Northwestern University

2001 Sheridan Road 580 Leverone Hall Evanston, IL 60208-2014 USA

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Kenneth L. Judd
Hoover Institution
Stanford University

Felix Kubler
Lehrstuhl für Wirtschaftstheorie
Universität Mannheim

Karl Schmedders
Kellogg School of Management
Northwestern University

www.kellogg.northwestern.edu/research/math

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Bond Portfolios and Two-Fund Separation in the Lucas Asset-Pricing Model*

Kenneth L. Judd
Hoover Institution
Stanford University
judd@hoover.stanford.edu

Felix Kubler
Lehrstuhl für Wirtschaftstheorie
Universität Mannheim
fkubler@rumms.uni-mannheim.de

Karl Schmedders
Kellogg School of Management
Northwestern University
k-schmedders@northwestern.edu

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Abstract

The two-fund separation theorem from static portfolio analysis generalizes to dynamic Lucas-style asset models only when a consol is present. If all bonds have finite maturity and do not span the consol, then equilibrium will deviate, often significantly, from two-fund separation even with the classical preference assumptions. Furthermore, equilibrium bond trading volume is unrealistically large, particularly for long-term bonds, and would be very costly in the presence of transaction costs. We demonstrate that investors choosing two-fund portfolios with bond ladders that approximately replicate consols do almost as well as traders with equilibrium investment strategies. This result is enhanced by adding bonds to the collection of assets even if they are not necessary for spanning. In light of these results, we argue that transaction cost considerations make portfolios using two-fund separation and bond laddering nearly optimal investment strategies.

Keywords: Dynamically complete markets, general equilibrium, consol, bonds, interest rate fluctuation, reinvestment risk, bond laddering.

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1 Introduction

How should investors allocate their wealth? Classic two-fund separation theory derives conditions under which they need only decide how to divide wealth between the safe asset and the market portfolio of risky assets. Even though real world portfolio decisions are made in a dynamic market, the static results are often applied, implicitly saying that the allocation between safe and risky assets should be constant over time for each investor; see, for example, Canner et al. (1997). We show that strong separation results from static portfolio theory do not generalize to dynamic Lucas-style asset market models. The mutual fund monetary separation theorems of static portfolio analysis hold in dynamic general equilibrium asset market models only when a consol is present, either explicitly or implicitly through dynamic trading of finite-maturity bonds. In the absence of a consol (or equivalent replication strategy) equilibrium portfolios will deviate from two-fund separation even with the classical preference assumptions. In particular, relative allocations of wealth across the risky assets may differ across investors because of the price risk associated with short-term bonds. We also find that the implied asset trading volume bears no relation to actual asset markets; instead, equilibrium positions and trading volume in bonds, particularly long-term bonds, are implausibly large in the absence of a consol. The Lucas asset model has some more sensible predictions as the number of bonds and their duration increase; in particular, the deviations from two-fund separation in equity positions disappear. However, the implications remain implausible for long-term bonds. These results show that standard equilibrium analysis of asset portfolios and trading cannot represent actual markets.

As an alternative, we consider a common bond investment strategy – the bond ladder – for investors in a Lucas-style asset market. A bond ladder strategy buys a fixed amount of the longest term bond in each period and holds it until maturity. The bond ladder is like a consol in that it creates a constant stream of revenue, but it differs from a consol since the cost of reestablishing the bond ladder in each period is risky due to price fluctuations in bonds. We find that bond ladders combined with a mutual fund of risky assets are an excellent alternative to the "equilibrium" investment strategy. However, the portfolio weights between the ladder and the mutual fund will differ from the weights between a consol and the mutual fund because of the risk in bond prices even when the time to maturity of the longest bond is similar to that observed in actual markets. In addition, we find a role for redundant bonds since adding more long-term bonds will improve the performance of bond ladder strategies even though the new bonds do not improve the span of the traded assets. Finally, while we do not explicitly model transaction costs we motivate the idea of bond ladders as a strong response to transaction costs. Equilibrium investment strategies imply enormous trading volume in the bond markets which would be very costly in the presence of transaction costs. On the contrary, bond ladders minimize transaction costs since the only transaction costs are those borne at the time the bonds are issued. Therefore, asset redundancy is desirable since it improves the performance of bond ladder strategies which in turn help investors economize on transaction costs. In summary, we conclude that conventional Lucas-style asset market models cannot represent actual portfolios and trading, but that the basic two-fund separation results are valid for dynamic environments when investors have access to – possibly redundant – long-term bonds that allow them to build bond ladders that approximately replicate a consol. We argue that such portfolio strategies are particularly sensible in the presence of transaction costs.

Beginning with Tobin (1958), many financial market models imply that portfolio allocation decisions can be reduced to a two-stage process: first decide the relative allocation of assets across the risky assets, and second decide how to divide total wealth between the risky assets and the safe asset. This is called *monetary separation*. The two-fund monetary separation result and its m-fund generalizations rely on special assumptions on either returns or tastes. In particular, Cass and Stiglitz (1970) have shown that two-fund monetary separation holds if an investor has HARA utility; more precisely, if an investor has HARA utility then his allocation of wealth in risky asset i relative to risky asset j does not depend on his total wealth. Separation theorems have strong implications for equilibrium. Rubinstein (1974) showed that if all investors have equi-cautious HARA utility, then all portfolios are, in equilibrium, a convex combination of the market portfolio of risky assets and the safe asset. The argument is simple: if all investors have equi-cautious HARA utility then each individual's consumption in any Pareto efficient allocation is a linear function of aggregate consumption, investors' demands for assets have linear income expansion paths, and asset demands can be aggregated.

These analyses use static models to prove two-fund separation theorems. Merton (1973) generalizes m-fund separation theory to dynamic contexts but only weakly. He assumes only intertemporally separable preferences and shows that at each moment each investor's portfolio will be a convex combination of a small number of mutual funds, as in Tobin (1958). However, the composition of those mutual funds may change and lead to substantial churning in portfolios. In this paper, we look for conditions under which mutual fund separation leads to far less churning, corresponding to the more common application of two-fund separation to real world investing.

Even though the classic two-fund separation theorems rely on assumptions about tastes, these assumptions are realistic enough that many argue that the implied investment policies are good ones to follow in general. For example, Canner et al. (1997) interpret two-fund separation to imply that investors should divide wealth among only stocks, bonds, and money, and do so in a time-invariant manner. They then note that many investment advisors violate the recommendations of two-fund separation, leading to an "asset allocation puzzle."

This paper shows that the two-fund monetary separation theorems generalize to dynamic general equilibrium models only under unrealistic assumptions about the bond market even assuming equi-cautious HARA investor utilities. We use the heterogeneous investor asset model of Judd et al. (2003), which is essentially a Lucas asset pricing model with heterogeneous investors and dynamically complete asset markets. The key result of Judd et al. is that no trader's portfolio changes in equilibrium; that is, after an initial period of trading, each investor holds a constant number of units of each asset over all times and states. Maturation of bonds will cause some trading. Judd et al. show that each investor has a fixed end-of-period holding of each t-year maturity bond; therefore, when a t-year bond becomes a $t-1$-year bond, some of those bonds are sold (or bought) to get back to the target portfolio. Since stocks have infinite maturity, there is no trading of stocks. This is a natural model to use to study monetary separation in dynamic economies since equilibrium already has a buy-and-hold nature. The remaining key issue is then whether the time-invariant equilibrium asset allocations are consistent with the predictions of static models implying monetary separation.

We show that the static monetary separation results hold in dynamic settings only if a consol can be constructed from the bonds. The intuition is clear. In our dynamic model, there is no change in equilibrium portfolios over time. So, if the span of the bonds includes a perfectly safe stream of payments, the dynamic problem reduces to the static problem. Otherwise, if bond trading cannot implement a consol, then the bond portion of an investor's portfolio will include some risk associated with fluctuating interest rates (bond prices), and the portfolio weights of the risky equity assets need to be altered to achieve the Pareto efficient allocation of consumption. This is particularly clear if there is only a one-period bond. An investor can use that bond to buy a sure delivery of one dollar in the next period, but bond price risk implies that next period's price of a one-period bond is uncertain, making it impossible to choose bond investment today that will deliver a dollar for sure two periods hence. The only way to hedge against the interest rate fluctuations is to alter the composition, not just the magnitude, of the stock portfolio. Since different investors have differing demands for a safe asset, those stock portfolio adjustments will be different for different investors. In some sense, portfolio separation still occurs. We can still construct two different mutual funds, one implementing the consol and one implementing the risky portion of investors' portfolios, and find that investors will always hold a convex combination of those two funds. While this general notion of portfolio separation holds, it is not the notion of monetary separation that many people have in mind when they talk about two-fund separation, see for example Canner et al. (1997) and Elton and Gruber (2000), where the conversation is about allocation between a safe asset and the market basket of risky assets.

We present examples showing that all investors will approximately hold the market portfolio of stocks if bonds with very long maturities are constantly issued. However, the presence of many bonds does not yield plausible equilibrium predictions; in particular, almost all reasonable specifications of tastes and endowment processes imply bond trading volume that is many orders of magnitude beyond anything sensible. The trading volumes we find are clearly irrational for any trader with even tiny transaction costs. This realization forces one to move away from conventional general equilibrium modeling and instead focus on the performance of simple strategies. In particular, we examine bond laddering strategies. A bond ladder is a portfolio with b units of a zero-coupon bond of each maturity. In each period, the maturing bonds will produce cash, and the laddering strategy takes some of those proceeds to reestablish the ladder by buying b units of the longest maturity bond, and consumes the rest. If the price of the longest maturity bond were constant then this laddering strategy would implement a consol. However, bond prices (and interest rates) will fluctuate and the cost of reinvestment will be risky for generic asset returns processes, particularly if the maximum maturity is small. To evaluate the rationality of bond laddering strategies, we look at models with long maturity bonds, such as the 30-year bonds we see in bond markets. We demonstrate that bond laddering strategies will be approximately optimal when the number of bonds of different maturities becomes sufficiently large.

While the results are mixed the pattern is clear. If there is a consol then we get the classic two-fund separation results. Without a consol, equilibrium portfolios will either have investors' equity portfolios deviating from the market basket or executing crazy bond trading strategies; in either case, equilibrium does not display the simple portfolio patterns like those in the static two-fund separation literature. However, when we take a pragmatic view of the problem, we find that when the market includes long maturity bonds, then two-fund separation is an approximately optimal response to equilibrium prices and is more plausible and more robust to transaction costs than the exact asset market equilibrium. Of course, all this depends critically on the HARA assumption for preferences. Future work will reexamine these problems when there is realistic heterogeneity in preferences.

Tobin (1958) first presented the two-fund idea in his analysis of portfolio demand in a mean-variance analysis with quadratic utility. Two-fund separation has been examined in the CAPM model of asset market equilibrium; see for example Black (1972). Cass and Stiglitz (1970) provide conditions on agents' preferences that ensure two-fund separation whereas Ross (1978) presents conditions on asset return distributions under which two-fund separation holds. Russell (1980) presents a unified approach of Cass and Stiglitz and Ross. Ingersoll (1987) provides a detailed overview of various separation results and highlights the distinction between restrictions on utility functions and restrictions on asset return distributions. See also Huang and Litzenberger (1988) for another textbook summary of portfolio separation theory. We stay away from analyses that rely on distributional assumptions about asset prices since we focus on equilibrium prices and portfolios, and there is no reason to believe that equilibrium asset prices fall into any of the special families that produce portfolio separation. We examine a Lucas-style asset market model where two-fund separation may be implied by investor preferences.

The numerous studies of portfolio allocations, and, for example, the many attempts to explain the asset allocation puzzle exposited in Canner et al. (1997), generally consider only one or two bonds and seldom pursue an equilibrium analysis. Our dynamic asset market model allows for a rich array of bonds, enabling us to stay away from results driven by the small number of assets and giving us a framework for examining bond ladder strategies in a dynamic model. Despite the popularity of bond ladders as a strategy for managing investments in fixed-income securities (for example, see Bohlin and Strickland, 2004), there is surprising little reference to this subject in the finance literature on modern portfolio theory. The aforementioned classical portfolio literature on two-fund separation, such as, among many others, Tobin (1958), Cass and Stiglitz (1970), and Black (1972), examines investors' portfolio decisions in one-period models, which by their very nature cannot examine bond ladders. To this day, and despite the early criticism by Merton (1973), the results of this static portfolio theory are often applied to dynamic contexts, see, for example, Canner et al. (1997) or Elton and Gruber (2000).

The last decade has seen a growing literature on optimal asset allocation in stochastic environments, but generally with partial equilibrium models and a small number of bonds. One string of this literature builds on the general dynamic continuous-time framework of Merton (1973) and assumes exogenously specified stochastic processes for stock returns or the interest rate. Recent examples of this literature include Brennan and Xia (2000) and Wachter (2003) among many other papers. A second string of literature uses discrete-time factor models to examine optimal asset allocation, see for example Campbell and Viceira (2001, 2002). Most of these papers focus on aspects of the optimal choice of the stock-bond-cash mix but do not examine the details of a stock or bond portfolio. A particular feature of these factor models is that only very few assets are needed for security markets to be complete. For example, the model of Brennan and Xia (2002) can exhibit complete security markets with only four securities, only two of which are bonds. Also Campbell and Viceira (2001) report computational results on portfolios with only 3-month and 10-year bonds. Because of the small number of bonds, the described portfolios in these models do not include bond ladders. Analyzing more bonds in these models would certainly be possible, but additional bonds would be redundant securities since markets are already complete. As a result there would be continua of optimal asset allocations and so any further analysis of particular bond portfolios would depend on quite arbitrary modeling choices. On the contrary, our model allows for a large number of exogenous states and so we can examine equilibria with a large number of non-redundant bonds. Furthermore, models with a small number of bonds make unrealistic assumptions about bond trading. Continuous-time models with only a short-term safe bond imply that bonds are created and liquidated at an infinite rate. Models that assume only 3-month and 10-year bonds imply strange trading behavior at the beginning of each month. The 3-month bonds purchased in the previous month become 2-month bonds; therefore, the investor will liquidate these bonds and buy 3-month bonds. Similarly for the 10-year bonds purchased in the previous month. This description of bond markets is unrealistic in many ways. Such trading behavior is costly in the presence of any transaction costs. Real bond markets have bonds of many maturities.

