Convergence of a Dynamic Matching and Bargaining Market with Two-sided Incomplete Information to Perfect Competition

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Abstract

Consider a decentralized, dynamic market with an infinite horizon in which both buyers and sellers have private information concerning their values for the indivisible traded good. Time is discrete, each period has length δ, and each unit of time a large number of new buyers and sellers enter the market to trade. Within a period each buyer is matched with a seller and each seller is matched with zero, one, or more buyers. Every seller runs a first price auction with a reservation price and, if trade occurs, both the seller and winning buyer exit the market with their realized utility. Traders who fail to trade either continue in the market to be rematched or become discouraged with probability $\delta\mu$ (μ is the discouragement rate) and exit with zero utility. We characterize the steady-state, perfect Bayesian equilibria as δ becomes small and the market—in effect—becomes large. We show that, as δ converges to zero, equilibrium prices at which trades occur converge to the Walrasian price and the realized allocations converge to the competitive allocation.

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1 Introduction

Asymmetric information and strategic behavior interferes with efficient trade. Nevertheless economists have long believed that for private goods' economies the presence of many traders overcomes both these imperfections and results in convergence to perfect competition. This paper contributes to a burgeoning literature that shows the robust ability of simple market mechanisms to elicit almost accurate cost and value information from buyers and sellers even as at it allocates the available supply almost efficiently. It shows how a completely decentralized market with two-sided incomplete information converges to a competitive outcome as each trader's ability to contact other traders serially increases. Thus a market that for each trader is big over time—as opposed to big at a moment in time—overcomes the difficulties of asymmetric information and strategic behavior. This is another step towards a full understanding of why price theory with its assumptions of complete information and price-taking works as well as it does even in markets where the validity of neither of these assumptions is self-evident.

These ideas may be made concrete by considering a bilateral bargaining situation in which the single buyer has a value $v \in [0,1]$ for an indivisible good and the single seller has a cost $c \in [0,1]$. They should trade only if $v \geq c$, but neither knows the other's value/cost. Instead each regards the other's value/cost as drawn from $[0,1]$ in accordance with a distribution $G(\cdot)$. Myerson and Satterthwaite (1983) showed that no individually rational, budget balanced mechanism exists that both respects the incentive constraints the asymmetric information imposes and prescribes trade only if $v \geq c$. Bilateral trade with two-sided incomplete information is intrinsically inefficient.

An instructive example of this phenomenon is the linear equilibrium Chatterjee and Samuelson (1983) derived for the bilateral $\frac{1}{2}$-double auction when G is the uniform distribution on $[0,1]$. The rules of this double auction are that buyer and seller simultaneously announce a bid $B(v)$ and offer $S(c)$ and they trade at price $p = \frac{1}{2}(B(v) + S(c))$ only if the buyer's bid is greater than the seller's offer. In their linear equilibrium trade occurs only if $v - c \geq \frac{1}{4}$, i.e., the asymmetric information and resulting misrepresentation of value/cost inserts an inefficient "wedge" of thickness $\frac{1}{4}$ into the double auction's outcome. Moreover the magnitude of this wedge is irreducible. Myerson and Satterthwaite (1983) showed that subject to budget balance, individual rationality, and incentive constraints this equilibrium maximizes the ex ante expected gains from trade and therefore is ex ante efficient.

A sequence of papers on the static, multi-lateral k-double auction in the independent private values environment have confirmed economists' intuition that increasing the number of traders causes this wedge to shrink and ultimately vanish in the limit. In the multilateral double auction there are n sellers each supplying one unit and n buyers each demanding one unit. Each trader's cost/value is private and, from the viewpoint of every other trader, independently drawn from [0,1] with distribution G. Sellers and buyers submit offers/bids simultaneously, a market clearing price p is computed, and the n units of supply are allocated at price p to those n traders who revealed through their offers/bids that they most value the available supply. Satterthwaite and Williams (1989) and Rustichini, Satterthwaite, and Williams (1994) established that as n increases the thickness of the wedge and the relative inefficiency associated with each equilibrium are $O(1/n)$ and $O(1/n^2)$ respectively. Relative inefficiency is the expected gains that the traders would realize if the market were perfectly competitive divided into the expected gains that the traders *fail* to realize in the equilibrium of the double auction market.

Thus, quite quickly, the static double auction market with independent private values converges to ex post efficiency—that is, perfect competition—as the number of traders grows.[1] This is despite dispensing with the technically important, but often unrealistic assumption of auction theory that the seller's cost is common knowledge among all participants. These results acknowledge that the effectiveness of a trading institution depends as much on its ability to elicit accurate information from both sides of the market as it does on its ability to allocate goods efficiently in accordance with that information.

These results, however, are derived under three restrictive assumptions: costs/values are independently drawn private signals, sellers have unit supply and buyers have unit demand, and the timing of the market is a one-shot static game. Papers by Fudenberg, Mobius, and Szeidl (2003), Cripps and Swinkels (2003), and Reny and Perry (2003) relax the first two assumptions. Specifically, Fudenberg, Mobius, and Szeidl show that for large markets in an environment with correlated private costs/values an equilibrium to the static double auction exists and traders misrepresentation of their true values is $O\left(\frac{1}{n}\right)$. Cripps and Swinkels, using a somewhat more general model of correlated private values, additionally dispense with the unit supply/unit demand assumption and show that the relative inefficiency of the static double auction is $O\left(\frac{1}{n^{2-\varepsilon}}\right)$ where ε is arbitrarily small. Reny and Perry loosen the first assumption more dramatically, allowing traders' cost/values to have a common value component and their private signals to be affiliated. They show, if the market is large enough, that an equilibrium exists, is almost ex post efficient, and almost fully aggregates the traders' private information, i.e., the double auction equilibrium is almost the unique, fully revealing rational expectations equilibrium that exists in the limit.

This paper, while retaining the independent private values and unit supply/unit demand assumptions, eliminates the third assumption that traders are playing a one-shot game in which, if they fail to trade now, they never have a later opportunity to trade. Commonly a trader who fails to trade now can enter into a new negotiation within a short time, perhaps even within minutes. To account for this possibility we consider a dynamic matching and bargaining model in which trades are consummated in a decentralized matter and traders who do not trade in the current period are rematched in the next period and try again. Gale (1987) and Mortensen and Wright (2002) study models of this

[1] Indeed Satterthwaite and Williams (2001) show that for this environment converges as fast as possible in the sense of worst case asymptotic optimality.

type and show that, as the friction that impedes traders' abilities to search for a favorable price goes to zero, the outcome approaches the competitive outcome. Their models, however, assume complete information and therefore can not test if such markets can succeed in eliciting traders' private information even as they allocate supply.

A reasonably complete, though somewhat informal description of our model and result is this. An indivisible good is traded in a market in which time progresses in discrete periods of length δ and generations of traders overlap. The parameter δ is the exogenous friction in our model that we take to zero. Every active buyer is randomly matched with an active seller each period. Depending on the luck of the draw, a seller may end up being matched with several buyers, a single buyer, or even no buyers. Each seller solicits a bid from each buyer with whom she is matched and, if the highest of the bids is satisfactory to her, she sells her single unit of the good and both she and the successful buyer exit the market. A buyer or seller who fails to trade remains in the market, is rematched the next period, and tries again to trade unless he should become discouraged and decide spontaneously to exit the market without trading.

Each unit of time a large number of potential sellers (formally, measure 1 of sellers) enters the market along with a large number of potential buyers (formally, measure a of sellers). Each potential seller independently draws a cost c in the unit interval from a distribution G_S and each potential buyer draws independently a value v in the unit interval from a distribution G_B. Individuals' costs and values are private to them. A potential trader only enters the market if, conditional on his private cost or value, his equilibrium expected utility is positive. Potential traders who have zero probability of profitable trade in equilibrium elect not to participate.

If trade occurs between a buyer and seller at price p, then they exit with utilities $v - p$ and $p - c$ respectively discounted back at rate r to the time that they entered. As in McAfee (1993) unsuccessful active traders may become discouraged and exit. This occurs for each trader each period with probability $\delta\mu > 0$ where μ is the discouragement rate per unit of time. If δ is large (i.e., periods are long), then a trader who enters the market is impatient, seeking to consummate a trade and realize positive utility amongst the first few matches he realizes. Otherwise he is likely to become discouraged and exit with zero utility. If, however, δ is small (i.e., periods are short), then a trader can wait through many matches looking for a good price with little concern about first becoming discouraged and exiting with no gain.

Buyers with higher values find it worthwhile to submit higher bids than buyers with lower values. At the extreme, a buyer with a value 0.1 will certainly not submit a bid greater than 0.1 while a buyer with a value 0.95 certainly might. The same logic applies to sellers: low cost sellers may be willing to accept lower bids than are higher cost sellers. This means high value buyers and low cost sellers tend quickly to realize a match that results in trade and exit. Low value buyers and high cost sellers may take a much longer time on average to trade and are likely to exit through discouragement rather than trade. Consequently, among the buyers and sellers who are active in the market in a given period t, low value buyers and high cost sellers may be overrepresented relative to the entering distributions G_B and G_S.

We characterize subgame perfect Bayesian equilibria for the steady state of this market and show that, as the period length goes to zero, all equilibria of the market converge to the Walrasian price and the competitive allocation. The Walrasian price p_W in this market is the solution to the equation

$$G_S(p_W) = a(1 - G_B(p_W)), \tag{1}$$

i.e., it is the price at which the measure of entering sellers with costs less than p_W equals the measure of entering buyers with values greater than p_W. If the market were completely centralized with every active buyer and seller participating in an enormous exchange that cleared each period's bids and offers simultaneously, then p_W would be the market clearing price each period. Our result is this. Given a $\delta > 0$, then each equilibria induces a trading range $\left[\underline{p}_\delta, \bar{p}_\delta\right]$. It is the range of offers that sellers of different types make, the range of bids that buyers make, and the range of prices at which trades are actually consummated in this equilibrium. We show that $\lim_{\delta\to 0} \underline{p}_\delta = p_W$ and $\lim_{\delta\to 0} \bar{p}_\delta = p_W$, i.e., the trading range converges to the competitive price. That the resulting allocations give traders the expected utility they would realize in a perfectly competitive market follows as a corollary.

This result, both intuitively and in its proof, is driven by two phenomena: local market size and global market clearing.[2] By local market size we mean the number of other traders with whom each individual trader interacts. This contrasts with global market size—the total number of traders active in the entire market—which is always large in our model. Thus as the time period δ shrinks each trader expects to match an increasing number of times before becoming discouraged and exiting. Each trader's local market becomes big over time as opposed to big at a point in time as is the case in the standard model of perfect competition or in the centralized k-double auction. This creates a strong option value effect for every trader. Even if a buyer has a high value, he has an increasing incentive as δ decreases to bid low and hold out for an offer near the low end of the offer distribution. Therefore all serious buyers bid within an increasingly narrow range just above the minimum offer any seller makes. A parallel argument applies to sellers, with the net effect being, as δ becomes small, all bids and offers concentrate within an interval of decreasing length, i.e., the trading range converges to a single price.

Local market size only forces the market to converge to a single price, not necessarily to the Walrasian price. It is global market clearing that forces convergence to the Walrasian price. To see this, suppose the market converges to a price p that is less than the Walrasian price. At this price more buyers want to buy than there are sellers who want to sell. Buyers are rationed through discouragement, for even a high value buyer may fail to be matched with a seller who wants to sell at p before he becomes discouraged and exits. This, however,

[2] De Fraja and Sákovics (2001) introduced these distinctions.

is inconsistent with equilibrium: the high value buyer can increase his bid above p and guarantee that he will trade and not be rationed out. This increases his expected utility and contradicts the hypothesis that the equilibrium converges to the price p rather than the Walrasian price.

A substantial literature exists that investigates the non-cooperative foundations of perfect competition using dynamic matching and bargaining games.[3] Most of the work of which we are aware has assumed complete information in the sense that each participant knows every other participant's values (or costs) for the traded good. The books of Osborne and Rubinstein (1990) and Gale (2000) contain excellent discussions of both their own and others' contributions to this literature. Papers that have been particularly influential include Mortensen (1982), Rubinstein and Wolinsky (1985, 1990), Gale (1986, 1987) and Mortensen and Wright (2002). Of these, our paper is most closely related to Gale (1987) and Mortensen and Wright (2002). The two main differences between their work and our's is that (i) when two traders meet they reciprocally observe the other's cost/value and (ii) each trader pays a small participation fee.[4] The first difference—full versus incomplete information—is fundamental, for the purpose of our paper is to determine if a decentralized market can elicit private valuation information at the same time it uses that information to assign the available supply efficiently.[5] The second difference prevents traders who have low or zero probabilities of successfully trading from entering and accumulating in the market. In our model the two assumptions that serve the same purpose are (i) only traders with positive expected utility enter and (ii) all active traders become discouraged with probability μ per unit time.

