, 1b is true for $\lfloor W^X \rfloor_\varepsilon = 0$ and any $C^{X\ell}$, $C^{Y\ell}$, and $\overline{U}^\ell$. Suppose it is true for $\lfloor W^X \rfloor_\varepsilon \leq K\varepsilon$ and for all $C^{X\ell}$, $C^{Y\ell}$, and $\overline{U}^\ell$, and consider $\lfloor W^X \rfloor_\varepsilon = (K+1)\varepsilon$. Let $\overline{U}$ be the sum promised by Y in its first move after ℓ. Clearly, $\overline{U} \geq \overline{U}^\ell$. Following its LEM strategy X promises some S such that $\varepsilon \leq S \leq \overline{U} - \overline{U}^\ell + \varepsilon$. After X's promise, at a node we denote by ℓ', we have $C^{Y\ell'} = C^{Y\ell} + \overline{U}$, $C^{X\ell'} = C^{X\ell} + S$, and $\overline{U}^{\ell'} = \varepsilon$. But this situation is equivalent to a configuration with $\overline{U}' = \varepsilon$, $C^{Y\ell'} = C^{Y\ell}$, $C^{X\ell'} = C^{X\ell}$ and with values $V'^Y = W^Y - \overline{U}$ and $V'^X = W^X - S \geq W^X - \left(\overline{U} - \overline{U}^\ell + \varepsilon\right)$. Since $\lfloor V'^X \rfloor_\varepsilon \leq K\varepsilon$, by the inductive assumption Y's overall expenditure will be at least $\lfloor V'^X \rfloor_\varepsilon - C^{X\ell} + C^{Y\ell} + \varepsilon + \overline{U}$. Now, this and $V'^X \geq W^X - \left(\overline{U} - \overline{U}^\ell + \varepsilon\right)$ imply that Y's overall expenditure is at least $\lfloor W^X \rfloor_\varepsilon - C^{X\ell} + C^{Y\ell} - \left(\overline{U} - \overline{U}^\ell + \varepsilon\right) + \varepsilon + \overline{U} = \lfloor W^X \rfloor_\varepsilon - C^{X\ell} + C^{Y\ell} + \bar{V}^\ell$.

For all $j = a, b, c$, Part $2j$ is the counterpart of $1j$. In particular, $2b$ is analogous to $1b$. Finally, $1c$ follows from $2b$. This completes the proof of the lemma.■

The existence of equilibrium follows from the following lemma.

Lemma **2** *LEM strategies for both parties constitute an equilibrium.*

Proof of Lemma 2: For $\lfloor W^Y \rfloor_\varepsilon \geq \lfloor W^X \rfloor_\varepsilon + \overline{U}$, $1a$ and $1b$ of Lemma 1 imply that Y's LEM strategy is best response against X's LEM strategy. $1c$ implies that X's

LEM strategy is best response against Y's LEM strategy. Analogously, $2a$–$2c$ of Lemma 1 imply that X's and Y's LEM strategies are mutual best responses when $\lfloor W^Y \rfloor_\varepsilon < \lfloor W^X \rfloor_\varepsilon + \overline{U}$. This demonstrates that LEM strategies constitute an equilibrium.■

To conclude the proof of Proposition 1, first observe that in any equilibrium there is a unique winner. To see this, suppose the contrary. Note that the equilibrium path only hits a finite number of nodes, as play will end in any subgame perfect equilibrium at any node where both players have made promises that exceed their values. Since there is not a unique winner, there must be a last node where some player mixes along the equilibrium path and is the winner along one path that follows and the loser along another path. Since a player's value is different from any level of payments that they could promise, the path that leads the player to be the winner must result in either a strictly positive or strictly negative payoff; while exiting results in a 0 payoff. This cannot be as the player will strictly prefer one of these pure outcomes. Next, note that in any subgame perfect equilibrium, no player will follow a strategy where they end up paying more than their value. Thus, by 1c and 2c and focusing on the initial node where $C^{k\ell} = 0$, Y can guarantee a win if $\lfloor W^Y \rfloor_\varepsilon \geq \lfloor W^X \rfloor_\varepsilon + \overline{U}$, and X can guarantee a win otherwise. Thus, given that the equilibrium is such that all equilibrium paths lead to the same winner, then the Proposition must hold, as then the player who has a strategy that guarantees a win against any subgame perfect equilibrium strategy of the other must have a positive utility and be the winner.■

PROPOSITION **2:** *Consider the campaign promises game with any full support distribution over values.*

1. *LEM strategies constitute an ex post perfect equilibrium.*

2. *In any ex post perfect equilibrium Y wins if $\lfloor W^Y \rfloor_\varepsilon \geq \lfloor W^X \rfloor_\varepsilon + \overline{U}$ and X wins otherwise.*

3. *In any ex post perfect equilibrium if Y wins then Y promises $\lfloor W^X \rfloor_\varepsilon + \overline{U}$ and if X wins then X promises* $\max\{\lfloor W^Y \rfloor_\varepsilon - \overline{U} + \varepsilon, 0\}$.