Recent papers on government financing (Angeletos, 2002, and Buera and Nicolini, 2004) have noted the strange portfolios implied by a fiscal authority's attempt to use bonds for hedging purposes. These models differ significantly from our analysis. Since these models assume a representative agent, the bond positions are large only because the government is using small fluctuations in bond prices to counter large exogenous shocks in government revenue needs. In our model, bond trading volumes are derived from individual investors' tastes and are orders of magnitude larger. Also, both papers restrict their attention to asset structures with only very few bonds. For example, Buera and Nicolini (2002) report numerical results for models with two and four bonds. And as a result they must allow for bonds with non-consecutive maturity structures that exhibit the problems we discussed above.

The remainder of this paper is organized as follows. Section 2 presents the basic dynamic general equilibrium asset market model and the classic assumptions on preferences. Section 3 discusses two-fund separation theory for our dynamic model, proves that the classic static result continues to hold when the safe asset is a consol, and argues that the classical result fails when only a short-maturity bond is available. In Section 4 we present numerous numerical examples, which motivate and guide our further analysis. In Section 5 we develop sufficient conditions for a small number of bonds of finite maturity to span the consol. In such economies portfolios exhibit two-fund separation for equity but bond portfolios typically look far different from the classic two-fund prediction. Section 6 examines a notion of approximate equilibrium. We argue that as the number of bonds with finite maturities increases the welfare loss from holding a non-optimal portfolio satisfying two-fund separation instead of the equilibrium portfolio tends to zero. Section 7 concludes.

2 The Asset Market Economy

We examine a standard Lucas asset pricing model with heterogeneous agents and complete asset markets. The exogenous dividend state follows a Markov chain with finite state space $\mathcal{Y} = \{1, 2, \ldots, Y\}$, $Y \geq 3$, and transition matrix Π. We assume a finite number of types $\mathcal{H} = \{1, 2, ..., H\}$ of infinitely-lived agents. There is a single perishable consumption good. Each agent h has a time-separable utility function

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u_h(c_t)\right\},$$

where c_t is consumption at time t. Our analysis of two-fund separation will assume particular parametric forms for the utility functions $u_h : \mathbb{R} \to \mathbb{R}$. In order to attain a simple stationary characterization of equilibrium, we assume that the discount factor $\beta \in (0, 1)$ is the same for all agents, and that all agents agree on the transition matrix Π.

The initial endowment of each agent consists only of shares in the firms. The firms distribute their output each period to its owners through dividends. Investors trade shares of firms and other securities in order to transfer wealth across time and states. We assume that there are $J \geq 2$ stocks, $j \in \mathcal{J} \equiv \{1, 2, \ldots, J\}$, traded on financial markets. A stock is an infinitely-lived asset ("Lucas tree") characterized by its state-dependent dividends. We denote the dividend of stock j by $d^j : \mathcal{Y} \to \mathbb{R}_+$, $j \in \mathcal{J}$, and assume that the dividend vectors d^j are linearly independent. Agent h has an initial endowment $\psi_j^{h,0}$ in stock $j \in \mathcal{J}$.

We assume that all stocks are in unit net supply, that is, $\sum_{h \in \mathcal{H}} \psi_j^{h,0} = 1$ for all $j \in \mathcal{J}$, and so the social endowment in the economy in state y is the sum of all firms' dividends in that state, so $e_y \equiv \sum_{j \in \mathcal{J}} d_y^j$ for all $y \in \mathcal{Y}$. We assume that all stocks have non-constant dividends and that the social endowment also has a positive variance.

Our model includes the possibility of two types of bonds. One type of bond we analyze is a consol. The consol pays one unit of the consumption good in each period in each state, that is, $d_y^c = 1$ for all $y \in \mathcal{Y}$. We also study finite-lived bonds. There are $K \geq 1$ bonds of maturities $1, 2, \ldots, K$ traded on financial markets. We assume that all finite-lived bonds are zero coupon bonds. (This assumption does not affect any results concerning stock investments since any other bond of similar maturity is equivalent to a sum of zero-coupon bonds.) A bond of maturity k delivers one unit of the consumption good k periods in the future. If at time t an agent owns a bond of maturity k and holds this bond into the next period, it turns into a bond of maturity $k - 1$. Agents do not have any initial endowment of the bonds. All bonds are thus in zero net supply.

For a more detailed description of the model and a definition of financial market equilibrium see Judd et al. (2003). Two results in Judd et al. (2003) are crucial for our analysis here. First, equilibrium is Markovian: individual consumption and asset prices depend only on the current state. Second, and more surprising, after one initial round of trading, each individual's portfolio is constant across states and time whenever all elements of the transition matrix are positive. The intuition is clear and follows directly from linear algebra and market completeness. Suppose the current dividend summarizes all information about future dividends; in that case, the dividend process is Markovian and we can identify the current state with the current dividend. Suppose that there are S states and S long-lived securities where each security's payoff depends solely on the current dividend. If utility is separable over time with constant discount rate (a common set of assumptions) then each agent's optimal consumption policy is a function of the state and is a vector of S numbers. If markets are dynamically complete then the state-contingent dividends from the S long-lived assets are S independent vectors. Therefore, any state-contingent consumption plan equals the returns generated by some unique *fixed and constant* combination of the S assets. If this target portfolio is not the agent's asset endowment, then he can obtain that portfolio through trading in the initial period. Therefore, any consumption plan can be implemented by some trade-once-and-hold-forever trading strategy. By concavity, there is a unique optimal consumption plan; hence, the trade-once-and-hold-forever strategy that implements the optimal consumption process must be the unique optimal trading strategy. This is true for each agent and for any price process. Therefore, it must hold in equilibrium. Judd et al. (2003) shows that this intuition generalizes to a mixture of long- and short-lived assets, finding that the holding of assets of any specific maturity is constant after initial trading when markets are dynamically complete.

These results allow us to express equilibrium in a simple manner. We do not need to express equilibrium values of all variables in the model as a function of time t. Instead, we let c_y^h denote consumption of agent h in state y. In addition, q_y^k denotes the price of bond k in state y, and the price of the consol is q_y^c. Similarly, p_y^j denotes the price of stock j in state y. The holdings of household h consist of θ_k^h bonds of maturity k or θ_c^h consols, and ψ_j^h units of stock j. If all bonds are of finite maturity then an agent's budget constraint is

$$c_y^h = \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h(1 - q_y^1) + \sum_{k=2}^{K} \theta_k^h (q_y^{k-1} - q_y^k). \tag{1}$$

If the economy has a consol but no short-lived bonds then the budget constraint after time 0 is

$$c_y^h = \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_c^h. \tag{2}$$

Cass and Stiglitz (1970) show that two-fund separation in economies with a riskless asset requires restrictions on investors' utility functions. In our dynamic general equilibrium model these restrictions amount to the assumption of equi-cautious HARA utility functions for all agents. We examine three special cases of the HARA utility functions: power utility, quadratic utility, and constant absolute risk aversion. We use the following notation for the utility function of household h.

power utility functions: $$u_h(c) = \begin{cases} \frac{1}{1-\gamma}(c - A^h)^{1-\gamma}, & \gamma > 0, \gamma \neq 1, \quad c > A^h \\ \ln(c - A^h), & \gamma = 1, \quad c > A^h \end{cases}$$

quadratic utility functions: $u_h(c) = -\frac{1}{2}(B^h - c)^2$

CARA utility functions: $u_h(c) = -\frac{1}{a^h} e^{-a^h c}$

For ease of reference we summarize the notation for the most important parameters and variables in the model.

d_y^j	dividend of stock j in state y
ψ_j^h	agent h's holding of stock j
θ_k^h	agent h's holding of maturity k bond
θ_c^h	agent h's holding of the consol
p_y^j	price of stock j in state y
q_y^k	price of maturity k bond in state y
q_y^c	price of the consol in state y

2.1 Linear Sharing Rules

The easiest way to describe equilibrium is to focus on the sharing rules that represent equilibrium consumption. This connection between equilibrium consumption sharing rules (as exposited in Wilson, 1968) and asset market equilibrium was made in Rubinstein (1974). We follow the same approach in our dynamic economy. We say that equilibrium consumption for household h in state y, c_y^h, follows a linear sharing rule if there exists real numbers

m^h, b^h so that

$$c_y^h = m^h e_y + b^h \qquad \forall h \in \mathcal{H},\ y \in \mathcal{Y}.$$

If investors have equi-cautious HARA utility, then this is true for all agents $h \in \mathcal{H}$, and $\sum_{h=1}^{H} m^h = 1$ and $\sum_{h=1}^{H} b^h = 0$ in equilibrium.

By the first and second theorem of welfare economics any efficient equilibrium maximizes some weighted sum of utilities. Let λ^h be the Negishi weight of agent h; we normalize $\lambda^1 = 1$. Judd et al. (2003) describe a Negishi approach to calculate equilibrium consumptions for our model. Appendix A contains a brief summary of the method. Using this approach and straightforward algebra we can calculate the linear sharing rules for the three families of utility functions under consideration.

For power utility functions the linear sharing rule is

$$c_y^h = e_y \cdot \left(\frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i\in\mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \right) + \left(A^h - \frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i\in\mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \sum_{i\in\mathcal{H}} A^i \right) = m^h e_y + b^h. \qquad (3)$$

Note that for CRRA utility functions, $A^h = 0$ for all $h \in \mathcal{H}$, the sharing rule has zero intercept, $b^h = 0$, and household h consumes a constant fraction

$$m^h = \left(\frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i\in\mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \right)$$

of the total endowment. For quadratic utility functions, we obtain

$$c_y^h = e_y \cdot \left(\frac{(\lambda^h)^{-1}}{\sum_{i\in\mathcal{H}} (\lambda^i)^{-1}} \right) + \left(B^h - \frac{(\lambda^h)^{-1}}{\sum_{i\in\mathcal{H}} (\lambda^i)^{-1}} \sum_{i\in\mathcal{H}} B^i \right). \qquad (4)$$

For CARA utility functions the linear sharing rules are

$$c_y^h = e_y \cdot \frac{\tau^h}{\sum_{i\in\mathcal{H}} \tau^i} + \left(\tau^h \ln(\lambda^h) - \frac{\tau^h}{\sum_{i\in\mathcal{H}} \tau^i} \sum_{i\in\mathcal{H}} \tau^i \ln(\lambda^i) \right), \qquad (5)$$

where $\tau^h = 1/a^h$ is the constant absolute risk tolerance of agent h.

3 Two-Fund Separation with a Consol

In this section we review agents' portfolios in economies with a consol. Classical two-fund monetary separation (see, for example, Cass and Stiglitz (1970), Ingersoll (1987), Huang and Litzenberger (1988)) states that investors who must allocate their wealth between a number of risky assets and a riskless security should all hold the same mutual fund of risky assets. An investor's risk aversion only affects the proportions of wealth that (s)he invests in the risky mutual fund and the riskless security. But the allocation of wealth across the different risky assets does not depend on the investor's preferences. For our dynamic general equilibrium model with several heterogeneous agents this property states that the proportions of wealth invested in any two stocks are the same for all agents in the economy.

Definition 1 *We say that portfolios exhibit two-fund monetary separation if*

$$\frac{\psi_j^h \, p_y^j}{\psi_{j'}^h \, p_y^{j'}} = \frac{\psi_j^{h'} \, p_y^j}{\psi_{j'}^{h'} \, p_y^{j'}}$$

for all stocks j, j' and all agents $h, h' \in \mathcal{H}$ in all states $y \in \mathcal{Y}$.

All stocks are in unit net supply and so market clearing and the requirement from the definition immediately imply that all agents' portfolios exhibit two-fund separation if and only if each agent has a constant share of each stock in the economy, that is, $\psi_j^h = \psi_{j'}^h$ for all stocks j, j' and all agents $h \in \mathcal{H}$. This constant share typically varies across agents. In the remainder of this paper we identify two-fund monetary separation with this constant-share property. Note that the ratio of wealth invested in any two stocks j, j' equals the ratio $p_y^j / p_y^{j'}$ of their prices and thus depends on the state $y \in \mathcal{Y}$.

A consol is a bond paying one unit of consumption in each period indefinitely. Notice that this is not the same as a constant interest rate since the value of a consol may vary over time as may the interest rate. Under the assumption that agents can only trade stocks and a consol we recover the classical two-fund monetary separation result of Cass and Stiglitz (1970) in our dynamic equilibrium context. The consol is the truly riskless asset in an infinite-horizon dynamic economy.

