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OPTIMAL CAPITAL ACCUMULATION
AND
DURABLE GOODS PRODUCTION

by

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ABSTRACT

Production of a durable good by a long-run profit maximizing monopolist
is studied under the assumption that capital investment is irreversible.
The class characterized by this analysis includes durable goods production
with the aid of specialized equipment, durable goods marketing involving
substantial advertising effort, and generation of new technological informa-
tion requiring investment in human capital.

Among the conclusions of the analysis two deserve special attention.
First the marginal condition for purchase of labor resembles the common
marginal condition for employment of capital in production of a nondurable.
This is a consequence of labor's productive contribution being imbedded in
the stock of a durable that has an associated holding cost. Thus, labor
assumes the features of a fixed factor of production when it is employed in
the production of a durable good. Second, in a stationary environment the
stock of the durable accumulates from zero to a peak value above the eventual
steady state level, towards which it then declines. This is a consequence
of the firm's exploiting the initial suitability of its capital stock to
large scale production. Irreversibility of capital investment induces an
expansionary cycle in the stock of a durable good and a reverse cycle in
its rental price. Regulation of the rental price of the durable eliminates
the cyclical behavior of the durable good's stock.

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Introduction

In this paper we analyze a monopolist's production of a durable good
under the supposition that capital investment is irreversible. Situations
characterized by this analysis include production of a durable with specialized
capital equipment, durables for which advertising expenditures constitute
a large share of marketing costs, and production of new information through
investment in human capital; see Arrow [1, pp. 40-41].

Irreversibility of investment constitutes one source (the other being
adjustment costs) of what Jorgenson [5] terms a recursive production
technology -- one for which production possibilities at any point in time
depend on the accumulated stock of capital. Consequences of adjustment costs
on investment behavior have been investigated by Gould [4] and Rothschild
[16] among others while irreversibility has been explored by Arrow [2] and
Nickell [11,12]. In all these analyses it is assumed that current revenue
derives from contemporaneous production rather than from the accumulated
stock of the good, implying that it is nondurable. Recent investigations
of durable goods production, including Feldstein and Rothschild [3],
Kleiman and Ophir [7], Levhari and Srinivasan [9], Levhari and Peles [8],
Schmalensee [17], Parks [13], Ramm [14], and Swan [20,21], have been based,
at least tacitly, on the supposition that the production technology is

decomposable -- one for which production possibilities are independent of the current capital stock. Only Sieper and Swan [19], and Kamien and Schwartz [6] have addressed the question of durable goods production with a recursive production technology. In both analyses a single irreversible investment in an infinitely durable capital input was posited. Sieper and Swan concluded that irreversibility might influence the relationship between market structure and product durability while Kamien and Schwartz found that it did.

Analysis of durable goods production in the presence of irreversible capital investment appears neither to be a trivial extension of previous work in regard to economic content nor does it involve straightforward application of earlier methodology. Our present study focuses on the optimal factor employment policies and output profile of a long run profit maximizing producer of a durable good. The consequences of regulation by means of an imposed product price on these variables is also investigated. Choice of product durability and its relationship to market structure is not addressed.

We employ the common assumption that the firm retains ownership of its product and collects a rental fee for the good's services. This assumption is realistic for durable goods such as housing, computers, copying machines, communications equipment and certain transportation modes. For analytic purposes it avoids the complication of potential purchasers having to correctly assess the good's future price movements. The rental price of the durable is inversely related to its stock. The durable is subject to decay at a constant proportionate rate and is produced with a variable factor, labor, and a fixed factor, capital. The productive capacity

of the capital stock is supposed to deteriorate at a fixed proportionate
rate through time and may be replenished by investment. The purchase
price of capital is assumed to be independent of the level of purchase.

Among the conclusions of our analysis, two deserve special attention.
First, the marginal condition for purchase of labor resembles the marginal
condition for purchase of a fixed factor for producing a nondurable good.
The intuitive reason for this is that labor's productive
contribution becomes imbedded in the stock of the durable which has an
associated holding cost. Thus, since labor is involved in the production
of a durable good, labor's implicit price to the firm, its effective rental,
differs from its purchase price. This point is elaborated below in connection
with expression (16). Second, in a stationary environment the stock of
the consumer durable accumulates from zero to a peak value above the steady
state level, towards which it then declines. Correspondingly, the rental
price of the durable begins high, declines to a trough below its steady
state value and then rises toward its stationary value. The intuitive
reason is that the firm exploits the suitability of its capital stock to
large scale production initially. By contrast, if capital is
infinitely durable, the stock of the consumer durable approaches its
steady state value monotonically from below, see [6], for in that case
its suitability to different levels of production is invariant through time.

Our investigation begins within a nonstationary environ-
ment in which factor prices and the demand for the durable are permitted
to vary through time. We then specialize our study to a stationary
environment and require the production function to be homogeneous of
degree one. This specialization enables us to carry out a phase diagram
analysis to deduce the temporal profile of the stock of durable good.
The results of this analysis are then contrasted with those of the polar
cases of infinitely durable capital and instantaneously vanishing capital,
i.e. when it is a variable factor of production. Certainty regarding
present and future values of all relevant parameters is assumed throughout.
The final section contains a review of our results and suggestions for
further work.

Nonstationary Environment

Production of the durable good at rate $Y(t)$ is governed by a dif-
ferentiable production function

(1) $\quad Y = F(L,K)$

where L denotes the variable factor ("labor") and K the stock of
productive capital. The rate of purchase of new productive capital is
$I(t)$ and capital in place is assumed to deteriorate at a constant exponential
rate δ so that

(2) $\quad K'(t) = I(t) - \delta K(t), \qquad K(0) = K_0$

The firm may select its initial capital stock, as well as subsequent
investment, at unit purchase price $c(t)$ at time t. The unit cost of
labor at time t is $w(t)$.

