The Good, The Bad, and The Ugly:
An Inquiry into the Causes and Nature of Credit Cycles

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Abstract

This paper builds models of *nonlinear* dynamics in the aggregate investment and borrower net worth and uses them to study the causes and nature of *endogenous* credit cycles. The basic model has two types of projects: *the Good* and *the Bad*. The Bad is highly productive, but, unlike the Good, it generates less aggregate demand spillovers and contributes little to improve borrower net worth. Furthermore, it is relatively difficult to finance externally due to the agency problem. With a low net worth, the agents cannot finance the Bad, and much of the credit goes to finance the Good, even when the Bad projects are more profitable than the Good projects. This over-investment to the Good creates a boom and generates high aggregate demand spillovers. This leads to an improvement in borrower net worth, which makes it possible for the agents to finance the Bad. This shift in the composition of the credit from the Good to the Bad at the peak of the boom causes a deterioration of net worth. The whole process repeats itself. Endogenous fluctuations occur, as the Good breeds the Bad, and the Bad destroys the Good.

The model is then extended to add a third type of the projects, *the Ugly*, which are unproductive but easy to finance. With a low net worth, the Good competes with the Ugly, creating the credit multiplier effect; with a high net worth, the Good competes with the Bad, creating the credit reversal effect. By combining these two effects, this model generates *intermittency* phenomena, i.e., relatively long periods of small and persistent movements punctuated intermittently by seemingly random-looking behaviors. Along these cycles, the economy exhibits asymmetric fluctuations; it experiences a long and slow process of recovery from a recession, followed by a rapid expansion, and possibly after a period of high volatility, plunges into a recession.

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1. Introduction.

It is commonly argued that an economic expansion often comes to an end as a result of the changing nature of credit and investment at the peak of the boom. According to the popular argument, more credit is extended to finance "socially unproductive" activities. Such an expansion of credit causes volatility and destabilizes the economy. (See Kindleberger 1996 for a review of the popular argument.) Central bankers indeed seem concerned that financial frenzies that emerge after a period of economic expansion might lead to misallocation of credit, thereby pushing the economy into a recession, and they often attempt to take precautionary measures to cool down the boom and to achieve a soft landing of the economy.

This paper develops dynamic general equilibrium models of endogenous credit cycles, which provide a theoretical support for the view that changing compositions of credit and of investment are responsible for creating instability and fluctuations. Furthermore, the equilibrium dynamics display some features reminiscent of the popular argument. Contrary to the popular argument, however, the agents are assumed to be fully rational and instability is not caused by "irrational exuberance." Indeed, fluctuations are not at all driven by the expectations of the agents, whether they are rational or not. In the models developed below, the equilibrium path is unique, and the cycles are purely deterministic. Endogenous fluctuations occur when the unique steady state of the time-invariant, deterministic nonlinear dynamical system loses its stability. They are based on neither 'sunspots" nor "bubbles," nor any form of indeterminacy or self-fulfilling expectations.[2]

Behind instability in our models is the heterogeneity of investment projects. Investment projects differ in many dimensions. They differ not only in profitability. They differ also in the severity of agency problems, and hence are subject to differing self-financing requirements. In addition, they differ in the input requirements, so that they have different general equilibrium effects, with different degrees of aggregate demand spillovers, or "backward linkages," to use Hirschman (1958)'s terminology. As a result, not all the profitable investments contribute equally to the overall balance sheet condition of the economy.

[2]For broad surveys on endogenous cycles, see Boldrin and Woodford (1990) and Guesnerie and Woodford (1992).

For example, suppose that there are two types of profitable investment projects, which we shall call *the Good* and *the Bad*. The Good improves the net worth of the other borrowers in the economy, because it generates demand for their endowment (or, put it differently, the capital created by this type of projects is complementary with their input endowment). The Bad may be more profitable than the Good. Unlike the Good, however, the Bad is "*socially* unproductive" in the sense that they generate less demand for the endowment held by other borrowers (or, the capital created by this type of projects is *not* complementary with their input endowment), hence it contributes little to improve the net worth of other borrowers. In addition, suppose that the Bad projects are subject to self-financing requirements, due to some agency problems. When the net worth is low, the agents are unable to finance the Bad projects, and much of the credit goes to finance the Good projects, even when the Bad projects may be more profitable than the Good projects. This over-investment to the Good projects generates high aggregate demand spillovers, creating a boom and leading to an improvement in borrower net worth. In a boom, with an improved net worth, the agents are now able to finance the profitable-yet-difficult-to-finance, Bad projects. The credit is now redirected from the Good to the Bad. This change in the composition of credit and of investment at the peak of the boom causes a deterioration of borrower net worth. The whole process repeats itself. Along these cycles, the Good breeds the Bad, and the Bad destroys the Good, as in ecological cycles driven by predator-prey or host-parasite interactions.[3] We call these two types of projects the Good and the Bad, not because of their welfare implications. We call them the Good and the Bad, because of the roles they play in the propagation mechanism through their differential general equilibrium price effects.[4] Crucial for generating endogenous fluctuations are: a) some profitable investments contribute little to improve borrower net worth than other profitable investments; and b) these investments are subject to agency problems, which are neither too big nor too small, so that the agents can finance them when and only when their net worth is sufficiently high.

[3]While the intuition behind fluctuations is similar with that of predator-prey cycles in biology, our models are quite different from what mathematical biologists call the predator-prey models (see, e.g., Murray 1990).

[4]Note that we do not assume any negative technological externalities associated with the Bad. We simply define the Good (or the Bad) as the profitable projects that generate (or does not generate) aggregate demand spillovers. In other words, we capture the term "socially unproductive" in the popular argument by the (relative) absence of positive pecuniary externalities of the Bad. No moral connotation is intended by the terms, the Good and the Bad.

Many recent studies in macroeconomics of imperfect credit markets have investigated the role of borrower net worth in the propagation mechanisms of business cycles. Among the most influential is Bernanke and Gertler (1989). Their study, as well as many others, focused on the *credit multiplier* mechanism: how the borrowing constraints introduce *persistence* into the aggregate investment dynamics. In the absence of exogenous shocks, there would be no recurrent fluctuations in their model.[5] The present study, on the other hand, emphasizes the *credit reversal* mechanism: how borrowing constraints introduce *instability* into the dynamics, which causes recurrent fluctuations even in the absence of any external shock. It should be pointed out that the present study and Bernanke-Gertler both share the observation that, in the presence of credit market frictions, saving does not necessarily flow into the most profitable investment projects, and that this problem can be alleviated (aggravated) by a higher (lower) borrower net worth. The two studies differ critically in the assumption on the set of profitable investment projects that compete in the credit market. In the Bernanke and Gertler model, all the profitable investments contribute equally to improve net worth of other borrowers. It is assumed that the only alternative use of saving in their model, storage, is unprofitable, subject to no borrowing constraint, and generates no aggregate demand spillovers. This means that, when an improved net worth allows more saving to flow into the profitable investments, saving is redirected *towards* the investments that generate aggregate demand spillovers, which further improve borrower net worth. This is the mechanism behind the credit multiplier effect in their model (and many others in the literature). The present study departs from Bernanke and Gertler in that *not all* the profitable investments have the same general equilibrium effect. *Some* profitable investments, which are subject to the borrowing constraints, do not help to improve the net worth of other borrowers. This means that, when an improved net worth allows more saving to flow into such profitable investments, saving may be redirected *away from* the investments that generate aggregate demand spillovers, which causes a deterioration of borrower net worth. This is the mechanism behind the credit reversal effect.

[5]In one variation of their models, Kiyotaki and Moore (1997; Section III) demonstrated that the equilibrium dynamics display oscillatory convergence to the steady state, which is why they called their paper, "Credit Cycles." However, these oscillations occur because they added the assumption that the investment opportunity arrives stochastically to each agent. The borrowing constraints in all of their models work only to amplify the movement

Needless to say, these two mechanisms are not mutually exclusive and can be usefully combined. We will indeed present a hybrid model, which allows for three types of projects, *the Good*, *the Bad*, and *the Ugly*. Only the Good generates aggregate demand spillovers and helps to improve the net worth of other borrowers; neither the Bad nor the Ugly improves borrower net worth. Unlike the Bad, the Ugly is not subject to any borrowing constraint, but the Ugly is not as profitable as the Bad. Thus, when the net worth is low, the Good competes with the Ugly, but not with the Bad, so that the credit multiplier mechanism becomes operative, and when the net worth is high, the Good competes with the Bad, but not with the Ugly, so that the credit reversal mechanism become operative. By combining the two mechanisms, the hybrid model generates *intermittency* phenomena. That is to say, relatively long periods of small and persistent movements are punctuated intermittently by seemingly random-looking behaviors. Along these cycles, the economy exhibits asymmetric fluctuations; it experiences a long, slow process of recovery from a recession, followed by a rapid expansion, and, possibly after a period of high volatility, plunges into a recession.

Before proceeding, mention should be made on the exposition. The phenomena analyzed in this paper, endogenous cycles and intermittency, are fundamentally nonlinear and dynamic in nature. The main challenge is to keep the dimensionality of the dynamical system down to make a global analysis of nonlinear dynamics possible. We have also made efforts to minimize the number of the steps needed to derive the nonlinear maps that govern the equilibrium trajectory, and to reduce the notational and algebraic burden to the reader, because presenting nonlinear dynamics is inevitably long and intricate. Whenever specification decisions had to be made, the choice was made for the sake of brevity and simplicity and for the ease and clarity of presentation, even at the risk of giving false impressions that the results were special or empirically implausible. To offset such risk, "*Remarks*" are provided throughout the paper to discuss how the results would carry over under alternative specifications,[6] and how various

caused by shocks, instead of reversing it. In any case, in all of their models, the steady state is stable and any fluctuations will dissipate in the absence of exogenous shocks.

[6] Some of these alternative specifications are presented in companion papers, Matsuyama (2004b, c). We have found both pluses and minuses in these different specifications. First of all, each of these specifications proves to be convenient for highlighting some specific points. Second, although it is slightly simpler to derive the nonlinear maps of the models in the companion papers, these maps are discontinuous, unlike those derived in this paper. While the discontinuity makes it easier to construct *examples* of endogenous cycles, it makes it difficult to characterize the

variables and assumptions can be given alternative interpretations without affecting the formal analysis, even though they would sometimes change empirical implications of the models. The reader mainly interested in understanding the mechanics of the models may want to skip these "*Remarks*," at least at first reading.

Section 2 presents the model of the Good and Bad projects. Then, it derives the dynamical system that governs the equilibrium trajectory under the additional assumption that the Good has no agency problem and hence it is subject to no borrowing constraint. Section 3 characterizes the equilibrium for the full set of parameter values, which enables us to identify the condition under which the steady state loses its stability and endogenous fluctuations occur. The main conclusion is that, when the Bad is sufficiently profitable, instability and fluctuations occur when the agency problem for the Bad is neither too low nor too high. Section 4 presents some examples of chaotic dynamics. Section 5 reintroduces a borrowing constraint for the Good projects. Section 6 develops a model of the Good, the Bad, and the Ugly, which combines both credit multiplier and credit reversal effects and shows how intermittency and asymmetric fluctuations occur. Section 7 offers some concluding comments.

2. The Good and The Bad.

In the basic model, there are two types of investment projects: the Good and the Bad. (A third type, called the Ugly, is introduced in section 6.) The Good and the Bad differ in two dimensions. They have different general equilibrium effects on the net worth of other agents. They may also differ in the agency problem, and hence in the self-financing requirements. To capture these differences in a simple and tractable manner, the following modeling strategies have been adopted.