Theorem 1 (Two-Fund Separation Theorem) *Consider an economy with $J \leq Y - 1$ stocks and a consol. If all agents have equi-cautious HARA utilities then their portfolios exhibit two-fund monetary separation in an efficient equilibrium.*

Proof: The statement of the theorem follows directly from the budget constraint (2). Agents' sharing rules are linear, $c_y^h = m^h e_y + b^h$ for all $h \in \mathcal{H}$, $y \in \mathcal{Y}$, and so the budget constraints immediately yield $\theta_c^h = b^h$ and $\psi_j^h = m^h$ for all $j = 1, \ldots, J$. □

Note that due to linear sharing rules a market with stocks and a consol implements the complete-market equilibrium even though it does not have a complete set of assets. The agents' portfolios are unique since $J + 1 \leq Y$ and the vectors d^c and d^j, $j \in \mathcal{J}$, are linearly independent. If the number of states Y is smaller than the total number of assets, $J + 1$, then the dividend vectors and consol payoff are linearly dependent and there is a continuum of portfolios supporting the agents' linear sharing rules.

The theorem implies that we can read off agents' portfolios from their linear sharing rules. Observe that in the special case of CRRA utility functions, $A^h = 0$ for all $h \in \mathcal{H}$, and so the agents do not trade the consol. This is a corollary to the theorem: Whenever the intercept terms of the sharing rules are zero then agents do not trade the consol and the stock markets are dynamically complete without a bond market. However, Schmedders (2005) shows under the additional condition $\sum_{h \in \mathcal{H}} A^h \neq 0$ that sharing rules have a nonzero intercept for a generic set of agents' initial stock portfolios. That is, with the exception of a set of initial portfolios that has measure zero and is closed, sharing rules will have nonzero intercepts.

This fact that sharing rules have generically nonzero intercepts immediately implies that a one-period bond cannot serve as the riskless asset in the economy and so portfolios cannot exhibit monetary separation. The economic intuition for this fact follows directly from the budget equation for an economy with $J+1$ states, J stocks and a one-period bond,

$$m^h \cdot e_y + b^h = \eta^h \cdot e + \theta_1^h(1 - q_y^1) \quad \text{for all } h \in \mathcal{H}. \tag{6}$$

Contrary to the budget equations for an economy with a consol the bond price q_y^1 now appears. The agents have to reestablish their position in the short-lived bond in every period. As a result they face reinvestment risk due to fluctuating equilibrium interest rates of the short-term bond. Schmedders (2005) shows that, generically, fluctuations in the price of the one-period bond prohibit a solution to equations (6). The reinvestment risk affects agents' bond and thus stock portfolios and leads to a change of the portfolio weights that implement equilibrium consumption. (On the contrary, in an economy with a consol, the agent establishes a position in the consol at time 0 once and forever. Fluctuations in the price of the consol therefore do no affect the agent just like he is unaffected by stock price fluctuations. This fact allows him to hold a portfolio exhibiting two-fund monetary separation.)

Obviously the agents' portfolios do satisfy a generalized separation property. Consumption follows a linear sharing rule and so an agent's portfolio effectively consists of one fund generating the safe payoff stream of a consol and the second fund generating a payoff identical to aggregate dividends. Both funds have non-zero positions of stocks and of the bond. Agent h holds b^h units of the first fund and m^h units of the second fund. However, when many people talk about two-fund separation they don't have this generalized notion in mind but instead the monetary separation between a market portfolio and a safe payoff, see for example Canner et al. (1997) and Elton and Gruber (2000). In this paper we focus exclusively on monetary separation.

Real-world investors do not have access to a consol. (With the exception of some rare types of consols issued in previous centuries, infinite-horizon bonds do not exist and are no longer issued.) Bond markets instead enable trade in many bonds with varying finite maturity. Thus, we are naturally led to the question whether it is possible for agents to synthesize a consol by trading finite-maturity bonds. A related question is then whether the nonexistence of the consol and its substitution through portfolios of finite-maturity bonds has quantitatively significant effects on agents' overall portfolios. To answer these questions we next examine equilibrium portfolios for economies with several finite-maturity bonds.

4 Equilibrium Portfolios with Finite-Maturity Bonds

We begin with a very simple example to show how we can easily compute equilibria for our model. Subsequently we examine larger models. The insights from these fairly extensive examples provide useful guidance for our further analysis.

4.1 Introductory Example

Consider an economy with $H = 3$ agents who have CARA utility functions with coefficients of absolute risk aversion of 1, 2, and 3, respectively. The agents' discount factor is $\beta = 0.95$. There are two independent stocks with identical 'high' and 'low' dividends of 1.02 and 0.98, respectively. The dividends of the first stock have a persistence probability of 0.8, that is, if the current dividend level is high (low), then the probability of having a high (low) dividend in the next period is 0.8. The corresponding probability of the second stock equals 0.6. As a result of this dividend structure the economy has $S = 4$ exogenous states of nature. The dividend vectors are

$$d^1 = (1.02, 1.02, 0.98, 0.98) \quad \text{and} \quad d^2 = (1.02, 0.98, 1.02, 0.98).$$

The Markov transition matrix for the exogenous dividend process is

$$\Pi = \begin{bmatrix} 0.48 & 0.32 & 0.12 & 0.08 \\ 0.32 & 0.48 & 0.08 & 0.12 \\ 0.12 & 0.08 & 0.48 & 0.32 \\ 0.08 & 0.12 & 0.32 & 0.48 \end{bmatrix}.$$

The economy starts in state $y_0 = 1$. The agents' initial holdings of the two stocks are identical, so $\psi_j^{h,0} = \frac{1}{3}$ for $h = 1, 2, 3$, $j = 1, 2$.

Using the formulas of Judd et al. (2003) (see Appendix A) we compute consumption allocations.

$$\begin{aligned} c^1 &= (0.688, 0.666, 0.666, 0.644), \\ c^2 &= (0.678, 0.667, 0.667, 0.656), \\ c^3 &= (0.674, 0.667, 0.667, 0.660). \end{aligned}$$

Note that the fluctuations of agents' consumption allocations across the four states is fairly small. The reason for this small variance is the small dividend variance of the two stocks. The state-contingent stock prices are

$$\begin{aligned} p^1 &= (19.43, 19.01, 18.98, 18.58), \\ p^2 &= (19.40, 18.98, 19.01, 18.60). \end{aligned}$$

Now suppose that the third asset in this economy is a consol. Note that markets are complete despite the lack of a fourth asset. The price vector of the consol is

$$q^c = (19.40, 18.99, 19.01, 18.61).$$

The economy satisfies the conditions of Theorem 1 and so agents' portfolios exhibit two-fund monetary separation.

$$
\begin{aligned}
(\psi_1^1, \psi_2^1, \theta_c^1) &= (\frac{6}{11}, \frac{6}{11}, -0.425), \\
(\psi_1^2, \psi_2^2, \theta_c^2) &= (\frac{3}{11}, \frac{3}{11}, \ \ 0.121), \\
(\psi_1^3, \psi_2^3, \theta_c^3) &= (\frac{2}{11}, \frac{2}{11}, \ \ 0.304).
\end{aligned}
$$

Now suppose markets are completed with two finite-maturity bonds. First we compute bond prices for bonds of various maturity.

$$
\begin{aligned}
q^1 &= (0.963, 0.946, 0.954, 0.938), \\
q^2 &= (0.918, 0.899, 0.906, 0.887), \\
q^5 &= (0.790, 0.773, 0.775, 0.758), \\
q^{10} &= (0.612, 0.599, 0.599, 0.586), \\
q^{25} &= (0.284, 0.277, 0.277, 0.271), \\
q^{50} &= (0.079, 0.077, 0.077, 0.075).
\end{aligned}
$$

The agents' equilibrium portfolios now depend on which two of these bonds are chosen to complete the market. The most natural choice is an economy with a one- and two-period bond but in the literature (see, for example, Campbell and Viceira, 2001) sometimes other combinations are chosen. For several choices of bonds we report equilibrium portfolios in Table I and the corresponding end-of-period wealth in the four assets in Table II.

bonds		Agent 1				Agent 2				Agent 3			
k_1	k_2	ψ_1^1	ψ_2^1	θ_1^1	$\theta_{k_2}^1$	ψ_1^2	ψ_2^2	θ_1^2	$\theta_{k_2}^2$	ψ_1^3	ψ_2^3	θ_1^3	$\theta_{k_2}^3$
1	2	0.467	0.191	-1.029	1.249	0.295	0.374	0.294	-0.357	0.238	0.435	0.735	-0.892
1	5	0.603	1.878	0.835	-45.582	0.256	-0.107	-0.238	12.977	0.141	-0.771	-0.598	32.605
1	10	0.519	0.395	-0.647	-6.830	0.280	0.316	0.184	1.945	0.201	0.290	0.463	4.885
1	25	0.518	0.381	-0.660	-13.739	0.281	0.319	0.188	3.912	0.201	0.299	0.472	9.827
1	50	0.518	0.381	-0.660	-49.528	0.281	0.319	0.188	14.101	0.201	0.299	0.472	35.427

Table I: Equilibrium Portfolios with Two Bonds

Agents' portfolios never exhibit two-fund monetary separation. The equilibrium portfolios depend on the set of bonds available to the investors. Any arbitrary choice of bond maturities in the model will greatly affect equilibrium outcomes. We believe the most natural choice of bonds is a family of bonds of consecutive maturities. Otherwise an agent would be artificially forced to sell bonds whenever a bond changed its remaining maturity to a level that is not permitted by the model. (For example, an agent bought a k-period bond that then in the next period has a remaining maturity of $k-1$ periods. If the model doesn't

bonds		Agent 1				Agent 2				Agent 3			
k_1	k_2	ψ_1^1	ψ_2^1	θ_1^1	$\theta_{k_2}^1$	ψ_1^2	ψ_2^2	θ_1^2	$\theta_{k_2}^2$	ψ_1^3	ψ_2^3	θ_1^3	$\theta_{k_2}^3$
1	2	9.07	3.71	−0.99	1.15	5.73	7.25	0.28	−0.33	4.63	8.44	0.71	−0.82
1	5	11.71	36.45	0.80	−36.02	4.98	−2.07	−0.23	10.26	2.74	−14.97	−0.58	25.77
1	10	10.08	7.66	−0.62	−4.18	5.45	6.12	0.18	1.19	3.90	5.62	0.45	2.99
1	25	10.07	7.40	−0.64	−3.90	5.45	6.20	0.18	1.11	3.91	5.81	0.45	2.79
1	50	10.07	7.40	−0.64	−3.90	5.45	6.20	0.18	1.11	3.91	5.81	0.45	2.79

Table II: End-of-period Wealth across Assets in State 1

allow for bonds of this maturity the agent would have to close that position.) Therefore, in the remainder of this paper, we will only consider economies with the property that if a bond of maturity k is present, then also bonds of maturity $k-1, k-2, \ldots, 1$ will be available to investors.

4.2 Equilibrium Portfolios with Many States and Bonds

The small example in the previous section showed that equilibrium portfolios in economies with a consol are very different from those in economies with finite-maturity bonds. We saw that stock portfolios varied greatly from stock portfolios prescribed by two-fund separation. In addition, the bond portfolios did not just serve to synthesize the consol but also had to account for the variations in the stock portfolios. The bond portfolios appeared somewhat unintuitive since they required rather large trades on the bond market.

Now we analyze these issues further by examining economies with much larger number of states. In order to complete markets agents now have access to larger families of bonds with different maturities. The purpose of these examples is to learn more details about the structure of equilibrium portfolios that then will guide our theoretical analysis in subsequent sections.

We consider economies with $H = 2$ agents with power utility functions. Setting $A^1 = -A^2 = b$ results in the linear sharing rules[1] $c^1 = m^1 \cdot e + b \cdot 1_Y$ and $c^2 = (1-m^1) \cdot e - b \cdot 1_Y$. We normalize stock dividends so that the expected aggregate endowment equals 1. Then we set $m^1 = \frac{1}{2} - b$ so that both agents consume on average half of the endowment. For the subsequent examples we use $b = 0.2$, $\gamma = 5$ and $\beta = 0.95$. The agents' sharing rules are then

$$c^1 = 0.3 \cdot e + 0.2 \cdot 1_Y \quad \text{and} \quad c^2 = 0.7 \cdot e - 0.2 \cdot 1_Y.$$

[1] To simplify the analysis we do not compute linear sharing rules for some given initial portfolios but instead take sharing rules as given and assume that the initial endowment is consistent with the sharing rules. There is a many-to-one relationship between endowments and consumption allocations, and it is more convenient to fix consumption rules.

We consider economies with $J \in \{3, 4, 5, 6, 7\}$ independent stocks. Each stock $j \in \mathcal{J}$ in the economy has only two dividend states, a "high" and a "low" state, resulting in a total of 2^J possible states in the economy. We define the persistence parameters ξ^j for each stock j and denote the dividend's 2×2 transition matrix by

$$\Xi = \begin{bmatrix} \frac{1}{2}(1+\xi^j) & \frac{1}{2}(1-\xi^j) \\ \frac{1}{2}(1-\xi^j) & \frac{1}{2}(1+\xi^j) \end{bmatrix}$$

with $\xi^j \in (0, 1)$. The Markov transition matrix $\Pi = \bigotimes_{j=1}^{J} \Xi$ for the entire economy is a Kronecker product of the individual transition matrices, see Appendix B.2. Table III reports the parameter values for our examples. These parameter values cover a reasonable range of

stock	1	2	3	4	5	6	7
high d	1.02	1.23	1.05	1.2	1.09	1.14	1.1
low d	0.98	0.77	0.95	0.8	0.91	0.86	0.9
ξ	0.1	0.62	0.22	0.48	0.32	0.4	0.36
$\frac{1}{2}(1+\xi)$	0.55	0.81	0.61	0.74	0.66	0.7	0.68

Table III: Stock Characteristics

persistence and variance in stock dividends. The varying dividend values and persistence probabilities are chosen so that the examples display generic behavior. (We calculated hundreds of examples showing qualitatively similar behavior.) To keep the expected social endowment at 1 we always normalize the dividend vectors. For this reason we multiply the dividend vectors by $1/J$ for the economy with J stocks. However, as we show below this normalization is not really necessary.

