Approximate Versus Exact Equilibria*

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Abstract

This paper develops theoretical foundations for an error analysis of approximate equilibria in dynamic stochastic general equilibrium models with heterogeneous agents and incomplete financial markets. While there are several algorithms which compute prices and allocations for which agents' first order conditions are approximately satisfied ('approximate equilibria'), there are few results on how to interpret the errors in these candidate solutions and how to relate the computed allocations and prices to exact equilibrium allocations and prices. We give a simple example which illustrates that approximate equilibria might be very far from exact equilibria. We then interpret approximate equilibria as equilibria for close-by economies, that is, for economies with close-by individual endowments and preferences. We provide sufficient conditions which ensure that approximate equilibria are close to exact equilibria of close-by economies.

We give a detailed discussion of the error analysis for two models which are commonly used in applications, an OLG model with stochastic production and an asset pricing model with infinitely lived agents. We illustrate the analysis with some numerical examples. In these examples the derived bounds are at most one order of magnitude larger than maximal errors in Euler equations.

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1 Introduction

The computation of equilibria in dynamic stochastic general equilibrium models with heterogeneous agents has become increasingly important in finance, macroeconomics and public finance. Many economic insights can be obtained by analyzing quantitative features of realistically calibrated models (prominent examples in the literature include, among others, Rios-Rull (1996), Heaton and Lucas (1996) or Krusell and Smith (1997)).

Unfortunately there are often no theoretical foundations for algorithms which claim to compute competitive equilibria in models with incomplete markets or overlapping generations. In particular, since all computation suffers from truncation and rounding errors it is obviously impossible to numerically verify (as some applied researchers claim) that the optimality and market clearing conditions are satisfied and that a competitive equilibrium is found. The fact that the equilibrium conditions are approximately satisfied generally does not yield any implications on how well the computed solution approximates an exact equilibrium. Computed allocations and prices could be arbitrarily far from competitive equilibrium allocations and prices.

In this paper we develop an error analysis for the computation of competitive equilibria in models with heterogeneous agents where equilibrium prices are infinite dimensional. We define an ϵ-equilibrium as a collection of finite sets of choices and prices such that there exists a process of prices and choices which takes values exclusively in these sets and for which the relative errors in agents' Euler equations and the errors in market clearing conditions are below some small ϵ at all times.

Existing algorithms for the computation of equilibria in dynamic models can be interpreted as computing ϵ-equilibria and the finiteness of ϵ-equilibria allows us to computationally verify if a given collection of endogenous variables (i.e. a candidate solution) constitutes an ϵ-equilibrium. In order to give an economic interpretation of the concept we follow Postlewaite and Schmeidler's (1981) analysis for finite economies and interpret ϵ-equilibria as approximating exact equilibria of a close-by economy.

In finite economies the problem of interpreting ϵ-equilibria is easiest illustrated in a standard Arrow-Debreu exchange economy. Scarf (1967) proposes a method which 'approximates' equilibria for any given finite economy in the following sense: Given individual endowments e^i for individuals $i = 1, \ldots, I$ and an aggregate excess demand function $\xi(p, (e^i))$, and given an $\epsilon > 0$, the methods finds a $\bar{p}$ such that $\|\xi(\bar{p}, (e^i))\| < \epsilon$. As Richter and Wong (1999) point out[1] this fact does not imply that it is possible to find a $\bar{p}$ such that $\|\bar{p} - p^*\| < \epsilon$ for some exact equilibrium price vector p^*.

[1]They examine the problem of the computation of equilibria from the viewpoint of computable analysis as developed by Turing (1936) and point out that while Scarf's algorithm generates a sequence of values converging to a competitive equilibrium knowing any finite initial sequence might shed no light at all on the limit.