First, following Bernanke and Gertler (1989), we adopt the Diamond (1965) two-period overlapping generations (OG) model as a basis of our analysis. In the Diamond model, a new generation of agents arrives to the scene in each period with an endowment, called "labor." This gives us a simple way of modeling differential aggregate demand spillovers, or general

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dynamics for the full set of the parameter values, because the standard techniques in mathematics are available only for the continuous maps. The specification in the present paper is the simplest one that we found, which maintains the continuity of the maps, enabling us to applying the standard techniques.

equilibrium price effects between the Good and the Bad, by assuming that labor is used in the former, but not in the latter. What is important is that the agents have some endowments, whose equilibrium values depend on the composition of the current investments. "Labor" in our model should not be literally interpreted. Instead, it should be interpreted more broadly to include "human capital," "land," "patents" or any other endowments or assets held by the potential borrowers, who could sell them or use them as collaterals to satisfy the self-financing requirements of their investments. The two-period OG framework also allows us to abstract from the complication that arises from the presence of the wealth-constrained investments in the intertemporal maximization problem. A "period" in our model should be interpreted as the time it takes to complete a typical investment project.

Second, we introduce the borrowing constraints by assuming that the borrowers may not be able to credibly commit to make a full repayment to the lenders. More specifically, it is assumed, as in Matsuyama (2000a,b, 2004a), that they can pledge only up to a fraction of the project revenue for the repayment. The results do not depend on the particular microeconomic story used to justify the borrowing constraints. One could have instead relied on informational asymmetry, for example, as in the standard moral hazard model, where the success of the investment depends on the hidden effort by the agent, or as in the costly-state-verification approach used by Bernanke and Gertler (1989). These alternatives, however, would require that investments would be subject to idiosyncratic shocks and that some projects would fail and the defaults would occur in positive probability. While these features might make the model descriptively more attractive, they are not an essential part of the story. The present specification has been chosen because it drastically simplifies the exposition and reduces the notational burden, and hence has the advantage of not distracting the reader's attention away from the main objective of this paper, i.e., dynamic general equilibrium implications of credit market frictions.

The detailed description of the model can now be stated.

Time is discrete and extends from zero to infinity ($t = 0, 1, 2, ...$). The economy is populated by overlapping generations of two-period lived agents. Each generation consists of a continuum of agents with unit mass. There is one final good, which is taken as the numeraire and can be either consumed or invested. In the first period, each agent is endowed with one unit of labor, which is supplied to the business sector. The agents consume only in the second. Thus, the aggregate labor supply is $L_t = 1$, and the equilibrium value of their labor endowment, w_t, is also the net worth of the young at the end of period t. The young in period t need to allocate their net worth to finance their consumption in period t+1. The following options are available to them.

First, all the young agents can lend a part or all of the net worth in the competitive credit market, which earns the gross return equal to r_{t+1} per unit. If they lend the entire net worth, their second-period consumption is equal to $r_{t+1}w_t$. Second, some young agents have access to an investment project and may use a part or all of the net worth to finance it. There are two types of projects, both of which come in discrete units. Each young agent has access to at most one type of the project, and each young agent can manage at most one project. More specifically,

The Good: A fraction μ_1 of the young knows how to start a firm in the business sector. Let us call them *entrepreneurs*. Setting up a firm requires one unit of the final good invested in period t. This enables these agents to produce $\phi(n_{t+1})$ units of the final good in period t+1 by employing n_{t+1} units of labor endowment supplied by the younger generation at the competitive wage rate, w_{t+1}. The production function satisfies $\phi(n) > 0$, $\phi'(n) > 0$ and $\phi''(n) < 0$ for all $n > 0$. Maximizing the profit, $\phi(n_{t+1}) - w_{t+1}n_{t+1}$, yields the demand for labor per firm, $w_{t+1} = \phi'(n_{t+1})$. The equilibrium profit from running a firm in period t+1 can thus be expressed as an increasing function of the equilibrium employment, $\pi_{t+1} = \pi(n_{t+1}) \equiv \phi(n_{t+1}) - \phi'(n_{t+1})n_{t+1}$ with $\pi'(n_{t+1}) = -\phi''(n_{t+1})n_{t+1} > 0$.

If $w_t < 1$, these agents need to borrow by $1 - w_t > 0$ in the competitive credit market to start the project. If $w_t > 1$, they can start the project and lend by $w_t - 1 > 0$. In either case, the second-period consumption is equal to $\pi_{t+1} - r_{t+1}(1-w_t)$ if they start the project, which is greater than $r_{t+1}w_t$ (the second-period consumption if they simply lend the entire net worth in the credit market) if and only if

$$(1) \qquad \pi_{t+1} \geq r_{t+1}.$$

The entrepreneurs want to (or are at least willing to) set up firms if and only if the profitability condition, (1), holds.

The Bad: A fraction $\mu_2 \leq 1 - \mu_1$ of the young have access to a project, which requires m units of the final good to be invested in period t and generates Rm units of the final good in period t+1. Let us call them *traders*. Note that, unlike the entrepreneurs, their capital does not require the use of "labor" as the complementary input. We may thus interpret their activities as hoarding the final good for one period to earn the gross return equal to R per unit, without generating any input demand.

If $w_t < m$, these agents need to borrow by $m - w_t > 0$ to start the project. If $w_t > m$, they can start the project and lend by $w_t - m > 0$. Hence, their second-period consumption is equal to $Rm - r_{t+1}(m - w_t)$ as a trader, which is greater than $r_{t+1}w_t$ if and only if

(2) $\quad R \geq r_{t+1}$.

The traders are willing to start their operation if and only if (2) holds.

Remark 1: It is not essential that different agents have access to different projects. This assumption was made solely for the expositional convenience. One could alternatively assume that all the agents are homogenous and have access to both types of projects. As long as it is assumed that no agent can invest both projects simultaneously and that the creditor can observe the type of the investment made by the borrower, the results would carry over, even though it would make the derivation of the equilibrium condition far more complicated. Nor is it essential that each agent can manage at most one project. This assumption reduces the agent's investment decision to a binary choice, which greatly simplifies the analysis. (It does, however, introduce the need for additional parameter restrictions; see (A2) and (A3), as well as *Remark 5*, later.) The assumption of the minimum investment requirement is essential. Without the nonconvexity, the borrowing constraint introduced later would never be binding.[7] The assumption that the two projects may have different minimum requirements does not play any essential role in this paper.[8]

[7] This is partly due to the assumption that all the agents have the same net worth. If there are sufficient mismatches between those who own the endowment and those who have access to the projects, the borrowing constraint could be binding even when the projects are divisible. However, introducing such wealth heterogeneity within each generation would increase the dimensionality of the nonlinear dynamical system, making it impossible to solve it analytically. We have chosen the model with the indivisible projects, because this makes it possible for the borrowing constraint to be binding, even when all the agents have the same wealth, thereby keeping the system one-dimensional.

[8] This is in contrast to the model in Matsuyama (2004b), in which differences in the minimum requirements play a critical role in generating cycles.

Remark 2: Although we associate the Good with setting up business firms conducted by entrepreneurs, and the Bad with the commodity trading conducted by traders, this is solely for the sake of concreteness, and the designations should not be taken literally.[9] The key difference here is in their input requirements, which implies different general equilibrium price effects. Nor should one interpret the "labor" intensity of the production as the key distinction between the Good and the Bad. In the present setting, the young agents are equally endowed with the single input. For example, one could alternatively assume that different young agents are endowed with different types of inputs, which are imperfect substitutes and enter symmetrically in the production as in the Dixit-Stiglitz monopolistic competition model; each young agent, as a sole supplier of its input, sells it to the firms set up by the old entrepreneurs. Then, the Good improves the net worth of the young through an increase in the monopoly profit. Aside from adding more notations and paragraphs, this alternative specification would not affect the dynamical analysis. More generally, one could assume that there are many sectors with different input requirements; the young agents differ in their endowments or assets, which they can use to finance their investments; they may also differ in quality as entrepreneurs or as traders. In such a general setting, the key feature of the projects that determine aggregate demand spillovers that improve borrower net worth would not be the "labor" intensity.[10]

Remark 3: It is possible to give yet another interpretation to the Bad projects. Both the Good and the Bad set up firms in the business sector, but the latter merely generates "private" consumption of Rm to the agents, without producing any (transferable) output. As long as such project does not generate any demand for the endowment held by others, the formal analysis would not need to change. However, it does change empirical implications, because a shift from the Good to the Bad leads to a decline in the measured TFP in the business sector according to this interpretation. For this reason, some readers may prefer this interpretation.[11] Nevertheless,

[9] Of course, we could have simply called them Type-1 and Type-2 projects as well as Type-1 and Type-2 agents. However, we are certain that that would make it harder for the reader to follow the argument.

[10] It is also possible to generate endogenous cycles without different input requirements. For example, in the model of Matsuyama (2004b), the projects are not different in the input requirements. The key difference there is the minimum investment requirement.

[11] Even with this interpretation, one should not automatically jump to the conclusion that the Bad projects have negative welfare implications. That those projects satisfy merely the ego of the agents or produce solely "private" consumption to the agents simply mean that the output of those projects is not transferable; it does not necessarily mean that they are inefficient.

it should be remembered that the key conceptual distinction between the Good and the Bad here is *not* the measured "productivity." Rather, it is the extent to which they improve the net worth of the other borrowers.[12]

The Borrowing Constraints:

The credit market is competitive in the sense that both lenders and borrowers take the equilibrium rate of return, r_{t+1}, given. It is not competitive, however, in the sense that one may not be able to borrow any amount at the equilibrium rate. The borrowing limit exists because the borrowers can pledge only up to a fraction of the project revenue for the repayment. More specifically, the entrepreneurs would not be able to credibly commit to repay more than $\lambda_1\pi_{t+1}$, where $0 \leq \lambda_1 \leq 1$. Knowing this, the lenders would allow the entrepreneurs to borrow only up to $\lambda_1\pi_{t+1}/r_{t+1}$. Thus, the entrepreneurs can start their businesses only if

(3) $\quad w_t \geq 1 - \lambda_1\pi_{t+1}/r_{t+1}$.

The borrowing constraint thus takes a form of the self-financing requirement. The entrepreneurs set up their firms, only when both (1) and (3) are satisfied. Note that (3) implies (1) if $w_t \leq 1 - \lambda_1$ and that (1) implies (3) if $w_t \geq 1 - \lambda_1$. In other words, the profitability is a relevant constraint when $w_t > 1 - \lambda_1$, while the self-financing requirement is a relevant constraint when $w_t < 1 - \lambda_1$. Likewise, the traders would not be able to credibly commit to repay more than λ_2Rm, where $0 \leq \lambda_2 \leq 1$. Knowing this, the lender would allow the traders to borrow only up to λ_2Rm/r_{t+1}. Thus, they cannot start their operations unless

(4) $\quad w_t \geq m[1 - \lambda_2R/r_{t+1}]$.

The traders invest in their operations, only when both (2) and (4) are satisfied. Note that (4) implies (2) if $w_t \leq (1 - \lambda_2)m$ and that (2) implies (4) if $w_t \geq (1 - \lambda_2)m$. Again, the borrowing constraint (4) can be binding only if $w_t \leq (1 - \lambda_2)m$.

The two parameters, λ_1 and λ_2, capture the agency problems associated with the two types of projects and the resulting credit market frictions in a parsimonious way. If they are equal to

[12] For example, one could imagine that some firms that are set up merely to satisfy the ego of the founders might require more inputs, say larger offices and corporate jets, so that they would help to improve the net worth of the suppliers of those inputs. In such a case, it is possible that the projects with a lower "measured" productivity can be the Good, in that sense that they have higher aggregate demand spillovers.

zero, the agents are never able to borrow and hence must self-finance their projects entirely. If they are equal to one, (3) and (4) are never binding, so that they can entirely rely on external finance. By setting these parameters between zero and one, we can deal with the whole range of intermediate cases between these two extremes. The reader may thus want to interpret this formulation simply as a black box, a convenient way of introducing the credit market frictions in a dynamic macroeconomic model, without worrying about the underlying causes of imperfections.[13]

As it turns out, the borrowing constraint for the Good is not essential for generating the credit reversal mechanism that causes instability and fluctuations. We will therefore set $\lambda_1 = 1$ and drop the subscript from λ_2 and let $\lambda_2 = \lambda < 1$ until section 4. This greatly minimizes the notational and algebraic burdens, without changing the results fundamentally. It will be shown in section 5 that, for any fixed $\lambda_2 < 1$, the results are robust to a small reduction in λ_1 from $\lambda_1 = 1$. Allowing $\lambda_1 < 1$ would be crucial for the extension in section 6, which introduces the credit multiplier effect.