The economy has J stocks and 2^J states of nature. Therefore we need $2^J - J$ bonds to complete the market. We now ask how much portfolios in such an economy deviate from two-fund separation. Table IV reports equilibrium portfolios of agent 1.

We make several observations about the agents' portfolios. The larger the number of stocks J and so the larger the number of states 2^J and number of bonds $2^J - J$, the closer the agents' stock portfolios get to the slope m^h of the linear sharing rules. For $J \in \{4, 5, 6, 7\}$ the first 6 digits of stock holdings and slopes are already identical (to keep the table small we report fewer than 6 digits.). For $J = 4$ the agents' holdings of the bonds of maturity 1 and 2 match b^h for the first 6 digits. For $J = 5$ there is a corresponding match already for the first 12 bonds. The longer the maturity of the bonds the greater the deviations of holdings from b^h (with the exception of just the holdings of bonds with very long maturities). In addition, once holdings deviate significantly from b^h they alternate in sign[2]. In summary,

[2]The nature of these bond holdings, namely the very large positions of alternating signs, may remind some readers of the unrelated literature on the optimal maturity structure of noncontingent government debt, see, for example, Angeletos (2002) and Buera and Nicolini (2004). Buera and Nicolini report very high debt positions from numerical calculations of their model with four bonds. The reason for their highly sensitive large debt positions is the close correlation of bond prices.

$(J,K) =$	$(3,5)$	$(4,12)$	$(J,K) =$	$(5,27)$	$(J,K) =$	$(6,58)$	$(J,K) =$	$(7,121)$
ψ_1^1	0.431	0.30	ψ_1^1	0.30	ψ_1^1	0.30	ψ_1^1	0.30
ψ_2^1	0.351	0.30	ψ_2^1		ψ_2^1		ψ_2^1	
ψ_3^1	0.387	0.30	ψ_3^1	⋮	ψ_3^1	⋮	ψ_3^1	⋮
ψ_4^1	–	0.30	ψ_4^1		ψ_4^1		ψ_4^1	
ψ_5^1	–	–	ψ_5^1	0.30	ψ_5^1		ψ_5^1	⋮
ψ_6^1	–	–	ψ_6^1	–	ψ_6^1	0.30	ψ_6^1	
ψ_7^1	–	–	ψ_7^1	–	ψ_7^1	–	ψ_7^1	0.30
θ_1^1	0.152	0.20	θ_1^1	0.20	θ_1^1	0.20	θ_1^1	0.20
θ_2^1	−0.184	0.20	θ_2^1		θ_2^1		θ_2^1	
θ_3^1	2.337	0.20	θ_8^1		θ_3^1		θ_{56}^1	⋮
θ_4^1	−7.498	0.20	θ_9^1	⋮	θ_4^1	⋮	θ_{57}^1	
θ_5^1	8.074	0.20	θ_{10}^1		θ_{25}^1		θ_{58}^1	0.20
θ_6^1	–	0.27	θ_{11}^1		θ_{26}^1		θ_{95}^1	0.20
θ_7^1	–	−0.66	θ_{12}^1	0.20	θ_{27}^1	0.20	θ_{100}^1	0.18
θ_8^1	–	6.33	θ_{15}^1	0.20	θ_{40}^1	0.20	θ_{110}^1	−29675
θ_9^1	–	−26.23	θ_{20}^1	−5.2	θ_{50}^1	1179	θ_{115}^1	504548
θ_{10}^1	–	66.16	θ_{25}^1	556	θ_{56}^1	10177	θ_{119}^1	157907
θ_{11}^1	–	−86.58	θ_{26}^1	−423	θ_{57}^1	−4627	θ_{120}^1	−50013
θ_{12}^1	–	46.58	θ_{27}^1	146	θ_{58}^1	998	θ_{121}^1	7670

Table IV: Portfolio of Agent 1

the more states and bonds are present in the economy, the closer the stock portfolios come to two-fund separation. At the same time an agent's holdings of bonds of a fixed maturity T comes closer to the constant components of his sharing rule. These observations lead us to the conjecture that as the number of bonds increases the portfolios should approach two-fund separation for stocks (holding a portion of the market portfolio) and constant holdings of the short-maturity bonds.

To check this conjecture we next examine the deviations of the equilibrium portfolios from two-fund separation more closely. In order to measure these deviations, define

$$\Delta^S \equiv \max_{j=1,2,\ldots,J} |\psi_j^1 - m^1|$$

to be the maximal deviation of agents' equilibrium stock holdings from the two-fund separation portfolio where we maximize the difference across all stocks (due to market-clearing it suffices to calculate the difference for the first agent). Similarly, we define

$$\Delta^k \equiv |\theta_k^1 - b^1|$$

to be the maximal difference between agents' holdings in bond k and the intercepts of their linear sharing rules. We regard this as the deviation from two-fund separation since, if the market contained a bond of maturity T for all T (a condition equivalent to a consol) then two fund separation would have an individual hold the same number b^h of bonds of all maturities and lead to no trading in bonds.

Table V reports deviations in stock holdings and the first five bonds, and Table VI reports deviations in bond positions for some selected longer maturity bonds. The results are clear. First, the equilibrium stock portfolios are close to classic two-fund separation and are practically the same when there are several stocks and many states and bonds. Second, the deviations in the bond portfolios from two-fund separation are negligible for short-maturity bonds but then explode as we increase the number of stocks and bonds. In particular, the bonds with very long maturities differ significantly from the two-fund separation predictions.

J	K	Δ^S	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
4	12	4.5 (−9)	1.3 (−9)	3.5 (−8)	2.0 (−6)	1.1 (−4)	3.7 (−3)
5	27	3.5 (−33)	6.3 (−34)	8.3 (−31)	8.3 (−28)	4.6 (−25)	1.6 (−22)
6	58	9.6 (−88)	4.2 (−85)	3.1 (−81)	1.1 (−77)	2.1 (−74)	3.0 (−71)
7	121	2.0 (−222)	4.9 (−214)	1.8 (−209)	3.0 (−205)	3.2 (−201)	2.4 (−197)

Table V: Deviations in Stock Holdings and First Five Bonds from Two-Fund Separation

k	(5, 27)	(6, 58)	(7, 121)
6	3.5 (−20)	3.0 (−68)	1.4 (−193)
7	5.3 (−18)	2.4 (−65)	6.3 (−190)
10	3.0 (−12)	2.9 (−57)	2.0 (−179)
11	1.5 (−10)	9.9 (−55)	4.5 (−176)
12	5.4 (−9)	2.9 (−52)	8.9 (−173)
20	5.37	7.5 (−35)	3.5 (−148)
25	555.6	1.1 (−25)	3.9 (−134)
26	423.4	5.3 (−24)	2.0 (−131)
27	145.8	2.4 (−22)	9.1 (−129)
40	–	3.7 (−5)	1.0 (−96)
50	–	1179.3	4.3 (−75)
56	–	10178	3.0 (−63)
57	–	4627.2	2.3 (−61)
58	–	998.2	1.7 (−59)

Table VI: Deviations in Bond Holdings from Two-Fund Separation

Remark: Computing the results in Tables V and VI requires us to solve the agents' budget equations (1). Although these equations are linear, solving them numerically is very difficult. The prices of bonds with very long maturity k are nearly perfectly correlated. This fact makes the equilibrium equations nearly singular and thus difficult to solve. One cannot solve them using a regular linear equation solver on a computer using 16 decimal digits of precision. To handle this difficulty, we used Mathematica with up to 1024 decimal digits of precision.

We tried many different examples and always had the same results: with many bonds, the stock portfolios are extremely close to satisfying two-fund separation. The holdings of bonds with short maturity are close to the value of the safe portion of the consumption stream. But the equilibrium holdings of long bonds are highly volatile, implying that investors are making dramatically large trades in long bonds in each period. Our results are surprisingly invariant to the size of the stock dividends and the utility parameter γ. And again, despite the crazy structure of the bond portfolios, we observe the recurring theme that the agents use the finite-maturity bonds to generate the safe portion of their consumption stream.

We can learn some details by closely examining the underlying budget equations, which lead to these portfolios. Recall that agent h's budget constraint in an economy with finite-maturity bonds is

$$c_y^h = \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^{K} \theta_k^h (q_y^{k-1} - q_y^k). \tag{7}$$

Suppose the agent's consumption allocation follows the linear sharing rule $c^h = m^h \cdot e + b^h 1_Y$ and that there are enough bonds so that markets lead to an efficient equilibrium implementing this consumption rule. If the agent's portfolio also exhibits two-fund separation, then there must exist stock weights $\psi_j^h \equiv \eta^h$ for all $j \in \mathcal{J}$ such that the budget constraint can be written as

$$m^h \cdot e + b^h 1_Y = \eta^h \cdot e + \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k). \tag{8}$$

Rearranging (8) yields

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + \sum_{k=1}^{K-1} (\theta_k^h - \theta_{k+1}^h) q^k + \theta_K^h q^K = 0. \tag{9}$$

As we have seen in our numerical examples, once the number of states and bonds gets sufficiently large, for all practical purposes $m^h \approx \eta^h$ and $b^h \approx \theta_1^h$. For example, in an economy with $J = 5$ stocks and $K = 27$ bonds the two deviations are $\Delta^S < 10^{-32}$ and $\Delta^1 < 10^{-33}$. Thus, equations (9) lead to

$$\sum_{k=1}^{K-1} (\theta_k^h - \theta_{k+1}^h) q^k + \theta_K^h q^K \approx 0. \tag{10}$$

We show in Appendix A that the price of a bond of maturity k is

$$q_y^k = \frac{\beta^k (\Pi^k)_{y\cdot} P}{u_1'(c_y^1)} = \frac{\beta^k \sum_{z=1}^{Y} (\Pi^k)_{yz} P_z}{u_1'(c_y^1)}.$$

where $P = \left(u_1'(c_y^1)\right)_{y \in \mathcal{Y}}$ is the vector of prices $u_1'(c_y^1)$ for consumption in state y. Substituting these expressions into equations (10) and denoting the vector of relative prices $P/u_1'(c_y^1)$ by R we obtain

$$\left(\sum_{k=1}^{K-1} (\theta_k^h - \theta_{k+1}^h)(\beta\Pi)^k + \theta_K^h (\beta\Pi)^K \right) R \approx 0. \tag{11}$$

This approximate equation must hold for all agents $h \in \mathcal{H}$. A sufficient conditions for this approximate equation to hold is that the matrix sum (in the large brackets) is approximately zero,

$$\sum_{k=1}^{K-1} (\theta_k^h - \theta_{k+1}^h)(\beta\Pi)^k + \theta_K^h (\beta\Pi)^K \approx 0. \tag{12}$$

In our example with $J = 5$ and $K = 27$ the largest element in the matrix on the left-hand side is smaller than 10^{-31} and many elements are zero. This matrix equation solely depends on the transition matrix Π (and the discount factor β). A consequence of this observation is that the asset prices matter very little for the portfolio decisions. The stock prices do not appear in the budget equations since agents do not trade the stocks after the initial period. And the state price ratios, the elements of R, also have no first-order effects on the bond portfolios. The agent's utility functions really only matter for determining the linear sharing rules in equilibrium. In the end, the bond positions θ_k^h, $k = 1, \ldots, K$, must (approximately) satisfy a system of equations that only depends on the transition matrix Π and the discount factor β. And the portfolios that have this property have positions in bonds of short maturity that get extremely close to portfolios exhibiting two-fund separation.

Our examples show an approximate separation between the stock market and the bond market once the number of states and bonds becomes sufficiently large. In addition, the stock portfolio becomes extremely close to the portfolio prescribed by two-fund separation (a portion of the market portfolio). These observations lead us to the analysis of conditions under which (i) exact separation between the stock and bond market holds, (ii) stock portfolios display exactly the (static) nature of constant holdings across all stocks, and (iii) a portfolio of finite-maturity bonds exactly spans the consol. We examine these issues in the next section.

4 Multiple Finite-Maturity Bonds Span the Consol

First we motivate the necessity of special conditions and argue that two-fund separation is generically impossible if we have too few bonds. Recall equation (9), the rearranged budget equation for agent h given his linear sharing rule,

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + \sum_{k=1}^{K-1} (\theta_k^h - \theta_{k+1}^h) q^k + \theta_K^h q^K = 0. \tag{13}$$

Equation (13) states that the $K+2$ vectors $e, 1_Y$ and $q^1, \ldots, q^K$ in $\mathbb{R}^Y$ are linearly dependent in this fashion. If the number of states Y exceeds $K+2$ then this condition cannot be satisfied for general economies. For example, if the total number of stocks and bonds $J+K$ equals the number of states Y (so markets are generically dynamically complete), and there are $J \geq 3$ stocks then the system (9) has more equations than unknowns. Using a genericity argument along the lines of those in Schmedders (2005) we can show equations (9) have no solutions unless parameters lie in some measure zero space.