However, if individual endowments are interior and if the value of the excess demand function at $\bar{p}$, $\|\xi(\bar{p}, (e^h))\|$, is small, then $\bar{p}$ is an equilibrium price for a close-by economy. Homogeneity of aggregate excess demand implies trivially that if $\bar{p} \cdot \xi(\bar{p}, (e^h)) = 0$ then $\|(\bar{p}, (e^h)) - (p^*, (\tilde{e}^h))\| < \epsilon$ with $\xi(p^*, (\tilde{e}^h)) = 0$. It is possible that $\bar{p}$ is not a good approximation for the equilibrium price of the given economy. However researchers rarely know the exact individual endowments of agents anyway, and if close-by specifications of exogenous variables lead to vastly different equilibria it will be at least useful to know one possible equilibrium for one realistic specification of endowments. As Postlewaite and Schmeidler (1981) put it, "If we don't know the characteristics, but rather, we must estimate them, it is clearly too much to hope that the allocation would be Walrasian with respect to the estimated characteristics even if it were Walrasian with respect to the true characteristics."

This issue has been well understood for a long time from a viewpoint of computational mathematics. In general, sources of errors in computations can be classified in three categories:

1. Errors due to the theory: The economic model contains many idealizations and simplifications.

2. Errors due to the specification of exogenous variables: The economic model depends on parameters which are themselves computed approximately, the results of experimental measurements or the results of statistical procedures.

3. Truncation and rounding errors: each limiting process must be broken off at some finite stage, computers usually use floating point arithmetic resulting in round-off errors.

In contrast to standard error analysis, which aims to bound the distance of the approximate solution to the exact solution, 'backward error analysis' exploits a trade-off between 2 and 3 and examines how much the given problem would have to be perturbed in order for the calculated solution to be an exact solution of the perturbed problem (see e.g. Wilkinson (1963) or Higham (1996)). While in the applied economic literature which uses computations there is a large debate about the trade-off between 1 and 3, there is surprisingly little discussion about a possible trade-off between 2 and 3. This paper explores how this latter trade-off can be used to interpret approximate solutions to dynamic general equilibrium models via backward error analysis.

We examine two concrete applications where we take as given that standard algorithms compute values for the endogenous variables for any possible sequence of exogenous shocks. We describe a method to construct an ϵ-equilibrium from the computer output. Although our definition of recursive equilibrium is discrete, it turns out to be very useful to use continuous algorithms to compute the ϵ-equilibria in practice. In particular we examine algorithms which assume that approximate policy and pricing functions are smooth. We show that in these applications our methods lead to reasonable and economically meaningful error bounds.

For models with a single agent, Santos and his co-authors have developed such sufficient conditions and give explicit error bounds both on policy functions and on allocations (Santos and Vigo (1998), Santos (2000), and Santos and Peralta-Alva (2002)). While even in their framework these conditions do not hold for all interesting specifications of the model, in applications, the conditions can often be verified. Under these conditions, error bounds on allocations can be derived from Euler equation residuals. However, most of these results do not generalize to models with heterogeneous agents and incomplete markets. No sufficient conditions are known which allow the derivation of error bounds on computed equilibrium prices and allocations in the models considered in this paper.

Backward error analysis is a standard tool in numerical analysis that was developed in the late 1950s and 1960s, see Wilkinson (1963). It is surprising that it has, to the best of our knowledge, not been widely used in economics. (Judd's textbook (1998), for example, mentions backward error analysis and provides a citation from the numerical analysis literature but never applies the concept to an economic problem.) The only somewhat related concept in economics is "backsolving" which was introduced by Sims (1989) for solving nonlinear, stochastic systems. Ingram (1990) describes backsolving from an econometric viewpoint. The endogenous variables in a stochastic dynamic optimization problem are affected by random shocks. Instead of taking a distribution of shocks as given and then solving for the distribution of the endogenous variables, backsolving begins by specifying a convenient or intuitive distribution for some of the endogenous variables and then attempts to find underlying distributions of random shocks and other variables that would yield the assumed distributions of the endogenous variables. Note that this approach is different from backward error analysis because it does not address the question how far away the exogenous distribution is from some desired or estimated one. In backward error analysis exogenous parameters are given, then an approximate solution is computed, and then the necessary perturbations in exogenous parameters are determined. Clearly, we always would like to have very small backward errors. In fact, the focus of our analysis of popular models in Sections 5 and 6 of this paper is the calculation of backward errors. Due to the nature of economic problems we cannot perform "pure" backward error analysis and only perturb exogenous parameters. Instead, we will compute bounds on perturbations of both exogenous parameters and endogenous equilibrium values. Higham (1993) calls this "mixed" backward error analysis.