Butters (circa 1979), Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano(2002) are the most important dynamic bargaining and matching models that incorporate incomplete information, albeit one-sided in the cases of Wolinsky and of De Fraja and Sákovics.[6] Of these four papers, only Butters considers the same problem as us. In an incomplete manuscript, he analyzes almost the identical two-sided incomplete information model that we study and makes a

[3] There is a related literature that we do not discuss here concerning is the micro-structure of intermediaries in markets, e.g., Spulber (1999) and Rust and Hall (2002). These models allow entry of an intermediary who posts fixed ask and offer prices and is assumed to be large enough to honor any size buy or sell order without exhausting its inventory or financial resources.

[4] Mortenson and Wright assume a small per period participation fee.

[5] Gale (1987, p. 31) comments that within a limit theorem such as the one he proves there and the one we prove here the complete information assumpiton is "restrictive." This, somewhat surprisingly is not true within "theorems in the limit." Then, he observes, "there is no scope for inferring an agent's type from his willingness to delay. The only effect of assuming incomplete information is to force an agent to treat all other agents symmetrically. For example, in [Gale (1985)] it is strictly easier to show the bargaining equilibrium is Walrasian under incomplete information than complete information."

[6] The models of Peters and Severinov (2002, 2003) also have two-sided incomplete information, are not one-shot games, and do robustly converge to perfect competition, but are intermediate in structure between the full centralization of the static double auction and the radical decentralization of our model. In particular, their model includes a centralized authority that makes all bids and offers public to all traders and allows traders to scan the available prices and seek actively the best available price.

great deal of progress towards proving a variant of the theorem that we prove here.[7]

Wolinsky (1988) considers the steady state of a market in which each seller sets a reservation price on his single unit of an indivisible differentiated good and the randomly matched buyers bid for his unit supply. The cost of each seller's good is zero. Each buyer each period independently draws his idiosyncratic value of the good for which he is bidding from a distribution G. Traders time discount their expected gains so they are impatient. Wolinsky shows that if the discount rate approaches zero, which implies that buyers can almost costlessly search a very large number of sellers, then the equilibrium price in the market converges to zero even if the ratio of buyers to sellers in the market is much larger than one. The reason for this non-Walrasian result is the differentiated product assumption that is central to his model. For small discount rates each buyer patiently waits until he is matched with a seller whose good is almost a perfect match for him, i.e., he waits until he realizes a value from the extreme right of G's tail. When he does obtain such a draw he, somewhat paradoxically, can bid close to zero even if he is bidding against several other buyers. The reason is that the other buyers almost certainly are not well matched with the seller's good; consequently they bid essentially zero and wait for a better match.[8]

The major difference between the models of Wolinsky (1988) and De Fraja and Sákovics (2001) is that in the latter model the good traded is homogeneous, not differentiated. Therefore, as in our model, each buyer's value remains constant when he fails to trade and is rematched in the next period to a new seller. They ensure a degree of competition in the local market by assuming that each seller each period has positive probability that two buyers will be matched with her and have to compete with each other. The entry/exit dynamics are that if a buyer of value v succeeds in trading, then both the buyer and seller—who always has cost zero—exit and are immediately replaced with a buyer of the same cost v and a seller. This latter assumption is both technical and substantive. It is technical in that it guarantees a steady state in their market. But it is also substantive in that it both exogenously fixes the distribution of buyer types that are active in the market and allows the distribution of entering traders to adjust endogenously in order to clear the market. As a consequence a multiplicity of prices may be supported as the discount rate approaches zero. Given the manner in which they define the Walrasian price, their conclusion is that the price distribution only converges to the Walrasian price as discounting vanishes if the parameters of the matching process are chosen fortuitously.[9]

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[7] We thank Asher Wolinsky for bringing Butters' manuscript to our attention in April 2003 after we had completed an earlier version of this paper.

[8] Coles and Muthoo (1998) have extended Wolinsky's model and results to a market in which buyers do not randomly search, but rather preferentially seek out newly entering traders.

[9] More specifically, because each pair of traders who consummates a trade is replaced by an identical pair, if there is a single price in the limit, then this price, by construction, clears the market of entering traders. Thus, if Gale's (1987) concept of flow equilibrium is adopted (and this is the concept we use), then by construction the market converges to the Walrasian price. If, on the other hand, one follows De Fraja and Sákovics in defining the Walrasian price in terms of the steady state "stock" of traders in the market rather than the "flow" of traders

Serrano (2002) studies a one-time entry market with two-sided incomplete information. He assumes discrete distributions of trader types. Each period traders are randomly matched in pairs. Trade between each matched pair is mediated by a discrete 0.5-double auction: traders may only announce prices from a set of three pre-specified prices. Traders who do trade leave the market and, because no new traders are entering, the market runs down over time until trade ceases among the remaining buyers and sellers. Serrano finds that, as discounting is removed, "equilibria with Walrasian and non-Walrasian features persist."

Stepping back from the details of Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano (2002), the simplest explanation why they fail to converge robustly to the Walrasian price and allocation is that the information/allocation problem each attempts to solve is different than the problem that large, static double auctions solve robustly. Think of the baseline problem as being this. Each unit of time measure 1 sellers and measure a buyers enter the market, each of whom has a private cost/value for a single unit of the homogeneous good. The sellers' units of supply need to be reallocated to those traders who most highly value them. Whatever mechanism that is employed must both induce the traders to reveal their costs/valuations and carry out the reallocation. The static double auction literature shows that an enormous, centralized, double auction held once per unit time solves this problem essentially perfectly by closely approximating the Walrasian price and then using that price to mediate trade.

Our model considers the same problem, but within a maximally decentralized market structure in which each period each buyer is randomly matched with one seller. If the discouragement rate and period length are both set equal to unity ($\mu = 1$ and $\delta = 1$), then every trader is certain to become discouraged prior to the next period. Our model is then just a sequence of small, decentralized, one shot markets that solves the problem quite poorly. But, as δ becomes small—the period length becomes short—our model becomes one of small, decentralized markets that are tightly connected over time through the rematching of traders who were initially unsuccessful. Our result is that, as the period length approaches zero, these small, connected markets solve the information/allocation problem equally as well as static double auctions that are held once per unit time.

Given this definition of the problem that both the static double auction and our matching and bargaining market solve, the reason why Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano (2002) do not obtain competitive outcomes as the frictions in their models vanish is clear: the problem their models address are different and, as their results establish, not intrinsically perfectly competitive even when the market becomes almost frictionless. Wolinsky's model relaxes the homogeneous good assumption and does not fully analyze the effects of entry/exit dynamics. De Fraja and Sákovics' model's entry/exit dynamics do not specify fixed measures of buyers and sellers entering the market

through it, then they show that the limiting price may or may not be Walrasian.

each unit of time and therefore have no force moving the market towards a supply-demand equilibrium. Serrano's model is a market that may initially be large but, as buyers and sellers successfully trade, becomes small and non-competitive over time, an effect that the discreteness of its prices aggravates.

The next section formally states the model and our main result establishing that the Walrasian price robustly emerges as the market becomes increasing frictionless. Section 3 derives basic properties of equilibria and presents a computed example illustrating our result. Section 4 proves our result and section 5 concludes with a discussion of possible extensions.

2 Model and theorem

We study the steady state of a market with two-sided incomplete information and an infinite horizon. In it heterogeneous buyers and sellers meet once per period ($t = \ldots, -1, 0, 1, \ldots$) and trade an indivisible, homogeneous good. Every seller is endowed with one unit of the traded good that she is willing to trade if the price she can obtain is at least her cost $c \in [0, 1]$. This cost is private information to her; to other traders it is an independent random variable with distribution G_S and density g_S. Similarly, every buyer seeks to purchase one unit of the good if the price he can obtain is at most his value $v \in [0, 1]$. This value is private; to others it is an independent random variable with distribution G_B and density g_B. Our model is therefore the standard independent private values model. We assume that the two densities are bounded away from zero: a $\underline{g} > 0$ exists such that, for all $c, v \in [0, 1]$, $g_S(c) > \underline{g}$ and $g_B(v) > \underline{g}$.

The length of each period is δ. Each unit of time a large number of potential sellers and a large number of potential buyers consider entering the market; formally each unit of time measure 1 of potential sellers and measure a of potential buyers consider entry where $a > 0$. This means that each period measure δ of potential sellers and measure $a\delta$ of potential buyers consider entry. Only those potential traders whose expected utility from entry is positive actually elect to enter and become active traders.[10] Active buyers and sellers who did not leave the market the previous period carry over. Let ζ be the endogenous steady state ratio of active buyers to active sellers in the market. A period consists of three steps:

1. Every buyer is matched with one seller. His match is equally likely to be with any seller and is independent of the matches other buyers realize. Since there are a continuum of buyers and sellers the matching probabilities are Poisson: the probability that a seller is matched with

[10] In an earlier version of this paper we assumed that potential traders whose expected utility is zero did enter the market and become active. These traders had zero probability of trading and ultimately exited the market through discouragement. Our convergence result (theorem 2 below) still holds under this alternative assumption, though the proofs of claims 16 and 17. are somewhat more complicated because of the presence among active traders of traders who have zero probability of trading.

$k = 0, 1, 2, \ldots$ buyers is[11]

$$\xi_k = \frac{\zeta^k}{k!\, e^{\zeta}}. \tag{2}$$

Consequently a seller may end up being matched with zero buyers, one buyer, two buyers, etc. Each seller, at the time she decides which if any bids she accepts, knows how many buyers with whom she is matched. Each buyer, at the time he submits his bid, only knows the endogenous steady-state probability distribution of how many buyers with whom he is competing.

2. The matched traders bargain in accordance with the rules of the buyers' bid double auction: each buyer simultaneously announces a take-it-or-leave-it offer $B(v)$ to the seller with whom they are bargaining. The seller accepts the highest offer she receives provided it is at least as large as her reservation value $S(c)$.[12] If two or more buyers tie with the highest bid, then the seller uses a fair lottery to choose between them. If a type v buyer trades in period t, then he leaves the market with utility $v - B(v)$. If a type c seller trades at price p, then she leaves the market with utility $p - c$ where p is the bid she accepts. Each seller, thus, runs an optimal auction; moreover their commitment to this auction is credible since the reservation value each sets stems from their dynamic optimization.[13]

3. Every active buyer and active seller who fails to trade decides for exogenous reasons if he will remain in the market the next period. Let $\mu > 0$ be the discouragement parameter. Each period with probability $\delta\mu > 0$ each active trader's situation may change sufficiently that he becomes discouraged and decides to exit the market to pursue other opportunities.

Traders discount their expected utility at the rate $r \geq 0$ per unit time. Together μ and r induce impatience in each trader. Let $\beta = \mu + r$ denote traders overall degree of impatience; $\delta\beta$ is therefore the rate per period at which each trader discounts his utility.

Step 3's assumption that a trader every period has probability $\delta\mu$ of becoming discouraged, turning to another sort of opportunity, and spontaneously exiting is important because every trader who enters must have a probability of either trading or exiting that, per unit of time, is bounded away from zero. Otherwise traders whose probability of trading is infinitesimal but positive would accumulate in the market and jeopardize the existence of a steady state. An alternative assumption for accomplishing the same purpose is the assumption that Gale (1987) and Mortensen (2002) employ in their full information models.

[11] In a market with M sellers and ζM buyers, the probability that a seller is matched with k buyers is $\xi_k^M = \binom{\zeta M}{k} \left(\frac{1}{M}\right)^k \left(1 - \frac{1}{M}\right)^{\zeta M - k}$. Poisson's theorem (see, for example, Shiryaev, 1995) shows that $\lim_{M\to\infty} \xi_k^M = \xi_k$.

[12] See Satterthwaite and Williams (1989) and Williams (1991) for a full analysis of the properties of the buyers-bid double auction.

[13] We do not know if these auctions are the equilibrium mechanism that would result if we tried to replicate McAfee's analysis (1993) within our model.

They assume that each trader both discounts utility and incurs a small participation cost for being active within the market. This causes any potential trader who has a low or zero probability of trading to refuse entry while traders who have non-negligible probabilities of trade only exit through trade, no matter how many periods it takes.