Although agents' portfolios typically do not exhibit two-fund separation in economies with only finite-maturity bonds we may ask whether there are special (non-generic) but economically reasonable conditions that do lead to two-fund separation in such economies. We next develop such sufficient conditions and begin with economies having i.i.d. dividends. We then generalize the insights from this simple class of economies to broader ones.

5.1 Equilibrium Portfolios with IID Dividends

We examine a simple case in which equations (9) do have a solution and portfolios exhibit two-fund separation even if there are only two bonds.

Proposition 1 *Consider an economy with J stocks, a one-period and a two-period bond and $Y \geq J+2$ dividend states. Suppose further that the Markov transition probabilities are state-independent, so all rows of the transition matrix Π are identical. If all agents have equi-cautious HARA utility functions then agents' portfolios satisfy monetary separation in an efficient equilibrium.*

Proof: Under the assumption that all states are i.i.d. the Euler equations (23, 24) imply that the price of the two-period bond satisfies $q^2 = \beta q^1$, that is, the prices of the two bonds are perfectly correlated. Then condition (9) of agent h becomes

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + (\theta_1^h - \theta_2^h) q^1 + \theta_2^h \beta q^1 = 0,$$

which is equivalent to

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + (\theta_1^h - (1-\beta)\theta_2^h) q^1 = 0,$$

These equations have the unique solution

$$\eta^h = m^h, \quad \theta_1^h = b^h, \quad \theta_2^h = \frac{b^h}{1-\beta}.$$

□

Under the condition of Proposition 1 the system (9) has a very special solution since the two price vectors q_1 and q_2 are linearly dependent. The key fact is that for i.i.d. dividend transition probabilities the two bonds are sufficient to span the consol. The interest rate fluctuation does not prohibit the agents from holding portfolios exhibiting two-fund separation.

Note that under the assumptions of Proposition 1 agents have a position of $\theta_2^h = \frac{b^h}{1-\beta}$ in the risky two-period bond. The more risk-averse agent h is, the larger the intercept term b^h of his linear sharing rule becomes, see the example in Section 4. So more risk-averse agents hold more risky bonds relative to stocks than less risk-averse agents. This fact does not violate two-fund separation. The holding of the two (risky) bonds yields exactly the constant part of the linear sharing rule.

This last point is a recurring theme in the remainder of our paper and deserves further elaboration. Even though the short-lived bonds are risky the agents' positions in these bonds serve the purpose of creating the safe portion of their respective consumption streams. And so among equally wealthy agents the more risk-averse ones with a larger safe portion b^h must hold larger bond positions. In this model, therefore, bonds should not be viewed as part of the risky portfolio, but as part of a portfolio generating a safe payoff stream. In light of this observation the asset allocation "puzzle" of Canner et al. (1997) disappears. More risk-averse agents should indeed invest a larger portion of their wealth in bonds.

5.2 Spanning the Consol

The discussion of economies with i.i.d. dividends revealed that a sufficient condition for agents' portfolios to satisfy two-fund separation is that the finite-maturity bonds span the consol. In that case the payoff of the bond portfolio delivers the safe part b^h of an agent's consumption stream. Then a stock portfolio exhibiting two-fund separation, namely holdings of size m^h of all stocks, delivers the risky part $m^h e$ of the agent's consumption stream. This observation leads us naturally to the question whether we can generalize this insight to more general Markov chains of dividends. For the presentation of such a generalization we need the following technical lemma. Its proof doesn't provide any economic intuition and we relegate it to Appendix B.1.

Lemma 1 *Suppose the $Y \times Y$ transition matrix $\Pi >> 0$ governing the Markov chain of exogenous states in the economy has only real eigenvalues. Further assume that Π is diagonalizable and has L $(\leq Y)$ distinct eigenvalues. Then the following statements are true.*

(1) If all eigenvalues are nonzero then the matrix equation $I_Y + \sum_{k=1}^{L} a_k \, \Pi^k = 0$ has a unique solution $(a_1^*, \ldots, a_L^*)$. Moreover, $\sum_{k=1}^{L} a_k^* = -1$.

(2) If zero is an eigenvalue of Π then the matrix equation $\sum_{k=1}^{L} a_k \, \Pi^k = 0$ has a nontrivial solution. Moreover, any solution $(a_1^*, \ldots, a_L^*)$ satisfies $\sum_{k=1}^{L} a_k^* = 0$.

Theorem 2 *Suppose the $Y \times Y$ transition matrix $\Pi >> 0$ governing the Markov chain of exogenous states in the economy has only real eigenvalues. Further assume that Π is diagonalizable and has L $(\leq Y)$ distinct eigenvalues. Then the consol is spanned by bonds of maturities $k = 1, 2, \ldots, L$.*

Proof: If L bonds of maturity $k = 1, 2, \ldots, L$ span the consol then there must be a portfolio $(\theta_1, \ldots, \theta_L)$ of these bonds such that

$$1_Y = \theta_1(1_Y - q^1) + \sum_{k=2}^{L} \theta_k(q^{k-1} - q^k). \tag{14}$$

This system of equations is equivalent to

$$(1 - \theta_1) \cdot 1_Y + \sum_{k=1}^{L-1} (\theta_k - \theta_{k+1}) q^k + \theta_L q^L = 0. \tag{15}$$

Recall that the price of a bond of maturity k is

$$q_y^k = \frac{\beta^k (\Pi^k)_y P}{u_1'(c_y^1)} = \frac{\beta^k \sum_{z=1}^{Y} (\Pi^k)_{yz} P_z}{u_1'(c_y^1)},$$

where $P = \left(u_1'(c_y^1)\right)_{y \in \mathcal{Y}}$ is the vector of prices $u_1'(c_y^1)$ for consumption in state y. Substituting these expressions into equations (15) and denoting the vector of relative prices $P/u_1'(c_y^1)$ by R we obtain

$$\left((1 - \theta_1) I_Y + \sum_{k=1}^{L-1} (\theta_k - \theta_{k+1})(\beta\Pi)^k + \theta_L(\beta\Pi)^L \right) R = 0, \tag{16}$$

where I_Y denotes the $Y \times Y$ identity matrix. A sufficient condition for these equations to have a solution is that the matrix equation

$$(1 - \theta_1) I_Y + \sum_{k=1}^{L-1} \left((\theta_k - \theta_{k+1})\beta^k \right) \Pi^k + (\theta_L \beta^L) \Pi^L = 0 \tag{17}$$

has a solution. If Π has only nonzero eigenvalues, then that fact follows from part (1) of Lemma 1. If Π has a zero eigenvalue, then that fact follows from part (2) of Lemma 1. In this case $\theta_1 = 1$. □

The proof of Theorem 2 makes the intuitive and approximate reasoning at the end of Section 4.2 precise for economies with the special transition matrices satisfying the theorem's assumptions. The proof depicts why a small number of finite-maturity bonds spans the consol when Π is diagonalizable. The spanning issues reduce to properties of Π, independent of the actual prices P, the initial endowments, and the dynamic evolution of the distribution of wealth. While this may appear strange, it follows from the fact that investors' Euler equations tell us that the relative prices of zero coupon bonds are determined by Π, not marginal utility. This fact reduces the spanning issue (equation 14) to the algebraic properties of Π (equation 17); in particular, the issue is how many powers of Π do you need in order to span $I = \Pi^0$. In the case where Π is diagonalizable (our examples below show that to be a reasonable assumption), Lemma 1 ensures that the number of distinct eigenvalues is the minimal number to accomplish that span.

The following corollary to Theorem 2 characterizes equilibrium portfolios.

Corollary 1 *[Corollary to Theorem 2] Suppose the economy's transition matrix Π satisfies the assumptions of Theorem 2. Suppose further that all agents have equi-cautious HARA utilities. If there are bonds of maturities $k = 1, 2, \ldots, L$ in the economy then there is an efficient equilibrium in which agents' portfolios satisfy monetary separation. Moreover, the bond portfolios in this equilibrium satisfy the following properties.*

(a) *If the transition matrix Π has only nonzero eigenvalues, then agent h's holdings of the bonds of maturity $j = 1, 2, \ldots, L$ are*

$$\theta_j^h = \frac{b^h}{M_a}\left(\sum_{k=j}^{L} \beta^{L-k} a_k^*\right)$$

where $M_a = \beta^L + \sum_{k=1}^{L} \beta^{L-k} a_k^*$ and $(a_1^*, a_2^*, \ldots, a_L^*)$ is the unique solution to the matrix equation $I_Y + \sum_{k=1}^{L} a_k \, \Pi^k = 0$.

(b) *If the transition matrix Π has a zero eigenvalue, then agent h holds $\theta_1^h = b^h$ and has holdings of the bonds of maturity $j = 1, 2, \ldots, L$ of*

$$\theta_j^h = \frac{b^h}{M_b}\left(\sum_{k=j}^{L} \beta^{L-k} a_k^*\right)$$

where $M_b = \sum_{k=1}^{L} \beta^{L-k} a_k^*$ and $(a_1^*, a_2^*, \ldots, a_L^*)$ is a nontrivial solution to the matrix equation $\sum_{k=1}^{L} a_k \, \Pi^k = 0$.

Appendix B.1 contains the proof of this corollary. A close examination of the statements of Corollary 1 leads us to a number of observations.

1. Proposition 1 is a simple consequence of Corollary 1, Part (b). With i.i.d. beliefs the Markov transition matrix Π has only $L = 2$ distinct eigenvalues, namely 1 and 0. Case (b) then states that 2 bonds are sufficient to span the consol. Moreover, since $\Pi = \Pi^2$ the pair $a^* = (a_1^*, a_2^*) = (-1, 1)$ is a solution to the matrix equation of Lemma 1, Part (2) leading to $M_b = 1 - \beta$ and so to portfolio holdings of $\theta_1^h = b^h$ and $\theta_1^h = \frac{b^h}{1-\beta}$.

2. Another extreme case is a transition matrix Π with the maximal number of $L = Y$ distinct eigenvalues. In that case the sufficient condition of Theorem 2 and Corollary 1 states that the number of bonds needed to span the consol is exactly the number of states Y . Of course, then the economy with J stocks would have a total of $J + Y$ assets, which exceeds the number of states Y. As a result optimal portfolios will be indeterminate. The portfolio exhibiting two-fund separation is then just one point in the manifold of equilibrium portfolios.

3. As $\beta \to 1$ it follows that $M_a \to 0$ and $M_b \to 0$. It can also be easily seen that $\left(\sum_{k=j}^{L} \beta^{L-k} a_k^*\right) \not\to 0$ for all j. Thus, $|\theta_j| \to \infty$ for all j in case (a) and all $j \geq 2$ in case (b). That is, as the discount factor tends to 1 the bond holdings spanning the consol become unboundedly large.

4. Observe that an agent's bond holdings are proportional to the constant portion of the agent's consumption stream. So, if for two agents $b^2 > b^1 > 0$ then the second agent will have larger positions (in absolute value) of all bonds in the economy.

Next we examine some economically motivated applications of the results from this section.

5.3 Identical Persistence Across Stocks and States

Consider an economy with J stocks that have independent dividend processes. Each stock $j \in \mathcal{J}$ in the economy has the same number D of dividend states. Since the individual dividend processes are independent there is a total of $Y = D^J$ possible states in this economy. The dividends may vary across stocks, but the stocks' $D \times D$ dividend transition matrices, Ξ, are identical.[3] We assume that Ξ has only real nonzero eigenvalues, is diagonalizable, and has l distinct eigenvalues. The Markov transition matrix Π for the economy is then the J-fold Kronecker product (see Appendix B.2) of the individual transition matrix for the dividend states of an individual stock, $\Pi = \Xi \bigotimes \Xi \bigotimes \cdots \bigotimes \Xi = \bigotimes_{j=1}^{J} \Xi$.

Theorem 3 *Consider an economy (as just described) with J independent stocks that each have D (stock-dependent) dividend states with identical diagonalizable transition matrices Ξ having only real nonzero eigenvalues. The matrix Ξ has l distinct eigenvalues. Then bonds of maturities $k = 1, 2, \ldots, L$ span the consol, where $L = \binom{J+l-1}{l-1}$. In the presence of these L bonds, and if all agents have equi-cautious HARA utilities, there exists an efficient equilibrium in which agents' portfolios satisfy two-fund separation.*

Proof: Lemma 2 in Appendix B.2 states that the matrix $\Pi = \bigotimes_{j=1}^{J} \Xi$ has only real nonzero eigenvalues, $L = \binom{J+l-1}{l-1}$ of which are distinct, and is diagonalizable. Theorem 2 and Corollary 1 then imply the statements of the theorem. □

We illustrate Theorem 3 with an example. Each stock $j \in \mathcal{J}$ in the economy has only two dividend states, a "high" and a "low" state. The high and the low dividends may vary across stocks, but the dividend processes have a common transition matrix. We denote the dividend's 2×2 transition matrix by

$$\Xi = \begin{bmatrix} \frac{1}{2}(1 + \xi_H) & \frac{1}{2}(1 - \xi_H) \\ \frac{1}{2}(1 - \xi_L) & \frac{1}{2}(1 + \xi_L) \end{bmatrix}$$

with $\xi_H, \xi_L \in (0, 1)$. This matrix Ξ has $D = 2$ distinct eigenvalues, 1 and $\xi = (\xi_H + \xi_L)/2 < 1$. The Markov transition matrix $\Pi = \bigotimes_{j=1}^{J} \Xi$ for the entire economy has only real nonzero eigenvalues, $J + 1$ of which are distinct. The eigenvalues are $1, \xi, \xi^2, \ldots, \xi^J$. (See Appendix B.2.)