The durable product is held by the firm and its services rented.
The services of the durable are assumed to decay exponentially at a constant
rate b. Thus the effective stock of durable obeys the differential equation

(3) $Q'(t) = Y(t) - bQ(t) = F(L,K) - bQ$, $Q(0) = 0$

where dependence on time t has been suppressed on the right. The firm

is assumed to begin with no durable accumulated; we are studying a new

firm, or one contemplating a new venture. The rental revenue generated

by an effective stock of durable of $Q(t)$ at time t is $R(Q(t),t)$; we

assume that $R_{QQ} < 0$, where subscripts indicate partial derivatives.

Let

(4) $\rho(t) = \int_0^t r(s)ds$ so $\rho'(t) = r(t)$

where $r(t)$ is the interest rate at t. Then the profit maximizing firm's

problem is to select a nonnegative initial capital stock K_0 and non-

negative purchases of labor $L(t)$ and additional capital $I(t)$ to

(5) maximize $\int_0^\infty e^{-\rho(t)} [R(Q(t),t) - w(t)L(t) - c(t)I(t)]dt - c(0)K_0$

subject to (2) and (3). This problem may be viewed as one of optimal

control, with state variables Q and K and control variables L and

I. Letting $\lambda_1(t)$ and $\lambda_2(t)$ denote the current value multipliers associated

with (3) and (2) respectively, the current value Hamiltonian is

$H = R(Q,t) - wL - cI + \lambda_1(F(L,K) - bQ) + \lambda_2(I - \delta K)$

with dependence of variables on t suppressed. Together with (2) and (3),

necessary conditions for optimality are:

(6) $\partial H/\partial L = -w + \lambda_1 F_L \leq 0$; $L \partial H/\partial L = 0$

so that either $L > 0$ and $w = \lambda_1 F_L(L,K)$

or $L = 0$ and $w \geq \lambda_1 F_L(0,K)$

(7) $\partial H/\partial I = -c + \lambda_2 \leq 0$; $I \partial H/\partial I = 0$

so that either $I > 0$ and $c = \lambda_2$

or $I = 0$ and $c \geq \lambda_2$

(8) $\lambda_1' = (r + b)\lambda_1 - R_Q$

(9) $\quad \lambda_2' = (r + \delta)\lambda_2 - \lambda_1 F_K$

(10) $\quad \lambda_2(0) = c(0)$

The cases of interest are those in which the firm finds it profitable to produce the durable good; we shall assume throughout that the maximum in (5) is positive but bounded above. If further the production function is concave, then the necessary conditions are also sufficient for optimality provided that also $\lambda_1 \geq 0$, since then the Hamiltonian is jointly concave in the state and control variables. The multiplier λ_1 is the marginal valuation of a unit of durable product and will typically be positive. Similarly λ_2 is the marginal valuation to the firm of a unit of productive capital.

The necessary conditions characterize the optimal solution. During a time interval of continuous investment, we have $\lambda_2(t) = c(t)$ from (7). Since this equation must hold throughout the interval of investment, $\lambda_2' = c'$ on the interval. Then (9), upon substitution from the above, becomes

(11) $\quad \lambda_1 F_K = c(r + \delta) - c' \qquad$ while $I > 0$

Now expression (8) together with the transversality condition $\lim_{t \to \infty} e^{-rt} \lambda_1(t) = 0$ (that we assume to hold, for sake of discussion in this section) implies

(12) $\quad \lambda_1(t) = \int_t^{\infty} e^{-(r + b)(s - t)} R_Q(s) ds$

so that, as in [6], $\lambda_1(t)$ is the value at time t of the increment to rentals attributable to a marginal unit produced at t. Thus according to (11) the (total) value of the marginal product of capital equals the rental on capital less capital gains, as in Jorgenson[5]. The marginal valuation of capital's product includes the rental stream it will generate over the future, discounted and taking into account its decay.

A further interpretation of the marginal condition for capital can be achieved. Assuming $|R_Q(\infty)| < \infty$, integration by parts of (12) yields

(13) $\lambda_1(t) = R_Q(t)/(r + b) + \int_t^\infty e^{-(r + b)(s - t)}(dR_Q/ds)ds/(r + b)$

Since $(r + b) \int_t^\infty e^{-(r + b)(s - t)}ds = 1$, we can define the average anti-cipated change in marginal revenue from t forward as

$$\overline{dR_Q(t)/ds} = (r + b) \int_t^\infty e^{-(r + b)(s - t)}(dR_Q/ds)ds$$

which upon substitution into (13) yields

(14) $\lambda_1(t) = R_Q(t)/(r + b) + \overline{dR_Q(t)/ds}/(r + b)^2$

Finally, substitution from (14) into (11) and rearrangement of terms yields

(15) $F_K R_Q = [c(r + \delta) - c'](r + b) - F_K \overline{dR_Q/ds}/(r + b)$

According to (15), along the optimal path the instantaneous marginal revenue product of capital is equated with a modified rental cost of capital minus the present value of its anticipated average change. We observe that the ordinary rental-cost-of-capital term that appears in the square bracket on the right hand side of (15) is itself compounded by a term, $(r + b)$, that we will argue below reflects a holding cost associated with the durable good. The intuitive reason for this compounding is that the productive contribution of capital becomes imbedded in the consumer durable.

Over an interval of time during which labor is purchased, $w = \lambda_1 F_L$ from (6). Differentiating this equation with respect to time and using it and (8) to eliminate λ_1 and λ_1' from the result yields

(16) $F_L R_Q = w(r + b) - w' + \frac{w}{F_L} \frac{dF_L}{dt}$, while $L > 0$

This says that the instantaneous marginal revenue product of a unit of labor equals the current interest rate plus product decay rate multiplied by the current wage, less the change in the wage rate, plus the change in the value of the marginal product of labor. Labor is a fully variable factor of production; yet because its product is durable, the marginal productivity condition for its employment bears some resemblence to the

standard condition for employment of a durable factor; see e.g. Jorgenson

and (11) above. Thus, labor has associated with it a user cost that

reflects the cost of holding the durable to whose production it contributes.