We do not adopt this approach for two reasons. The first reason is substantive: casual empiricism suggests that discouragement—or distraction by alternative opportunities—is a real phenomenon. In any case, discouragement (i.e., spontaneous exit) is a venerable assumption to make within theories of bargaining. See, for example, Binmore, Rubinstein, and Wolinsky's (1986) careful discussion of the distinction between impatience stemming from an exogenous probability of bargaining breakdown and impatience stemming from time preference. Also see the model of McAfee (1993) that incorporates an exogenous probability of exit into a similar model that has one-sided incomplete information. The second reason is expedient: equilibria of double auctions with participation costs are not yet well understood, so tractability argues for incorporating discouragement rather than participation costs into the model.[14]

As explained in the introduction, a seller who has low cost tends to trade within a short number of periods of her entry because most buyers with whom she might be matched have a value higher than her cost and therefore tend to agree to trade. A high cost seller, on the other hand, tends not to trade as quickly or not at all. As a consequence, in the steady state among the population of sellers who are active, high cost sellers are relatively common and low cost seller are relatively uncommon. Exactly parallel logic implies that, in the steady state, low value buyers are relatively common and high value buyers are relatively uncommon. Moreover, this tendency of traders to wait several periods before trading or exiting implies that the total measure of traders active within the market may be larger—perhaps much larger—than the total measure $(1+a)\delta$ of potential traders who consider entry each period.

To formalize the fact that the distribution of trader types within the market's steady state is endogenous, let T_S be the measure of active sellers in the market at the beginning of each period, T_B be the measure of active buyers, F_S be the distribution of active seller types, and F_B be the distribution of active buyer types. The corresponding densities are f_S and f_B and, establishing useful notation, the right-hand distributions are $\bar{F}_S \equiv 1 - F_S$ and $\bar{F}_B \equiv 1 - F_B$. Let, in the steady state, the probability that in a given period a type c seller trades be $\rho_S\left[S(c)\right]$ and the let the probability that a type v buyer trades be $\rho_B\left[B(v)\right]$. Define $W_S\left(c\right)$ and $W_B\left(v\right)$ to be the beginning-of-period steady-state net payoffs

[14] Jianjun Wu at Northwestern University is exploring the properties of static double auctions as part of his dissertation.

to a seller of type c and the buyer of type v, respectively. Let

$$\begin{aligned}
\underline{c} &\equiv S(0), \\
\bar{c} &\equiv \sup_c \{c \mid W_S(c) > 0\}, \\
\underline{v} &\equiv \inf_v \{v \mid W_B(v) > 0\}, \text{ and} \\
\bar{v} &\equiv B(1),
\end{aligned} \tag{3}$$

Since a trader only becomes active in the market if his expected utility from participating is positive, no seller enters whose cost exceeds $\bar{c}$ and no buyer enters whose value is less than $\underline{v}$. We show in the next section that active sellers' equilibrium bids all fall in the interval $[\underline{c}, \bar{c}]$, active buyers' equilibrium offers all fall in $[\underline{v}, \bar{v}]$, and that these intervals are equal: $[\underline{c}, \bar{c}] = [\underline{v}, \bar{v}] \equiv [\underline{p}, \bar{p}]$.

Our goal is to establish sufficient conditions for symmetric, steady state equilibria to converge to the Walrasian price and competitive allocation as the period length in the market goes to zero. By a symmetric, steady state equilibrium we mean one in which every seller in every period plays an anonymous, time invariant strategy $S(\cdot)$, every buyer plays an anonymous, time invariant strategy $B(\cdot)$, and both these strategies are always optimal. Formally, given the friction δ, a perfect Bayesian equilibria consists of strategies $\{B, S\}$, ratio ζ, and distributions $\{F_B, F_S\}$ such that (i) $\{B, S\}$, ζ, and $\{F_B, F_S\}$ generate ζ and $\{F_B, F_S\}$ as their steady state and (ii) no type of trader can increase his or her expected utility by a unilateral deviation from the strategies $\{B, S\}$.[15] We assume that:

A1. Sellers always bid their full dynamic opportunity cost.

A2. For each $\delta > 0$ an equilibrium satisfying A1 exists in which each potential trader's ex ante probability of trade is positive.

Assumption A1 is natural given that in the buyer's bid double auction the sellers can never affect the price at which trade occurs. It also makes possible a simple proof that the strategy S of sellers is increasing, a feature that is intuitive and necessary in the proofs. Assumption A2 states that well behaved equilibria exist in which trade occurs. This is necessary for two reasons. First, a no-trade equilibrium always exists in which neither buyers nor sellers enter the market. Second, it is an open question whether such non-trivial equilibria always exist, though numerical experiments (see section 3.5) suggest that they do for well behaved distribution G_S and G_B.

In order to state our theorem we must define admissible sequences of equilibria. An admissible equilibrium sequence rules out sequences in which the buyer-seller ratio ζ_δ goes to either 0 or ∞ as δ goes to 0. Consider for example a sequence of equilibria indexed by δ such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$ and $\zeta_\delta \to \infty$. Such a sequence is uninteresting because it violates the spirit of assumption A2's requirement that each trader's ex ante probability of trading is

[15] Requirement (ii) that each period each active trader must solve his dynamic optimization problem guarantees that all equilibriia are subgame perfect.

positive. Specifically, the number of buyers with which each seller is matched grows unboundedly. Therefore each seller is sure to sell to a buyer whose value v is arbitrarily close to 1 and each buyer whose value is significantly less than 1 is certain not to trade. In fact, if $P_{B\delta}^{\mathrm{EA}}$ denotes the ex ante probability that a potential buyer will trade, then in such sequences $\lim_{\delta \to 0} P_{B\delta}^{\mathrm{EA}} = 0$ because the probability of a buyer drawing value $v = 1$ is zero.

Definition 1 *A sequence of equilibria indexed by δ such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$ is admissible if a $\bar{\zeta} > 0$ exists such that the equilibrium for each δ_n exists, satisfies A1, gives each trader an ex ante positive probability of trade, and $\zeta_\delta \in (1/\bar{\zeta}, \bar{\zeta})$.*

We are now ready to state our main result.

Theorem 2 *Fix any admissible sequence of equilibria and let $\{S_\delta, B_\delta\}$ be the strategies associated with the equilibrium that δ indexes, let $[\underline{c}_\delta, \bar{c}_\delta] = [\underline{v}_\delta, \bar{v}_\delta]$ be the offer/bid ranges and buyer seller ratios respectively those strategies imply, and let $W_{S\delta}(c)$ and $W_{B\delta}(v)$ be the resulting interim expected utilities of the sellers and buyers respectively. Then both the bidding and offering ranges converge to p_W :*

$$\lim_{\delta \to 0} \underline{c}_\delta = \lim_{\delta \to 0} \bar{c}_\delta = \lim_{\delta \to 0} \underline{v}_\delta = \lim_{\delta \to 0} \bar{v}_\delta = p_W. \tag{4}$$

In addition, each trader's interim expected utility converges to the utility he would realize if the market were perfectly competitive:

$$\lim_{\delta \to 0} W_{S\delta}(c) = \max[0, p_W - c] \tag{5}$$

and

$$\lim_{\delta \to 0} W_{B\delta}(v) = \max[0, v - p_W]. \tag{6}$$

A word of explanation may be helpful here concerning the theorem's second half. If the market were completely centralized and cleared each period at the Walrasian price—that is, if it were perfectly competitive—then each buyer of type v who traded would realize utility $v - p_W$ and each seller who traded of type c would realize utility $p_W - c$. Participants who failed to trade would exit with zero utility.

The proof of the theorem is contained in section 4.

3 Basic properties of equilibria

In this section we derive basic properties that equilibria of our model satisfy. These properties—formulas for probabilities of trade, the strict monotonicity of strategies, and necessary conditions for a strategy pair (S, B) to be an equilibrium—enable us to compute the examples of equilibria and provide the foundations for the proof of our main result. We assume throughout both this section and section 4 that that δ and the equilibrium it indexes is an element of an admissible equilibrium sequence. We also assume that δ is sufficiently small so that $\delta\mu < 1$, i.e., active traders have a positive probability of continuing in the market if they fail to trade.

3.1 Discounted ultimate probability of trade

An essential construct for the analysis of our model is the discounted ultimate probability of trade. It allows a trader's expected gains from participating in the market to be written as simply as possible because it incorporates both the possibility that a trader may become discouraged prior to consummating a trade and the effect time discounting has on the value of a trade that may occur several periods into the future. Define recursively $P_B(\lambda)$ to be a buyer's discounted ultimate probability of trade if he bids λ:

$$\begin{aligned} P_B(\lambda) &= \rho_B(\lambda) + \bar{\rho}_B(\lambda)(1-\delta\mu)(1-\delta r)P_B(\lambda) \\ &\approx \rho_B(\lambda) + \bar{\rho}_B(\lambda)(1-\delta\beta)P_B(\lambda) \\ &= \rho_B(\lambda) + \bar{\rho}_B(\lambda)(1-\delta\beta)\{\rho_B(\lambda) + \bar{\rho}_B(\lambda)(1-\delta\beta)[\rho_B(\lambda) + \cdots]\} \end{aligned}$$

recalling that $\rho_B(\lambda)$ is the probability that the trader will trade in a given period if he bids λ, $\bar{\rho}_B(\lambda) = 1 - \rho_B(\lambda)$, and $\beta = \mu + r$. Observe that the formula incorporates traders' time discounting into the probability calculation. Also observe that the approximation becomes increasingly good as $\delta \to 0$; we use this approximation throughout the paper because our interest is the small δ case.

This construct is useful within a steady state equilibrium because it converts the buyer's dynamic decision problem into a static decision problem. Specifically, if successfully trading gives the buyer a gain U, then for small δ his discounted expected utility W_B from following the stationary strategy of bidding λ is

$$\begin{aligned} W_B(\lambda, U) &= \rho_B(\lambda)U + \bar{\rho}_B(\lambda)(1-\delta\beta)\{\rho_B(\lambda)U + \bar{\rho}_B(\lambda)(1-\delta\beta)[\rho_B(\lambda)U + \cdots]\} \\ &= (\rho_B(\lambda) + \bar{\rho}_B(\lambda)(1-\delta\beta)\{\rho_B(\lambda) + \bar{\rho}_B(\lambda)(1-\delta\beta)[\rho_B(\lambda) + \cdots]\})U \\ &= P_B(\lambda)U. \end{aligned}$$

Solving this recursion gives the explicit formula:

$$P_B(\lambda) = \frac{\rho_B(\lambda)}{\delta\beta + (1-\delta\beta)\rho_B(\lambda)}. \tag{7}$$

The parallel recursion for sellers implies that

$$P_S(\lambda) = \frac{\rho_S(\lambda)}{\delta\beta + (1-\delta\beta)\rho_S(\lambda)}. \tag{8}$$

In section 3.3 we derive explicit formulas for $\rho_B(\cdot)$ and $\rho_S(\cdot)$.

3.2 Strategies are strictly increasing

This subsection demonstrates the most basic property that our equilibria satisfy: equilibrium strategies are strictly increasing. As a preliminary, we first characterize the set of traders that are active in the market. We then turn to the monotonicity results.

Claim 3 *In any equilibrium* $\underline{v} < 1$, $\overline{c} > 0$, *and*

$$(\underline{v}, 1] \subseteq \{v | W_B(v) > 0\}, \tag{9}$$
$$[0, \overline{c}) \subseteq \{c | W_S(c) > 0\}. \tag{10}$$

Proof. If an equilibrium has positive ex ante probability of trade for each potential trader, then $T_B \int_{\underline{v}}^{1} \rho_B[B(v)] f_B(v) dv > 0$ and $T_S \int_0^{\overline{c}} \rho_S[S(c)] f_S(c) dc > 0$. This is true only if $\underline{v} < 1$ and $\overline{c} > 0$. By bidding $B(v)$ in every period, a buyer gets an equilibrium payoff $W_B(v) = vP_B[B(v)] - D_B(B(v))$ where $D_B(v)$ is his discounted expected equilibrium payment. By Milgrom and Segal's (2002) theorem 2,

$$W_B(v) = W_B(\underline{v}) + \int_{\underline{v}}^{v} P_B[B(x)] dx,$$

so $W_B(\cdot)$ is non-decreasing on $(\underline{v}, 1]$. Assume, contrary to (9), that $W_B[B(v')] = 0$ for some $v' \in (\underline{v}, 1]$. It then follows by the monotonicity of $W_B(\cdot)$ that $W_B(v) = 0$ for all $v \in (\underline{v}, v')$, contradicting the definition of $\underline{v}$. Therefore $W_B(v) > 0$ for all $v \in (\underline{v}, 1]$, establishing (9). The proof of (10) is exactly parallel and is omitted.■

Claim 4 B *is strictly increasing on* $(\underline{v}, 1]$.