[3] Actually, it would be sufficient for all the individual transition matrices to have the same eigenvalues. The matrices do not have to be identical.

In this economy $J+1$ bonds span the consol. We now examine the weights of the bonds in the portfolio that spans the consol. The formulas of Corollary 1, Part (a), yield closed-form solutions for the individual bond holdings, but they are difficult to assess. (For completeness, we display the closed-form solution for $(a_1^*, \ldots, a_{J+1}^*)$ for small values of J in Appendix B.2.) Therefore, we calculate their numerical values for a few selected values. Tables VII and VIII display the portfolios of finite-maturity bonds that span one unit of the consol for $\beta = 0.95$ and $\beta = 0.99$, respectively.

ξ	0.2				0.5			
J	2	3	4	5	2	3	4	5
θ_1	1.18	1.00	1.00	1.00	6.35	0.28	1.05	1.00
θ_2	−4.57	1.22	0.998	1.00	−33.10	11.67	−0.44	1.09
θ_3	25.67	−4.80	1.23	1.00	49.96	−44.29	15.20	−0.92
θ_4		25.86	−4.85	1.23		56.69	−50.67	17.22
θ_5			25.90	−4.86			60.27	−54.06
θ_6				25.91				62.11

Table VII: Bond Portfolio Spanning one Unit of the Consol, $\beta = 0.95$

ξ	0.2				0.5			
J	2	3	4	5	2	3	4	5
θ_1	2.01	0.99	1.00	1.00	32.92	−3.51	1.30	0.99
θ_2	−29.55	2.26	0.99	1.00	−192.76	64.79	−8.01	1.59
θ_3	129.83	−30.82	2.31	0.99	263.15	−257.14	86.02	−11.01
θ_4		130.87	−31.08	2.32		300.31	−293.91	98.15
θ_5			131.07	−31.13			320.12	−313.51
θ_6				131.12				330.34

Table VIII: Bond Portfolio Spanning one Unit of the Consol, $\beta = 0.99$

The portfolios look similar to our computed results in the previous section. The holdings of bonds with short maturity are close to 1, the spanned position of the consol. But holdings of bonds with higher maturity are much larger and some bonds are even held in a short position. Moreover, as the eigenvalue stemming from the persistence parameters grows, these positions become even larger. The same is true when the discount factor increases. The reason for the (weird) form of the portfolio is that the bond price vectors q^k become more and more collinear as k grows. The spanning condition then requires increasingly larger (in absolute value) weights on these vectors that also have to alternate in sign.

We observe that the weight for the one-period bond converges quickly to 1, as the number of stocks, J, (and bonds, $J+1$) grows. The same is true, albeit at a slower pace, for the other bond weights. The weights are given by the formula of Corollary 1, Part (a).

$$\theta_j = \frac{1}{M_a}\left(\sum_{k=j}^{J+1} \beta^{(J+1)-k} a_k^*\right) = \frac{\sum_{k=j}^{J+1} \beta^{(J+1)-k} a_k^*}{\beta^{J+1} + \sum_{k=1}^{J+1} \beta^{(J+1)-k} a_k^*}.$$

Note that for $j = 1$ the denominator exceeds the numerator by β^{J+1} and so, as J grows, the ratio tends to 1. For $j = 2$ the difference is $\beta^{J+1} + \beta^J a_1^*$ which tends to zero and so the ratio θ_2 also tends to 1. We can make similar arguments for the other bond positions. For a better understanding of the portfolio structure we examine what happens when we let β tend to 1. As we observed before, M_a tends to 0 and the portfolio must explode. But we can compute the ratios of the bond weights.

Proposition 2 *The ratio of bond weights in the portfolio of finite-maturity bonds spanning the consol as β tends to 1 is as follows (for $J \in \{2, 3, 4, 5\}$).*

J	2	3	4	5
θ_2/θ_1	$-\frac{1+\xi}{\xi^2}$	$-\frac{1+\xi+\xi^2}{\xi^3}$	$-\frac{1+\xi+\xi^2+\xi^3}{\xi^4}$	$-\frac{1+\xi+\xi^2+\xi^3+\xi^4}{\xi^5}$
θ_3/θ_1	$\frac{1}{\xi^3}$	$\frac{1+\xi+\xi^2}{\xi^5}$	$\frac{1+\xi+2\xi^2+\xi^3+\xi^4}{\xi^7}$	$\frac{1+\xi+2\xi^2+2\xi^3+2\xi^4+\xi^5+\xi^6}{\xi^9}$
θ_4/θ_1		$-\frac{1}{\xi^6}$	$-\frac{1+\xi+\xi^2+\xi^3}{\xi^9}$	$-\frac{1+\xi+2\xi^2+2\xi^3+2\xi^4+\xi^5+\xi^6}{\xi^{12}}$
θ_5/θ_1			$\frac{1}{\xi^{10}}$	$\frac{1+\xi+\xi^2+\xi^3+\xi^4}{\xi^{14}}$
θ_6/θ_1				$-\frac{1}{\xi^{15}}$

Table IX: Weight Ratios as $\beta \to 1$

Proof: The ratios are given by $\lim_{\beta \to 1} \frac{\theta_j}{\theta_1}$, where θ_j, $j \in \mathcal{J}$ is given by Corollary 1. □

The numbers in Table IX point at the cause for the changing sign pattern in the bond portfolios of Tables VII and VIII. The weights have alternating signs and are very large. This leads to the (large) ratios of alternating signs. These ratios appear, only mitigated by the discount factor β, in the portfolio formulas and so cause the economically weird looking portfolios patterns for bonds of longer maturity in Tables VII and VIII.

6 Approximately Optimal Portfolios with Bond Ladders

The previous section argued that models with many bonds will imply that each investor will hold a fraction of the market portfolio of stocks, an approximately constant holding of short-maturity bonds, but the holdings of long bonds will differ substantially from a constant portfolio and involve large amounts of trading. The implications for long bonds are not intuitive. Also, if there were small transaction costs these large trades in bonds would be substantially reduced. In this section, we examine whether simple strategies can come very close to implementing equilibrium utility.

6.1 Bond Ladders and Asymptotic Two-fund Separation

For general economies, a small number of finite-maturity bonds will be insufficient to obtain two-fund separation. Real-world bond markets offer bonds across many maturities. If there are enough bonds then equilibrium will be Pareto efficient but equilibrium portfolios may deviate from the simple ones recommended by mutual fund separation. If there are many bonds then some assets will be redundant and there will be many portfolios that implement equilibrium consumption allocations. The question is how close can such a market come to producing classic separation results for the equilibrium portfolios. The next theorem states that a portfolio with constant stock holdings and constant bond holdings (consistent with the linear sharing rules) yields the equilibrium consumption allocation in the limit as the number of bonds tends to infinity. This portfolio is an example of the laddering strategy since the bond portfolio is reestablished at each state no matter what the bond prices (and interest rates) are.

Theorem 4 *Assume that there are Y states, J stocks and that the investors have equi-cautious HARA utility functions. Suppose that the economy $\mathcal{E}^B$ has B finite-maturity bonds, and that consumption in an efficient equilibrium follows the linear sharing rules $c^h = m^h e + b^h \cdot 1_Y$, $h \in \mathcal{H}$. Define the portfolios*

$$\begin{aligned} \psi_j^h &= m^h, \quad \forall j = 1, \ldots, J, \\ \theta_k^h &= b^h, \quad \forall k. \end{aligned}$$

Then in the limit as B increases

$$\lim_{B \to \infty} \left(\sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^{B} \theta_k^h (q_y^{k-1} - q_y^k) \right) = c_y^h.$$

Remark: As B increases the number of assets $J+B$ will exceed the fixed number of states, Y, and so the bond price vectors will be linearly dependent. As a result optimal portfolios will be indeterminate. Note that the theorem only examines one particular portfolio, namely one satisfying two-fund separation and bond laddering. (To avoid indeterminate optimal portfolios we could increase the number of states in the limit process in order to keep the number of states and assets identical.)

Proof: Asset prices for bonds and stocks will not depend on B since we are assuming that B is large enough so that the equilibrium in $\mathcal{E}^B$ implements the consumption sharing rules $c^h = m^h e + b^h \cdot 1_Y$ for all B. The budget constraint (1) yields the consumption allocation that is implied by a portfolio with $\psi_j^h = m^h \ \forall j = 1, \ldots, J$, $\theta_k^h = b^h \ \forall k = 1, \ldots, B$, namely

$$\begin{aligned} c_y^h &= \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^{B} \theta_k^h (q_y^{k-1} - q_y^k) \\ &= \sum_{j=1}^{J} m^h d_y^j + b^h (1 - q_y^1) + \sum_{k=2}^{B} b^h (q_y^{k-1} - q_y^k) \\ &= m^h e_y + b^h - b^h q_y^B . \end{aligned}$$

The price q_y^B of bond B is given by the formula (25), see Appendix A. Because $\beta < 1$, $q_y^B \to 0$ as $B \to \infty$ since $\beta^B \to 0$. Thus, $c_y^h \to m^h e_y + b^h$ and the statement of the theorem follows. □

Theorem 4 states that if we have a large number of finite-maturity bonds then the classic portfolio from static separation theory will come arbitrarily close to implementing the equilibrium sharing rule. This result leads us to the conjecture that a portfolio satisfying two-fund separation and a constant portfolio of a large finite number B of bonds of maturities $1, 2, \ldots, B$ is approximately optimal once B becomes sufficiently large. To check this approximation we calculate the changes in agents' welfare from using such a portfolio as opposed to using the optimal portfolio.

6.2 Welfare Measure for Portfolios

Define a utility vector v^h by $v_y^h = u^h(c_y^h)$ for a consumption vector c^h, where c_y^h is the consumption of agent h in state $y \in \mathcal{Y}$. Next define

$$V^h(c^h) = \sum_{t=0}^{\infty} \beta^t \Pi^t v^h = [I - \beta\Pi]^{-1} v^h$$

to be the vector of (total) utility values. If the economy starts in state y_0 then an agent's objective function value over the infinite horizon equals $V_{y_0}^h(c^h)$. Now we can define $C_{y_0}^{h,*}$ to be the consumption equivalent of agent h's equilibrium consumption, which is defined by

$$\sum_{t=0}^{\infty} \beta^t u^h(C_{y_0}^{h,*}) = V_{y_0}^h(m^h e + b^h \cdot 1_Y) \iff C_{y_0}^{h,*} = (u^h)^{-1} \left((1 - \beta) V_{y_0}^h (m^h e + b^h \cdot 1_Y) \right).$$

Similarly, we define a consumption equivalent $C_{y_0}^{h,B}$ for the consumption process that agent h can achieve by holding a portfolio that satisfies two-fund separation and uses a bond laddering strategy for bonds of maturity $1, 2, \ldots, B$. The agent determines this portfolio by solving the maximization problem

$$\max_{\{m,b\}} V_{y_0}^h(me + b \cdot 1_Y) \quad \text{s.t.} \quad \left([I_S - \beta\Pi]^{-1} (P \otimes ((me + b \cdot 1_Y) - c^{h,B})) \right)_{y_0} = 0.$$

The agent is restricted to a two-fund strategy and a bond ladder but is allowed to choose an optimal stock weight $\hat{m}^h$ and bond holdings $\hat{b}^h$ subject to satisfying the infinite-horizon budget constraint (see equation (19) in Appendix A). The prices in the budget constraint are given by the equilibrium prices. We denote the consumption equivalent from this portfolio, which is optimal given the restrictions imposed on the agent, by

$$C_{y_0}^{h,B} = (u^h)^{-1}\left((1-\beta)V_{y_0}^h(\hat{m}^h e + \hat{b}^h \cdot 1_Y)\right).$$

For the welfare comparison of the portfolio with a bond ladder to an agent's equilibrium portfolio we compute the welfare gain of each of these two portfolios relative to the welfare of the agent's initial endowment of stocks. For this purpose we also define a consumption equivalent $C_{y_0}^{h,0}$ for the consumption vector that would result from constant initial stock holdings $\psi^{h,0} \equiv \psi_j^{h,0}$ for all $j \in \mathcal{J}$. Since in our examples we took sharing rules as given we need to calculate supporting initial stock endowments $\psi^{h,0}$ by solving the budget equations.

$$\left([I_S - \beta\Pi]^{-1}(P \otimes ((m^h e + b^h \cdot 1_Y) - \psi^{h,0} e))\right)_{y_0} = 0, \; h = 1, \ldots, H.$$

Again the prices in the budget equation are the equilibrium prices. We denote the consumption equivalent from this initial portfolio by

$$C_{y_0}^{h,0} = (u^h)^{-1}\left((1-\beta)V_{y_0}^h(\psi^{h,0} e)\right).$$

The welfare loss of the portfolio with constant bond holdings $\hat{b}^h$ relative to the optimal portfolio is then given by

$$\Delta C_{y_0}^h = 1 - \frac{C_{y_0}^{h,B} - C_{y_0}^{h,0}}{C_{y_0}^{h,*} - C_{y_0}^{h,0}} = \frac{C_{y_0}^{h,*} - C_{y_0}^{h,B}}{C_{y_0}^{h,*} - C_{y_0}^{h,0}}.$$

6.3 Portfolios with Bond Ladders

We calculate welfare losses for approximately optimal portfolios. In order to connect our results to our previous examples we choose some of the same model specifications as before. We use the power utility functions from Section 4.2 with the resulting linear sharing rules

$$c^1 = \left(\frac{1}{2} - b\right) \cdot e + b \cdot 1_Y \quad \text{and} \quad c^2 = \left(\frac{1}{2} + b\right) \cdot e - b \cdot 1_Y.$$

As before, we normalize stock dividends so that the expected aggregate endowment equals 1 and both agents consume on average half of the endowment. The dividend vectors of the $J = 4$ independent stocks are as follows,

stock	1	2	3	4
high d	1.05	1.08	1.12	1.15
low d	0.95	0.92	0.88	0.85

We let the economy start in state $y_0 = 7$ (since $c_7^1 = c_7^2 = 0.5$). The transition probabilities for all four stocks are those of Section 5.3, that is, all four stocks have identical 2×2 transition matrices. Our analysis in Section 5.3 then implies that markets are complete with $J + 1 = 5$ bonds. The equilibrium portfolios for this economy then follow directly from Theorem 3 and Tables VII and VIII.