4. *In any ex post perfect equilibrium only voters between $\hat{m} = \{\min i : \lceil U_i \rceil^\varepsilon = \lceil U_m \rceil^\varepsilon\}$ and $\hat{n} = \{\min i : U_i > -\varepsilon\}$ can receive positive payments.*

Proof of Proposition 2:

Part 1 follows from Lemma 2.

Part 2 follows from the definition of ex post perfect equilibrium and Proposition 1,.

Part 3. Assume to the contrary that in some ex post perfect equilibrium Y wins and with some probability promises less than $\lfloor W^X \rfloor_\varepsilon + \overline{U}$, say $\bar{W}$. Consider then the case where Y with value W^Y such that $\lfloor W^Y \rfloor_\varepsilon = \bar{W}$ plays against X with value W^X. In an ex post perfect equilibrium, the strategies of Y with such a value against X with value W^X are an equilibrium of that complete information game, and Y loses with certainty. However, by mimicking the strategy of the higher type that wins (only up to any node where the promises do not exceed $\bar{W}$), Y would win with positive probability and never pay more than value and end up with strictly positive utility against X with value W^X. This is a contradiction.

Now assume to the contrary that in equilibrium Y with value W^Y wins and promises more than $\lfloor W^X \rfloor_\varepsilon + \overline{U}$, say $\hat{U}$. Consider the case where Y with value W^Y such that $\lfloor W^Y \rfloor_\varepsilon = \lfloor W^X \rfloor_\varepsilon + \overline{U}$ plays against X with value W^X. Note that in any equilibrium Y does not pay more than $\lfloor W^Y \rfloor_\varepsilon$. By Part 2 this W^Y wins and as was just noted it does not pay more than $\lfloor W^X \rfloor_\varepsilon + \overline{U}$. Thus, by mimicking W^Y against the strategy of W^X, type W^Y would win and pay $\lfloor W^X \rfloor_\varepsilon + \overline{U}$ which is less than what W^Y is paying in the supposed equilibrium (since Y is always paying at least this much by the above argument, and sometimes more by supposition), leading to a contradiction.

We now show *Part 4*.

Definition 1 *Assume the minimal amount needed for Y to obtain a majority is $\overline{U} > 0$ and it is Y's turn to make an offer. Y's offer in the amount $c > \overline{U}$ is wasteful if $c - \overline{U} > \overline{U}' - \varepsilon$ where $\overline{U}'$ is the minimal amount needed for X to obtain a majority after Y offered c.*

To understand this, note that an offer by Y can attain two objectives: achieving a majority, and increasing the amount $\overline{U}'$ that X will subsequently need to offer in order to obtain a majority. An offer is wasteful if it is greater than the minimal amount needed to achieve majority plus the amount by which it increases $\overline{U}'$. The definition of a wasteful offer by X is analogous.

We now show that a wasteful offer can only be made as the last offer in any ex post perfect equilibrium.

Lemma **3** *In an ex post perfect equilibrium no party ever makes a wasteful offer.*

Proof of Lemma 3: Assume to the contrary that the ex post perfect equilibrium strategies lead to Y or X making a wasteful offer at some penultimate stage in the game when Y has value W^Y and X has value W^X, with $\lfloor W^X \rfloor_\varepsilon \leq \lfloor W^Y \rfloor_\varepsilon - \overline{U}$. Assume that along this path Y is the first to make a wasteful offer. Such an offer must occur before X has offered $\lfloor W^X \rfloor_\varepsilon \leq \lfloor W^Y \rfloor_\varepsilon - \overline{U}$ (If not then the first wasteful offer of Y is made after X has offered $\lfloor W^X \rfloor_\varepsilon$ and hence after Y has offered $\lfloor W^X \rfloor_\varepsilon + \overline{U}$ and then, Y wins with a promise of more than $\lfloor W^X \rfloor_\varepsilon + \overline{U}$, contradicting part 3.) Now consider the case where Y faces an X with value W^X such that $\lfloor W^X \rfloor_\varepsilon = \lfloor W^Y \rfloor_\varepsilon - \overline{U}$. Such an X should lose against W^Y by part 2. But if W^X mimics W^X until Y makes the wasteful offer, and then continues with LEM, then according to lemma 1 X will win, leading to a contradiction. ■