The analysis in our paper is, from a theoretical perspective, perhaps closest to Mailath et al.'s (2003) discussion of ϵ-equilibria in dynamic games. An important difference is that they allow for perturbations in the instantaneous pay-off functions of the game. In our framework this can lead to preferences over payoff streams that are far away from the original preferences. Therefore, we cannot consider these as admissible.

The paper is organized as follows. In Section 2 we illustrate the main intuition in a simple two-period example. Section 3 outlines an abstract dynamic model and defines what we mean by close-by economies. Section 4 develops the theoretical foundations of our method. In Section 5 we apply this method to a model with overlapping generations and production. In Section 6 we apply the methods to a version of Lucas' (1978) asset pricing model with heterogeneous agents.

2 The Main Intuition in a Two-period Economy

In this section we demonstrate the main themes of this paper in a simple two-period model. We first show how competitive equilibria can be characterized by a system of equations that relates endogenous variables in one period to endogenous variables of the next period. These equations, which we will refer to as the equilibrium equations, enable us later in the paper to describe infinite equilibria with finite sets. Secondly, we define an ϵ-equilibrium and provide an example that shows that ϵ-equilibrium prices and allocations can be a terrible approximation to exact equilibria. We show that in the example perturbations in individual endowments can rationalize ϵ-equilibria as exact equilibria.

We consider a simple pure exchange economy with two agents, two time periods and no uncertainty. There is a single commodity in each period, agents' endowments are (e_0^i, e_1^i) for $i = 1, 2$. Agents can trade a bond which pays one unit in the second period, the price of the bond is denoted by q. Agents' bond holdings are θ^i, $i = 1, 2$. Agents preferences are represented by time-separable utility

$$U^i(x_0, x_1) = v_i(x_0) + u_i(x_1), \quad i = 1, 2,$$

for increasing, differentiable and concave functions $v_i, u_i : \mathbb{R}_+ \to \mathbb{R}$. A competitive equilibrium is a collection of choices $(c^i, \theta^i)_{i=1,2}$ and a bond price q such that both agents maximize utility and markets clear, i.e. $\theta^1 + \theta^2 = 0$ and for both $i = 1, 2$,

$$(c^i, \theta^i) \in \arg \max_{c \in \mathbb{R}_+^2, \theta \in \mathbb{R}} U^i(c) \quad \text{s.t.} \quad c_0 = e_0^i - q\theta, \quad c_1 = e_1^i + \theta.$$

In order to represent equilibria for infinite horizon models we want to derive a system of equations that links endogenous variables (i.e. choices and prices) today to endogenous variables next period and which is necessary and sufficient for equilibrium. In this simple example, we define the vector of relevant endogenous variables to consist of current consumption, current portfolios and current prices, $z = ((c^i, \theta^i)_{i=1,2}, q)$. (Even though agents do not trade the bond in the second period we include zero bond holdings and a zero price for the bond in the state variable z_1 for that period. This set-up has the advantage that the resulting equilibrium expressions look very similar to those in the infinite-horizon problems that we examine in the main part of the paper.)