For our first set of examples we set $\xi = 0.2$ and so have a persistence probability for a stock's dividend state of 0.6 (see Section 5.3). The discount factor is $\beta = 0.95$. We vary the utility parameters b and γ. Tables X reports the maximal welfare loss (always rounded upwards) across agents, $\Delta C = \max_{h\in\{1,2\}} \Delta C_1^h$, from holding a two-fund portfolio with a bond ladder of length B. (We performed these welfare calculations with standard double precision. Numbers that are too close to computer machine precision to be meaningful are not reported and instead replaced by "≈ 0".)

γ	1		3		5		10	
$B\backslash b$	0.05	0.3	0.05	0.3	0.05	0.3	0.05	0.3
1	1.4 (−4)	1.4 (−4)	1.4 (−3)	1.4 (−3)	7.2 (−3)	7.2 (−3)	4.3 (−2)	4.8 (−2)
2	5.0 (−6)	5.0 (−6)	3.0 (−3)	3.0 (−3)	1.3 (−2)	1.3 (−2)	6.7 (−2)	7.5 (−2)
5	2.4 (−10)	2.4 (−10)	3.2 (−3)	3.2 (−3)	1.4 (−2)	1.4 (−2)	7.1 (−2)	8.1 (−2)
10	8.3 (−13)	1.1 (−13)	2.6 (−3)	2.6 (−3)	1.2 (−2)	1.[illegible] (−2)	6.7 (−2)	7.5 (−2)
30	≈ 0	6.3 (−13)	7.7 (−4)	7.7 (−4)	5.1 (−3)	5.[illegible] (−3)	4.2 (−2)	4.7 (−2)
50	≈ 0	≈ 0	1.6 (−4)	1.6 (−4)	1.4 (−3)	1.[illegible] (−3)	1.7 (−2)	1.9 (−2)
100	≈ 0	≈ 0	1.2 (−6)	1.2 (−6)	1.3 (−5)	1[illegible] (−5)	2.7 (−4)	2.8 (−4)

Table X: Welfare Loss from Bond Ladder ($\xi = 0.2$)

As expected the relative welfare losses decrease to zero as the number B of bonds increases. However, the losses do not decrease monotonically to zero. Recall that the equilibrium portfolios exhibit holdings close to b for the one-period bond but already very different holdings for bonds of other short maturity. A trivial bond ladder of length 1 prescribes a bond holding that is not too far off from the equilibrium holding of approximately b. On the contrary, a bond ladder of length 5, for example, forces a portfolio upon an agent that is very different from the equilibrium portfolio in the holdings of these bonds. At the same time the length of the bond ladder is still too short for the limiting behavior of Theorem 4 to set in. These facts result in the increased welfare losses for $\gamma \geq 3$. Once the ladder gets long enough the welfare losses decrease monotonically to zero. Observe that welfare losses continue to decrease even after sufficiently many bonds are present to ensure market completeness. With $J = 4$ stocks and $Y = 16$ states only 12 bonds are needed to complete the markets. The addition of more long-term bonds improves the performance of bond ladder strategies even though the new bonds do not improve the span of the traded assets. The longer the time to maturity of the longest bond the smaller are both its prices across states and the standard deviation of these prices. The decreasing reinvestment risk results in smaller welfare losses of the bond ladder.

Table XI reports the restricted portfolio weights $(\hat{m}^1, \hat{\psi}^1)$ for agent 1. The last row in the table shows the coefficients of the linear sharing rule which correspond to the holdings of stocks and the consol in an economy with a consol. The agent's holdings deviate

γ	1		3		5		10	
$B\backslash b$	0.05	0.3	0.05	0.3	0.05	0.3	0.05	0.3
1	(.497, .062)	(.482, .370)	(.499, .021)	(.496, .129)	(.500, .013)	(.500, .077)	(.501, .006)	(.507, .036)
2	(.495, .052)	(.470, .311)	(.498, .019)	(.492, .111)	(.500, .012)	(.498, .067)	(.501, .005)	(.505, .031)
5	(.489, .050)	(.433, .300)	(.496, .020)	(.476, .118)	(.498, .012)	(.488, .073)	(.500, .006)	(.501, .034)
10	(.480, .050)	(.380, .300)	(.491, .023)	(.447, .137)	(.495, .015)	(.469, .088)	(.498, .007)	(.491, .042)
30	(.461, .050)	(.265, .300)	(.473, .035)	(.336, .210)	(.479, .027)	(.376, .161)	(.488, .016)	(.432, .095)
50	(.454, .050)	(.223, .300)	(.460, .043)	(.260, .260)	(.465, .038)	(.290, .229)	(.474, .029)	(.346, .172)
100	(.450, .050)	(.202, .300)	(.451, .049)	(.205, .297)	(.451, .049)	(.209, .293)	(.453, .047)	(.218, .284)
(m^1, b)	(.45, .05)	(.2, .3)	(.45, .05)	(.2, .3)	(.45, .05)	(.2, .3)	(.45, .05)	(.2, .3)

Table XI: $(\hat{m}^1, \hat{b}^1)$ for Table X

considerably from these coefficients even when the welfare loss is already very small. For example, if $\gamma = 5$, $b = 0.3$ and $B = 30$, the holdings are $(\hat{m}^1, \hat{b}^1) = (.376, .161)$ instead of $(m^1, b) = (.2, .3)$ even though the welfare loss is only just above 0.5%. This deviation is caused by the reinvestment risk in the longest bond. So, even though a ladder of, for example, 30 bonds comes very close to implementing the equilibrium allocation it uses portfolio weights different from the stock and consol weights to do so.

We recalculated all numbers in Tables X and XI for various sets of parameters. For completion we report in Appendix C results for a larger level of the persistence parameter ($\xi = 0.5$). The results do not change qualitatively. Similarly, changing the discount factor does not result in qualitatively different results.

In summary, holding a portfolio of a bond ladder together with a mutual fund of risky assets gives investors almost the same welfare as the equilibrium investment strategy. The portfolio weights between the mutual fund and the bond ladder differ from the weights between the mutual fund and a consol because of the risk in bond prices even when the time to maturity of the longest bond is similar to that observed in actual markets. We observe an important role for redundant bonds since adding more long-term bonds improves the performance of bond ladder strategies even though the new bonds do not improve the span of the traded assets. And although we did not explicitly model transaction costs we motivate the construction of bond ladders as a sensible investment approach in the face of transaction costs. As we have seen, equilibrium investment strategies imply enormous trading volume in the bond markets which would be very costly in the presence of transaction costs. On the contrary, bond ladders minimize transaction costs since the only transaction costs are those borne at the time the bonds are issued. Therefore, asset redundancy is desirable since it improves the performance of bond ladder strategies which in turn help investors economize on transaction costs.

7 Conclusion

We have reexamined the classical two-fund separation theory in a dynamic general equilibrium model and found that the static results fail to generalize to a dynamic world unless a consol is present, either explicitly or implicitly through dynamic trading of finite-maturity bonds. If a consol does not exist then economies with families of finite-maturity bonds typically exhibit an approximate separation of equity and bond markets but equilibrium portfolios that imply unrealistically large trading volumes in bonds.

We then analyzed the welfare properties of portfolios with bond ladders, a popular investment strategy for fixed-income investments. Welfare losses from (non-equilibrium) portfolios exhibiting classic asset allocations (with bond ladders mimicking a consol) approach zero as the length of the bond ladder increases. In light of these results, we argue that transaction cost considerations make portfolios using two-fund separation and bond laddering nearly optimal investment strategies in dynamic markets.

Appendix

A Equilibrium in Dynamically Complete Markets

We use the Negishi approach (Negishi (1960)) of Judd et al. (2003) to characterize efficient equilibria in our model. Efficient equilibria exhibit time-homogeneous consumption processes and asset prices, that is, consumption allocations and asset prices only depend on the last shock y. Define the vector $P = \left(u_1'(c_y^1)\right)_{y\in\mathcal{Y}} \in \mathbb{R}^S_{++}$ to be the vector of prices for consumption across states $y \in \mathcal{Y}$. We denote the $S \times S$ identity matrix by I_S, Negishi weights by λ^h, $h = 2, \dots, H$, and use $\otimes$ to denote element-wise multiplication of vectors.

If the economy starts in the state $y_0 \in \mathcal{Y}$ at period $t = 0$, then the Negishi weights and consumption vectors must satisfy the following equations.

$$
\begin{aligned}
u_1'(c_y^1) - \lambda^h u_h'(c_y^h) &= 0,\ h = 2, \dots H,\ y \in \mathcal{Y}, && (18)\\
\left([I_S - \beta\Pi]^{-1}(P \otimes (c^h - \sum_{j=1}^{J} \psi_j^{h,0} d^j))\right)_{y_0} &= 0,\ h = 2, \dots, H, && (19)\\
\sum_{h=1}^{H} c_y^h - e_y &= 0,\ y \in \mathcal{Y}. && (20)
\end{aligned}
$$

Once we have computed the consumption vectors we can give closed-form solutions for asset prices and portfolio holdings. The price vector of a stock j is given by

$$q^j \otimes P = [I_S - \beta\Pi]^{-1}\beta\Pi(P \otimes d^j). \tag{21}$$

Similarly, the price of a consol is given by

$$q^c \otimes P = [I_S - \beta\Pi]^{-1}\beta\Pi P. \tag{22}$$

We calculate the price of finite-maturity bonds in a recursive fashion. First, the price of the one-period bond in state y is

$$q_y^1 = \frac{\beta \Pi_{y\cdot} P}{u_1'(c_y^1)} = \frac{\beta \sum_{z=1}^{Y} \Pi_{yz} P_z}{u_1'(c_y^1)}, \tag{23}$$

where $\Pi_{y\cdot}$ denotes row y of the matrix Π. Then the price of the bond of maturity k is

$$q_y^k = \frac{\beta \Pi_{y\cdot} (P \otimes q^{k-1})}{u_1'(c_y^1)} = \frac{\beta \sum_{z=1}^{Y} \Pi_{yz} P_z q_z^{k-1}}{u_1'(c_y^1)}, \tag{24}$$

Repeated substitution yields the bond price formula

$$q_y^k = \frac{\beta^k (\Pi^k)_{y\cdot} P}{u_1'(c_y^1)} = \frac{\beta^k \sum_{z=1}^{Y} (\Pi^k)_{yz} P_z}{u_1'(c_y^1)}. \tag{25}$$

B Technical Details

B.1 Additional Proofs

Proof of Lemma 1: Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_Y$ be the eigenvalues of the matrix Π. Since Π is a transition matrix $\lambda_1 = 1$. Since Π is diagonalizable and nonsingular, $\Pi = C \Lambda C^{-1}$ where C is invertible and Λ is diagonal containing only, but all of, the eigenvalues λ_i. Furthermore, $C^{-1} \Pi^k C = \Lambda^k$ for any $k = 1, 2, \ldots$ (see, for example, Simon and Blume (1994, Theorem 23.7)).

Statement (1). Multiplying the statement's matrix equation by C^{-1} from the left and by C from the right leads to the equivalent system,

$$\sum_{k=1}^{L} a_k \, \Lambda^k = -I_Y.$$

Λ is diagonal and has only L distinct entries. As a result this last system is equivalent to the L-dimensional linear system

$$M \cdot (a_1, \ldots, a_L)^T = -(1_Y)^T,$$

where 1_Y is the Y-dimensional row vector of all ones and

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_2 & (\lambda_2)^2 & \cdots & (\lambda_2)^L \\ \lambda_3 & (\lambda_3)^2 & \cdots & (\lambda_3)^L \\ & & \ddots & \\ \lambda_L & (\lambda_L)^2 & \cdots & (\lambda_L)^L \end{bmatrix}.$$

where we assume w.l.o.g. that $\lambda_1 = 1, \lambda_2, \ldots, \lambda_L$ are the L distinct eigenvalues of Π. Column k contains the corresponding (distinct) eigenvalues of Π^k. The matrix M has full rank L since all eigenvalues are nonzero. Thus, the original matrix equation has a unique solution.

Note that the first equation requires $\sum_{k=1}^{L} a_k = -1$.