We now continue with the proof of part 4. After any offer is made a new function describing the advantage that X holds over Y for each voter emerges. Specifically, given $U = (U_i)_{i=1}^N$, if X makes offers of c_i^X, then the new advantage of X over Y is given by U', where $U'_i = U_i + c_i^X$. We now clarify and develop further some aspects of the notation. U_m is the amount by which the median voter prefers X over Y, when each voter i prefers X over Y by U_i (and this U_i incorporates the basic preferences of i and the difference in promises that i has received up to but not including c_i^X). If the median voter under U is, say, voter 7, then U'_m is *not* necessarily $U_7 + c_7^X$, as under U' the preferences are different, and the median voter may change. So U'_m denotes the utility of the *new* median voter when each i prefers X over Y by the amount U'_i. Similarly, since the advantage that X holds over Y is changing, we replace the symbols of $\bar{n}$, $\hat{n}$ and $\hat{m}$ with the following functions for any U. Let $\tilde{U}$ be a reordering of U that is decreasing. Then $\hat{N}(U) = \left\{\max i : \tilde{U}_i > -\varepsilon\right\}$, $\hat{M}(U) = \left\{\min i : \left\lceil \tilde{U}_i \right\rceil^\varepsilon = \left\lceil \tilde{U}_m \right\rceil^\varepsilon\right\}$, and $\bar{N}(U) = \left\{\max i : \tilde{U}_i > 0\right\}$.

>From the lemma we know that no party makes a wasteful offer during the game. We now use the fact that no wasteful offers are made to deduce that offers are made only to voters between $\hat{m}$ and $\hat{n}$. If $U_m > 0$ then there are three basic possibilities. If it is X's turn, then X quits. If it is Y's turn, Y can make an ineffective offer, c_i^Y, so that it remains the case that the median voter prefers X, that is $U'_i = U_i - c_i^Y$ we have $U'_m > 0$, so that X wins. The third possibility is where Y makes an effective offer, so that $U'_m < 0$. In this case, if Y's offer is not wasteful then the following claims hold.

Claim **1** *Y makes positive offers, $c_i^Y > 0$, only to $\hat{N}(U) - m + 1$ voters, and each voter i receiving an offer satisfies $-\varepsilon \leq U_i \leq \lceil U_m \rceil^\varepsilon$.*

This implies that if $\lceil U_i \rceil^\varepsilon > \lceil U_m \rceil^\varepsilon$ then $c_i^Y = 0$, and hence if $\lceil U'_i \rceil^\varepsilon = \lceil U_i \rceil^\varepsilon - c_i^Y$ is reordered to be decreasing, these individuals remain before $\hat{M}(U)$. It also implies that if $U_i < -\varepsilon$ then $c_i^Y = 0$, and hence if $\lceil U_i' \rceil^\varepsilon = \lceil U_i \rceil^\varepsilon - c_i^Y$ is reordered to be decreasing, these individuals remain after $\hat{N}(U)$.

Claim **2** *If* $c_i^Y > 0$ *then* $\lceil U_m \rceil^\varepsilon \geq \lceil U_i' \rceil^\varepsilon = \lceil U_i \rceil^\varepsilon - c_i^Y \geq \max\{\lceil U_i \rceil^\varepsilon : U_i < -\varepsilon\}$.

The above two properties imply that for individuals who get positive offers the advantage of X before and after Y's offers is in the range $[-\varepsilon, \lceil U_m \rceil^\varepsilon]$. They also imply $\hat{M}(U') \geq \hat{M}(U)$ and $\hat{N}(U') \leq \hat{N}(U)$.

Proof of claim 1: If Y makes an offer c_i^Y to any i where $c_i^Y < \lceil U_i \rceil^\varepsilon$ then it is wasteful. This is because this voter continues to prefer X so does not increase the amount that X needs to spend to get a majority and doesn't help Y obtain a majority.