Proof. $W_B(v) = \sup_{\lambda \geq 0}(v - \lambda)P_B(\lambda) = (v - B(v))P_B(B(v))$ is the upper envelope of a set of affine functions . It follows that $W_B(\cdot)$ is a continuous, increasing, and convex function that is differentiable almost everywhere.[16] Convexity implies that $W_B'(\cdot)$ is non-decreasing on $[\underline{v}, 1]$. By the envelope theorem $W_B'(\cdot) = P_B[B(\cdot)]$; $P_B[B(\cdot)]$ is therefore non-decreasing on $[\underline{v}, 1]$ at all differentiable points. Milgrom and Segal's (2002) theorem 1 implies that at non-differentiable points $v' \in [\underline{v}, 1]$

$$\lim_{v \to v'-} W_B'(v) \leq P_B(B(v')) \leq \lim_{v \to v'+} W_B'(v).$$

Thus $P_B[B(\cdot)]$ is everywhere non-decreasing on $[\underline{v}, 1]$.

Pick any v, $v' \in (\underline{v}, 1]$ such that $v < v'$. Since $P_B[B(\cdot)]$ is everywhere non-decreasing, $P_B[B(v)] \leq P_B[B(v')]$ necessarily. We first show that B is non-decreasing on $(\underline{v}, 1]$. Suppose, to the contrary, that $B(v) > B(v')$. The rules of the buyer's bid double auction imply that $P_B(\cdot)$ is non-decreasing; therefore $P_B[B(v)] \geq P_B[B(v')]$. Consequently $P_B[B(v)] = P_B[B(v')]$. But this gives v' incentive to lower his bid to $B(v')$, since by doing so he will buy with

[16] An increasing function is differentiable almost everywhere.

the same positive probability but pay a lower price. This contradicts B being an optimal strategy and establishes that B is non-decreasing. If $B(v') = B(v)$ $(= \lambda)$ because B is not strictly increasing, then any buyer with $v'' \in (v, v')$ will raise his bid infinitesimally from λ to $\lambda' > \lambda$ to avoid the rationing that results from a tie. This proves that B is strictly increasing.[17] ■

Claim 5 *S is continuous and strictly increasing on $[0, \bar{c})$.*

Proof. Assumption A1 states that since sellers in the market do not affect price, they bid their total opportunity cost:

$$S(c) = c + (1 - \delta\beta)W_S(c) \tag{11}$$

for all $c \in [0, \bar{c})$ where $W_S(c)$ is the equilibrium payoff to a seller with cost c. In a stationary equilibrium $W_S(c) = D(S(c)) - cP_S(S(c))$ where $P_S[S(c)]$ is her discounted ultimate probability of trading when her offer is $S(c)$ and $D(S(c))$ is the expected equilibrium payment to the seller with cost c. Milgrom and Segal's theorem 2 implies that $W_S(\cdot)$ is continuous and can be written, for $c \in [0, \bar{c}]$, as

$$W_S(c) = W_S(\bar{c}) + \int_c^{\bar{c}} P_S(S(x))dx \tag{12}$$

$$= \int_c^{\bar{c}} P_S(S(x))dx \tag{13}$$

where the second line follows from from the definition of $\bar{c}$ and the continuity of $W_S(\cdot)$. This immediately implies that $W_S(\cdot)$ is strictly decreasing (and therefore almost everywhere differentiable) because the definition of $\bar{c}$ implies that $P_S(S(c)) > 0$ for all $c \in [0, \bar{c})$. It, when combined with equation (11), also implies that $S(\cdot)$ is continuous. Therefore, for almost all $c \in [0, \bar{c})$,

$$S'(c) = 1 - (1 - \delta\beta)P_S[S(c)] > 0$$

because $W_S'(c) = -P_S[S(c)]$. Since $S(\cdot)$ is continuous, this is sufficient to establish that $S(\cdot)$ is strictly increasing for all $c \in [0, \bar{c})$. ■

Claim 6 $[\underline{c}, \bar{c}] = [\underline{v}, \bar{v}] = [S(0), S(\bar{c})] = [B(\underline{v}), B(1)] = [\underline{p}, \bar{p}]$.

Proof. Given that S is strictly increasing, $S(0) = \underline{c}$ is the lowest offer any seller ever makes. A buyer with valuation $v < \underline{c}$ does not enter the market since he can only hope to trade by submitting a bid at or above $\underline{c}$, i.e. above her valuation. S is continuous by claim 5, so a buyer with valuation $v > \underline{c}$ will enter the market with a bid $B(v) \in (\underline{c}, v)$ since he can make profit with positive probability. Therefore $\lim_{v \to \underline{c}+} B(v) = \underline{v} = \underline{c}$.

By definition $\bar{c} \equiv \sup_c\{c \,|\, W_S(c) > 0\}$. Equation (11) therefore implies that $S(\bar{c}) = \bar{c}$. A seller with cost $c > \overline{v} = B(1)$ will not enter the market, so $\bar{c} \leq B(1)$.

[17] Alternatively, one can use Theorem 2.2 in Satterthwaite and Williams (1989) with only trivial adaptations.

If $\bar{c} = S(\bar{c}) < \bar{v} \equiv B(1)$, then a seller with cost $c' \in (\bar{c}, B(1))$ can enter and, with positive probability, earn a profit with an offer $S(c') \in (c', B(1))$. This, however, is a contradiction:

$$\sup_c \{c \,|\, W_S(c) > 0\} \geq c' > \bar{c} \equiv \sup_c \{c \,|\, W_S(c) > 0\}.$$

Therefore $S(\bar{c}) = \bar{c} = \bar{v} = B(1)$.■

These findings are summarized as follows.

Proposition 7 *Suppose that $\{B, S\}$ is a stationary equilibrium. Then B and S are strictly increasing over their domains. They also satisfy the boundary conditions $\underline{p} = \underline{v} = \underline{c} = S(0) = B(\underline{v})$ and $\bar{p} = \bar{c} = \bar{v} = B(1) = S(\bar{c})$*

Note that strict monotonicity of B and S allows us to define their inverses, V and C: $V(\lambda) = \inf\{v : B(v) > \lambda\}$ and $C(\lambda) = \inf\{c : S(c) > \lambda\}$. These functions are used frequently below.

3.3 Explicit formulas for the probabilities of trading

Focus on a particular seller of type c who has in equilibrium has a positive probability of trade. In a given period she is matched with zero buyers with probability ξ_0 and with one or more buyers with probability $\bar{\xi}_0 = 1-\xi_0$. Suppose she is matched and v^* is the highest type buyer with whom she is matched. Since each buyer's bid function $B(\cdot)$ is increasing by proposition 7, she accepts his bid if and only if $B(v^*) \geq \lambda$ where λ is her offer. The density from which v^* is drawn is $f_B^*(\cdot)$; it is generated by the steady state density of buyer types $f_B(\cdot)$ and the distribution $\{\xi_0, \xi_1, \xi_2, \ldots\}$ specifying the probabilities with which each seller is matched with zero, one, two, or more buyers. Formally, the distribution F_B^* is conditional on the seller being matched and is defined as, for $v \in [\underline{v}, 1]$,

$$\begin{aligned} F_B^*(v) &= \frac{1}{\bar{\xi}_0} \sum_{i=1}^{\infty} \xi_i \left[F_B(v)\right]^i \\ &= \frac{e^{-\zeta \bar{F}_B(v)} - e^{-\zeta}}{1 - e^{-\zeta}}. \end{aligned}$$

The density f_B^* is the derivative of F_B^*. Notice that F_B^* exhibits first order stochastic dominance with respect to F_B, i.e., for all $v \in [\bar{v}, 1]$, $F_B^*(v) \leq F_B(v)$. Finally, it follows that if a seller offers λ, her probability of trading conditional on being matched with at least one buyer is

$$\hat{\rho}_S(\lambda) = \frac{1 - e^{-\zeta \bar{F}_B(V(\lambda))}}{1 - e^{-\zeta}}.$$

The unconditional probability of trade $\rho_S(\lambda)$ is related to the conditional probability by the formula $\rho_S(\lambda) = \bar{\xi}_0 \hat{\rho}_S(\lambda)$.

A similar expression obtains for $\rho_B(\lambda)$, the probability that a buyer submitting bid λ successfully trades in any given period. Focus on a particular buyer in the steady state and let ω_1 be the probability that no other buyers be matched with the same seller as he, ω_2 be the probability that one other buyer is matched with the same seller as he, ω_3 is the probability that two other buyers are matched with the same seller as he, etc. Recall that ξ_k is the probability that a seller will be matched with k buyers in a given period. Then

$$\omega_j = \frac{j\ \xi_j}{\sum_{k=1}^{\infty} k\ \xi_k} = \frac{j\ \xi_j}{\zeta};$$

that this correct can be seen by considering a large number of sellers and, given the probabilities $\{\xi_0, \xi_1, \xi_2, \ldots\}$, counting what proportion of buyers have no competition, what proportion have one competitor, and so forth.

A buyer who bids λ and is the highest bidder has probability $F_S(C(\lambda))$ of having his bid accepted; this is just the probability that the seller with whom the buyer is matched will have a low enough reservation value so as to accept his offer. Similarly if j buyers are matched with the seller with whom he is matched, then he has $j-1$ competitors and the probability that all $j-1$ will bid less than λ is $[F_B(V(\lambda))]^{j-1}$. Therefore the probability the bid λ is successful in a particular period is

$$\begin{aligned} \rho_B(\lambda) &= F_S(C(\lambda)) \sum_{j=1}^{\infty} \omega_j\ [F_B(V(\lambda))]^{j-1} \\ &= F_S(C(\lambda))\, e^{-\zeta \overline{F}_B(V(\lambda))}, \end{aligned}$$

where the second equality follows by direct calculation.

3.4 Necessary conditions for strategies and steady state distributions

In this subsection the goal is to write down a set of necessary conditions that are sufficiently complete so as to form a basis for calculating section 3.5's example and, also, to create a foundation for section 4's proof of theorem 2. We first derive fixed point conditions that traders' strategies must satisfy. Consider sellers first. Substituting (12),

$$W_S(c) = \int_c^{\bar{c}} P_S(S(x))dx \tag{14}$$

into (11) gives a fixed point condition sellers' strategies must satisfy:

$$S(c) = c + (1-\delta\beta) \int_c^{\bar{c}} P_S(S(x))dx. \tag{15}$$

The parallel expression for a buyer's expected utility is[18]

$$W_B(v) = \int_{\underline{v}}^{v} P_B[B(x)]\,dx \tag{16}$$

[18]Formally, theorem 2 of Milgrom and Segal (2002) justifies this standard expression.

for $v \in [\underline{v}, 1]$. Alternatively,

$$W_B(v) = \max_{\lambda \in [0,1]} (v - \lambda) P_B(\lambda) = (v - B(v)) P_B(B(v)).$$

Substituting (16) into this and solving gives a fixed point condition buyers' strategies must satisfy:

$$B(v) = v - \frac{1}{P_B[B(v)]} \int_{\underline{v}}^{v} P_B[B(x)]\, dx \text{ for } v > \underline{v}. \tag{17}$$

for $v \in [\underline{v}, 1]$.

In our model, the distributions $\{F_B, F_S\}$ are endogenously determined by traders' strategies. In any steady state, the numbers of entering and leaving traders must be equal. This gives rise to three necessary conditions. First, in the steady state, for each type $v \in [\underline{v}, 1]$, the density f_B must be such that the mass of buyers entering equals the mass of buyers leaving:

$$a\delta g_B(v) = T_B f_B(v) \{\rho_B[B(v)] + \bar{\rho}_B[B(v)]\, \delta\mu\} \tag{18}$$

where the left-hand side is the measure of type v buyers of who enter each period and the right-hand side, is the measure of type v buyers who exit each period. Note that it takes into account that within each period successful traders exit before discouraged traders. Second, the analogous steady state condition for the density f_S is, for $c \in [0, \bar{c}]$,

$$\delta g_S(c) = T_S f_S(c) \{\rho_S[S(c)] + \bar{\rho}_S[S(c)]\, \delta\mu\}. \tag{19}$$

Third, trade always occurs between pairs consisting of one seller and one buyer. Therefore, given a cohort of buyers and sellers who enter during a given unit of time, the mass of those buyers who ultimately end up trading must equal the mass of sellers who ultimately end up trading:

$$a \int_{\underline{v}}^{1} P_B(v)\, g_B(v)\, dv = \int_{0}^{\bar{c}} P_S(c)\, g_S(c)\, dc. \tag{20}$$

Together the fixed point conditions (15 and 17), the expected utility formulas (14 and 16), the steady state conditions (18 and 19), and the overall mass balance equation (20) form a useful set of necessary conditions for equilibria of our model.