Remark 4. The assumption that the Bad faces the tighter borrowing constraints than the Good is made mostly for the expositional reason, but can also be justified in a couple of ways. For example, those who invested in the business sector can pledge most of their project revenue, because they hire labor (and purchase other inputs) and operate in the formal sector, which leave enough of a paper trail of their activities, making it easy for the creditors to seize their revenue when they defaulted. On the other hand, the creditors can seize only a small fraction of the revenue from the trading operation, because it may require nothing but hoarding and stockpiling goods in a hidden place. Another possible interpretation of λ_1 and λ_2 is that the investment projects are partly motivated by the private benefits that accrue to the investors, and hence the lenders value the projects less than the borrowers. According to this interpretation, the Bad

[13] Nevertheless, it is possible to give any number of moral hazard stories to justify the assumption that the borrowers can pledge only up to a fraction of the project revenue. The simplest story would be that the borrowers strategically default, whenever the repayment obligation exceeds the default cost, which is proportional to the project revenue. Alternatively, each project is specific to the borrower, and the productivity of the project would be only a fraction without his services. Then, the borrower, by threatening to withdraw his services, can renegotiate the repayment obligation down. See Hart and Moore (1994) and Kiyotaki and Moore (1997). It is also possible to use the costly-state verification approach of Townsend (1979), used by Bernanke and Gertler (1989) or the standard ex-ante moral

includes the projects primarily driven by the empire-building motives of the investors. Again, the formal analysis would not need to change under this interpretation, except that λ_1 and λ_2 are no longer free parameters, because they are determined by the output structure of the projects (see Matsuyama 2004c). Of course, the empirical implications would be different under this interpretation, see *Remark 3*.

Equilibrium Wage and Business Profit:

Let $k_{t+1} \le \mu_1$ be the number of young entrepreneurs in period t that start their firms (hence it is the number of active firms in period t+1). Let $x_{t+1} \le \mu_2$ be the number of young traders in period t that start their operations. (The aggregate investment they make is thus equal to mx_{t+1}.) Since only the firms hire labor, the labor market equilibrium in period t+1 is $n_{t+1}k_{t+1} = 1$, from which $n_{t+1} = 1/k_{t+1}$. Thus, the equilibrium wage rate and the business profit per firm in period t+1 may be expressed as functions of k_{t+1}:

(5) $w_{t+1} = \phi'(1/k_{t+1}) \equiv W(k_{t+1})$

(6) $\pi_{t+1} = \pi(1/k_{t+1}) = \phi(1/k_{t+1}) - \phi'(1/k_{t+1})/k_{t+1} \equiv \Pi(k_{t+1})$,

where $W'(k_{t+1}) > 0$ and $\Pi'(k_{t+1}) < 0$. A higher business investment means a high wage and a lower profit. Note that the investment in the business sector, the Good, generates labor demand and drives up the wage rate, thereby improving the net worth of the next generation of the agents. In contrast, trading, the Bad, contributes nothing to the net worth of the next generation.

It is straightforward to show that these functions satisfy $\phi(1/k)k = k\Pi(k) + W(k)$ and $k\Pi'(k) + W'(k) = 0$ as the identities. In addition, we make the following assumptions.

(A1) There exists $K > 0$, such that $W(K) = K$ and $W(k) > k$ for all $k \in (0, K)$.

(A2) $K < \mu_1$.

(A3) $\max_{k \in [0, K]} \{W(k) - k\} < m\mu_2$.

(A4) $\lim_{k \to +0} \Pi(k) = +\infty$.

For example, let $\phi(n) = (Kn)^{\beta}/\beta$, with $K < \mu_1$ and $0 < \beta < 1$. Then, (A1), (A2) and (A4) are all satisfied. (A3) is also satisfied if $K < (m\mu_2)/\beta(1-\beta)^{(1-\beta)/\beta}$. (A1) is introduced only to rule out an

hazard problem, where the effort made by the borrower affects the success probability of the project, and the ex-post moral hazard problem used by Holmstrom and Tirole (1997).

uninteresting case, where the dynamics of k_t would converge to zero in the long run. It will be shown later that, if $k_t \in (0, K]$, $k_s \in (0, K]$ for all $s > t$, so that K may be interpreted as the upper bound for the number of firms that the economy could ever sustain. Thus, (A2) means that the economy never runs out of the potential supply of the entrepreneurs. In other words, (A2) ensures that it is not the scarcity of the entrepreneurial talents, but the scarcity of the saving and of the credit that will drive the dynamics of business formation in this economy. (A3) may be interpreted similarly. It ensures that the aggregate investment in trading is potentially large enough, so that there are always some inactive traders in the steady state. It turns out that dropping (A3) would not affect the results fundamentally, but would drastically increase the number of the cases that need to be examined. (A4) ensures that some entrepreneurs invest in equilibrium, $k_{t+1} > 0$.

Remark 5: (A2) and (A3) help to remove the unwanted implication of the assumption that each agent can manage at most one project. This assumption, which reduces the agent's investment choice to a zero-one decision, is made for the analytical simplicity. Both (A2) and (A3) would not be needed if the agents were allowed to invest at any scale, subject only the minimum investment requirement. It turns out, however, that such an alternative specification would make the model algebraic cumbersome. It should also be noted that these assumptions can be weakened significantly. (A2) can be replaced by $W(\min\{K, k_c\}) < \mu_1$ and (A3) by $W(k_{cc}) - k_{cc} < m\mu_2$, where k_c and k_{cc} are the values defined later. (A2) and (A3) are chosen simply because k_c and k_{cc} depends also on R and λ_2, hence the meanings of these alternative assumptions may not be immediately apparent to the reader.

The Investment Schedules:

Because we have set $\lambda_1 = 1$, the borrowing constraint for the entrepreneurs, (3), is never binding, whenever (1) holds, and (1) always holds because of (A4). If (1) holds with the strict inequality, all the entrepreneurs start firms. If (1) holds with the equality, they are indifferent. Therefore, the investment schedule by the entrepreneurs is given simply by the following complementarity slackness condition,

(7) $\quad 0 < k_{t+1} \leq \mu_1, \qquad \Pi(k_{t+1}) \geq r_{t+1},$

which is illustrated in Figures 1a through 1c. As shown below, (A1) and (A2) ensure that $k_{t+1} < \mu_1$ and $\Pi(k_{t+1}) = r_{t+1}$ in equilibrium. The investment demand schedule by the entrepreneurs is thus downward-sloping in the relevant range. Thus, the return to business investment declines when more firms are active.

We now turn to the investment schedule by the traders. First, let us define $R(w_t) \equiv R/\max\{(1 - w_t/m)/\lambda, 1\}$, so that

$$R(W(k_t)) = \begin{cases} \lambda R/[1 - W(k_t)/m] & \text{if } k_t < k_\lambda, \\ R & \text{if } k_t \geq k_\lambda, \end{cases}$$

where k_λ is defined implicitly by $W(k_\lambda) \equiv (1 - \lambda)m$. Figure 2 illustrates the function, $R(W(k_t))$. If $r_{t+1} < R(W(k_t))$, both (2) and (4) are satisfied with the strict inequality, so that all the traders start the trading operation. If $r_{t+1} > R(W(k_t))$, at least one of the conditions is violated, so that no one starts the trading operation. A fraction of the traders starts their operation, if and only if $r_{t+1} = R(W(k_t))$. In words, $R(W(k_t))$ is the rate of return that the lenders can expect from the credit extended to the trading operation. Note that $R(W(k_t))$ is constant and equal to R for $k_t \geq k_\lambda$, when the profitability constraint, (2), is more stringent than the borrowing constraint, (4). On the other hand, it is increasing in k_t for $k_t < k_\lambda$, when the borrowing constraint, (4), is more stringent than the profitability constraint, (2). With a higher net worth, the traders need to borrow less, which means that they can credibly offer a higher rate of return to the lender.

The investment schedule by the traders may thus be expressed as

$$(8) \qquad mx_{t+1} \begin{cases} = m\mu_2 & \text{if } r_{t+1} < R(W(k_t)), \\ \in [0, m\mu_2] & \text{if } r_{t+1} = R(W(k_t)), \\ = 0 & \text{if } r_{t+1} > R(W(k_t)). \end{cases}$$

In each of Figures 1a through 1c, eq. (8) is illustrated as a step function, which graphs $W(k_t) - mx_{t+1}$.

The Credit Market Equilibrium:

The credit market equilibrium requires that r_{t+1} adjust to equate the aggregate investment and the aggregate saving, i.e., $k_{t+1} + mx_{t+1} = w_t$, or equivalently

(9) $k_{t+1} = W(k_t) - mx_{t+1}$.

Figures 1a through 1c illustrate three alternative cases, depending on the value of k_t.[14]

In Figure 1a, $W(k_t)$ is sufficiently low that $R(W(k_t)) < \Pi(W(k_t))$. Thus, the net worth of the traders is so low that they cannot finance their investment ($x_{t+1} = 0$) and all the savings are channeled into the investment in the business sector ($k_{t+1} = W(k_t) < \mu_1$). The required rate of return in equilibrium is too high for the traders ($r_{t+1} = \Pi(W(k_t)) > R(W(k_t))$). This case occurs, when $k_t < k_c$, where k_c is defined uniquely by $R(W(k_c)) \equiv \Pi(W(k_c))$.

In Figure 1b, $\Pi(W(k_t)) < R(W(k_t)) < \Pi(W(k_t) - m\mu_2)$ and the equilibrium rate of return is equal to $r_{t+1} = R(W(k_t)) = \Pi(k_{t+1}) = \Pi(W(k_t) - mx_{t+1})$ and $0 < x_{t+1} < \mu_2$. This occurs when $k_c < k_t < k_{cc}$, where k_{cc} ($> k_c$) is defined uniquely by $R(W(k_{cc})) \equiv \Pi(W(k_{cc}) - m\mu_2)$. This is the case where some, but not all, traders invest. An increase in k_t thus has the effect of further increasing the investment in trading. Its effect on business investment depends whether k_t is higher or less than k_λ. If $k_t > k_\lambda$, the borrowing constraint of the traders is not binding, so that the rate of return is fixed at $R(W(k_t)) = R$. Thus, the investment in the business sector remains constant at $\Pi^{-1}(R)$. On the other hand, if $k_t < k_\lambda$, the borrowing constraint for the traders is binding, so that $R(W(k_t))$ increases with k_t. A higher net worth eases the borrowing constraint of the traders, so that they can guarantee a higher rate of return to the lenders. As a result, business investment is squeezed out. In short, k_{t+1} is a decreasing function of k_t if $k_c < k_t < k_{cc}$ *and* $k_t < k_\lambda$.

Finally, in Figure 1c, $W(k_t)$ is sufficiently high that $R(W(k_t)) > \Pi(W(k_t) - m\mu_2) = r_{t+1}$, hence $x_{t+1} = \mu_2$ and $k_{t+1} = W(k_t) - m\mu_2$. This occurs when $k_t > k_{cc}$. This is the case where the net worth is so high that all the traders invest. Given that the trading opportunities are exhausted, an increase in the saving translates to an increase in business investment. Hence, k_{t+1} increases with k_t in this range. This situation occurs as an unwanted by-product of the assumption that the traders can manage at most one trading operation, which was made to simplify the analysis of the trader's decision problem. Note, however, that we have imposed (A3) to ensure that $k_{t+1} = W(k_t) - m\mu_2 < k_t$ in this range, so that this situation would never occur in the neighborhood of the steady state.