Making an offer to more than $\hat{N}(U) - m + 1$ voters is wasteful because X need not buy them all back and not all were needed to obtain a majority. Specifically, if instead Y did not make an offer to any one of them then the amount offered would decrease, but $\overline{U} + \overline{U}'$ (the amount required to obtain a majority by Y, plus the amount that subsequently X is forced to spend to obtain a majority) is unchanged.

If Y makes an offer to $\bar{\imath}$ with $U_{\bar{\imath}} > \lceil U_m \rceil^\varepsilon$, consider the alternative where instead Y makes the offer of $c_{i'}^Y = c_{\bar{\imath}}^Y - (\lceil U_{\bar{\imath}} \rceil^\varepsilon - \lceil U_m \rceil^\varepsilon) + \varepsilon \leq c_{\bar{\imath}}^Y$ to some other voter i' with $\lceil U_m \rceil^\varepsilon \geq U_{i'} > 0$ to whom Y was not making an offer (which exists by the preceding arguments). Then Y obtains a majority, and the amount that X is required to spend to obtain a majority increases. This increase is because X would have had to offer $\bar{\imath}$ an amount $c_{\bar{\imath}}^Y - U_{\bar{\imath}}$ and has to offer i' an amount $c_{i'}^Y - \lceil U_{i'} \rceil = c_{\bar{\imath}}^Y - (\lceil U_{\bar{\imath}} \rceil^\varepsilon - \lceil U_m \rceil^\varepsilon) + \varepsilon - \lceil U_{i'} \rceil^\varepsilon > c_{\bar{\imath}}^Y - \lceil U_{\bar{\imath}} \rceil^\varepsilon$. So the original offer to $\bar{\imath}$ was wasteful.

Lastly, making an offer to $\bar{\imath}$ with $U_{\bar{\imath}} < -\varepsilon$ is wasteful. That such an offer does not help Y obtain a majority is obvious. It also does not increase the subsequent cost to X in obtaining a majority. The minimal cost majority for X will not result in an offer to $\bar{\imath}$ unless $\lceil U_i \rceil^\varepsilon - c_i^Y \leq \lceil U_{\bar{\imath}} \rceil^\varepsilon - c_{\bar{\imath}}^Y$ for all i with $\lceil U_m \rceil^\varepsilon \geq \lceil U_i \rceil^\varepsilon \geq -\varepsilon$, since otherwise X can obtain a majority by promising less to other voters. But if c_i^Y is such that all i with $\lceil U_m \rceil^\varepsilon \geq \lceil U_i \rceil^\varepsilon \geq -\varepsilon$ are brought to $U_i' \leq \lceil U_{\bar{\imath}} \rceil^\varepsilon - c_{\bar{\imath}}^Y$, then $c_i^Y > 0$ for all such i. Now, if for some such i, say $\hat{\imath}$, $\lceil U_{\hat{\imath}} \rceil^\varepsilon - c_{\hat{\imath}}^Y < \lceil U_{\bar{\imath}} \rceil^\varepsilon - c_{\bar{\imath}}^Y$ then X does not make an offer to $\hat{\imath}$. In that case, if Y were to lower the offer to $\hat{\imath}$ to $c_{\hat{\imath}}'^Y$(so that $\lceil U_{\hat{\imath}} \rceil^\varepsilon - c_{\hat{\imath}}'^Y < \lceil U_{\bar{\imath}} \rceil^\varepsilon - c_{\bar{\imath}}^Y$) then the cost to X in obtaining a subsequent majority would not change, and the cost to Y would be lower. Hence in that case the offer is wasteful. If there is no such $\hat{\imath}$ then for all i with $\lceil U_m \rceil^\varepsilon \geq \lceil U_i \rceil^\varepsilon \geq -\varepsilon$ we have $\lceil U_i \rceil^\varepsilon - c_i^Y = \lceil U_{\bar{\imath}} \rceil^\varepsilon - c_{\bar{\imath}}^Y$. In that case not making the offer to $\bar{\imath}$ will not effect the amount X must offer to obtain a majority. $\Diamond$

Proof of claim 2: The first inequality follows because offers by Y decrease U and are only made to the set of voters that receive positive offers as characterized in claim 1. The second inequality follows because making an offer that leads to $U'_{\bar{\imath}} < \max\{\lceil U_i \rceil^{\varepsilon} : U_i < -\varepsilon\}$ implies that the least expensive way for X to obtain a majority will involve X not making an offer to $\bar{\imath}$ (as making an offer to $\arg\max\{\lceil U_i \rceil^{\varepsilon} : U_i < -\varepsilon\}$ is less expensive). But then if Y decreases the offer to $\bar{\imath}$ the cost of offers decreases with Y retaining the majority and no change in the minimal cost for X to subsequently obtain a majority. $\Diamond$