In this two-period example, we define a system of equations $h(z_0, \kappa, z_1)$ such that $((\bar{c}^i, \bar{\theta}^i)_{i=1,2}, \bar{q}) \in \mathbb{R}^2_{++} \times \mathbb{R} \times \mathbb{R}^2_{++} \times \mathbb{R} \times \mathbb{R}_{++}$ is a competitive equilibrium if and only if there exist $\kappa = (\kappa^1, \kappa^2) \in \mathbb{R}^2_{+} \times \mathbb{R}^2_{+}$ such that $h(\bar{z}_0, \kappa, \bar{z}_1) = 0$, with $\bar{z}_0 = ((\bar{c}^i_0, \bar{\theta}^i)_{i=1,2}, \bar{q})$ and $\bar{z}_1 = ((\bar{c}^i_1, 0)_{i=1,2}, 0)$. The system is as follows:

$$h(z_0, \kappa, z_1) = \begin{cases} -q v_i'(c_0^i) + u_i'(c_1^i) - q_0\kappa_0^i + \kappa_1^i, & i = 1,2 \\ c_0^i - (e_0^i - q_0\theta_0^i), & i = 1,2 \\ c_1^i - (e_1^i - \theta_0^i), & i = 1,2 \\ \kappa_0^i c_0^i, & i = 1,2 \\ \kappa_1^i c_1^i, & i = 1,2 \\ \theta^1 - \theta^2 & \end{cases}$$

(In the analysis below we refer to $h(\cdot)$ as the equilibrium equations. In order to characterize equilibria of infinite economies, we require that for all periods t, endogenous variables at t and at $t+1$ satisfy the agents' first-order conditions and the market-clearing conditions, which we in turn summarize in a system h.)

An exact equilibrium is characterized by $h = 0$, but computational methods can rarely find exact solutions. All one can usually hope for is to find an ϵ-equilibrium, namely (z_0, z_1) such that

$$\min_{\kappa \in \mathbb{R}^4_+} \| h(z_0, \kappa, z_1) \| < \epsilon.$$

Unfortunately, even in this very simple framework, one can construct economies where ϵ-equilibria can be arbitrarily far from exact equilibria.

2.1 Approximate Equilibria can be far from Exact

Consider the following class of economies parameterized by $\delta > 0$.

$$v_1(x) = x, \quad u_1(x) = -\frac{1}{x}, \quad e^1 = (2, \delta).$$

$$v_2(x) = -\frac{1}{x}, \quad u_2(x) = x, \quad e^2 = (0, 2).$$

One can easily verify that a competitive equilibrium is given by

$$q = \frac{1}{(2+\delta)^2}, \quad \theta^1 = 2 = -\theta^2.$$

This equilibrium is unique for $\delta > 0$.

In addition, for $\delta < \frac{1}{\sqrt{4-\epsilon}} - \frac{1}{2}$, the following values of the asset price and holdings yield an ϵ-equilibrium.

$$q = 4, \theta^1 = -\theta^2 = \frac{1}{2}.$$

All equations except for $h^1 = 0$ for agent 1 hold with equality. The error in this equation is below ϵ by construction.

This example shows that even for very small $\epsilon > 0$ we can construct an economy and an ϵ-equilibrium which is far from an exact equilibrium both in allocations and prices. Furthermore it is worth noting that agents' welfare levels differ significantly between the exact equilibrium and the ϵ-equilibrium. For very small δ, utility levels in the exact equilibrium are approximately $(U^1, U^2) \approx (1, -2)$ while in the ϵ-equilibrium they are approximately $(U_\epsilon^1, U_\epsilon^2) \approx (-2, 1)$. No matter how one looks at it, the ϵ-equilibrium is evidently a terrible approximation for the exact equilibrium[2]. This observation motivates us to interpret ϵ-equilibria as approximate equilibria for 'close-by' economies.