Remark 6: A Digression on Credit Rationing: For the case shown in Figure 1b, where $r_{t+1} = \Pi(k_{t+1}) = R(W(k_t))$, only a fraction of the traders starts their operation. When $k_t \geq k_\lambda$, $r_{t+1} = R$ holds in equilibrium, and (2) is thus satisfied with equality. Some traders invest while others do not, simply because they are indifferent. When $k_t < k_\lambda$, $r_{t+1} = \lambda R/[1 - W(k_t)/m] < R$, hence (4) is binding, while (2) is satisfied with strict inequality. In other words, all the traders strictly prefer borrowing to invest, rather than lending their net worth to others. Therefore, the equilibrium allocation necessarily involves credit rationing, where a fraction of the traders are denied the credit. Those who denied the credit cannot entice the potential lenders by promising a higher rate of return, because the lenders would know that the borrowers would not be able to keep the promise. It should be noted, however, that equilibrium credit rationing occurs in this model due to the homogeneity of the traders. The homogeneity means that, whenever some traders face the borrowing constraint, all the traders face the borrowing constraint, so that coin tosses or some random devices must be evoked to determine the allocation of the credit. [15] Suppose instead that the traders were heterogeneous in some observable characteristics. For example, suppose each young trader receives, in addition to the labor endowment, the final goods endowment, y, which is drawn from G, a cumulative distribution function with no mass point. Then, there would be a critical level of y, $Y(w_t, r_{t+1}) \equiv m(1 - \lambda R/r_{t+1}) - w_t$, such that only the traders whose endowment income exceed $Y(w_t, r_{t+1})$ would be able to finance their investment. This makes the aggregate investment in trading, $mx_{t+1} = m[1 - G(Y(w_t, r_{t+1}))]$, smoothly decreasing in r_{t+1}, and increasing

[14]Figures 1a-1c are drawn under the assumption, $W(k_t) < \mu_1$, which ensures $k_{t+1} < \mu_1$ in equilibrium. This assumption will be verified later. These figures are also drawn such that $W(k_t) > m\mu_2$. In the cases of Figures 1a and 1b, this need not be the case, but it does not affect for the discussion in the text.

[15] While some authors use the term, "credit-rationing," whenever some borrowing limits exist, here it is used to describe the situation that the aggregate supply of credit falls short of the aggregate demand, so that some borrowers cannot borrow up to their borrowing limit. In other words, there is no credit rationing if every borrower can borrow up to its limit. In such a situation, their borrowing may be constrained by their net worth, which affects the borrowing limit, but not because they are credit-rationed. This is consistent with the following definition of credit rationing by Freixas and Rochet (1997, Ch.5), who attributed it to Baltensperger: "some borrower's demand for credit is turned down, even if this borrower is willing to pay all the price and nonprice elements of the loan contract."

in w_t. Thus, the borrowing constraint would be enough to determine the allocation of the credit, and credit rationing would not occur. What is essential for the following analysis is that, when the borrowing constraint is binding for marginal traders, an increase in the net worth of the traders increases the aggregate investment in trading, for each r_{t+1}. Therefore, it is the borrowing constraint, not the equilibrium credit rationing per se, that matters. The equilibrium credit rationing is nothing but an artifact of the homogeneity assumption, which is imposed to simplify the analysis.

The Equilibrium Trajectory:

As should be clear from Figures 1a-1c, $k_{t+1} = W(k_t)$ if and only if $k_t \leq k_c$; $\Pi(k_{t+1}) = R(W(k_t))$ if and only if $k_c \leq k_t \leq k_{cc}$; and $k_{t+1} = W(k_t) - m\mu_2$ if and only if $k_t \geq k_{cc}$. These observations can be summarized as follows:

$$(10) \qquad k_{t+1} = \Psi(k_t) \equiv \begin{cases} W(k_t) & \text{if } k_t \leq k_c, \\ \Pi^{-1}(R(W(k_t))) & \text{if } k_c < k_t \leq k_{cc}, \\ W(k_t) - m\mu_2 & \text{if } k_t > k_{cc}. \end{cases}$$

Equation (10) determines k_{t+1} uniquely as a function of k_t. Since $k_t \leq K$ implies $k_{t+1} = \Psi(k_t) = W(k_t) - mx_{t+1} \leq W(k_t) \leq W(K) = K$, Ψ maps $(0,K]$ into itself. Thus, for any $k_0 \in (0, K]$, this map defines a unique trajectory in $(0, K]$. Furthermore, $k_t \leq K$ and (A2) mean that $\mu_1 > K = W(K) \geq W(k_t)$, as has been assumed.

The equilibrium trajectory of the economy can thus be solved for by applying the map (10), Ψ, iteratively, starting with the initial condition, $k_0 \in (0, K]$. This completes the description of the model. We now turn to the characterization of the equilibrium dynamics.

3. The Dynamic Analysis.

It turns out that there are five generic cases of the equilibrium dynamics, as illustrated by Figure 3a through Figure 3e.[16] Figure 3a depicts the case, where $k_c \geq K$, so that $k_{t+1} = W(k_t)$ for all $k_t \in (0, K]$. Thus, from the monotonicity of W and (A1), k_t converges monotonically to $k^* = K$ for any $k_0 \in (0,K]$. This is the case, where the traders never become active and all the saving goes to the investment in the business sector. The condition, $k_c \geq K$, can be rewritten as $\Pi(K) \geq R(W(K)) = R(K)$, or equivalently

(11) $\quad R \leq \Pi(K)\,\mathrm{Max}\{(1 - K/m)/\lambda, 1\}$.

With a sufficiently small R, the trading operation is not profitable and never competes with business investment for the credit. When $W(K) = K < m$, the condition (11) is also met when λ is sufficiently small for any R. This is because the traders must borrow to start their operations even when the net worth reaches its highest possible value. If λ is sufficiently small, they can never borrow, and hence they can never invest, and hence all the saving goes to business investment, even when $k_{t+1} = W(k_t) > \Pi^{-1}(R)$ so that the trading operation is more profitable than the business investment.

In the other four cases, $k_c < K$ holds, so that some traders become eventually active; $x_{t+1} > 0$ for $k_t \in (k_c , K]$. Figure 3b depicts the case, where $k_\lambda \leq k_c$ or equivalently, $W(k_c) \geq (1 - \lambda)m$, which can be rewritten as

(12) $\quad R \leq \Pi((1 - \lambda)m)$.

Under this condition, $W(k_t) > (1 - \lambda)m$ and $R(W(k_t)) = R$ for all $k_t > k_c$. This means that the borrowing constraint is not binding for the traders, whenever they are active. Eq. (10) is thus simplified to

(13) $$k_{t+1} = \Psi(k_t) = \begin{cases} W(k_t), & \text{if } k_t \leq k_c \\ \Pi^{-1}(R) = W(k_c), & \text{if } k_c < k_t \leq \min\{k_{cc}, K\} \\ W(k_t) - m\mu_2. & \text{if } k_{cc} < k_t \leq K. \end{cases}$$

[16] Figure 3a through Figure 3e are drawn such that W(0) = 0 and W is concave. These need not be the case. (A1) assumes only that $W(k) > k$ for all $k \in (0, K]$ and $W(K) = K$.

As shown in Figure 3b, the map has a flat segment, over (k_c , min{k_{cc}, K}), but it is strictly increasing elsewhere. Furthermore, (A3) ensures $k_{cc} > W(k_{cc}) - m\mu_2$, so that the steady state is located at the flat segment.[17] The dynamics of k_t hence converges monotonically to the unique steady state, $k^* = \Pi^{-1}(R) = W(k_c)$. As the business sector expands, borrower net worth improves and the profitability of business investment declines. As soon as the equilibrium rate of return drops to R, the traders start investing, because they do not face the binding borrowing constraint. Thus, the equilibrium rate of return stays constant at R, and business investment remains constant at $\Pi^{-1}(R)$.

In the three cases depicted by Figure 3c through 3e, $k_c < k_\lambda$ holds. As in Figure 3b, for $k_t > k_\lambda$, the borrowing constraint is not binding for the traders, so that the equilibrium rate of return is equal to R, and hence $k_{t+1} = \Psi(k_t) = \Pi^{-1}(R)$. In contrast to Figure 3b, however, all these figures show the intervals below k_λ, in which $k_{t+1} = \Psi(k_t) > \Pi^{-1}(R)$ holds, suggesting an *over-investment* in the business sector, $\Pi(k_{t+1}) < R$. Inside these intervals, below k_c, the saving continues to flow only into the business investment; $k_{t+1} = W(k_t) > \Pi^{-1}(R)$. For $k_c < k_t < k_\lambda$, on the other hand, the saving starts flowing into the trading operation, even though the traders are still constrained by the low net worth, and the equilibrium rate of return remains strictly below R. Thus, we have

(14) $$k_{t+1} = \Psi(k_t) = \Pi^{-1}(\lambda R/[1 - W(k_t)/m])$$

for $k_c < k_t < \min\{k_\lambda, k_{cc}, K\}$. Note that (14) is decreasing in k_t. In other words, the map has a downward-sloping segment, when neither (11) nor (12) hold.

It should be clear why an increase in k_t leads to a lower k_{t+1} when trading is active but borrowing constrained. A higher k_t, by improving the net worth of the traders, eases their borrowing constraint, which enables them to make a credible commitment to generate a higher return to the lenders. This drives up the equilibrium rate of return. To keep the investment in the business sector profitable, the business sector must shrink. Thus, more saving is channeled into the investment in trading at the expense of the investment in the business sector.

Figure 3c depicts the case where the borrowing constraint for trading is not binding in the steady state. That is, the map intersects with the 45° line at a flat segment, i.e., over the interval,

[17] In both Figures 3b and 3c, $k_{cc} > K$. This need not be the case, nor is it essential for the discussion in the text.

$(k_\lambda, \min\{k_{cc}, K\})$. The condition for this is $k_\lambda \leq k^* = \Pi^{-1}(R) < k_{cc}$. Since (A3) ensures $k^* < k_{cc}$, this occurs whenever $k_\lambda \leq \Pi^{-1}(R)$, or equivalently, $W(\Pi^{-1}(R)) \geq (1 - \lambda)m$, which can be further rewritten to

(15) $\quad R \leq \Pi(W^{-1}((1 - \lambda)m))$.

When (15) holds but (11) and (12) are violated, the dynamics of k_t converges to $k^* = \Pi^{-1}(R) < W(k_c)$, as illustrated in Figure 3c. The dynamics is not, however, globally monotone. Starting from $k_0 < k_\lambda$, the dynamics of k_t generally overshoots k^* and approaches k^* from above.[18]

For the cases depicted by Figures 3d and 3e, (11) and (15) are both violated, which also implies the violation of (12).[19] Thus, the map intersects with the 45° line at the downward sloping part, $(k_c, \min\{k_\lambda, k_{cc}, K\})$. Therefore, the traders face the binding borrowing constraint in a neighborhood of the steady state. By setting $k_t = k_{t+1} = k^*$ in (14), the steady state is given by

(16) $\quad \Pi(k^*)[1 - W(k^*)/m] = \lambda R$.

Both in Figure 3d and Figure 3e, the dynamics around the steady state is oscillatory. The two figures differ in the stability of the steady state, which depends on the slope of the map at k^*. Differentiating (14) and then setting $k_t = k_{t+1} = k^*$ yield,

$$\Psi'(k^*) = W'(k^*)\Pi(k^*)/\Pi'(k^*)[m - W(k^*)] = -\, k^*\Pi(k^*)/[m - W(k^*)],$$

where use has been made of (16) and $W'(k^*)+k^*\Pi'(k^*) = 0$. From $k^*\Pi(k^*) + W(k^*) = k^*\phi(1/k^*)$, $|\Psi'(k^*)| < 1$ if and only if

(17) $\quad k^*\phi(1/k^*) < m$.

Note that the LHS of (17) is increasing in k^*, while the LHS of (16) is decreasing in k^*. Hence, (17) can be rewritten to

(18) $\quad \lambda R > \Pi(h(m))[1 - W(h(m))/m]$,

where $h(m)$ is defined implicitly by $h\phi(1/h) \equiv m$. This case is illustrated in Figure 3d. When (18) holds, the steady state, k^*, is asymptotically stable; the convergence is locally oscillatory.

On the other hand, if

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[18] The qualified "generally" is needed, because the equilibrium trajectory is monotone, if $k_0 \in \{W^{-T}(k^*) \mid T = 0, 1, 2,...\}$, which is at most countable and hence of measure zero.

[19] Figures 3d and 3e are drawn such that $k_\lambda < K$. This need not be the case, nor is it essential for the discussion in the text.