The above properties jointly imply that after a move by Y leading to U' from U, then a reordering of U' and U as decreasing functions has them coincide where either has values above $\lceil U_m \rceil^{\varepsilon}$ or below $-\varepsilon$, and hence when one is in between those values, so is the other. For similar reasons the same is true after a move by X, which implies that this holds throughout the process: the only offers are to individuals with values in the intermediate group. This concludes the proof of part 4, and hence of Proposition **2**. $\square$

PROPOSITION **3:** *In the uniform-offer up-front vote-buying game, if $W^Y > W^X + (m+1)\varepsilon$, $j \neq k = X, Y$, then party j wins in (every) equilibrium and j's total payments are bounded above by $m\varepsilon$.*

The proof is based on the following lemma.

LEMMA **4 :** *Consider a subgame starting with a move by party i. If i increases its standing offer with positive probability, then it must be that in the equilibrium continuation $j \neq i$ drops out with positive probability at the next node.*

Note that this implies that, in any equilibrium, the only node on the equilibrium path where the current bidder (if he has not won already) has a strictly positive expected payoff is at the first node. Note also, that if the bidding were to continue past the first node, it must involve mixing or dropping out completely at any subsequent node on the equilibrium path.

Proof of Lemma 4: Suppose to the contrary that j stays in at the next move for sure. Let us go to the first subsequent node where some agent drops out with positive probability (such a node exists as the value of the infinite play is negative infinity). That bidder must have 0 expected utility at that node. That node is reached with probability 1 on the continuation. If that bidder is i, then i has a negative expected utility conditional on making a bid now. If that bidder is j, then j has a negative expected utility conditional on making a bid at the next turn. Thus, we reach a contradiction in both cases.■

Proof of Proposition 3: Assume that $W^Y > W^X$, and let ε be sufficiently small to satisfy $W^Y > W^X + (m+1)\varepsilon$.

Let p^ℓ denote the last price offered by $\ell = X, Y$. We first prove that, if $p^Y > p^X$, then X quits. This is obviously true if $p^Y > W^X$. Suppose therefore that $p^Y < W^X$. The proof proceeds by induction as follows.

Observe that when $mp^Y > (m-1)p^X + W^X$, X quits since beating Y would require X to increase its commitment by $mp^Y - (m-1)p^X > W^X$.

We next establish that if X quits when $p^Y > p^X$ and $mp^Y > (m-1)p^X + k\varepsilon$, then X also quits when $p^Y > p^X$ and $mp^Y > (m-1)p^X + (k-1)\,\varepsilon$.

So, suppose that $p^Y > p^X$ and $mp^Y > (m-1)p^X + (k-1)\varepsilon$. Let q^ℓ denote the next price offer by party $\ell = X, Y$. Clearly,

$$mq^X \leq (m-1)p^X + W^X \tag{2}$$

as otherwise it is better for X to quit. Substituting from $mp^Y > (m-1)p^X + (k-1)\,\varepsilon$, it follows that

$$mq^X < mp^Y - (k-1)\,\varepsilon + W^X \tag{3}$$

or

$$mp^Y > mq^X - W^X + (k-1)\,\varepsilon. \tag{4}$$

Consider now a price offer q^Y that responds to q^X by increasing Y's total commitment by more than $W^Y - m\varepsilon$ but less than W^Y. That is, q^Y satisfies

$$(m-1)p^Y + W^Y - m\varepsilon \leq mq^Y < (m-1)p^Y + W^Y. \tag{5}$$

We have $mq^X \leq (m-1)p^X + W^X < (m-1)p^Y + W^Y - (m+1)\varepsilon \leq mq^Y$ and hence $q^Y > q^X$ (where the first inequality follows from (2), the second from the hypotheses $W^Y > W^X + (m+1)\varepsilon$ and $p^Y > p^X$, and the third from (5). We also have

$$\begin{aligned} mq^Y &\geq (m-1)p^Y + W^Y - m\varepsilon = mp^Y + W^Y - m\varepsilon - p^Y > mq^X - p^Y - W^X + (k-1)\,\varepsilon + W^Y - m\varepsilon \\ &\geq (m-1)q^X + (k-1)\,\varepsilon + \left(W^Y - W^X\right) - m\varepsilon > (m-1)q^X + k\varepsilon, \end{aligned}$$