3.5 A computed example

These necessary conditions (14-20) supplemented with boundary conditions enable us to compute an illustrative example of an equilibrium for our model and to show how, as δ is reduced, the equilibrium converges towards the perfectly competitive limit. The boundary conditions are

$$\begin{aligned} S(0) &= \underline{c}, S(\bar{c}) = \bar{c}, W_S(\bar{c}) = 0 \\ B(\underline{v}) &= \underline{v}, B(1) = \bar{v}, W_B(\underline{v}) = 0 \end{aligned}$$

where $\bar{c} = \bar{v} = \bar{p}$ and $\underline{v} = \underline{c} = \underline{p}$. Our computation specifies that traders' private values are drawn from the uniform distribution ($g_S(c) = g_B(v) = 1$) on the unit interval, the mass of buyers entering each unit of time exceeds the mass of sellers entering by 10% ($a = 1.1$), the discouragement rate is one per unit time ($\mu = 1.0$), and the discount rate is zero ($r = 0.0$). The Walrasian price for these parameter values is $p_W = 0.524$. We computed the equilibrium by fitting sixth degree Chebyshev polynomials to the set of conditions using the method of colocation.

Figure 1 graphs equilibrium strategies S, B and steady state densities f_S, f_B for these parameter values.[19] The left column of the figure graphs strategies and densities for period length $\delta = 0.2$; the right column does the same for period length $\delta = 0.1$. Visual inspection of these equilibria shows the flattening of strategies that occurs as the period length shortens and each trader's option to wait another period for a better deal becomes more valuable. Thus, as δ is cut in half, the trading range $[\underline{p}, \bar{p}]$ narrows from [0.375, 0.574] down to [0.445, 0.550], which is almost a halving of its width from 0.199 to 0.105. In both equilibria the buyer-seller ratio is $\zeta = 1.570$. Observe that for both period lengths the trading range includes the Walrasian price. Inspection of the densities shows that, as the period length shortens, sellers with costs just below $\bar{c}$ and buyers with values just above $\underline{v}$ tend to accumulate within the market.

Given that for the static double auction's relative inefficiency converges to zero at a quadratic rate, of particular interest from our computations is that cutting δ in half only cuts the relative inefficiency I of the equilibrium by slightly less than half: $I = 0.106$ for $\delta = 0.2$ and $I = 0.559$ for $\delta = 0.1$. Thus, even though each trader's market size is doubled in the sense that his expected number of matches before becoming discouraged doubles, there is only a linear decrease in the relative inefficiency. The reason for this slow rate appears to be that the matching market's structure each period forces $\delta\mu$ proportion of each trader type to leave the market discouraged, irrespective of their potential gains from trade. By contrast, the quadratic convergence of the k-double auction is achieved because the "wedge" in that market excludes from trade only those traders who have the smallest gains from trade to realize. This difference in exclusion mechanics, evidently, accounts for the static double auction's much faster, quadratic rate.

This raises an important question that is well beyond the scope of this paper, but that nevertheless requires brief comment. If this dynamic market's rate is so much slower, then why do we observe in the world dynamic markets with continuous trading so much more often than we observe markets organized with periodic, static double auctions? An answer to this question would appear to require an accounting for both the benefits and costs of continuous trade. First, on the benefit side, we are not aware of any good story explaining why markets value continuous trading as much as appears to be the case. Second, the costs may in fact be very small. If in our example δ were reduced to 0.01 or even 0.001—not unreasonable lengths if the unit of time is one year—then

[19] We do not know if this equilibrium is unique.

the relative inefficiency would be very close to zero. In addition, our model may be identifying the slowest possible rate of convergence for a dynamic market because the matching technology it uses is as dumb as possible—buyers are randomly matched with any active seller. Smarter, more efficient matching technologies can easily be imagined that likely would speed up the convergence rate.

4 Proof of the theorem

4.1 A restriction on the shape of B

Our purpose in this subsection is to show that buyers' equilibrium strategies $B(\cdot)$ must be within $\delta^{1/3}$ of either $\underline{v}$ or $\bar{v}$ except within some interval contained in $[\underline{v}, 1]$ that has length no greater than $\delta^{1/3}$. The first claim we establish is a preliminary restriction on the shape of $B(\cdot)$.

Claim 8 *In equilibrium, for all $c \in [0, \bar{c}]$,*

$$\int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B(x)\, dx \leq 2\bar{\zeta}\beta\delta. \tag{21}$$

Proof. $W_S(c)$, a seller's expected utility can be written recursively as the sum of the seller's expected gains from trade in the current period plus her expected continuation value if she fails to trade in the current period:

$$\begin{aligned} W_S(c) &= \bar{\xi}_0 \int_{V(S(c))}^{1} B(x)\ f_B^*(x)\, dx - \bar{\xi}_0 \bar{F}_B^*(V(S(c))) \\ &\quad + \left\{\xi_0 + \bar{\xi}_0 F_B^*(V(S(c)))\right\}(1-\delta\beta) W_S(c). \end{aligned}$$

where $F_B^*(V(S(c)))$ is the probability that, conditional on at least one buyer being matched with her, she fails to trade in the current period. Move all terms involving $W_S(c)$ to the left-hand-side (LHS) and insert the expression $-S(c) + c + (1-\beta\delta) W_S(c) = 0$, which is equation (11) rewritten, into its RHS:

$$\begin{aligned} & W_S(c)\left\{1 - (1-\delta\beta)\xi_0 - (1-\delta\beta)\bar{\xi}_0 F_B^*(V(S(c)))\right\} = \\ &\qquad \bar{\xi}_0 \int_{V(S(c))}^{1} B(x)\ f_B^*(x)\, dx - \bar{\xi}_0 c \bar{F}_B^*(V(S(c))) \\ &\qquad\qquad + \bar{\xi}_0 \bar{F}_B^*(V(S(c)))\left\{-S(c) + c + (1-\delta\beta) W_S(c)\right\}. \end{aligned}$$

Cancel two terms on the RHS and move terms to the LHS to get

$$\begin{aligned} & W_S(c)\left\{\begin{array}{l} 1 - (1-\beta\delta)\xi_0 \\ \qquad -(1-\delta\beta)\bar{\xi}_0\left[F_B^*(V(S(c))) + \bar{F}_B^*(V(S(c)))\right] \end{array}\right\} = \\ &\qquad\qquad \bar{\xi}_0 \int_{V(S(c))}^{1} B(x)\ f_B^*(x)\, dx - \bar{\xi}_0 \bar{F}_B^*(V(S(c)))\, S(c). \end{aligned}$$

Recall that $F_B^*(v) + \bar{F}_B^*(v) = 1$ and $\xi_0 + \bar{\xi}_0 = 1$. Then

$$\delta\beta W_S(c) = \bar{\xi}_0 \int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B^*(x)\, dx, \tag{22}$$

i.e., in equilibrium, for a type c seller, the expected marginal cost of waiting an additional period to trade is equal to the expected marginal expected gain from waiting.

Rearranging (22) gives

$$\int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B^*(x)\, dx = \frac{\beta\delta}{\bar{\xi}_0} W_S(c) \le \frac{\beta\delta}{\bar{\xi}_0}$$

because $W_S(c) \le 1$. First order stochastic dominance implies that

$$\int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B(x)\, dx \le \int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B^*(x)\, dx;$$

Therefore

$$\int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B(x)\, dx \le \frac{\beta\delta}{\bar{\xi}_0} \tag{23}$$

for all $c \in [0, \bar{c})$. The probability that a seller will not be matched with any buyer is

$$\xi_0 = \frac{\zeta^0}{0!\, e^{\zeta}} = \frac{1}{e^{\zeta}} < e^{-1/\bar{\zeta}} \tag{24}$$

because the equilibrium is an element of an admissible sequence and therefore $\zeta \in [1/\bar{\zeta}, \bar{\zeta}]$. A bound on the the complementary probability is

$$\bar{\xi}_0 > 1 - e^{-1/\bar{\zeta}} > \frac{1}{2\bar{\zeta}}$$

because $1 - e^{-x} = x - \frac{x^2}{2} + \ldots$ and $\frac{1}{2\bar{\zeta}}$ is both small and positive. Using this observation, we conclude from (23):

$$\int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B(x)\, dx \le 2\bar{\zeta}\beta\delta. \blacksquare \tag{25}$$

The bound (25) does not have any bite if $f_B(x)$ becomes small as δ becomes small. Therefore in the next claim we establish a lower bound on $f_B(v)$ that is independent of δ.

Claim 9 *For all $v \in [\underline{v}, 1]$, $f_B(v) \ge \underline{g}(\bar{c} - B(v))$.*

Proof. Consider the highest type buyer, $v = 1$. In equilibrium he bids $B(1)$ instead of some $\lambda < B(1)$. His expected gain from following this strategy is

$P_B(B(1))(1-B(1))$. If he bids $\lambda < B(1)$, then his expected gain is $P_B(\lambda)(1-\lambda)$. Revealed preference implies $P_B(B(1))(1-B(1)) \geq P_B(\lambda)(1-\lambda)$. Therefore

$$P_B(\lambda) \leq \frac{P_B(B(1))(1-B(1))}{1-\lambda} = \frac{P_B(\bar{c})(1-\bar{c})}{1-\lambda}. \tag{26}$$

Note also that, for $\lambda < B(1)$, $\rho_B[B(1)] \geq \rho_B(\lambda)$ because ρ_B is a non-decreasing function.

Inequality (26) permits us to bound $\rho_B(\lambda)$ from above. It and formula (7) imply the following sequence of inequalities

$$\begin{aligned} P_B(\lambda) = \frac{\rho_B(\lambda)}{\rho_B(\lambda) + \delta\beta[1-\rho_B(\lambda)]} &\leq \frac{P_B(\bar{c})(1-\bar{c})}{1-\lambda}, \\ \frac{\rho_B(\lambda)}{\rho_B(\lambda)+\beta\delta} &\leq \frac{P_B(\bar{c})(1-\bar{c})}{1-\lambda}, \\ \rho_B(\lambda) &\leq \frac{\beta\delta}{\frac{1-\lambda}{P_B(\bar{c})(1-\bar{c})}-1}, \\ \rho_B(\lambda) &\leq \frac{\beta\delta}{\frac{1-\lambda}{(1-\bar{c})}-1}, \text{ and} \\ \rho_B(\lambda) &\leq \frac{\beta\delta(1-\bar{c})}{\bar{c}-\lambda} \end{aligned} \tag{27}$$

where the second line follows from dropping the less than unity factor $(1-\rho_B(\lambda))$, the third line from solving the inequality, the fourth line from $P_B(\bar{c}) \leq 1$, and the fifth line from simplifying the fourth line.

Inequality (27) allows us to establish the desired lower bound on $f_B(v)$ provided we have an upper bound on T_B, the mass of buyers active in the market. Suppose all potential buyers (measure a each period) entered and became active, none successfully traded, and all left the market only due to discouragement. The total mass of active buyers in the market would then be $T_B = a/\beta$. Since many buyers leave as a result of successful trade an upper bound on the mass of sellers in the market is $T_B \leq a/\beta$. Solve (18) for $f_B(v)$

and substitute in the bound on $\rho_B(\lambda)$ to get

$$\begin{aligned}
f_B(v) &= \frac{1}{T_B}\frac{a\delta\, g_B(v)}{\rho_B[B(v)]+\bar{\rho}_B[B(v)]\,\delta\mu} && (28)\\
&\geq \frac{1}{T_B}\frac{a\delta\, g_B(v)}{\rho_B[B(v)]+\bar{\rho}_B[B(v)]\,\delta\beta}\\
&\geq \frac{1}{T_B}\frac{a\delta\, g_B(v)}{(1-\beta\delta)\frac{\beta\delta(1-\bar{c})}{\bar{c}-\lambda}+\beta\delta}\\
&\geq \frac{g_B(v)}{(1-\beta\delta)\frac{1-\bar{c}}{\bar{c}-\lambda}+1}\\
&\geq \frac{\underline{g}}{(1-\beta\delta)\frac{1-\bar{c}}{\bar{c}-\lambda}+1}\\
&\geq \frac{\underline{g}}{\frac{(1-\bar{c})}{\bar{c}-\lambda}+1}=\frac{\underline{g}}{\lambda}(\bar{c}-\lambda)\\
&\geq \underline{g}(\bar{c}-B(v))
\end{aligned}$$

where $\beta = r+\mu \geq \mu$ implies the second line, (27) implies the third line, $T_B < a/\beta$ implies the fourth line, $\underline{g}$ being the lower bound on the densities g_B and g_S implies the fifth line, $(1-\beta\delta) \leq 1$ implies the sixth line, and $\lambda \leq 1$ implies the seventh line.■

We now use the bounds established in claims 8 and 9 to place a strong restriction on the shape of B. Figure 1 shows the construction used in the next claim and shows how the claim's conclusion confines $B(\cdot)$ to a narrow band of width proportional to $\delta^{1/3}$.