(19) $\lambda R < \Pi(h(m))[1 - W(h(m))/m]$,

then $|\Psi'(k^*)| > 1$ and hence the steady state, k^*, is unstable, as illustrated in Figure 3e. For any initial condition, the equilibrium trajectory will eventually be trapped in the interval, $I \equiv [\max\{\Psi(W(k_c)), \Psi(\min\{k_\lambda, k_{cc}\})\}, W(k_c)]$, as illustrated by the box in Figure 3e.[20] Furthermore, if $k_\lambda \geq \min\{k_{cc}, K\}$, k_t fluctuates indefinitely except for a countable set of initial conditions. If $k_\lambda < \min\{k_{cc}, K\}$, k_t fluctuates indefinitely except for a countable set of initial conditions for a generic subset of the parameter values satisfying (19) and violating (11) and (15).[21] In other words, the equilibrium dynamics exhibit permanent endogenous fluctuations almost surely.

To summarize,

Proposition 1. Let $\lambda_1 = 1$ and $\lambda_2 = \lambda \in (0,1)$. Then,

A. Let $R \leq \Pi(K)\max\{(1 - K/m)/\lambda, 1\}$ or equivalently, $k_c \geq K$. Then, $x_{t+1} = 0$ for all $t \geq 0$ and the dynamics of k_t converges monotonically to the unique steady state, K.

B. Let $\Pi(K) < R \leq \Pi((1 - \lambda)m)$, or equivalently, $k_\lambda \leq k_c < K$. Then, the dynamics of k_t converges monotonically to the unique steady state, $k^* = \Pi^{-1}(R) = W(k_c)$. Some traders eventually become active and never face the binding borrowing constraint.

C. Let $\Pi((1 - \lambda)m) < R \leq \Pi(W^{-1}((1 - \lambda)m))$ or equivalently, $k_c < k_\lambda \leq \Pi^{-1}(R)$. Then, the dynamics of k_t converges to the unique steady state, $k^* = \Pi^{-1}(R) < W(k_c)$. Some traders are active and do not face the binding borrowing constraint in the neighborhood of the steady state.

D. Let $R > \Pi(W^{-1}((1 - \lambda)m))$, $\Pi(h(m))[1 - W(h(m))/m]/\lambda$. Then, the dynamics of k has the unique steady state, $k^* \in (k_c, \min\{k_\lambda, k_{cc}, K\})$, satisfying $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$.

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[20] In Figure 3e, $k_\lambda < W(k_c) < K < k_{cc}$. Hence, $I = [\Psi(k_\lambda), W(k_c)] = [\Pi^{-1}(R), W(k_c)]$.

[21] To see this, let $C \subset (0, K]$ be the set of initial conditions for which k_t converges. Let $k_\infty = \lim_{t\to\infty} \Psi^t(k_0)$ be the limit point for $k_0 \in C$. From the continuity of Ψ, $\Psi(k_\infty) = \lim_{t\to\infty} \Psi(k_t) = \lim_{t\to\infty} k_{t+1} = k_\infty$. Hence, $k_\infty = k^*$. Since k^* is unstable, k_t cannot approach it asymptotically. It must be mapped to k^* in a finite iteration. That is, there exits T such that $\Psi^T(k_0) = k^*$, or $C = \{\Psi^{-T}(k_0) \mid T = 0,1,2,...\}$. If $k_\lambda \geq \min\{k_{cc}, K\}$, the map has no flat segment and hence the preimage of Ψ is finite and hence C is at most countable. If $k_\lambda < \min\{k_{cc}, K\}$, the map has a flat segment, at which it is equal to $\Pi^{-1}(R)$. Thus, C is at most countable unless $\Pi^{-1}(R) \in \{\Psi^{-T}(k_0) \mid T = 0,1,2,...\}$, which occurs only for a nongeneric set of parameter values. (As clear from this proof, it is easy to show that, even when $k_\lambda < \min\{k_{cc}, K\}$, if $W(k_c) < \min\{k_\lambda, k_{cc}\}$, the flat segment does not belong to I. Hence, if we restrict the initial condition in I, k_t fluctuates indefinitely for almost initial conditions in I for all the parameter values satisfying (19) and violate (11) and (15).)

The traders face the binding borrowing constraint in the neighborhood of the steady state. The steady state is asymptotically stable. The convergence is locally oscillatory.

E. Let $\Pi(K)(1 - K/m)/\lambda$, $\Pi(W^{-1}((1 - \lambda)m))) < R < \Pi(h(m))[1 - W(h(m))/m]/\lambda$. Then, the dynamics of k has the unique steady state, $k^* \in (k_c, \min\{k_\lambda, k_{cc}, K\})$, satisfying $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$. The traders face the binding borrowing constraint in the neighborhood of the steady state. The steady state is unstable. Every equilibrium trajectory will be eventually trapped in the interval, $I \equiv [\max\{\Psi(W(k_c)), \Psi(\min\{k_\lambda, k_{cc}\})\}, W(k_c)]$. Furthermore, the equilibrium dynamics exhibits permanent, endogenous fluctuations for almost all initial conditions.

In order to avoid a taxonomical exposition, let us focus on the case where $K < m < K\phi(1/K)$ in the following discussion.[22] Proposition 1 is illustrated by Figure 4, which divides the parameter space, (λ, R), into five regions, where Region A satisfies the conditions given in Proposition 1A, Region B satisfies those given in Proposition 1B, etc. The borders between B and C and between C and D are asymptotic to $\lambda = 1$. The borders between D and E and between A and E are hyperbolae and asymptotic to $\lambda = 0$.

If the economy is in Region A, the traders remain inactive and hence have no effect on the dynamics of business formation, and the model behaves just as the standard one-sector neoclassical growth model. There are two ways in which this could happen. First, if the trading operation is unprofitable, not surprisingly, it never competes with business investment in the credit market. More specifically, this occurs if $R \leq \Pi(K)$, i.e., when the rate of return in trading is always dominated by business investment. Second, even if $R > \Pi(K)$, so that the trading operation becomes eventually as profitable as business investment, the traders would not be able to borrow if they suffer from the severe agency problem (a small λ).

If the economy is in Region B, the trading operation eventually becomes as profitable as business investment, because $R > \Pi(K)$. Furthermore, the agency problem associated with the trading operation is so minor (λ is sufficiently high) that the traders can finance their investments

[22]Note $K < K\phi(1/K)$ for any K, because $K\phi(1/K) = K\Pi(K) + W(K) > W(K) = K$. Matsuyama (2001) offers a detailed discussion for the cases where $m < K$ and $m > K\phi(1/K)$.

as soon as the equilibrium rate of return drops to R. As a result, business investment stays constant at $\Pi^{-1}(R)$. In these cases, trading changes the dynamics of business formation, but it is simply because the credit market allocates the saving to the most profitable investments. Furthermore the dynamics always converges to the unique steady state.

The presence of the profitable trading operation has nontrivial effects on the dynamics when the economy is in Region C, D, or E, i.e., when λ is neither too high nor too low. In particular, in the cases of D and E, the traders face the binding borrowing constraint in the neighborhood of the steady state. The agency problem associated with the trading operation is significant enough (i.e., λ is not too high) that the credit continues to flow into the business sector, even if its rate of return is strictly less than R. Of course, the traders are eager to take advantage of the lower equilibrium rate of return, but some of them are unable to do so, because of their borrowing constraint. If λ is not too low, an improvement in net worth would ease the borrowing constraint, which drives up the equilibrium rate. This is because, with a higher net worth, they need to borrow less, and hence they would be able to guarantee the lender a higher rate of return. A rise of the equilibrium rate of return in turn causes a decline in the investment in the business sector, which reduces the net worth of the agents in the next period. When λ is relatively high (i.e., if the economy is in Region D), this effect is not strong enough to make the steady state unstable. When λ is relatively low (i.e., if the economy is in Region E), this effect is strong enough to make the steady state unstable and generates endogenous fluctuations.[23] Thus,

Corollary 1.

Suppose $K < m < K\phi(1/K)$. For any $R > \Pi(K)$, endogenous fluctuations occur (almost surely) for an intermediate value of λ.

This corollary is the main conclusion of the basic model. *Endogenous credit cycles occur when the Bad project is sufficiently profitable (a high R) and when the agency problem associated with the Bad project is big enough that the agents cannot finance it when their net worth is low, but small enough that the agents can finance it when their net worth is high.*

Region D is also of some interest, because the local convergence toward the steady state is oscillatory, and the transitional dynamics is cyclical. If the economy is hit by recurrent shocks, the equilibrium dynamics exhibit considerable fluctuations.[24] A quick look at Proposition 1D (and Figure 4) verifies that a sufficiently high R ensures that the economy is in Region D. Thus,

Corollary 2.

For any $\lambda \in (0,1)$, the dynamics around the steady state is oscillatory for a sufficiently high R.

The intuition behind this result is easy to grasp. In the presence of the agency problem, the trader's borrowing constraint becomes binding, if they are sufficiently eager to invest, i.e., if the trading operation is sufficiently profitable.

4. Some Examples of Chaotic Dynamics

Propositions 1D and 1E give the conditions under which the model generates locally oscillatory convergence and endogenous fluctuations for almost all initial conditions. To be able to say more about the nature of global dynamics, let us impose some specific functional forms.

Example 1:

Let $\phi(n)= 2(Kn)^{1/2}$, with $K < \mu_1$, $4m\mu_2$, which satisfies (A1) through (A4). If $R > K/(1-\lambda)m$, and $R > (1- K/m)/\lambda$, the economy is either in Region D (for $\lambda R > K/m$) or in Region E (for $\lambda R < K/m$). Furthermore, in order to avoid a taxonomical exposition, let us focus on the case, where $W(k_c) < \min \{k_\lambda, k_{cc}\}$ so that the map is strictly decreasing in $(k_c, W(k_c)]$.[25] Some algebra can show that, by defining $z_t \equiv (k_t/K)^{1/2}$, the equilibrium dynamics over this range can be expressed by the map: ψ: $(0, \psi(z_c)] \rightarrow (0, z_c]$, defined by

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[23] Technically speaking, as the economy crosses $\lambda R = \Pi(h(m))[1 - W(h(m))/m]$ from Region D to Region E, the dynamical system experiences a *flip bifurcation*.

[24] In addition, it is possible that there may be endogenous fluctuations in Region D. When the parameters satisfy the conditions given in Proposition 1D, we do know that the local dynamics converges, but little can be said of the nature of global dynamics. For example, if the flip bifurcation that occurs at the boundary of D and E is of *subcritical* type, there are (unstable) period-2 cycles in the neighborhood of k* near the boundary on the side of Region D: see Guckenheimer and Holmes (1983, Theorem 3.5.1).

[25] For example, $K < (1-\lambda)m(1-\lambda+\lambda R)$ ensures $k_\lambda > W(k_c)$; $K > mR^2(1-\lambda-\mu_2)$ ensures $k_{cc} > k_\lambda$, hence $k_{cc} > W(k_c)$.

$z_{t+1} = \psi(z_t) \equiv \min\ \{z_t^{1/2}, [1 - (K/m)z_t]/(\lambda R)\}$,

where $z_c \equiv (k_c/K)^{1/2} < 1$, which satisfies $\psi(z_c) = z_c^{1/2} = [1 - (K/m)z_c]/(\lambda R)$. The map is unimodal: it is strictly increasing in $(0, z_c)$ and strictly decreasing in $I_- \equiv (z_c, \psi(z_c)]$. Furthermore, the slope is constant in I_-. In Region D, where $\lambda R > K/m$, the slope in I_- is less than one in absolute value. Therefore, the economy converges to the steady state, $z^* = 1/(\lambda R+K/m) \in I_-$, for any initial condition. In Region E, the case illustrated in Figure 5a, the slope in I_- is greater than one in absolute value. This means that, if $z_t \neq z^*$, the equilibrium trajectory will escape I_- after a finite iteration. However, it will never leave $I = [\psi^2(z_c), \psi(z_c)]$, because the map is strictly increasing in $I_+ \equiv [\psi^2(z_c), z_c)$. Therefore, the equilibrium trajectory visits both I_+ and I_- infinitely often, for almost all initial conditions in I (i.e., except for a countable set of initial conditions in I, for which the equilibrium trajectory is mapped into z^* in a finite iteration). Furthermore, if $\lambda R > 2(1 - K/4m)$, then $z_c < 1/4$, which ensures that the slope of the map is strictly greater than one in absolute value anywhere in $I_+ \cup I_-$.[26] This means that there are period cycles of every period length, all of which are unstable, and the equilibrium trajectory does not converge to any periodic cycle for almost all initial conditions. In short, the map is chaotic.[27]

In the previous example, the functional form is chosen so that the slope of the map is constant when $z_t > z_c$. This guarantees that there exist no periodic cycles that stay entirely above z_c. In the next example, the functional form is chosen so that the slope of the map is constant also when $z_t < z_c$.