where the second inequality follows from (4), the third follows from $q^X \geq p^Y$, and the fourth from the hypothesis $W^Y > W^X + (m+1)\varepsilon$. Now, by the inductive hypothesis $q^Y > q^X$ and $mq^Y > (m-1)q^X + k\varepsilon$ leads X to quit. This implies that X does not win with positive probability after q^X: if this were the case, then by Lemma 4 Y must quit with positive probability after q^X, but as we have just seen Y can do better. Thus, it has been established by induction that, if $p^Y > p^X$, then X quits.

In any equilibrium party X's first price offer must be no more than W^X/m. But, if party Y responds by offering $p^Y = p^X + \varepsilon$, it follows from the above that it will win, and this will be profitable for Y since $mp^Y = m(p^X + \varepsilon) \leq W^X + m\varepsilon < W^Y$. Therefore, if party X moves first, it will offer 0 in equilibrium. If Y moves first, it will offer price ε and X will not match.

The case of $W^Y < W^X$ is almost identical. $\square$

PROPOSITION 4: Assume that the W^k's are independent, that they have a common finite support, that $\Pr\left(W^k = \tilde{W}^k\right) \geq 1 - \eta$, and that $\tilde{W}^Y > \tilde{W}^X + (m+1)\,\varepsilon$. For any $\delta > 0$, there exists $\eta(\delta) > 0$ such that if $\eta < \eta(\delta)$, then there is an undominated PBE with an outcome that coincides with the complete information outcome (i.e., Y wins paying no more than $m\varepsilon$) with probability at least δ.

Proof of Proposition 4: Without loss of generality, suppose that X has to move first. Consider an auxiliary game in which X's initial price offers are restricted to be either 0 or $\bar{p}^X$ such that $m\bar{p}^X \geq \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$, and select a PBE of this game. Existence can be seen as follows. The only weakly dominated strategies for a type are to increase its bid by more than its value. So consider an extensive-form game with such strategies removed for all types. What remains is a finite extensive-form game, so a PBE exists. Consider extending a PBE of that game in any way to the original game (which only means describing continuation strategies at nodes precluded by one's own earlier actions). This will be an undominated PBE of the original game.

Construct now an undominated PBE for the full game as follows. If $W^X \leq \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$, then X yields immediately. If $W^X > \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$, then X follows the equilibrium strategy of the auxiliary game. If p^X is such that $mp^X < \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$, then Y's belief is that W^X is the maximum between W^X and the smallest value of U greater than mp^X, both of which are below $\tilde{W}^Y - \varepsilon$, and Y plays the same strategy it would play in the equilibrium of the complete information game against that type of X. If p^X is such that $mp^X \geq \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$, then Y's belief and strategy is the same as in the selected equilibrium of the auxiliary game. By construction, in either case Y's behavior is a best response to its beliefs. To establish that this is an equilibrium, it has to be shown that, if $W^X > \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$, it is not beneficial to X to deviate to $\underline{p}^X \geq \varepsilon$ such that $m\underline{p}^X < \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$. Observe that, after $\underline{p}^X$, X's payoff is at most

$$(1-\eta)(W^X - m\underline{p}^X - \tilde{W}^Y) + \eta W^X \leq (1-\eta)(W^X - m\varepsilon - \tilde{W}^Y) + \eta W^X \qquad (6)$$

since in the event $W = \tilde{W}^Y$, which occurs with probability $1 - \eta$, Y will continue after $\underline{p}^X$ under the belief that $W^X < \tilde{W}^Y - \varepsilon$ and in order to win X will have to increase its bid later by at least $\left\lfloor \tilde{W}^Y \right\rfloor_{\varepsilon}$.

If instead X offers $\left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} - m\varepsilon$, its payoff will be at least

$$(1-\eta)(W^X - \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} + m\varepsilon) + \eta(- \left\lceil \tilde{W}^Y \right\rceil^{\varepsilon} + m\varepsilon) \tag{7}$$

since in the event $W = \tilde{W}^Y$, X will win immediately.

Clearly When η is sufficiently small (7) is larger (6). Hence, the above construction is indeed an equilibrium. It is immediate that, η is sufficiently small, the equilibrium outcome is near the complete information outcome. □