Statement (2). Multiplying the statement's matrix equation by C^{-1} from the left and by C from the right implies,

$$\sum_{k=1}^{L} a_k \, \Lambda^k = 0.$$

The diagonal matrix Λ has only $L-1$ distinct nonzero entries. As a result this last system is equivalent to the $(L-1)$-dimensional linear system

$$M' \cdot (a_1, \ldots a_L)^T = 0,$$

where

$$M' = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_2 & (\lambda_2)^2 & \cdots & (\lambda_2)^L \\ \lambda_3 & (\lambda_3)^2 & \cdots & (\lambda_3)^L \\ & & \ddots & \\ \lambda_{L-1} & (\lambda_{L-1})^2 & \cdots & (\lambda_{L-1})^L \end{bmatrix}.$$

where we assume w.l.o.g. that $\lambda_1 = 1, \lambda_2, \ldots \lambda_{L-1}$ are the $L-1$ distinct nonzero eigenvalues of Π. The matrix M' has full row rank $L-1$. Thus, the original matrix equation must have a nontrivial solution. (In fact, the system has a one-dimensional linear solution manifold.) Note that the first equation requires $\sum_{k=1}^{L} a_k = 0$. $\square$

Proof of Corollary 1: In an economy with bonds of maturities $k = 1, 2, \ldots, L$, budget constraint (8) becomes

$$m^h \cdot e + b^h 1_Y = \eta^h \cdot e + \theta_1^h (1_Y - q^1) + \sum_{k=2}^{L} \theta_k^h (q^{k-1} - q^k). \tag{26}$$

A sufficient condition for two-fund separation is $m^h = \eta^h$ for all agents $h \in \mathcal{H}$. (This condition is only sufficient but not necessary since there could be other stock weights $\bar{\eta}^h \neq m^h$.) For this condition to hold agent h's bond portfolio must satisfy

$$b^h 1_Y = \theta_1^h (1_Y - q^1) + \sum_{k=2}^{L} \theta_k^h (q^{k-1} - q^k), \tag{27}$$

that is, the L bonds must span the consol. That fact follows from Theorem 2.

In the proof of Theorem 2 we showed that a sufficient condition for the previous system of equations to have a solution is that the matrix equation

$$(b^h - \theta_1^h) I_Y + \sum_{k=1}^{L-1} (\theta_k^h - \theta_{k+1}^h)(\beta\Pi)^k + \theta_L^h (\beta\Pi)^L = 0. \tag{28}$$

has a solution. Note that the coefficients satisfy $(b^h - \theta_1^h) + \sum_{k=1}^{L-1} (\theta_k^h - \theta_{k+1}^h) + \theta_L^h = b^h$.

Case (a). Suppose the transition matrix Π has only nonzero eigenvalues. Multiply equations $I_Y + \sum_{k=1}^{L} a_k \Pi^k = 0$ (Lemma 1, Part (1)) by β^L to obtain $\beta^L I_Y + \sum_{k=1}^{L} \beta^{L-k} a_k^* (\beta\Pi)^k =$

0 and define the sum of the (new) coefficients to be $M_a = \beta^L + \sum_{k=1}^{L} \beta^{L-k} a_k^*$. Then multiplying through by $\frac{b^h}{M_a}$ yields the expression

$$(\frac{b^h}{M_a}\beta^L) I_Y + \sum_{k=1}^{L} (\frac{b^h}{M_a}\beta^{L-k} a_k^*)\ (\beta\Pi)^k = 0, \tag{29}$$

where the sum of the coefficients $\frac{b^h}{M_a}(\beta^L + \sum_{k=1}^{L} \beta^{L-k} a_k^*)$ equals b^h. Matching the coefficients in equations (28) and (29) gives the expressions of the corollary.

Case (b). Suppose the transition matrix Π has a zero eigenvalue. Multiply equations $\sum_{k=1}^{L} a_k\ \Pi^k = 0$ (Lemma 1, Part (2)) by β^L to obtain $\sum_{k=1}^{L} \beta^{L-k} a_k^*\ (\beta\Pi)^k = 0$ and define the sum of the (new) coefficients to be $M_b = \sum_{k=1}^{L} \beta^{L-k} a_k^*$. Then multiplying through by $\frac{b^h}{M_b}$ yields the expression

$$\sum_{k=1}^{L} (\frac{b^h}{M_b}\beta^{L-k} a_k^*)\ (\beta\Pi)^k = 0, \tag{30}$$

where the sum of the coefficients $\sum_{k=1}^{L} \frac{b^h}{M_b}\beta^{L-k} a_k^*$ equals b^h. Matching the coefficients in equations (28) and (30) yields $\theta_1^h = b^h$ and the other expressions of the corollary. □

B.2 Kronecker Products

Let A be an $n \times p$ matrix and B be an $m \times q$ matrix. Then the Kronecker or direct product $A \bigotimes B$ is defined as the $nm \times pq$ matrix

$$A \bigotimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1p}B \\ a_{21}B & a_{22}B & \cdots & a_{2p}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{np}B \end{bmatrix}.$$

Langville and Stewart (2004) list many useful properties of the Kronecker product. For our purposes we need the following properties.

1. If A and B are stochastic (Markov matrices) then $A \bigotimes B$ is stochastic.
2. $\text{rank}(A \bigotimes B) = \text{rank}(A)\text{rank}(B)$.
3. Let A and B be two square matrices. Let λ (μ) be an eigenvalue of A (B) and x_A (x_B) be the corresponding eigenvector. Then $\lambda\mu$ is an eigenvalue of $A \bigotimes B$ and $x_A \bigotimes x_B$ is the corresponding eigenvector. Every eigenvalue of $A \bigotimes B$ arises as a product of eigenvalues of A and B.
4. If A and B are diagonalizable then $A \bigotimes B$ is diagonalizable.
5. $(PDP^{-1}) \bigotimes (PDP^{-1}) = (P \bigotimes P)(D \bigotimes D)(P^{-1} \bigotimes P^{-1})$

In Sections 4.2 and 5.3 we defined economies with special transition matrices that are J-fold Kronecker products of $D \times D$ transition matrices Ξ, so $\Pi = \Xi \bigotimes \Xi \bigotimes \cdots \bigotimes \Xi = \bigotimes_{j=1}^{J} \Xi$. Property 1 of Kronecker products implies that Π is a stochastic matrix (Markov transition matrix). The following properties of Π follow from the characteristics of Ξ and the listed properties of Kronecker products.

Lemma 2 *Let the transition matrix Π be a J-fold Kronecker product of the matrix Ξ, which has only real nonzero eigenvalues, is diagonalizable, and has l distinct eigenvalues. Then Π has the following properties.*

1. $rank(\Pi) = (rank(\Xi))^J$.
2. *The matrix Π has D^J real nonzero eigenvalues, $\binom{J+l+1}{l-1}$ of which are distinct.*
3. *The matrix Π is diagonalizable, that is, the eigenvector matrix C of Π has full rank D^J.*

In Sections 4.2 and 5.3 we encountered the special case of the 2×2 transition matrix by

$$\Xi = \begin{bmatrix} \frac{1}{2}(1+\xi_H) & \frac{1}{2}(1-\xi_H) \\ \frac{1}{2}(1-\xi_L) & \frac{1}{2}(1+\xi_L) \end{bmatrix}$$

with $\xi_H, \xi_L \in (0,1)$. This matrix Ξ has $D = 2$ distinct eigenvalues, 1 and $\xi = (\xi_H + \xi_L)/2 < 1$. For the computation of bond portfolios we need to find $(a_1^*, a_2^*, \ldots, a_{J+1}^*)$ where

$$(a_1^*, a_2^*, \ldots, a_{J+1}^*)^T = -M^{-1} \cdot 1_{J+1}$$

and

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi & \xi^2 & \cdots & \xi^{J+1} \\ \xi^2 & \xi^4 & \cdots & \xi^{2(J+1)} \\ & & \ddots & \\ \xi^J & \xi^{2J} & \cdots & \xi^{J(J+1)} \end{bmatrix}.$$

We give the solution for $J = 2, 3, 4, 5$ and leave all other cases to the reader and Mathematica. For $J = 2$ the unique weights $a^* = (a_1^*, a_2^*, a_3^*)$ are as follows.

$$a = \left(-\frac{1+\xi+\xi^2}{\xi^2}, \frac{1+\xi+\xi^2}{\xi^3}, -\frac{1}{\xi^3}\right).$$

And for $J = 3$ the unique weights $a^* = (a_1^*, a_2^*, a_3^*, a_4^*)$ are

$$a = \left(-\frac{1+\xi+\xi^2+\xi^3}{\xi^3}, -\frac{1+\xi+2\xi^2+\xi^3+\xi^4}{\xi^5}, -\frac{1+\xi+\xi^2+\xi^3}{\xi^6}, \frac{1}{\xi^6}\right).$$

For $J = 4$ the unique weights $a^* = (a_1^*, a_2^*, a_3^*, a_4^*, a_5^*)$ are

$$\begin{aligned} a \;=\; & \left(-\frac{1+\xi+\xi^2+\xi^3+\xi^4}{\xi^4}, \frac{1+\xi+2\xi^2+2\xi^3+2\xi^4+\xi^5+\xi^6}{\xi^7},\right. \\ & \left. -\frac{1+\xi+2\xi^2+2\xi^3+2\xi^4+\xi^5+\xi^6}{\xi^9}, \frac{1+\xi+\xi^2+\xi^3+\xi^4}{\xi^{10}}, -\frac{1}{\xi^{10}}\right). \end{aligned}$$

Finally, for $J = 5$ the unique weights $a^* = (a_1^*, a_2^*, a_3^*, a_4^*, a_5^*, a_6^*)$ are

$$\begin{aligned} a \;=\; \Bigg(& -\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5}{\xi^5}, \frac{1+\xi+2\xi^2+2\xi^3+3\xi^4+2\xi^5+2\xi^6+\xi^7+\xi^8}{\xi^9}, \\ & -\frac{1+\xi+2\xi^2+3\xi^3+3\xi^4+3\xi^5+3\xi^6+2\xi^7+\xi^8+\xi^9}{\xi^{12}}, \\ & \frac{1+\xi+2\xi^2+2\xi^3+3\xi^4+2\xi^5+2\xi^6+\xi^7+\xi^8}{\xi^{14}}, -\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5}{\xi^{15}}, \frac{1}{\xi^{15}} \Bigg). \end{aligned}$$

C Additional Results for Section 6.2

For the examples in Section 6.2, Tables XII and XII report the analog results for Tables X and XI when the persistence parameter is $\xi = 0.5$. This parameter change does not result in any qualitatively different results. (Again numbers that are too close to computer machine precision to be meaningful are not reported and instead replaced by "≈ 0".)

γ	1		3		5		10	
$B\backslash b$	0.05	0.3	0.05	0.3	0.05	0.3	0.05	0.3
1	7.8 (−4)	7.9 (−4)	5.3 (−6)	5.3 (−6)	9.0 (−4)	9.0 (−4)	1.2 (−2)	1.3 (−2)
2	1.9 (−4)	1.9 (−4)	8.6 (−4)	8.6 (−4)	5.6 (−3)	5.6 (−3)	3.6 (−2)	3.9 (−2)
5	2.2 (−6)	2.3 (−6)	2.8 (−3)	2.8 (−3)	1.3 (−2)	1.3 (−2)	6.6 (−2)	7.4 (−2)
10	1.3 (−9)	1.4 (−9)	2.6 (−3)	2.6 (−3)	1.2 (−2)	1.2 (−2)	6.7 (−2)	7.5 (−2)
30	1.2 (−11)	8.8 (−14)	7.7 (−4)	7.7 (−4)	5.1 (−3)	5.1 (−3)	4.2 (−2)	4.7 (−2)
50	≈ 0	≈ 0	1.6 (−4)	1.6 (−4)	1.4 (−3)	1.4 (−3)	1.7 (−2)	1.9 (−2)
100	≈ 0	≈ 0	1.2 (−6)	1.2 (−6)	1.3 (−5)	1.3 (−5)	2.7 (−4)	2.8 (−4)

Table XII: Welfare Loss from Bond Ladder ($\xi = 0.5$)

γ	1		3		5		10	
$B\backslash b$	0.05	0.3	0.05	0.3	0.05	0.3	0.05	0.3
1	(.496, .095)	(.472, .570)	(.499, .034)	(.492, .203)	(.500, .205)	(.498, .123)	(.501, .010)	(.506, .060)
2	(.494, .064)	(.463, .387)	(.498, .024)	(.489, .141)	(.499, .014)	(.496, .086)	(.501, .007)	(.505, .041)
5	(.489, .051)	(.431, .307)	(.496, .020)	(.475, .121)	(.498, .012)	(.487, .075)	(.500, .006)	(.500, .035)
10	.480, .050)	(.380, .300)	(.491, .023)	(.447, .137)	(.495, .015)	(.469, .088)	(.498, .007)	(.491, .042)
30	.461, .050)	(.265, .300)	(.473, .035)	(.336, .210)	(.479, .027)	(.376, .161)	(.488, .016)	(.431, .095)
50	(.454, .050)	(.223, .300)	(.460, .043)	(.260, .260)	(.465, .038)	(.290, .229)	(.474, .029)	(.345, .172)
100	(.450, .050)	(.202, .300)	(.451, .049)	(.205, .297)	(.451, .049)	(.209, .293)	(.453, .047)	(.218, .284)
(m^1, b^1)	(.45, .05)	(.2, .3)	(.45, .05)	(.2, .3)	(.45, .05)	(.2, .3)	(.45, .05)	(.2, .3)

Table XIII: $(\hat{m}^1, \hat{b}^1)$ for Table XII

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