Example 2:

Let $\phi(n) = 2(Kn)^{1/2}$ if $n \le 1/k_c$; $= 2(z_c)^{-1/2} + \log(k_c n)/z_c$, if $n > 1/k_c$, which satisfies (A1) through (A4) with $K < \mu_1, 4m\mu_2$. As in Example 1, let $R > K/(1-\lambda)m$, and $R > (1- K/m)/\lambda$, so that the economy is either in Region D (for $\lambda R > K/m$) or in Region E (for $\lambda R < K/m$), and impose the

[26] For example, $\mu_1 = 0.2$, $\mu_2 = 0.8$, $K = 0.1$, $m = 0.05$, $\lambda = 0.25$, $R = 7.8$ satisfy the last condition, as well as all the other conditions imposed earlier.

[27] See, for example, Devaney (1987, Chapter 1.6 and 1.7). The set of initial conditions for which the trajectory is eventually periodic is a Cantor set, i.e., it is uncountable, but contains no interior or isolated points. Furthermore, this chaotic map is structually stable. (For an introduction to the chaotic dynamical system written for economists, see Grandmont 1986).

same restrictions on the parameters to ensure $W(k_c) < \min\{k_\lambda, k_{cc}\}$. Then, the dynamics is now given by

$$z_{t+1} = \psi(z_t) \equiv \min\{(z_c)^{-1/2} z_t, [1 - (K/m)z_t]/(\lambda R)\},$$

on $(0, \psi(z_c)]$, as illustrated in Figure 5b. This map differs from Example 1 in that the slope of the map is constant in $(0, z_c)$, which is greater than one because $z_c < 1$. Therefore, for $\lambda R < K/m$, the slope of the map is greater than one in absolute value anywhere in $I_+ \cup I_-$. Thus, with this functional form, the map is chaotic whenever the parameters satisfy the conditions in Proposition 1E.

5. Reintroducing the Borrowing Constraint in the Business Sector

So far, we have analyzed the equilibrium trajectory under the assumption that $\lambda_1 = 1 > \lambda_2 = \lambda$. We are now going to show that, for any $\lambda_2 = \lambda < 1$, a small reduction in λ_1 from $\lambda_1 = 1$ would not affect the equilibrium trajectory.

Recall that the entrepreneurs start firms when both (1) and (3) are satisfied. (A4) ensures that some entrepreneurs are active, $k_{t+1} > 0$, hence both (1) and (3) hold in equilibrium. Furthermore, $k_t \leq K$ ensures that $k_{t+1} = W(k_t) - mx_{t+1} \leq W(k_t) \leq W(K) = K < \mu_1$. Therefore, at least (1) or (3) must be binding, hence

(20) $\quad \Pi(k_{t+1})/\max\{[1 - W(k_t)]/\lambda_1, 1\} = r_{t+1}$.

The credit market equilibrium is given by (8), (9) and (20). It is easy to see that, given k_t, these equations jointly determine k_{t+1} uniquely.

Let us find the condition under which the map given in eq. (10) solves the credit market equilibrium, determined by (8), (9), and (20). First, for any $k_t \geq k_c$, eq. (10) solves the credit market equilibrium if and only if the entrepreneurs do not face the binding borrowing constraint, that is, when (20) is $\Pi(k_{t+1}) = r_{t+1}$, i.e., $W(k_t) \geq 1 - \lambda_1$ for all $k_t \geq k_c$. The condition for this is $\lambda_1 \geq 1 - W(k_c)$. Then, in order for (10) to be the equilibrium, it suffices to show that $x_{t+1} = 0$ and $k_{t+1} = W(k_t)$ solve (8), (9) and (20) for $k_t < k_c$. This condition is given by

$$(21)\quad R/\max\{[1-W(k_t)/m]/\lambda_2, 1\} \leq \begin{cases} \lambda_1\Pi(W(k_t))/[1-W(k_t)] & \text{if } k_t < k_{\lambda 1} \\ \Pi(W(k_t)) & \text{if } k_{\lambda 1} \leq k_t < k_c, \end{cases}$$

where $k_{\lambda 1}$ is defined implicitly by $W(k_{\lambda 1}) \equiv 1 - \lambda_1$ and satisfies $k_{\lambda 1} < k_c$. Eq. (21) is illustrated by Figure 6a (for $k_c < k_\lambda$) and Figure 6b (for $k_c > k_\lambda$). By definition of k_c, the LHS of (21) is strictly less than $\Pi(W(k_t))$ for all $k_t < k_c$. Since the RHS of (21) converges to $\Pi(W(k_t))$, as λ_1 approaches one, there exists $\lambda_1' < 1$ such that eq. (21) holds for $\lambda_1 \in [\lambda_1', 1]$. Since the LHS of (21) weakly increases with λ_2, the lowest value of λ_1 for which (21) holds, λ_1', is weakly increasing in λ_2. It is also easy to see that (21) is violated for a sufficiently small λ_1, hence, $\lambda_1' > 0$. Furthermore, for any $\lambda_1 > 0$, (21) holds for a sufficiently small $\lambda_2 > 0$. Thus, λ_1' approaches zero with λ_2. One can thus conclude

<u>Proposition 2.</u>

For any $\lambda_2 = \lambda \in (0,1)$, there exists $\Lambda(\lambda_2) \in (0,1)$, such that, for $\lambda_1 \in [\Lambda(\lambda_2), 1]$, the equilibrium dynamics is independent of λ_1.[28] Λ is nondecreasing in λ_2 and satisfies $\Lambda(\lambda_2) \geq 1 - W(k_c)$, and $\lim_{\lambda 2\to 0} \Lambda(\lambda_2) = 0$.

Proposition 2 thus means that the analysis need not be changed, as long as λ_1 is sufficiently high. In particular, Proposition 1, their corollaries, as well as Examples 1 and 2 are all unaffected.

Even with a weaker condition on λ_1, the possibility of endogenous fluctuations survives. When $\lambda_1 < \Lambda(\lambda_2)$, the map depends on λ_1, but shifts continuously as λ_1 changes. Therefore, as long as the reduction is small enough, k* is unaffected and remains the only steady state of the map. Therefore, as long as $\lambda_2 = \lambda$ satisfies the condition given in Proposition 1E, the map generates endogenous fluctuations, because its unique steady state is unstable.

The above analysis thus shows that the key mechanism in generating endogenous fluctuations is that an improved economic condition eases the borrowing constraints for the Bad more than those for the Good, so that the saving is channeled into the former at the expense of the latter. The assumption made earlier that the Good faces no borrowing constraint itself is not crucial for the results obtained so far.

6. The Good, The Bad and The Ugly: Introducing Credit Multiplier

This section presents an extension of the model of section 5, which serves two purposes. First, recent studies in macroeconomics, such as Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), have emphasized the role of credit market imperfections in propagation mechanisms of business cycles. In particular, they stressed a *credit multiplier* effect. An increase in net worth stimulates business investment by easing the borrowing constraint of the entrepreneurs, which further improves their net worth, leading to more business investment. This introduces *persistence* into the system. The model developed above has no such a credit multiplier effect.[29] Quite the contrary, the mechanism identified may be called a *credit reversal* effect, because an increase in net worth stimulates trading at the expense of business investment, leading to a deterioration of the net worth. This introduces *instability* into the system. This does not mean, however, that these two mechanisms are mutually exclusive. Combining the two is not only feasible but also useful because it adds some realism to the equilibrium dynamics. In the model shown below, both credit multiplier and reversal effects are present and the equilibrium dynamics exhibit persistence at a low level of economic activities and instability at a high level.

Second, in the previous models, trading, which is the only alternative to the business investment, not only generates less aggregate demand spillovers than the other, but also faces tighter borrowing constraints. This might give the reader a false impression that these two features, less spillovers and tighter borrowing constraints, must go together to have instability and fluctuations. The extension presented below will show that this need not be the case, by adding another investment opportunity, which generates less spillovers and face less borrowing

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[28] The function, Λ, also depends on other parameters of the model, m, R, K, as well as the functional form of ϕ.

[29] In the model above, an increase in the net worth leads to an increase in business investment when $k_t < k_c$. This occurs because an increase in the net worth leads to an increase in the aggregate savings, all of which are used to finance the investment in the business sector. The aggregate investment in the business sector is independent of whether the entrepreneurs face the borrowing constraint. Therefore, it should not be interpreted as the credit multiplier effect.

constraints. What is needed for endogenous fluctuations is that *some* profitable projects have less spillovers than others, and can be financed only at a high level of economic activities.

The model discussed in the last section is now modified to allow the young agents to have access to a storage technology, which transforms one unit of the final good in period t into ρ units of the final good in period t+1. The storage technology is available to all the young. Furthermore, it is divisible, so that the agents can invest, regardless of their level of net worth. It is assumed that the gross rate of return on storage satisfies $\rho \in (\lambda_2 R, R)$. This restriction ensures that storage dominates trading when net worth is low, while trading dominates storage when it is high. That is, the economy now has the following three types of the investment: i) *The Good* (Business Investment), which is profitable, relatively easy to finance and generates demand for the labor endowment held by the next generation of the agents; ii) *The Bad* (Trading), which is profitable, relatively difficult to finance, and generates no demand for the labor endowment; and iii) T*he Ugly* (Storage), which is unprofitable, has no need for being financed, and generates no demand for the endowment.

Let s_t be the total units of the final good invested in storage at the end of period t. Then, the credit market equilibrium condition is now given by

$$(8) \qquad mx_{t+1} \begin{cases} = m\mu_2 & \text{if } r_{t+1} < R(W(k_t)), \\ \in [0, m\mu_2] & \text{if } r_{t+1} = R(W(k_t)), \\ = 0 & \text{if } r_{t+1} > R(W(k_t)). \end{cases}$$

$$(20) \qquad \Pi(k_{t+1})/\max\{[1 - W(k_t)]/\lambda_1, 1\} = r_{t+1}.$$

$$(22) \qquad s_t \begin{cases} = 0, & \text{if } r_{t+1} > \rho \\ \geq 0, & \text{if } r_{t+1} = \rho \\ = \infty, & \text{if } r_{t+1} < \rho \end{cases}$$

$$(23) \qquad k_{t+1} = W(k_t) - mx_{t+1} - s_t.$$

Eqs. (8) and (20) are reproduced here for easy reference. Introducing the storage technology does not make any difference in the range where $r_{t+1} > \rho$. If the storage technology is used in equilibrium, the equilibrium rate of return must be $r_{t+1} = \rho$.

Characterizing the credit market equilibrium and the equilibrium trajectory determined by (8), (20), (22) and (23) for the full set of parameter values require one to go through a large number of cases. Furthermore, in many of these cases, the presence of the storage technology does not affect the properties of the equilibrium dynamics fundamentally. In what follows, let us report one representative case, in which the introduction of the storage technology creates some important changes. More specifically, let us consider the case, where the following conditions hold. First, R and $\lambda_2 = \lambda$ satisfy the conditions given in Proposition 1E. This ensures that $k_c < k^* < k_\lambda$. Second, ρ is not too low nor too high so that $k_c < k_\rho < k^*$, where k_ρ is implicitly defined by $R(W(k_\rho)) \equiv \rho$. Third, λ_1 is large enough that $k_{\lambda 1} < k_\rho$, and small enough that the RHS of (21) is greater than ρ for $k_t < k'$ and smaller than ρ for $k_t > k'$. (It is feasible to find such λ_1 because $k_c < k_\rho$.) These conditions are illustrated in Figure 7.[30]

Then, for $k_t < k'$, the business profit is so high that all the saving goes to the investment in the business sector, and $x_{t+1} = s_t = 0$. For $k' < k_t < k_\rho$, some saving goes to the storage, $s_t > 0$, and hence $r_{t+1} = \rho > R(W(k_t))$, and the trading remains inactive, $x_{t+1} = 0$. Within this range, the borrowing constraint is binding for the entrepreneurs when $k' < k_t < k_{\lambda 1}$, and the profitability constraint is binding for the entrepreneurs when $k_{\lambda 1} < k_t < k_\rho$. For $k_\rho < k_t < \min\{k_\lambda, k_{cc}, K\}$, the storage technology is not used, $s_t = 0$. The entrepreneurs, whose borrowing constraint is not binding, compete for the credit with the traders who become active, and face the binding borrowing constraint, and the interest is given by $r_{t+1} = R(W(k_t)) > \rho$. The unstable steady state, k^*, shown in Proposition 1E, is located in this range.