To justify our assertion that $(r + b)$ may be viewed as a holding

cost per dollar value of the durable good, we define $P(Q(t),t)$ as the

unit sale price of the durable given a total stock at time t of $Q(t)$

and $p(Q(t),t)$ as the corresponding rental price per unit time. Under

complete certainty regarding the course of future events, known to all,

the sale price of a unit of durable at t

$$(17) \quad P(Q(t),t) = \int_t^\infty e^{-(r + b)(s - t)} p(Q(s),s)ds$$

equals the discounted value of potential rental revenues derivable from

the durable from t forward. Differentiation of both sides of (17)

with respect to t, yields

$$(18) \quad dP(Q(t),t)/dt = -p(Q(t),t) + (r + b)P(Q(t),t)$$

According to (18) momentary postponement of purchase of the durable's

stock results in an instantaneous decline in its value equal to the loss

of rental revenue and an appreciation equal to the opportunity cost of

holding funds in this way. We can, therefore, interpret $(r + b)$ as the

holding cost per dollar value of the consumer durable.

If both labor and investment goods are purchased during some time

interval, then (6) may be employed to eliminate λ_1 from (11), resulting in

$$(19) \quad F_K/F_L = (c(r + \delta) - c')/ w \qquad \text{while } I > 0 \text{ and } L > 0$$

according to which the usual condition prevails that factors are employed

so the ratio of their marginal products equals the ratio of their current

costs. Thus, the fact that the product is a durable does not affect the

optimal capital-labor ratio.

Neither productive factor is purchased unless its marginal value equals

marginal cost. To find conditions that must hold over intervals during

which a factor is not purchased, we follow the procedure suggested by
Arrow and elucidated by Nickell [12]. Suppose that $I(t) = 0$ optimally
for $t_0 < t < t_1$ but that capital is purchased both immediately before
and subsequent to this period. Then, from (7),

$$(20) \quad e^{-\rho(t) - \delta t} (c(t) - \lambda_2(t)) \geq 0 \quad \text{for} \quad t_0 \leq t \leq t_1$$

with equality holding at t_0 and at t_1

Since

$$d \{e^{-\rho(t) - \delta t}(c - \lambda_2)\}/dt = e^{-\rho(t) - \delta t}[c' + \lambda_1 F_K - (r + \delta)c]$$

it follows by the fundamental theorem of integral calculus that

$$(21) \quad \int_{t_0}^{t} e^{-\rho(t) - \delta t}(c' - (r + \delta)c + \lambda_1 F_K) \, dt \geq 0 \qquad t_0 \leq t \leq t_1$$

$$(22) \quad \int_{t_0}^{t_1} e^{-\rho(t) - \delta t}(c' - (r + \delta)c + \lambda_1 F_K) dt = 0$$

Relations (21) and (22) also imply

$$(23) \quad \int_{t}^{t_1} e^{-\rho(t) - \delta t}(\lambda_1 F_K - (r + \delta)c + c') dt \leq 0, \qquad t_0 \leq t \leq t_1$$

These conditions and the interpretations to follow are analogous to those
found by Arrow and by Nickell between periods of investment for the respective
cases of competitive and monopolistic sellers of a nondurable good. Over
the interval between investment periods, the integral of discounted cost
of capital employment is equal to the discounted marginal value of its
product, where the weighting (discounting) term includes both the interest
rate and the decay rate of the productive factor. Put another way, the
marginal cost - marginal benefit condition (11), that must hold at each
moment of investment must also hold on average over intervals of no investment.
It follows from (21) that it would be profitable to rent another unit
of capital at t_0 for any term short of t_1, while from (23) it is clear
that it would not be worthwhile to rent an additional unit of capital at
any time t, $t_0 < t < t_1$, for use until t_1. The difference between our

conditions and those of earlier investigators lies in the marginal valuation

of capital's product. For nondurable goods sold, the value is just the

marginal revenue; for durable goods rented, the value is the stream of

marginal rentals generated by the durable good.

Likewise suppose labor is employed up to time n_0 and after time

n_1 but not during the interval $n_0 < t < n_1$. Then, from (6) we have

$$(24) \quad e^{-\rho(t) - bt}(w - \lambda_1 F_L) \geq 0 \qquad \text{for } n_0 \leq t \leq n_1$$

with equality at n_0 and at n_1

Since

$$d\{e^{-\rho(t) - bt}(w - \lambda_1 F_L)\}/dt =$$
$$e^{-\rho(t) - bt}[R_Q F_L - (r + b)w + w' - \lambda_1 dF_L/dt]$$

it follows that

$$(25) \quad \int_{n_0}^{t} e^{-\rho(t) - bt}[R_Q F_L - (r + b)w + w' - \lambda_1 dF_L/dt]dt \geq 0, \quad n_0 \leq t \leq n_1$$

and

$$(26) \quad \int_{n_0}^{n_1} e^{-\rho(t) - bt} [R_Q F_L - (r + b)w + w' - \lambda_1 dF_L/dt]dt = 0$$

While labor is not being purchased, its marginal revenue product is on

average equal to its average cost. The components of cost have been

discussed earlier in connection with (16). Note also that the weighting

term includes the rate of decay of the product as well as the rate of interest.

The durability of the product of the variable factor, therefore, plays

a central role in this condition as it did in connection with the condition,

(16), that obtains when labor is employed.

Stationary Environment

The necessary conditions, (2), (3), (6) through (10), in the general

case do not appear to lend themselves to analysis beyond interpretation,

as presented above. Further insight into the solution may be obtained

by assuming a stationary environment and a linearly homogeneous production

function, as we do in the remainder of the paper. Thus we suppose that

$R(Q,t) = R(Q)$ is stationary and w, c, r are constant. Further, define

f by

(27) $F(L,K) = KF(L/K,1) \equiv Kf(X)$ with $X \equiv L/K$

We have

(28) $f(0) = 0, \quad F_L = f'(X) > 0, \quad F_K = f(X) - Xf'(X) > 0$

$f''(X) < 0, \quad \lim_{X \to 0} f'(X) = \infty, \quad \lim_{X \to \infty} f'(X) = 0$

Since $f'' < 0$, we can define the inverse function

(29) $g \equiv (f')^{-1}$ with $g' < 0$

Combining (6), (28) and (29) gives

(30) $L(t) = \begin{cases} 0 & \text{when} \quad \lambda_1(t) \leq 0 \\ \\ K(t)g(w/\lambda_1(t)) & \text{when} \quad \lambda_1(t) > 0 \end{cases}$