Claim 10 *Suppose $\bar{c}-\underline{v} \geq 2\delta^{1/3}$. For given $\delta > 0$, let $v^* = V\left(\underline{v}+\delta^{1/3}\right)$ and $v^{**} = V\left(\bar{c}-\delta^{1/3}\right)$. Then*

$$v^{**}-v^* \leq \frac{2\bar{\zeta}}{\underline{g}}\,\beta\delta^{1/3} \tag{29}$$

Proof. Substituting inequality (28) into (25) gives

$$\underline{g}\int_{V(S(c))}^{1}(B(x)-S(c))(\bar{c}-B(x))\,dx \leq 2\bar{\zeta}\beta\delta.$$

The special case of this inequality in which $c = 0$ gives the restriction on the buyers' strategy $B(\cdot)$:

$$\int_{\underline{v}}^{1}(B(x)-\underline{v})(\bar{c}-B(x))\,dx \leq \frac{2\bar{\zeta}\beta\delta}{\underline{g}} \tag{30}$$

because $S(0) = \underline{c} = \underline{v}$ and $V(\underline{v}) = \underline{v}$.

Note that, for $x \in [v^*, v^{**}]$, the following inequalities are true: $B(x) - \underline{v} \geq \delta^{1/3}$ and $(\bar{c} - B(x)) \geq \delta^{1/3}$. Therefore

$$\int_{v^*}^{v^{**}} (B(x) - \underline{v})(\bar{c} - B(x))\, dx \geq \delta^{2/3}(v^{**} - v^*).$$

This inequality together with the observation that the integrand of (30) is positive for the whole interval of integration $[\underline{v}, 1]$ implies

$$\begin{aligned} \frac{a\beta\delta}{\underline{g}} &\geq \int_{\underline{v}}^{1} (B(x) - \underline{v})(\bar{c} - B(x))\, dx \\ &\geq \int_{v^*}^{v^{**}} (B(x) - \underline{v})(\bar{c} - B(x))\, dx \\ &\geq \delta^{2/3}(v^{**} - v^*). \end{aligned}$$

The first and last terms of this sequence of inequalities imply (29).■

4.2 The law of one price

In this subsection, we demonstrate that $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = 0$. Since all trades occur at prices within the interval $[\underline{v}_\delta, \bar{c}_\delta]$ this means that as the period length approaches zero all trades occur at essentially one price. Intuitively this is driven by increasing local market size and the resulting option value: as δ becomes small each trader can safely wait for a very favorable offer/bid.

Proposition 11 *Consider any $\overline{\zeta}$-admissible sequence of equilibria $\delta_n \to 0$. Then*

$$\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = 0.$$

Proof. Suppose a sequence of equilibria indexed by δ exists such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$ and $\overline{\lim}_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$. We show that this is a contradiction: therefore, necessarily, $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = 0$. From now on, fix a subsequence such that $\lim_{n\to\infty}(\bar{c}_\delta - \underline{v}_\delta) = \eta$ and $\bar{c}_\delta - \underline{v}_\delta > \eta$.

Pick a small δ from the subsequence and let the strategies $\{S, B\}$, probabilities $\{\xi_0, \xi_1, \xi_{2,} \ldots\}$, and distributions $[F_S, F_B]$ characterize the equilibrium associated with it. Recall that $S(0) = \underline{c} = \underline{v} = B(\underline{v})$ and $B(1) = \bar{v} = \bar{c} = S(\bar{c})$. Also recall above from above the definitions of v^* and v^{**}. Define in addition

$$\tilde{v} = \bar{v} - \frac{1}{4}(\bar{v} - \underline{v}),\ b = B(\tilde{v}), b' = b + \delta^{1/3},\ \text{and}\ \tilde{v}' = V(b')$$

as shown in figure 2. We prove the proposition with a sequence of four claims, the last of which has the proposition as a corollary. The first of these claims derives three intermediate inequalities.

Claim 12 *Given the construction of $\tilde{v}$, v^*, v^{**}, b, and b' and given that, by assumption,* $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$, *if δ is sufficiently small, then*

$$\tilde{v} \le v^*, \tag{31}$$

$$\inf_{v\in[\tilde{v},\tilde{v}']} (\bar{v} - B(v)) \ge \frac{1}{2}\eta, \tag{32}$$

$$\tilde{v}' - \tilde{v} \ge \frac{1}{8}\eta. \tag{33}$$

Proof. We begin with three observations:

O1 The assumption that $\bar{v} - \underline{v} \ge \eta$ for all δ in the sequence and the definition $\tilde{v} = \bar{v} - \frac{1}{4}(\bar{v} - \underline{v})$ imply $\tilde{v} + \frac{1}{4}\eta < \bar{v}$.

O2 The definition $B(v^{**}) = \bar{v} - \delta^{1/3}$ and the inequality $B(v^{**}) \le v^{**}$ imply that $\bar{v} - \delta^{1/3} \le v^{**}$. That $B(v^{**}) \le v^{**}$ follows from the fact that $v^{**} \in (\underline{v}, \bar{c})$ and therefore $\rho_B(v^{**}) > 0$; hence bidding $\lambda \in (\underline{v}, v^{**})$ generates a positive payoff and bidding $\lambda \in (v^{**}, 1)$ generates a negative payoff.

O3 Recall from (29) that $v^{**} \le v^* + \frac{2\bar{\zeta}}{\underline{g}}\ \beta\delta^{1/3}$.

To derive (31), note that O2 and O3 imply

$$\bar{v} \le v^* + \left(1 + \frac{2\bar{\zeta}\beta}{\underline{g}}\right)\delta^{1/3}$$

Combining this with O1 gives

$$\tilde{v} \le v^* - \frac{1}{4}\eta + \left(1 + \frac{2\bar{\zeta}\beta}{\underline{g}}\right)\delta^{1/3}.$$

Thus, for small enough δ,

$$\tilde{v} \le v^*. \tag{34}$$

Turning to (32), that $B(\cdot)$ is increasing, $\tilde{v} \le v^*$, $B(\tilde{v}) = b$, $B(v^*) = \underline{v} + \delta^{1/3}$, $b' = b + \delta^{1/3}$, and $B(v^{**}) = b'$ together imply that $b \in [\underline{v}, \underline{v} + \delta^{1/3}]$ and $b' \in [\underline{v} + \delta^{1/3}, \underline{v} + 2\delta^{1/3}]$. Consequently, for sufficiently small δ,

$$\begin{aligned} \inf_{v\in[\tilde{v},\tilde{v}']} (\bar{v} - B(v)) &\ge \bar{v} - b' \\ &\ge \bar{v} - \underline{v} - 2\delta^{1/3} \\ &\ge \eta - 2\delta^{1/3} \\ &\ge \frac{1}{2}\eta. \end{aligned} \tag{35}$$

This proves (32). Finally, to establish (33), note that by construction $v^* < \tilde{v}'$. Therefore

$$\begin{aligned}
\tilde{v}' - \tilde{v} &> v^* - \tilde{v} \\
&\geq v^{**} - \tilde{v} - \frac{2\bar{\zeta}}{\underline{g}}\ \delta^{1/3} \\
&\geq \bar{v} - \tilde{v} - \left(1 + \frac{2\bar{\zeta}}{\underline{g}}\right)\ \delta^{1/3} \\
&\geq \frac{1}{4}\eta - \left(1 + \frac{2\bar{\zeta}}{\underline{g}}\right)\ \delta^{1/3} \\
&\geq \frac{1}{8}\eta
\end{aligned} \tag{36}$$

where line two follows from $v^{**} \leq v^* + \frac{2\bar{\zeta}}{\underline{g}}\ \delta^{1/3}$, line three follows from $v^{**} > \bar{v} - \delta^{1/3}$, line four follows from $\bar{v} - \tilde{v} \geq \frac{1}{4}\eta$, and line five follows if δ is sufficiently small.$\blacksquare$

Claim 13 *Given* $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$, *if* δ *is sufficiently small, then a* $\gamma > 0$ *exists such that*

$$\frac{\rho_B(b')}{\rho_B(b)} > 1 + \gamma.$$

Proof. Since $V(b') = \tilde{v}'$ and $V(b) = \tilde{v}$, the ratio of $\rho_B(b')$ and $\rho_B(b)$ is

$$\begin{aligned}
\frac{\rho_B(b')}{\rho_B(b)} &= \frac{F_S(C(b'))\, e^{-\zeta \overline{F}_B(\tilde{v}')}}{F_S(C(b))\, e^{-\zeta \overline{F}_B(\tilde{v})}} \\
&\geq e^{\zeta\left[F_B(\tilde{v}') - F_B(\tilde{v})\right]} \\
&\geq 1 + \zeta\left[F_B(\tilde{v}') - F_B(\tilde{v})\right] \\
&\geq 1 + \frac{1}{\bar{\bar{\zeta}}}\int_{\tilde{v}}^{\tilde{v}'} f_B(x)\ dx,
\end{aligned}$$

where the second line follows from $b' > b$ and both F_S and C being increasing, the third line follows by $e^x \geq 1 + x$ $(x \geq 0)$, and the last line follows from $\zeta \geq 1/\bar{\zeta}$. Recall from (28) that $f_B(v) \geq \underline{g}(\bar{v} - B(v))$. Therefore

$$\begin{aligned}
\frac{\rho_B(b')}{\rho_B(b)} &\geq 1 + \frac{\underline{g}}{\bar{\bar{\zeta}}}\int_{\tilde{v}}^{\tilde{v}'} (\bar{v} - B(v))\ dx \\
&\geq 1 + \frac{\underline{g}}{\bar{\bar{\zeta}}}(\tilde{v}' - \tilde{v}) \inf_{v\in[\tilde{v},\tilde{v}']}(\bar{v} - B(v)) \\
&\geq 1 + \frac{\underline{g}}{\bar{\bar{\zeta}}}\left(\frac{1}{8}\eta\right)\left(\frac{1}{2}\eta\right) \\
&= 1 + \frac{1}{16}\frac{\underline{g}}{\bar{\bar{\zeta}}}\eta^2 \\
&= 1 + \gamma
\end{aligned}$$

where line three follows from (32) and (33) and line five follow from $\gamma = \frac{1}{16\zeta}\underline{g}\eta^2$.■

Claim 14 *Given* $lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$*, if* δ *is sufficiently small, then*

$$\frac{P_B(b')}{P_B(b)} \geq 1 + \gamma^*$$

where $\gamma^* = \frac{1}{4}\gamma\eta > 0$.

Proof. Direct calculation proves this. Recall from (7) the formula for $P_B(b)$. Therefore

$$\frac{P_B(b')}{P_B(b)} = \frac{\rho_B(b')}{\rho_B(b)} \frac{\beta\delta + \rho_B(b) - \beta\delta\rho_B(b)}{\beta\delta + \rho_B(b') - \beta\delta\rho_B(b')}.$$

Define x and y so that $\rho_B(b') = \beta\delta x$ and $\rho_B(b) = \beta\delta y$. Then, after some manipulation,

$$\begin{aligned} \frac{P_B(b')}{P_B(b)} &= \frac{1 + \frac{1}{y} - \beta\delta}{1 + \frac{1}{x} - \beta\delta} \\ &\geq \frac{1 + \frac{1+\gamma}{x} - \beta\delta}{1 + \frac{1}{x} - \beta\delta} \\ &= 1 + \frac{\frac{\gamma}{x}}{1 + \frac{1}{x} - \beta\delta} \\ &\geq 1 + \frac{\frac{\gamma}{x}}{1 + \frac{1}{x}} \\ &= 1 + \frac{\gamma\beta\delta}{\rho_B(b') + \beta\delta} \\ &\geq 1 + \frac{1}{2}\gamma(\bar{v} - b') \end{aligned}$$

where line two follows from claim 13's implication that $\frac{1}{y} \geq \frac{1+\gamma}{x}$, line four follows from $\beta\delta \in (0,1)$, line five follows from the definition of x, and line six follows from inequality (27) and $1 - b' < 1$. By construction $b' \in (\underline{v} + \delta^{1/3}, \underline{v} + 2\delta^{1/3})$. Hence, for δ sufficiently small,

$$\begin{aligned} \frac{P_B(b')}{P_B(b)} &\geq 1 + \frac{1}{2}\gamma(\bar{v} - \underline{v} - \delta^{1/3}) \\ &\geq 1 + \frac{1}{4}\gamma\eta \end{aligned}$$

because $\bar{v} - \underline{v} \geq \eta$.■

Claim 15 *Given* $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$*, if* δ *is sufficiently small, then a type* $\tilde{v}$ *buyer has an incentive to deviate from bidding* $B(\tilde{v}) = b$ *to bidding* $b' > b$.