The equilibrium dynamics is thus governed by the following map:

$$
(24) \qquad k_{t+1} = \Psi(k_t) \equiv \begin{cases} W(k_t), & \text{if } k_t \leq k', \\ \Pi^{-1}(\rho[1 - W(k_t)]/\lambda_1), & \text{if } k' < k_t \leq k_{\lambda 1}, \\ \Pi^{-1}(\rho), & \text{if } k_{\lambda 1} < k_t \leq k_\rho. \\ \Pi^{-1}(\lambda_2 R/[1 - W(k_t)/m]), & \text{if } k_\rho < k_t \leq \min\{k_\lambda, k_{cc}\}, \\ \Pi^{-1}(R), & \text{if } k_\lambda < k_t \leq k_{cc}, \\ W(k_t) - m\mu_2. & \text{if } k_t \geq k_{cc}, \end{cases}
$$

[30] In Figure 7, $k_{\lambda 1} < k_c$. This need not be the case, nor is it essential for the discussion in the text.

where k' is given implicitly by $\lambda_1\Pi(W(k'))/[1 - W(k')] \equiv \rho$. Eq. (24) differs from (10) for $k' < k_t < k_\rho$, where some saving go to the storage technology and the rate of return is fixed at ρ. In particular, for $k' < k_t < k_{\lambda 1}$, the investment in the business sector is determined by the borrowing constraint,

(25) $W(k_t) = 1 - \lambda_1\Pi(k_{t+1})/\rho$.

In this range, an increase in the net worth, $W(k_t)$, eases the borrowing constraint of the entrepreneurs, so that their investment demand goes up. Instead of pushing the equilibrium rate of return, the rise in the investment demand in the business sector is financed by redirecting the savings from storage. Intuitively enough, an increase in ρ/λ_1 shifts down the map in this range. The presence of the Ugly thus reduces the Good, which slows down the expansion processes. Unlike the Bad, however, the Ugly does not destroy the Good. And a higher business investment today leads to a higher business investment tomorrow. This mechanism is essentially identical with the one studied by Bernanke and Gertler (1989); see also Matsuyama (2004a, Section 5).

The crucial feature of the dynamics governed by (24) is that the credit multiplier effect is operative at a lower level of activities, while the credit reversal effect is operative at a higher level, including in the neighborhood of the unstable steady state, k^*. In this sense, this model is a hybrid of the model developed earlier and of a credit multiplier model à la Bernanke-Gertler.

Figure 8 illustrates the map (24) under additional restrictions, $\Psi(k_\rho) = \Pi^{-1}(\rho) \leq \min\{k_\lambda, k_{cc}\}$ and $k_{\lambda 1} > \Psi^2(k_\rho) = \Psi(\Pi^{-1}(\rho))$. The first restriction ensures that some traders remain inactive at $\Psi(k_\rho)$. This means that the trapping interval is given by $I \equiv [\Psi^2(k_\rho), \Psi(k_\rho)] = [\Psi(\Pi^{-1}(\rho)), \Pi^{-1}(\rho)]$.[31] The second restriction ensures that the trapping interval, I, overlaps with $(k', k_{\lambda 1})$, i.e., the range over which the credit multiplier effect is operative. Let us fix ρ and change λ_1. As λ_1 is reduced, $k_{\lambda 1}$ increases from $\Psi^2(k_\rho)$ to k_ρ, and at the same time, the map shifts down below $k_{\lambda 1}$. Clearly, the map has the unique steady state, k^*, as long as λ_1 is not too small (or $k_{\lambda 1}$ is sufficiently close to $\Psi^2(k_\rho)$). As λ_1 is made smaller (and $k_{\lambda 1}$ approaches k_ρ), the equilibrium

[31] Note that this restriction is weaker than the restriction, $W(k_c) \leq \min\{k_\lambda, k_{cc}\}$, because $k_c < k_\rho$ implies $\Pi(W(k_c)) = R(W(k_c)) < R(W(k_\rho)) = \rho$, hence $W(k_c) > \Pi^{-1}(\rho)$.

dynamics may have additional steady states in $(k', k_{\lambda 1})$.[32] The following proposition gives the exact condition under which that happens.

Proposition 3. Let k^* be the (unstable) steady state in Proposition 1E.

A. If $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 < \rho h(1)$, the equilibrium dynamics governed by (24) has, in addition to k^*, two other steady states, $k_1^{**}, k_2^{**} \in (k', k_{\lambda 1})$. They satisfy $k_1^{**} < h(1) < k_2^{**}$, and k_1^{**} is stable and k_2^{**} is unstable.

B. If $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 = \rho h(1)$, the equilibrium dynamics governed by (24) has, in addition to k^*, another steady state, $k^{**} = h(1) \in (k', k_{\lambda 1})$, which is stable from below and unstable from above.

C. Otherwise, k^* is the unique steady state of (24).

Proof. See Appendix.

If $\lambda_1 > 1 - W(h(1))$ or $\lambda_1 > \rho h(1)$, neither condition given in Proposition 3A or 3B hold, endogenous fluctuations clearly survive, because the map has a unique steady state, k^*, which is unstable. Even if $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 \leq \rho h(1)$, the equilibrium dynamics may still exhibit endogenous fluctuations in $I \equiv [\Psi^2(k_\rho), \Psi(k_\rho)]$. This is because, if $h(1) < \Psi^2(k_\rho)$, $k_2^{**} < \Psi^2(k_\rho)$ as long as λ_1 is not too much lower than $\rho h(1)$, and hence the map has a unique steady state in I, k^*, which is unstable, and, for any initial condition in I, the equilibrium trajectory never leaves I.

The above argument indicates that, as long as λ_1 is not too small (or ρ is not too large), the introduction of the credit multiplier effect does not affect the result that the borrowing-constrained investment in trading generates endogenous fluctuations. This does not mean, however, that the credit multiplier effect has little effects on the nature of fluctuations. The introduction of the credit multiplier effect, by shifting down the map below $k_{\lambda 1}$, can slow down an economic expansion, thereby creating asymmetry in business cycles. This is most clearly illustrated by Figure 9, which depicts the case where $\Psi^2(k_\rho) < h(1) < k_{\lambda 1}$. If $\lambda_1 = \rho h(1)$, as indicated in Proposition 3B, the map is tangent to the 45° line at h(1), which creates an additional

[32] Since $k_\rho < k^* < \Pi^{-1}(\rho)$, the map does not intersect with the 45° line in $[k_{\lambda 1}, k_\rho)$.

steady state, $k^{**} = h(1)$. It is stable from below but unstable from above, and there are *homoclinic orbits*, which leave from k^{**}, and converges to k^{**} from below. Starting from this situation, let λ_1 go up slightly. As indicated in Proposition 3A, such a change in the parameter value makes the steady state, k^{**}, disappear, and the map is left with the unique steady state, k^*, in its downward-sloping segment, which is unstable.[33] The credit multiplier effect is responsible for the segment, where the map is increasing and stays above but very close to the 45° line. Thus, the equilibrium dynamics display *intermittency*, as a tangent bifurcation eliminates the tangent point, k^{**}, and its homoclinic orbits. The equilibrium trajectory occasionally has to travel through the narrow corridor. The trajectory stays in the neighborhood of h(1) for possibly long time, as the economy's business sector expands gradually. Then, the economy starts accelerating through the credit multiplier effect. At the peak, the traders start investing. Then, the economy plunges into a recession (possibly after going through a period of high volatility, as the trajectory oscillates around k^*). Then, at the bottom, the economy begins its slow and long process of expansion. The map depicted in Figure 9 is said to display intermittency, because its dynamic behavior is characterized by a relatively long periods of small movements punctuated by intermittent periods of seemingly random-looking movements.[34]

Example 3.

As in Examples 1 and 2, let $\phi(n)= 2(Kn)^{1/2}$ with $K < \mu_1, 4m\mu_2$, and impose the same restrictions to ensure $W(k_c) \leq \min \{k_\lambda, k_{cc}\}$. This guarantees $\Psi(k_\rho) = \Pi^{-1}(\rho) < W(k_c) \leq \min \{k_\lambda, k_{cc}\}$. As seen in Examples 1 and 2, $1 - K/m < \lambda_2 R < K/m$, and $R > K/(1-\lambda_2)m$ ensure that the conditions in Proposition 1E are satisfied. Let us choose ρ such that $K/m\rho^2 < (1 - \lambda_2 R/\rho) < K/m\rho$ (this is feasible because $\lambda_2 R + K/m > 1$) and λ_1 such that $K/\rho^2 < 1 - \lambda_1 < m(1 - \lambda_2 R/\rho)$. Then, (24) can be rewritten in the relevant range as

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[33] Technically speaking, this is known as a *saddle-node* or *tangent bifurcation*.

[34] What is significant here is that the introduction of the credit multiplier effect can create the intermittency, regardless of the functional form of ϕ. Even without the credit multiplier effect, one can always choose a functional form of ϕ, so as to make the function $W(k) = \Psi(k)$ come close to the 45° line below k_c to generate the intermittency phenomenon. In this sense, the presence of the credit multiplier effect is not necessary for the intermittency. It simply makes it more plausible.

$$(26)\qquad z_{t+1} = \psi(z_t) \equiv \begin{cases} (z_t)^{1/2}, & \text{if } z_t \le z', \\ \lambda_1/\rho(1 - Kz_t), & \text{if } z' < z_t \le z_{\lambda 1}, \\ 1/\rho, & \text{if } z_{\lambda 1} < z_t \le z_\rho, \\ [1 - (K/m)z_t]/(\lambda_2 R), & \text{if } z_\rho < z_t \le 1/\rho, \end{cases}$$

where $z_t \equiv (k_t/K)^{1/2}$ and $z' \equiv (k'/K)^{1/2}$, $z_{\lambda 1} \equiv (k_{\lambda 1}/K)^{1/2}$, and $z_\rho \equiv (k_\rho/K)^{1/2}$ satisfy $(z')^{1/2} = \lambda_1/\rho(1 - Kz')$, $\lambda_1 = 1 - Kz_{\lambda 1}$, and $\lambda_2 R/\rho = [1 - (K/m)z_\rho]$, and $z' < z_{\lambda 1} < z_\rho < z^* = 1/(\lambda_2 R + K/m) < 1/\rho < 1$.

Let $\lambda_1 < 1/2$, or equivalently $z_{\lambda 1} > (h(1)/K)^{1/2} = 1/2K$. If $\lambda_1 \ge \rho/4K$, z^* is the unique steady state of (26). If $\lambda_1 < \rho/4K$, $z_1^{**} \equiv [1 - (1 - 4\lambda_1 K/\rho)^{1/2}]/2K$ and $z_2^{**} \equiv [1 + (1 - 4\lambda_1 K/\rho)^{1/2}]/2K$ are two additional steady states of (26). They satisfy $z_1^{**} < (h(1)/K)^{1/2} < z_2^{**}$. If $1/2K < \psi^2(z_\rho) = [1 - (K/m\rho)]/(\lambda_2 R)$, then z^* remains the unique steady state in $I \equiv [\psi^2(z_\rho), z_\rho]$, for all $\lambda_1 > \lambda_{1min}$, where λ_{1min} is defined by $[1 + (1 - 4\lambda_{1min}K/\rho)^{1/2}]/2K \equiv [1 - (K/m\rho)]/(\lambda_2 R)$. If $1/2K > \psi^2(z_\rho) = [1 - (K/m\rho)]/(\lambda_2 R)$, then a tangent bifurcation occurs at $\lambda_1 = \rho/4K$, and intermittency phenomena emerge for $\lambda_1 > \rho/4K$.