Consider first the case in which both L and I are positive over

some interval of time. Then from (19), with (28) and the stationarity

assumptions of this section, we have

(31) $[f(X) - Xf'(X)]/f'(X) = c(r + \delta)/w$

which uniquely determines X. But then λ_1 is a constant, determined through

(30):

(32) $\lambda_1 = w/f'(X)$

Hence, from (8), Q uniquely satisfies

(33) $R'(Q)/(r + b) = \lambda_1$

The foregoing, combined with (3), implies that K is stationary:

(34) $Kf(X) = bQ$

so

(35) $L = XK$

must be also. From (2)

(36) $I = \delta K$

Finally, from (7)

(37) $\lambda_2 = c$

Thus if both L and I are positive on some time interval, then all the variables of the problem are stationary there. These stationary values, satisfying (31) - (37), will be indicated by *.

In the stationary solution just defined, the usual marginal conditions hold. The marginal value of a unit of durable is the present value of the marginal revenue stream it generates, (33). Production maintains the stock of durable, (34), while investment keeps the productive capital intact, (36). This stationary solution cannot obtain initially since $Q(0) = 0$ would not be satisfied. Hence we next consider some initial interval of time $0 < t < t*$ during which no investment occurs (beyond the purchase of the initial plant K_0) but the variable factor is employed.

We now have $I(t) = 0$, for $0 < t < t*$ so (2) gives

(38) $K(t) = K_0 e^{-\delta t}$

and (7) gives

(39) $\lambda_2(t) \leq c$

Substitute (38), (27), and (30) into (3) to get

(40) $Q' = K_0 e^{-\delta t} f(g(w/\lambda_1)) - bQ, \qquad Q(0) = 0$

Equations (8) with $R(Q,t)$ replaced by $R(Q)$ and (40) form a pair of nonautonomous first order linear differential equations with nonconstant coefficients for the simultaneous determination of λ_1 and Q on $[0,t*]$. We suppose there is a solution tending toward the stationary state defined by (31) - (37) and turn next to the boundary conditions available for the solution of this differential equation system.

Since λ_1 and Q must be continuous functions of time,

(41) $\quad \lambda_1(t^*) = \lambda_1^*, \qquad Q(t^*) = Q^*$

Linearity of investment cost may, however, result in a finite upward

jump in K at t^* (with a momentarily infinite $I(t^*)$) so only

(42) $\quad K_0 e^{-\delta t^*} \leq K^*$

is required. Conditions (21), (22) specialize to

(43) $\quad \displaystyle\int_0^t e^{-(r+\delta)t} [\lambda_1(f - Xf') - (r+\delta)c]dt \geq 0 \qquad 0 \leq t \leq t^*$

(44) $\quad \displaystyle\int_0^{t^*} e^{-(r+\delta)t} [\lambda_1(f - Xf') - (r+\delta)c]dt = 0$

Thus, we have four boundary conditions for solution of (8) and (40),

namely $Q(0) = 0$, (41), and (44). (There are also two inequality con-

ditions to be observed, (42) and (43)). If there is a solution of the

problem converging to a stationary state, then these four boundary

conditions together will determine the two constants of integration and

the unknowns K_0 and t^*.

Qualitative Behavior for $0 \leq t \leq t^*$

To ascertain the qualitative behavior of the variables over $[0, t^*]$

a phase diagram will be helpful. Use of the diagram will be somewhat

more difficult than for autonomous problems however. The locus of points

(λ_1, Q) such that $\lambda_1' = 0$ is easiest to determine. From (8), $dQ/d\lambda_1 =$

$(r + b)/R''(Q) < 0$ so the locus is downward sloping. Its intercepts

satisfy $R'(Q) = 0$ when $\lambda_1 = 0$ and $R'(0)/(r+b) = \lambda_1$ when $Q = 0$. To the

right of the locus, $\lambda_1' > 0$, while $\lambda_1' < 0$ to its left.

According to (40), the locus of points (λ_1, Q) such that $Q' = 0$

changes over time. For t fixed, combinations (λ_1, Q) for which

$Q' = 0$ satisfy $b\partial Q/\partial\lambda_1 = -K_0 e^{-\delta t} f'g'w/\lambda_1^2 > 0$ so the locus is upward

sloping. Since $f(0) = 0$, it passes through the origin. The locus

rotates down towards the Q = 0 axis with passage of time. Above the
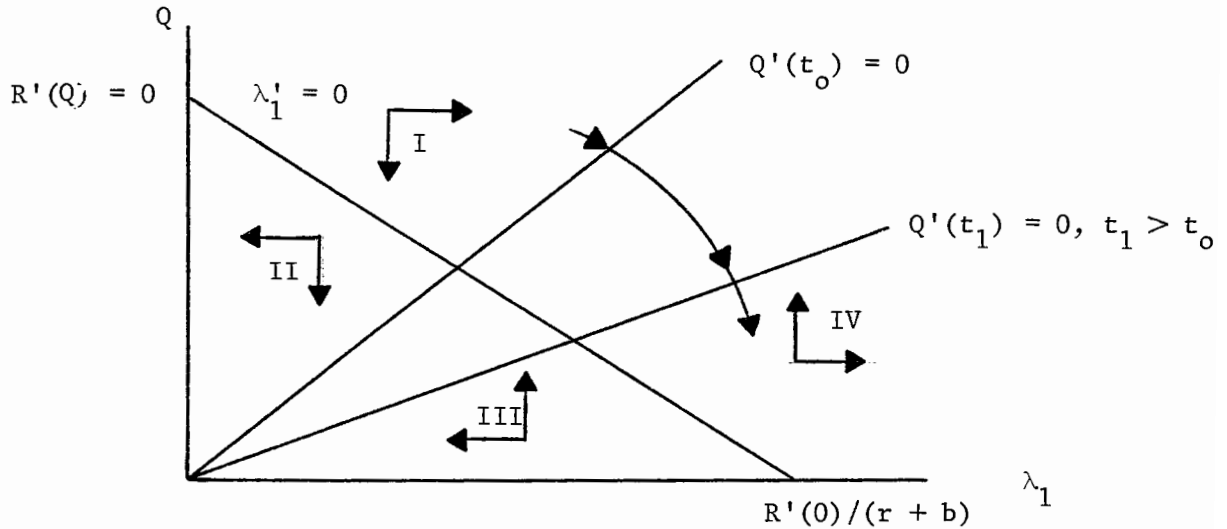
locus, Q' < 0, while Q' > 0 below it.