Proof. If we denote the expected utility of a type v buyer who bids λ as $\pi_B(\lambda, v)$, then to prove this we need to show that $\pi_B(b', \tilde{v}) - \pi_B(b, \tilde{v}) > 0$ for δ sufficiently small. First note that by construction $\tilde{v} = \bar{v} - \frac{1}{4}(\bar{v} - \underline{v})$ and $b < \underline{v} + \delta^{1/3}$. Therefore

$$\begin{aligned}
\tilde{v} - b &\geq \bar{v} - \frac{1}{4}(\bar{v} - \underline{v}) - \underline{v} - \delta^{1/3} \\
&= \frac{3}{4}(\bar{v} - \underline{v}) - \delta^{1/3} \\
&\geq \frac{3}{4}\eta - \delta^{1/3} \\
&\geq \frac{1}{2}\eta
\end{aligned}$$

for sufficiently small δ because $\bar{v} - \underline{v} > \eta$. Next observe that, for sufficiently small δ,

$$\begin{aligned}
\pi_B(b', \tilde{v}) - \pi_B(b, \tilde{v}) &= (\tilde{v} - b') P_B(b') - (\tilde{v} - b) P_B(b) \\
&\geq [(1 + \gamma^*)(\tilde{v} - b') - (\tilde{v} - b)] P_B(b) \\
&= \left[(1 + \gamma^*)\left(\tilde{v} - b - \delta^{1/3}\right) - (\tilde{v} - b)\right] P_B(b) \\
&= \left[\gamma^*(\tilde{v} - b) - (1 + \gamma^*)\delta^{1/3}\right] P_B(b) \\
&\geq \left[\frac{1}{2}\eta\, \gamma^* - (1 + \gamma^*)\delta^{1/3}\right] P_B(b) \\
&> 0
\end{aligned}$$

where line 2 follows from claim 14. ■

Claim 15 directly implies proposition 11 because it contradicts the maintained hypothesis that an admissible subsequence of equilibria exists such that $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$.

4.3 Convergence of the bidding and offering ranges to the Walrasian price

Recall that the Walrasian price p_W is the solution to the equation

$$G_S(p_W) = a\, \bar{G}_B(p_W); \tag{37}$$

it is just the price at which the measure of sellers entering the market with cost $c \leq p_W$ equals the measure of buyers entering the market with values $v \geq p_W$. This price would clear the market each period if there were a centralized market. In this subsection we prove our main result: as $\delta \to 0$ the bidding range $[\underline{v}, \bar{v}]$ collapses to the Walrasian price. More formally, for any sequence of equilibria indexed by δ such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$, both

$$\lim_{\delta\to 0} \underline{v}_\delta = p_W \text{ and } \lim_{\delta\to 0} \bar{v}_\delta = p_W. \tag{38}$$

We show this through the proof of two claims. Each of these claims uses the idea that if the price is not converging to the Walrasian price, then the market does not clear globally and an excess of traders builds up on one side or the other of the market. Traders on the long side then have an incentive to deviate from their prescribed bid/offer in order to trade before becoming discouraged.

Claim 16 $\underline{\lim}_{\delta\to 0}\bar{v}_\delta \geq p_W$.

Proof. Let $v_* = \underline{\lim}_{\delta\to 0}\bar{v}_\delta$ and assume, contrary to the statement in the claim, that $v_* < p_W$. For the remainder of this proof, fix a subsequence $\bar{v}_\delta \to v_*$. Let $\tilde{v}_\delta = \bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}$ (note that $\tilde{v}_\delta \in (\bar{v}_\delta, 1]$ for all small enough δ, by proposition 11). Revealed preference implies that

$$\begin{aligned} \pi_B\left(B_\delta\left(\tilde{v}_\delta\right), \tilde{v}_\delta\right) &\geq \pi_B\left(B_\delta\left(1\right), \tilde{v}_\delta\right) \\ \left[\tilde{v}_\delta - B_\delta\left(\tilde{v}_\delta\right)\right]\ P_{B\delta}\left[B_\delta\left(\tilde{v}_\delta\right)\right] &\geq \left[\tilde{v}_\delta - B_\delta\left(1\right)\right]\ P_{B\delta}\left[B_\delta\left(1\right)\right]. \end{aligned}$$

Therefore

$$\begin{aligned} P_{B\delta}\left[B_\delta\left(\tilde{v}_\delta\right)\right] &\geq \frac{\tilde{v}_\delta - B_\delta\left(1\right)}{\tilde{v}_\delta - B_\delta\left(\tilde{v}_\delta\right)} P_{B\delta}\left[B_\delta\left(1\right)\right] \\ &\geq \frac{\tilde{v}_\delta - B_\delta\left(1\right)}{\tilde{v}_\delta - \underline{v}_\delta} P_{B\delta}\left[B_\delta\left(1\right)\right], \end{aligned} \tag{39}$$

where the second inequality follows from the fact that $B_\delta\left(\cdot\right)$ is strictly increasing and therefore $B_\delta\left(\tilde{v}_\delta\right) \geq B_\delta\left(\underline{v}_\delta\right) = \underline{v}_\delta$. Note that

$$\begin{aligned} \frac{\tilde{v}_\delta - B_\delta\left(1\right)}{\tilde{v}_\delta - \underline{v}_\delta} &= \frac{\tilde{v}_\delta - \bar{v}_\delta}{\tilde{v}_\delta - \underline{v}_\delta} \\ &= \frac{\sqrt{\bar{v}_\delta - \underline{v}_\delta}}{\sqrt{\bar{v}_\delta - \underline{v}_\delta} + \bar{v}_\delta - \underline{v}_\delta}, \end{aligned} \tag{40}$$

where the equality in the first line follows from $B_\delta\left(1\right) = \bar{v}_\delta$, and the equality in the second line is the substitution of the definition $\tilde{v}_\delta = \bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}$. Combining (39) and (40) we get

$$P_{B\delta}\left[B_\delta\left(\tilde{v}_\delta\right)\right] \geq \frac{\sqrt{\bar{v}_\delta - \underline{v}_\delta}}{\sqrt{\bar{v}_\delta - \underline{v}_\delta} + \bar{v}_\delta - \underline{v}_\delta} P_{B\delta}\left[B_\delta\left(1\right)\right],$$

So in particular, $P_{B\delta}\left[B_\delta\left(1\right)\right] = 1$ and, by proposition 11, $\lim_{\delta\to 0}(\bar{v}_\delta - \underline{v}_\delta) = 0$ imply[20]

$$\lim_{\delta\to 0} P_{B\delta}\left[B_\delta\left(\tilde{v}_\delta\right)\right] = 1.$$

Mass balance, equation (20) above, states that

$$\int_{\underline{v}_\delta}^{1} a g_B(x) P_{B\delta}\left[B_\delta\left(x\right)\right] dx = \int_0^{\bar{v}_\delta} g_S(x) P_{S\delta}\left[S_\delta\left(x\right)\right] dx. \tag{41}$$

[20] A type 1 buyer always trades immediately because $B\left(1\right) = \bar{v} = \bar{c} = S\left(\bar{c}\right)$.

Given that $P_{B\delta}[B_\delta(\cdot)]$ is increasing and $\tilde{v}_\delta > \underline{v}$,

$$\int_{\underline{v}_\delta}^{1} ag_B(x)P_{B\delta}[B_\delta(x)] \geq P_{B\delta}[B_\delta(\tilde{v}_\delta)]\int_{\tilde{v}_\delta}^{1} ag_B(x)dx \geq P_{B\delta}[B_\delta(\tilde{v}_\delta)]\,a\overline{G}_B(\tilde{v}_\delta)$$

and

$$\int_0^{\bar{v}_\delta} g_S(x)P_{S\delta}[S_\delta(x)]\,dx \leq G_S[\bar{v}_\delta].$$

Therefore it follows from (41) that

$$P_{B\delta}[B_\delta(\tilde{v}_\delta)]\,a\overline{G}_B(\tilde{v}_\delta) \leq G_S[\bar{v}_\delta]$$

or, since $P_{B\delta}[B_\delta(\tilde{v}_\delta)] \leq 1$,

$$a\overline{G}_B(\tilde{v}_\delta) \leq G_S[\bar{v}_\delta]. \tag{42}$$

By taking limits in (42) as $\delta \to 0$ and invoking continuity of G_S and $\overline{G}_B$, we obtain

$$a\overline{G}_B\left(\lim_{\delta\to 0}\tilde{v}_\delta\right) \leq G_S\left(\lim_{\delta\to 0}\bar{v}_\delta\right). \tag{43}$$

The definition of v_* and proposition 11 imply $\lim_{\delta\to 0}\tilde{v}_\delta = \lim_{\delta\to 0}[\bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}] = \bar{v}_\delta$ and, by hypothesis, $\lim_{\delta\to 0}\bar{v}_\delta = v_*$. Therefore we obtain from (43):

$$a\overline{G}_B(v_*) \leq G_S(v_*).$$

This, however, is a contradiction because the the maintained assumption that $v_* < p_W$ implies that $a\overline{G}_B(v_*) > a\overline{G}_B(p_W) = G_S(p_W) > G_S(v_*)$.■

Claim 17 $\overline{\lim}_{\delta\to 0}\underline{c}_\delta \leq p_W$.

Proof. Verification of this claim follows the same logic as that of claim 16. Define $c_* = \overline{\lim}_{\delta\to 0}\underline{c}_\delta$ and suppose, contrary to the statement in the claim, that $c_* > p_W$. For the remainder of this proof, fix a subsequence $\underline{c}_\delta \to c_*$. Let $\tilde{c}_\delta = \underline{c}_\delta + \sqrt{\bar{c}_\delta - \underline{c}_\delta}$ noting that proposition 11 implies $\tilde{c}_\delta \in [0, \bar{c}_\delta)$ for all small enough δ. A seller who offers and succeeds in trading does not realize $S_\delta(v)$ as her revenue. She realizes something more because the bid she accepts is at least as great as $S_\delta(v)$. Therefore, for each δ sufficiently small, a function $\phi_\delta(\cdot) : [0, \bar{c}_\delta] \to [\underline{c}_\delta, \bar{c}_\delta]$ exists that maps, conditional on consummating a trade, the seller's offer into her expected revenue from the sale. Thus $\phi_\delta[S_\delta(c)]$ is a type c seller's expected revenue given that she offers $S_\delta(c)$. Take note that $\phi_\delta[S_\delta(c)] \in [S_\delta(c), \bar{c}_\delta]$ because the expected revenue can not be less than the seller's offer $S_\delta(c)$. Revealed preference implies that

$$\begin{aligned} \pi_S(S_\delta(\tilde{c}_\delta), \tilde{c}_\delta) &\geq \pi_S(S_\delta(0), \tilde{c}_\delta) \\ [\phi_\delta[S_\delta(\tilde{c}_\delta)] - \tilde{c}_\delta]\; P_{S\delta}[S_\delta(\tilde{c}_\delta)] &\geq [\phi_\delta[S_\delta(0)] - \tilde{c}_\delta]\; P_{S\delta}[S_\delta(0)]. \end{aligned} \tag{44}$$

Solving,

$$
\begin{aligned}
P_{S\delta}\left[S_\delta\left(\tilde{c}_\delta\right)\right] &\geq \frac{\phi_\delta\left[S_\delta(0)\right]-\tilde{c}_\delta}{\phi_\delta\left[S_\delta\left(\tilde{c}_\delta\right)\right]-\tilde{c}_\delta}\, P_{S\delta}\left[S_\delta(0)\right] \qquad (45)\\
&\geq \frac{\underline{c}_\delta-\tilde{c}_\delta}{\overline{c}_\delta-\tilde{c}_\delta}\, P_{S\delta}\left[S_\delta(0)\right]\\
&= \frac{\sqrt{\overline{c}_\delta-\underline{c}_\delta}}{\overline{c}_\delta-\underline{c}_\delta+\sqrt{\overline{c}_\delta-\underline{c}_\delta}}\, P_{S\delta}\left[S_\delta(0)\right]
\end{aligned}
$$

where the second line follows from the fact that $\phi_\delta\left[S_\delta(0)\right] \geq S_\delta\left(0\right) = \underline{c}_\delta$ and the third line follows by substitution of the definition for $\widetilde{c}_\delta$. So in particular, $P_{S\delta}\left[S_\delta(0)\right] = 1$ and, by proposition 11, $\lim_{\delta\to 0}(\overline{c}_\delta - \underline{c}_\delta) = 0$ together imply

$$\lim_{\delta\to 0} P_{S\delta}\left[S_\delta\left(\tilde{c}_\delta\right)\right] = 1. \qquad (46)$$