7. Concluding Remarks

This paper has presented dynamic general equilibrium models of imperfect credit markets, in which the economy fluctuates endogenously along its unique equilibrium path. The model is based on the heterogeneity of investment projects. In the basic model, there are two types of projects: the Good and the Bad. The Bad is highly profitable, but generates less aggregate demand spillovers than the Good. Hence, the former does not improve the net worth of other agents as much as the latter. Furthermore, the Bad is relatively difficult to finance externally, so that the agents need to have a high level of net worth to be able to start the Bad projects. When the net worth is low, the agents cannot finance the Bad, and much of the credit thus goes to the Good, even when the Bad is more profitable than the Good. This over-investment to the Good creates much aggregate demand spillovers, creating a boom. With an improved net worth, the agents are now able to invest into the Bad. This shift in the composition of the credit from the Good to the Bad at the peak of the boom causes a decline in net worth. The whole process repeats itself. Endogenous fluctuations occur because the Good breeds the Bad and the Bad destroys the Good, as in ecological cycles driven by predator-prey or host-parasite interactions. An extension of the basic model introduces a third type of the projects, the Ugly, which is unprofitable, contributes nothing to improve borrower net worth, but is subject to no borrowing constraint. In this extended model, when the net worth is low, the Good competes with the Ugly in the credit market. Thus, the credit multiplier mechanism is at work in recessions. When the net worth is high, the Good competes with the Bad. Therefore, the credit reversal mechanism is at work in booms. By combing the two mechanisms, this model generates asymmetric fluctuations, along which the economy experiences a long and slow process of recovery, followed by a rapid expansion, and then, possibly after periods of high volatility, it plunges into a recession.

Several cautions should be made when interpreting the message of this paper. First, the Good (the Bad) is defined as the profitable investment projects that contribute more (less) to improve the net worth of the next generation of the agents. These effects operate solely through changes in the competitive prices. They are based entirely on pecuniary externalities, not on technological externalities. Therefore, one should not interpret a shift of the credit from the Good to the Bad as a sign of inefficiency. Of course, more credit to the Bad means bad news for the next generation of the agents, but it is also a consequence of good news for the current generation of the agents, i.e., their net worth is high.

Second, one should not hold the Bad solely responsible for credit cycles. True, the presence of the Bad is essential for credit cycles. If the Bad were removed from the models (or if it is made irrelevant by reducing R or λ so as to move the economy from Region E to Region A of Figure 4), the dynamics monotonically converges, as in the standard neoclassical growth model. Furthermore, the credit reversal takes place when the saving begins to flow into the Bad. However, it is misleading to say that the credit extended to the Bad is the cause of credit cycles. This is because credit cycles can be eliminated also if more credit were extended to the Bad. Recall that, if the agency cost associated with the Bad is made sufficiently small (a large λ), the economy moves from Region E to Region B in Figure 4. One reason why endogenous fluctuations occur in Region E is that the agency problem associated with the Bad is significant enough that the saving continues to flow into the Good, even after the profitability of the Good becomes lower than that of the Bad. Without this overinvestment into the Good, there would not be a boom. And without the boom that precedes it, the credit reversal could not happen. Viewed this way, one might be equally tempted to argue that the credit extended for the Good is the cause of credit cycles. It is more appropriate to interpret that the heterogeneity of the investment projects and the changing composition of the credit are the causes of credit cycles, and it should not be attributed solely to the credit extended for the Good nor to the credit extended for the Bad.

Third, even though the credit market imperfections play a critical role in generating credit cycles, our analysis does not suggest that economies with less developed financial markets are more vulnerable to instability. Figure 4 suggests that endogenous cycles can occur for an intermediate range of the credit market imperfections. Thus, an improvement in the credit market could introduce instability into the system. Nor should one conclude that a significant improvement in the credit market could eliminate endogenous cycles. In the formal analysis, we have assumed, only for the convenience, that there is one type of the Bad projects. One could instead assume that there are many types of the Bad projects, and each type could generate instability for a different range of the credit market imperfection. Then, any further improvement in the credit market may simply replace some types of the Bad projects by other types of the Bad projects, in which case instability would never be eliminated.

Fourth, by demonstrating recurrent fluctuations through the iterations of the time-invariant deterministic nonlinear maps, this paper is not trying to argue that exogenous shocks are unimportant to understanding economic fluctuations. What it suggests is that exogenous shocks do not need to be large,--indeed, they can be arbitrarily small--, to generate large fluctuations. It would be interesting to extend the model to add some exogenous shocks and investigate the interplay between the shocks and internal destabilizing mechanism of the nonlinear system. For example, consider adding some exogenous recurrent technology shocks to the final goods production, which affects the profitability of the Good projects. Imagine, in particular, such an extension in the hybrid model of Section 6. That would shake the nonlinear map of eq. (24) up and down. Suppose that, for most of the times, the shocks are so small that the map satisfies the condition given in Proposition 3A, so that the equilibrium dynamics oscillate around the unique stable steady state, k_1^{**}, and hence can be described by the credit multiplier model a la Bernanke-Gertler. Every once in a while, the shocks are just large enough to push up the map so that it briefly satisfies the condition given in Proposition 3C. Then, after such shocks, the economy experiences a rapid expansion, and possibly after a period of high volatility, plunges into a recession, from which the economy recovers slowly to the old steady state. Such an extension may be useful for understanding why credit market imperfections, while introducing the persistence into the investment dynamics, also make the economy subject to intermittent episodes of "mania, panics, and crashes," as described in Kindleberger, without relying any irrationality. [35]

[35] See Matsuyama (2004d).

Appendix: Proof of Proposition 3.

Because the introduction of the storage technology changes the map only for (k',k_ρ), and since $k_\rho < k^* < \Pi^{-1}(\rho)$ implies $\Psi(k_t) > k_t$ in $[k_{\lambda 1}, k_\rho)$, the dynamical system, (24), could have additional steady states only in $(k', k_{\lambda 1})$, where it is given by

(*) $\quad k_{t+1} = \Psi(k_t) = \Pi^{-1}(\rho[1 - W(k_t)]/\lambda_1)$.

By differentiating (*) and then setting $k_t = k_{t+1} = k^{**}$, the slope of the map at a steady state in this range is equal to $\Psi'(k^{**}) = -\rho W'(k^{**})/\Pi'(k^{**})\lambda_1 = \rho k^{**}/\lambda_1$, which is increasing in k^{**}. Since Ψ is continuous, and $\Psi(k') > k'$ and $\Psi(k_{\lambda 1}) > k_{\lambda 1}$ hold, this means that either

i) the map intersects with the 45° line twice at k_1^{**} and $k_2^{**} > k_1^{**}$;

ii) it is tangent to the 45° line at a single point, $k^{**} \in (k', k_{\lambda 1})$ and $\Psi(k_t) > k_t$ in $(k', k_{\lambda 1})/\{k^{**}\}$; or

iii) $\Psi(k_t) > k_t$ in $(k', k_{\lambda 1})$.

Consider the case of ii). Then, $\rho k^{**}/\lambda_1 = 1$ and $k^{**} = \Pi^{-1}(\rho[1 - W(k^{**})]/\lambda_1)$, which imply that $\Pi(k^{**})k^{**} + W(k^{**}) = \phi(1/k^{**})k^{**} = 1$, or $k^{**} = h(1) = \lambda_1/\rho$. Thus, $\lambda_1 = \rho h(1)$ implies that (*) is tangent to the 45° line at $k^{**} = h(1)$. Furthermore, $h(1) = \Psi(h(1)) < W(h(1))$ implies that $\lambda_1\Pi(W(h(1)))/[1 - W(h(1))] < \lambda_1\Pi(h(1))/[1- W(h(1))] = \lambda_1/h(1) = \rho = \lambda_1\Pi(W(k'))/[1- W(k')]$, or equivalently, $k^{**} = h(1) > k'$, and $\lambda_1 < 1 - W(h(1))$ implies that $k^{**} = h(1) < k_{\lambda 1}$. This proves Proposition 3B. The case of i) can always be obtained by increasing ρ from the case of i), which shifts down the map to create a stable steady state at $k_1^{**} < h(1)$ and an unstable steady state at $k_2^{**} > h(1)$. This proves Proposition 3A. Otherwise, iii) must hold, i.e., the map must lie above 45° line over the entire range, in $(k', k_{\lambda 1})$, which completes the proof of Proposition 3.

References:

Bernanke, Ben, and Mark Gertler, "Agency Costs, Net Worth, and Business Fluctuations," American Economic Review, 79, March 1989, 14-31.

Boldrin, M. and M. Woodford, "Equilibrium Models Displaying Endogenous Fluctuations and Chaos," Journal of Monetary Economics, 1990, 25, 189-222.

Devaney, Robert L., An Introduction to Chaotic Dynamical Systems, New York, Addison Wesley, 1987.

Diamond, Peter, "Government Debt in a Neoclassical Gorwth Model," American Economic Review, December 1965, 55, 1126-1150.

Freixas, X., and J. R. Rochet, Microeconomics of Banking, Cambridge, MIT Press, 1997.

Grandmont, J.-M., "Periodic and Aperiodic Behavior in Discrete One-Dimensional Dynamical Systems," in Contributions to Mathematical Economics: In Honor of Gerard Debreu, ed. By W. Hildenbrand and A. Mas-Colell, Amsterdam, North-Holland, 1986.

Guckenheimer, John, and Philip Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Applied Mathematical Series, vol. 42, Corrected Second Printing, Springer-Verlag, New York, New York, 1986.

Guesnerie, R. and M. Woodford, "Endogenous Fluctuations," in Advances in Economic Theory: Six World Congress, Vol II., ed. by J. J. Laffont, New York, Cambridge University Press, 1992.

Hart, O. and J. H. Moore (1994), "A Theory of Debt Based on the Inalienability of Human Capital," Quarterly Journal of Economics, 109, 841-879.

Hirschman, Albert O., The Strategy of Economic Development, Yale University Press, 1958.

Holmstrom, B., and J. Tirole (1997), "Financial Intermediation, Loanable Funds, and the Real Sector," Quarterly Journal of Economics, 112, 663-691.

Kindleberger, Charles P., Manias, Panics, and Crashes: A History of Financial Crises, The Third Edition, New York, John Wiley & Sons, Inc., 1996.

Kiyotaki, Nobuhiro, and John H. Moore, "Credit Cycles," "Journal of Political Economy, 105, April 1997, 211-248.

Matsuyama, Kiminori, "Endogenous Inequality," Review of Economic Studies, 67, October 2000, 743-759, *a).*

Matsuyama, Kiminori, "On the Rise and Fall of Class Societies," 2000*b*). Latest version available at http://www.faculty.econ.northwestern.edu/faculty/matsuyama/.

Matsuyama, Kiminori, "Good and Bad Investment: An Inquiry into the Causes of Credit Cycles," Working Paper, Northwestern University, September 2001 version, available at http://www.faculty.econ.northwestern.edu/faculty/matsuyama/.

Matsuyama, Kiminori, "Financial Market Globalization, Symmetry-Breaking, and Endogenous Inequality of Nations," Econometrica, May 2004, 853-884. *a)*

Matsuyama, Kiminori, "Credit Traps and Credit Cycles," 2004*b*). Latest version available at http://www.faculty.econ.northwestern.edu/faculty/matsuyama/.

Matsuyama, Kiminori, "A Simple Model of Endogenous Credit Cycles," 2004*c*). in preparation.

Matsuyama, Kiminori, "Bubbleless Bubbles," in preparation.

Murray, J.D., Mathematical Biology, Corrected second printing, New York, Springer-Verlag, 1990.

Townsend, R.M. (1979), "Optimal Contracts and Competitive Markets with Costly State Verification," Journal of Economic Theory, 21, 221-272.