Figure 1

Although the boundaries of regions will change as the Q' = 0 locus moves,

it will be convenient to denote as regions I, II, III, and IV areas

where the direction of movement would be southeast, southwest, northwest,

and northeast respectively, depicted in Figure 1. But note that a given

point (λ_1, Q) that is in, say, region IV at one moment may later be in

region I as the boundaries change. Regions I and II expand while regions

III and IV shrink through time.

The steady-state point (λ_1^*, Q^*) lies on the $\lambda_1' = 0$ locus at its

intersection with the curve given by $K^*f(g(w/\lambda_1)) = bQ$. The path sought

begins with Q(0) = 0 and terminates at (λ_1^*, Q^*). To establish the

character of that path, we shall first show that a seemingly plausible

pattern is in fact not optimal.

Suppose the path were to approach (λ_1^*, Q^*) monotonically, with the

marginal value of an incremental unit of durable, λ_1, falling steadily

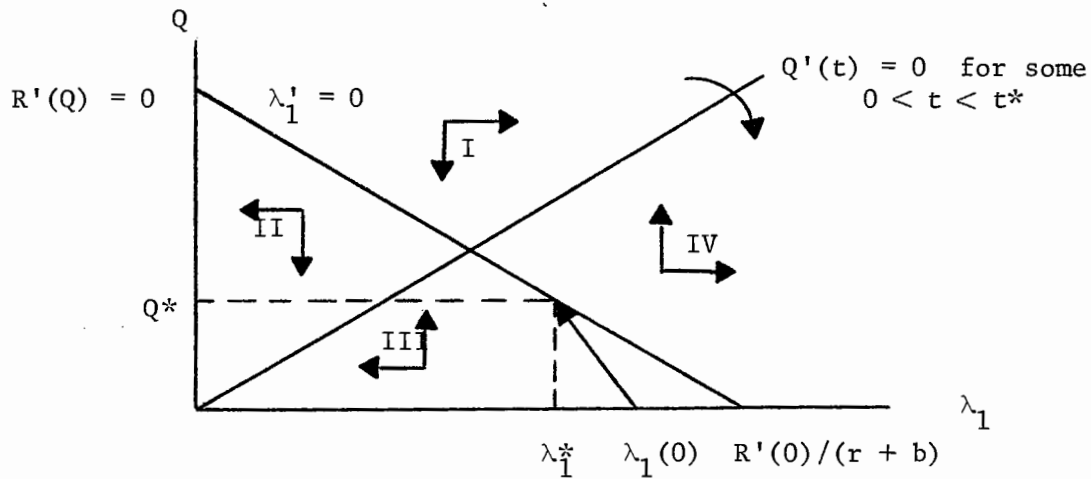as the durable accumulates. Such a path is illustrated in Figure 2.

Figure 2

If such a path were followed, then $\lambda_1'(t) < 0$ for $0 \le t \le t^*$.
From (29) and (30), it then follows that $X'(t) < 0$ for $0 \le t < t^*$.
Therefore, the value of the marginal product of capital, $\lambda_1(f - Xf')$,
must likewise fall over $0 \le t < t^*$, being thereby bounded below by
its value at t^* namely $c(r + \delta)$; see (31) and (32). (Note from
(6), with $L > 0$, that the required continuity of λ_1 implies continuity
of X). Thus

$$(45) \quad \int_0^{t^*} e^{-(r + \delta)t} \lambda_1(f - Xf')dt > \int_0^{t^*} e^{-(r + \delta)t} \lambda_1^*(f - Xf^*)dt =$$

$$= \int_0^{t^*} e^{-(r + \delta)t}(r + \delta)cdt$$

But this contradicts (44). Thus the path shown in Figure 2 cannot
be optimal. If the marginal valuation of capital is to be equal to
its steady state value on average over the period $0 \le t \le t^*$, then
a certain monotone function of the marginal valuation of a unit of
durable, λ_1, must likewise average its steady state value. Speci-
fically, $\lambda_1[f(g(w/\lambda_1)) - g(w/\lambda_1)f'(g)w/\lambda_1))]$ is an increasing function
of λ_1; the average value of this function over $[0,t^*]$ must equal
the steady state value of the function.