As in the proof of claim 16, the mass balance equation (41) must hold:

$$\int_{\underline{c}_\delta}^{1} a g_B(x) P_{B\delta}\left[B_\delta\left(x\right)\right] dx = \int_0^{\overline{c}_\delta} g_S(x) P_{S\delta}\left[S_\delta\left(x\right)\right] dx. \qquad (47)$$

Since

$$\int_0^{\overline{c}_\delta} g_S(x) P_{S\delta}\left[S_\delta\left(x\right)\right] dx \geq P_{S\delta}\left[S_\delta\left(\tilde{c}_\delta\right)\right] G_S(\tilde{c}_\delta)$$

and

$$\int_{\underline{c}_\delta}^{1} a g_B(x) P_{B\delta}\left[B_\delta\left(x\right)\right] dx \leq a\overline{G}_B(\underline{c}_\delta),$$

it follows from (47) that

$$a\overline{G}_B(\underline{c}_\delta) \geq P_{S\delta}\left[S_\delta\left(\tilde{c}_\delta\right)\right] G_S(\tilde{c}_\delta)$$

or, since $P_{S\delta}\left[S_\delta\left(\tilde{c}_\delta\right)\right] \leq 1$,

$$a\overline{G}_B(\underline{c}_\delta) \geq G_S(\tilde{c}_\delta). \qquad (48)$$

By taking limits in (48) as $\delta \to 0$ and invoking continuity of G_S and $\overline{G}_B$, we obtain

$$a\overline{G}_B\left(\lim_{\delta\to 0}\underline{c}_\delta\right) \geq G_S\left(\lim_{\delta\to 0}\tilde{c}_\delta\right). \qquad (49)$$

Since $\lim_{\delta\to 0}\tilde{c}_\delta = c_*$ and $\lim_{\delta\to 0}\underline{c}_\delta = c_*$ by proposition 11, (49) implies

$$a\overline{G}_B(c_*) \geq G_S\left(c_*\right).$$

This, however, is a contradiction because the maintained assumption $c_* > p_W$ implies $a\overline{G}_B(c_*) < a\overline{G}_B(p_W) = G_S\left(p_W\right) < G_S\left(c_*\right)$.■

Proof of the main theorem. Claims 16 and 17 together with $\inf_{\delta\to 0}(\overline{c}_\delta - \underline{v}_\delta) = 0$ establishes (38): prices realized in the market converge to the Walrasian price. The proofs of those two claims together show that an arbitrarily small deviation upward in a buyer's equilibrium bid or an arbitrarily small deviation downward in a seller's equilibrium offer can guarantee trade, provided δ is sufficiently small. This, together with the result that realized prices converge to the Walrasian price, establishes (5) and (6): equilibrium expected utility for both buyers and sellers approaches what they would expect if the market were competitive.■

5 Conclusions

In this paper we have proven that essentially all equilibria of a simple, dynamic matching and bargaining market in which both sellers and buyers have incomplete information converge to the Walrasian price and competitive allocation as the model's friction—the length of the matching period—goes to zero. This convergence is driven by the interaction of two forces within the model: local market size and global market clearing. The significance of our result is to show that in the presence of private information a fully decentralized market such as the one we model can deliver the same economic performance as a centralized market such as the k-double auction that Rustichini, Satterthwaite, and Williams (1994) studied. This is an important extension of the full information dynamic matching and bargaining models, for it shows that a decentralized market can handle the elicitation of private values and costs even as it allocates the market supply to the traders who most highly value that supply.

The limitations of our model and results immediately raise further questions. Six particularly stand out for future investigation:

- The numerical experiments reported in section 3.5 suggest that the k-double auction's quadratic rate of convergence towards efficiency does not carry over to this dynamic model and is in fact linear. How fast, in fact, does the market converge to the competitive allocation as δ decreases? Do other matching technologies achieve a faster rate of convergence?

- We only considered the buyer's bid double auction mechanism. Does convergence robustly hold for a broader class of trading mechanisms, e.g., the 0.5-double auction?

- Our model assumes independent private values and costs. We would like to know if our results generalize to both correlated costs/values and to interdependent values with a common component and affiliated private signals.

- It would be attractive to incorporate into our model the alternative assumption that every active trader pays a small participation cost as in the full information bargaining and matching models of Gale (1987) and Mortensen and Wright (2002).

- An important feature of the centralized k-double auction as studied by Rustichini, Satterthwaite, and Williams (1994) is that if the number of traders on both sides is large, then a trader who reports his cost or value truthfully as opposed to following his equilibrium strategy realizes almost no reduction in expected utility. In other words, for large markets the k-double auction is almost a strategy-proof mechanism.[21] Traders are not so fortunate in our bargaining and matching market even if the period length is very short. To play optimally—or even acceptably—they must have good knowledge of the distribution of prices at which trades are occurring. Nevertheless the following conjecture appears worth exploring: if our matching and bargaining market were populated with non-optimizing buyers and sellers, then a class of simple learning rules exists such that the market converges to the Walrasian price and competitive allocation. If this is true, then those rules would be almost-dominant strategies and the market as a whole would be almost strategy-proof.

- Kultti (2000) studies a complete information model in which each buyer, as part of his optimization, decides whether to be a searcher or waiter. Sellers do the same. Searching buyers are then randomly matched with waiting sellers and searching sellers are randomly matched with waiting buyers. Giving every trader the choice to be a searcher or waiter would be an elegant way of endogenizing the model's market structure rather than mandating that all buyers are searchers and all sellers are waiters.

If these and other questions can be answered adequately in future work, then this theory may become useful in designing and regulating decentralized markets with incomplete information in much the same way auction theory has become useful in designing auctions. The ubiquity of the Internet with its capability for increasing local market size and reducing period length makes pursuit of this goal attractive.

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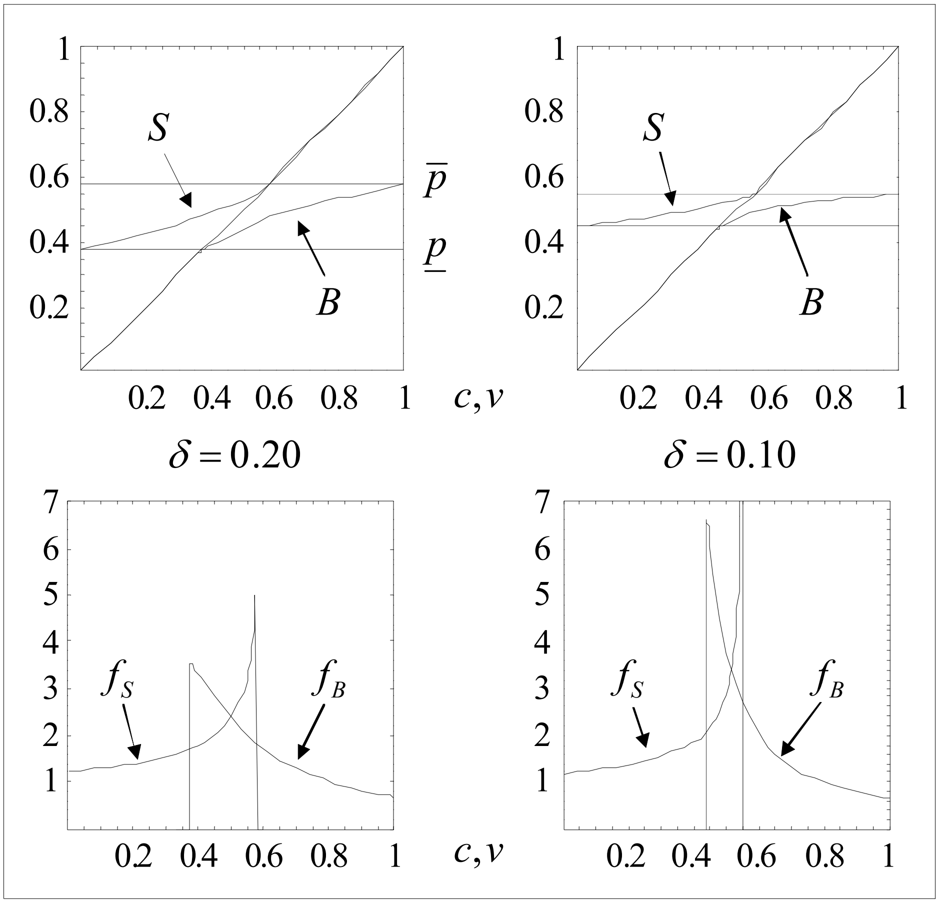

Figure 1: This figure graphs two equilibria for the case in which g_S and g_B are uniform, $a = 1.1$, $\mu = 1.0$, and $r = 0.0$. On the left side period length is $\delta = 0.20$.It has relative inefficiency $I = 0.106$ and masses of active traders $T_S = 0.199$ and $T_B = 0.313$. On the right side period length is $\delta = 0.10$. It has relative inefficiency $I = 0.0559$ and masses of active traders $T_S = 0.105$ and $T_B = 0.165$.

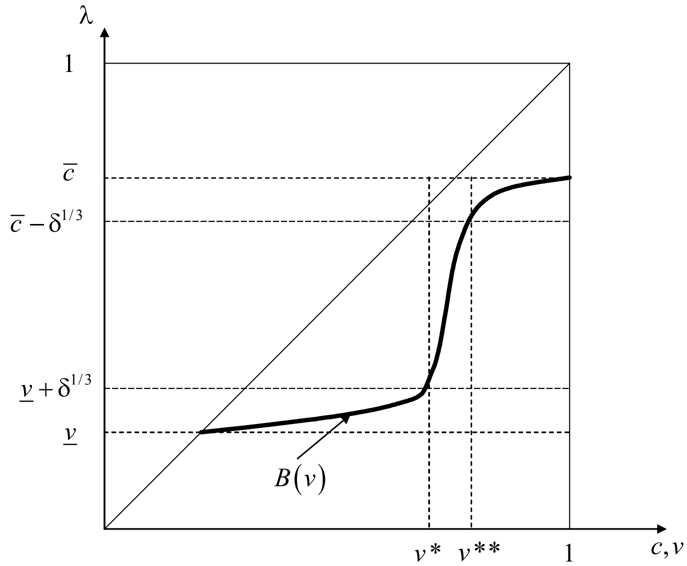

Figure 2: $\sqrt[3]{\delta}$ band that confines $B(\cdot)$

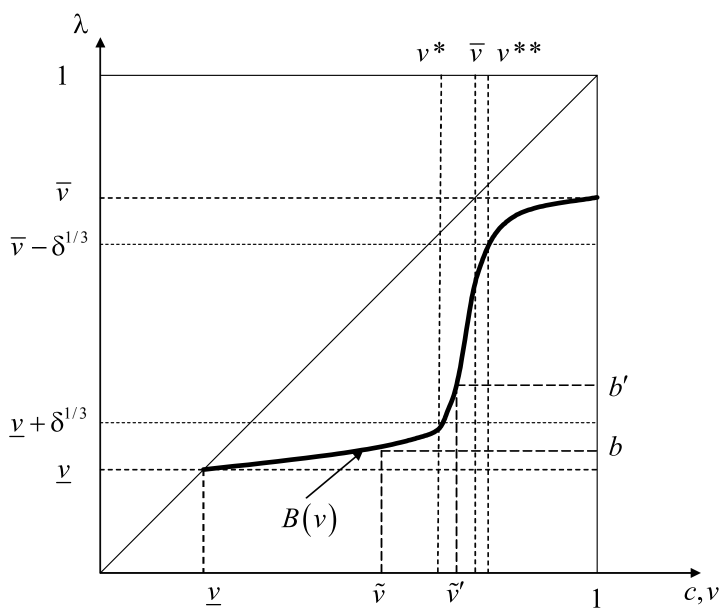

Figure 3: Construction of v^{*}, v^{**}, $\widetilde{v}$, $\widetilde{v}'$, b and b'.