It is evident, therefore, that an optimal path converging to (λ_1^*, Q_1^*) must begin in region III, with $Q(0) = 0$ and $\lambda_1(0) < R'(0)/(r+b)$ but must pass out of this region into regions II or IV, before convergence. If the path were to enter region II, it would remain there with both coordinates falling. Thus the path must pass from region III into region IV. A rising path in region IV must eventually touch the falling $Q' = 0$ locus. At the moment of intersection of these two curves, the optimal path of $Q(t)$ is stationary (since $Q' = 0$) while the boundary between regions I and IV continues to fall. Thus the boundary falls below the optimal path, which is then in region I. The path continues falling in region I until it reaches (λ_1^*, Q_1^*). At that moment, t^*, (35) is satisfied and a (possibly infinite) investment sufficiently large to bring the capital stock up to K^* is made. The shift in K will simultaneously bring about a shift in the $Q' = 0$ locus (to satisfy $K^*f(g(w/\lambda_1)) = bQ$), so the point (λ_1^*, Q_1^*) is stationary and the optimal path remains there. This is the only pattern consistent with the necessary conditions and convergence to (λ_1^*, Q_1^*). The path is illustrated in Fig. 3 below.
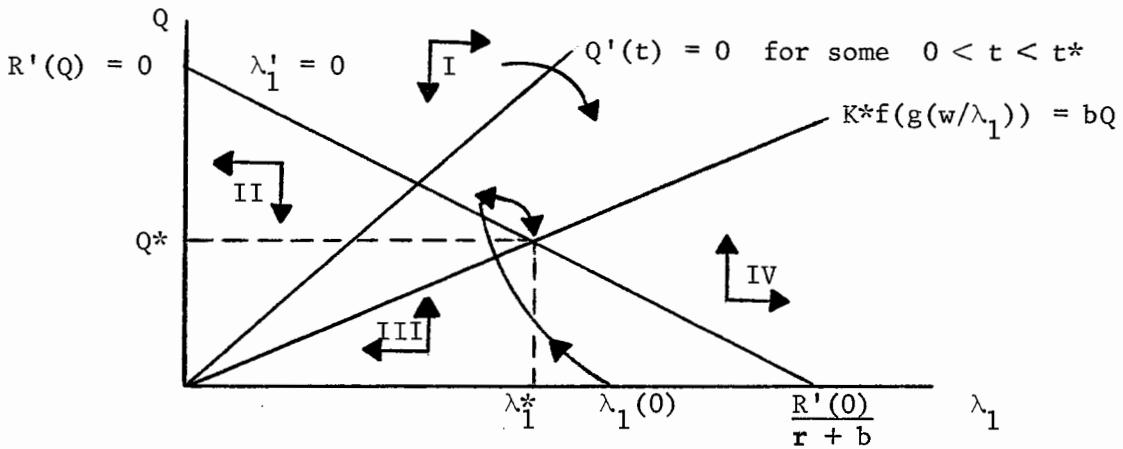

Figure 3

The approximate temporal pattern of $Q(t)$, $\lambda_1(t)$ can be shown as follows. Denote as t_1 the time at which λ_1 reaches its minimum (crossing the stationary $\lambda_1' = 0$ locus) and let t_2 denote the time at which Q achieves its maximum, touching the falling $Q' = 0$ locus.
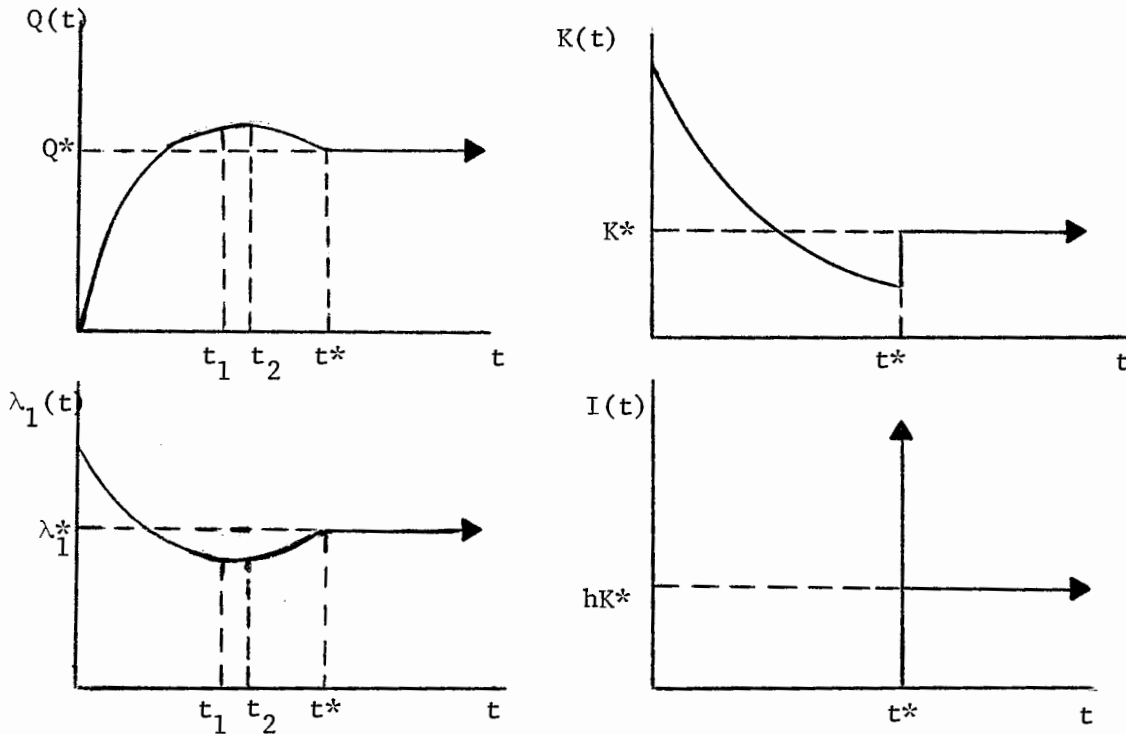

Figure 4

Only the direction of movement of Q and λ_1 is meant to be depicted; no inferences have been made about the concavity or convexity of these curves. Due to (29) - (30), the sketch of $\lambda_1(t)$ is also a sketch of $X(t) = L(t)/K(t)$, since these two variables will move together.

During $0 < t < t_1$ the stock of durable is rising while the marginal value of the durable falls. The output rate is also falling, since it follows from (1) that output falls whenever λ_1 does (but output can fall even if λ_1 is rising.) Over the next interval, $t_1 < t < t_2$, the stock of durable continues to accumulate while the marginal value of increments is rising. Then the stock of durable peaks

and declines towards its stationary value while the marginal value of the durable climbs towards its steady state value. Since the demand for service is a downward sloping function, the rental price varies inversely with the stock of durable.

Intuitively, the accumulation of the stock of durable above its eventual steady state level appears to result from the initial suitability of the capital stock to large scale production. Put another way, production economies of scale are available at the early stages of the accumulation process as a consequence of the large initial purchase of capital. These scale economies diminish through time as the capital stock deteriorates. To further highlight this result, we contrast this case with the polar cases of perfectly flexible capital and of infinitely durable capital.

Case of $\delta \to \infty$ (Perfectly Flexible Capital)

The limiting case of $\delta \to \infty$ is not readily analyzed from the conditions already developed, but the relevant necessary conditions are easily derived nevertheless. In this case, capital is a variable factor of production. The firm's problem can be written as

$$\underset{K \geq 0,\, L \geq 0}{\text{maximize}} \quad \int_0^\infty e^{-rt}[R(Q) - wL - CK]dt$$
$$\text{subject to (3)}$$

This is an optimal control problem with a single state variable Q and two control variables L, K. Form the current value Hamiltonian

$$H = R(Q) - wL - CK + \lambda_1[Kf(L/K) - bQ]$$

where λ_1 is the current value multiplier associated with (3). Necessary conditions for an optimum are

$$(46) \quad \partial H/\partial L = -w + \lambda_1 f' \leq 0; \qquad L(w - \lambda f') = 0$$

(47) $\partial H/\partial K = -C + \lambda_1(f - Xf') \leq 0;$ $K[C - \lambda_1(f - Xf')] = 0$

(48) $\lambda_1' = (r + b)\lambda_1 - R'(Q)$

In this case, both factors must be purchased continuously if output is

to be positive and continuous (recall (28)). If $L > 0$ and $K > 0$,

then from (46) and (47)

(49) $C/w = (f - Xf')/f'$

which has a unique solution for X.   Then

(50) $\lambda_1 = w/f'$

uniquely determines λ_1 so that

(51) $(r + b)\lambda_1 = R'(Q)$

indicating Q is constant also. Hence

(52) $Kf(X) = bQ$

can be solved for K and then

(53) $L = XK$

completes the determination of the variable values. Comparing the

foregoing with (31) - (37) makes it evident that the steady state

position is very nearly the same for $\delta < \infty$ as for $\delta = \infty$ except for

the calculation of the cost of using capital. If capital is durable,

then its cost of use may be spread over its useful life, while if capital

is nondurable its entire purchase price must be considered cost of use

since its useful life is momentary. Presumably, the durability of the

capital good would be reflected in its price, so that δ and c would

be inversely related. If in fact $(r + \delta)c$ were constant for varying

values of δ, with $(r + \delta)c = C$, then the steady state position would

be invariant with respect to δ.

We argued earlier with $\delta < \infty$, that the steady state position would

not be optimally attained immediately due to $Q(0) = 0$. In the present

instance, however, a finite jump of Q from its initial value of zero to its steady state value Q^* can be achieved through an infinite infusion of both K and L in the proportion $L/K = X^*$, where X^* is the solution of (49). Since the production function is linearly homogeneous, the unit cost of production is the same regardless of the output rate so long as the factor proportions is X^*. This case is effectively that discussed by Swan [20, 21].
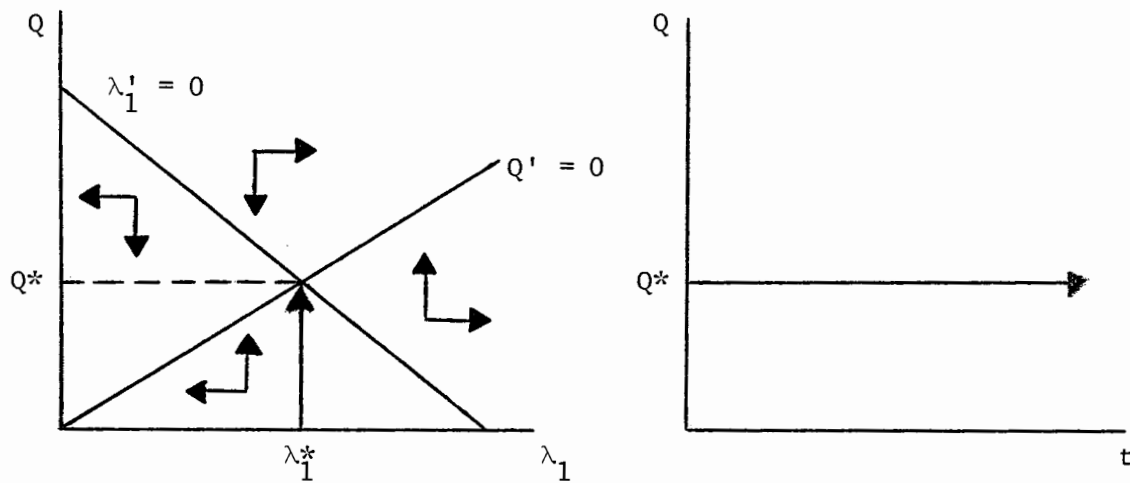

Figure 5

Case of $\delta = 0$ (Infinitely durable capital)

The other polar case is that of perfectly durable capital . The pair of differential equations specialized from (8) and (40) is

$$(8) \quad \lambda_1' = (r + b)\lambda_1 - R'(Q)$$

$$(54) \quad Q' = K_0 f(g(w/\lambda_1)) - bQ, \qquad Q(0) = 0$$

Since the $Q' = 0$ locus is stationary in this special case, the path cannot be of the form illustrated in Figure 3. The stationary solution of the pair (8), (54), denoted $\bar{Q}, \bar{\lambda}_1$ is the simultaneous solution of the equation system

$$(55) \quad R'(\bar{Q})/(r + b) = \bar{\lambda}_1$$
$$K_0 f(g(w/\bar{\lambda}_1)) = b\bar{Q}$$

In addition, from (44), we have

$$(56) \quad \int_0^{\bar{t}} e^{-rt}[\lambda_1(f - Xf') - rc]dt = 0$$

where $\bar{t} \leq \infty$ is the time at which the steady state $\bar{Q}, \bar{\lambda}_1$ is reached.
Thus in case $\delta = 0$, if there is an optimal solution path converging
to a steady state, it must be the solution to (8), (54), using the
boundary conditions $Q(0) = 0$, $Q(\bar{t}) = \bar{Q}$, $\lambda_1(\bar{t}) = \bar{\lambda}_1$ and (56) to find
the two constants of integration, K_0, and $\bar{t}$. Inequality (34) must
hold for $0 \leq t \leq \bar{t}$, with $\delta = 0$. The approach to this steady state
will be monotonic. See Figure 6.

With the aid of (56), we can compare $(\bar{Q}, \bar{\lambda}_1)$ with the values
$(Q*, \lambda_1^*)$ satisfying (31) - (37) with $\delta = 0$. As we argued earlier,
condition (56) requires that a certain increasing function of λ_1 have
an average over $0 \leq t \leq \bar{t}$ of its value at λ_1^*. Since $\lambda_1'(t) < 0$ over
$0 \leq t \leq \bar{t}$, this implies
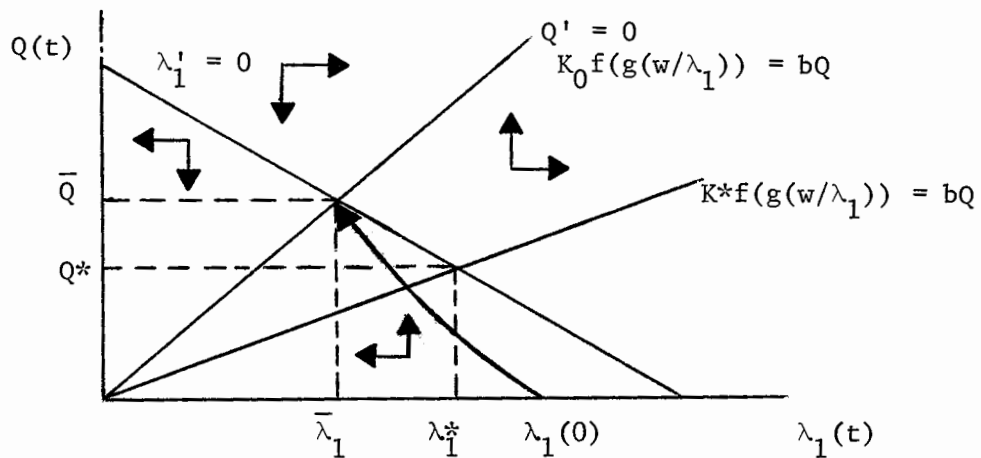
$$(57) \quad \bar{\lambda}_1 < \lambda_1^*$$

Figure 6

Hence, as is clear from Figure 6, we also have

$$(58) \quad \bar{Q} > Q*$$

$$\bar{K}_0 > K*$$

where K_0 denotes the optimal capital stock in this case. This is

effectively the case we discussed in Kamien and Schwartz [6].

Thus, perfect flexibility of capital together with the assumed

constancy of unit prices permits perfect adjustment of scale

to the current production rate and thereby immediate achievement of the

steady state. Infinitely durable capital is, on the other hand, ideal

for only a single rate of production. Since output initially must

exceed the steady state replacement output the capital stock purchased

initially tends to exceed the amount ideal for steady state production

when capital is flexible.

The Case $R'' = 0$

Finally, to gain insight into the role of monopoly power we

consider the optimality conditions for a firm facing a horizontal demand

curve, because of regulation. Thus we replace the assumption

$R'' < 0$ with the supposition that $R'' = 0$, so $R' = p$,

a constant to be specified shortly. In this case, (8) may be integrated

directly to yield

$$\lambda_1(t) = p/(r + b) + e^{(r + b)t}(\lambda_1(0) - p/(r + b))$$

from which it is apparent that either

(59) $\quad \lambda_1(0) = \lambda_1(t) = p/(r + b)$

or else the marginal value of the durable grows or declines without

bound. Let us assume the first situation obtains. (The other two are

cases in which the firm would expand indefinitely or would fail to

produce.) Then X is constant:

(60) $\quad X(t) = g(w(r + b)/p)) \qquad$ or, equivalently

$\qquad w/f'(X) = p/(r + b)$

With λ_1 and X known constants, (9) may be integrated to

$$\lambda_2(t) = A/(r + \delta) + e^{(r + \delta)t}(c - A/(r + \delta))$$

where

(61) $A = \lambda_1(f - Xf')$ with λ_1 and X evaluated at (59),(60).

Again, either

(62) $\lambda_2 = c = A/(r + \delta)$

or else the marginal value of capital grows or falls without bound.

Taking the first case as the one of interest, and combining with (59)

we have assumed in effect that

(63) $\lambda_1 = p/(r + b) = w/f'(X) = c(r + \delta)/(f - Xf')$

according to which the value of the rental stream generated by a unit

of durable is just equal to the marginal cost of production. (The basis

for the assertions above is now clear; if price were larger the firm

would immediately grow indefinitely large, while a smaller price would

bring forth no production.) Since X is constant, it follows that

$$L'/L = K'/K$$

which, combined with (2), implies

(64) $L' = (I/K - \delta)L$

In sum, if the price p is such that (63) can be satisfied, then

λ_1, λ_2 and X are all constants. The rate of investment I and

hence scale is unrestricted, except for the required nonnegativity.

Selection of I determines K through (2), and then L through (64)

and Q through (3).

These conclusions should not be surprising. If the firm operates

with a linearly homogeneous production function and a fixed price, it

either can make a positive unit profit and immediately expands indefin-

itely, or makes a loss per unit and produces nothing, or else it

makes just normal profits and finds its scale of operations a matter of indifference. This last case has been emphasized in this section. The firm's indifference to plant scale and immediate achievement of the steady state stem from the invariance of current demand with respect to past sales.

Summary

We have examined the optimal production of a consumer durable with the aid of a capital stock subject to deterioration. It was found that durability of the product caused labor to be treated by the firm somewhat like a fixed factor of production. This observation may have implications for interpreting current econometric studies of investment behavior and of labor employment, and be suggestive for new studies. The deterioration of capital induces a hump-like profile of the stock of the durable product, with the top of the hump above the eventual steady state level. The rental price of the consumer durable, therefore, falls from its initial level to a low from which it ascends to its steady state value. This implication might be amenable to empirical testing.

Among the possible extensions of this work might be an analysis of the relationship between market structure and the choice of durability along the lines of [6] with allowance for deterioration of the capital stock. Recent generalizations of the traditional model of capital accumulation to take into account obsolescence and maintenance, as in [18], [10] and [12], might also be applied to this model. Lastly, policy questions regarding the consequences of an investment credit tax addressed in [3] and [15] might also be investigated in the context of the model presented in this paper.

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