2.2 Perturbing Endowments Makes Approximate Equilibria Exact

In our example, we can easily explain the idea that an ϵ-equilibrium can be understood as approximating an exact equilibrium of a 'close-by' economy. If $\delta = 0$, we obtain an economy with close-by endowments. For this economy $q = 4$ and $\theta^1 = -\theta^2 = 1/2$ constitutes an exact equilibrium. We make this observation repeatedly in this paper and therefore describe explicitly how to find the necessary perturbation in endowments: At the ϵ-equilibrium $q = 4$, $\theta^1 = -\theta^2 = 1/2$ the only equilibrium equation that does not hold with equality is

$$h^1 = -q + \frac{1}{(e_1^1 + \theta)^2} = -4 + \frac{1}{(e_1^1 + 1/2)^2}.$$

If we replace the endowments e_1^1 by $\tilde{e}_1^1 = e_1^1 - o$ for some small o we can evidently set $h^1 = 0$ by using $o = -\delta$. The equilibrium equations imply directly which perturbations must be used. The ϵ-equilibrium is exact for the perturbed economy.

While this is the main idea underlying our error analysis in the infinite model, there is one additional complication which arises when agents live for many periods: Errors may propagate over time and no sensible bounds on perturbations in endowments can be derived by perturbing endowments every period. In Section 4, we will discuss this problem and a solution in great length. First we need to lay out the basic infinite horizon model.

3 A General Model

In this section we fix the main ideas in an abstract framework which encompasses both economies with overlapping generations and economies with infinitely lived agents as well as economies with and without production. In Sections 5 and 6 below we consider two standard models and show how to apply the methods developed in this and the next section.

[2] For finite economies there do exist sufficient conditions which relate approximate equilibria to exact equilibria (see for example Blum et al. (1998, chapter 8) and Anderson (1986)). However, these cannot be generalized to the infinite horizon economies we consider in this paper.

3.1 The Abstract Economy

Time and uncertainty are represented by a countably infinite tree Σ. Each node of the tree, $\sigma \in \Sigma$, is a finite history of shocks $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ for a given initial shock s_0. The process of shocks (s_t) is assumed to be a Markov chain with finite support $\mathcal{S}$. If $s^{t'}$ is a successor of s^t we write $s^{t'} \succ s^t$. The number of elements in $\mathcal{S}$ is S. Given an $S \times S$ transition matrix Π, we define probabilities for each node by $\pi(s_0) = 1$ and $\pi(s^t) = \Pi(s_t|s_{t-1})\pi(s^{t-1})$ for all $t \geq 1$.

There are L commodities, $l \in \mathcal{L}$, at each node. As it is commonly done in the dynamic GEI literature (see for example Magill and Quinzii (1994)) we take the commodity space to be

$$\ell_\infty(\Sigma, \mathcal{L}) = \{((x_1(\sigma), \ldots, x_L(\sigma)) : \sup_{(\sigma,l) \in \Sigma \times \mathcal{L}} |x_l(\sigma)| < \infty\}.$$

There are countably many individuals $i \in \mathcal{I}$ and countably many firms $k \in K$. An individual $i \in \mathcal{I}$ is characterized by his consumption set X^i, his individual endowments $e^i \in X^i \subset \ell_\infty$, his preferences $P^i \subset X^i \times X^i$ (where $P^i = \{(x, y) \in X^i \times X^i : x \succeq^i y\}$) and trading constraints. To simplify notation, we assume that the consumption sets are identical across agents and write $X = X^i$.

A firm $k \in K$ is characterized by its production set Y^k. An economy $\mathcal{E}$ is characterized by a demographic structure, assets, technologies and preferences, endowments and trading constraints. In the concrete models below we describe $\mathcal{E}$ explicitly.

The original economy is assumed to be Markovian. The number of agents active in markets at a given node is finite and time-invariant but may depend on the underlying shock, agents maximize time and state-separable utility, firms only make decisions on spot markets and all individual endowments, payoffs of assets, production sets of firms and spot utility functions of individuals are time-invariant functions of the shock, s, alone. In particular we assume that individual endowments depend only on the shock and can be written as $e^i(s^t) = \mathbf{e}^i(s_t)$. We define aggregate endowments $\mathbf{e}(s) = \sum_{i \in \mathcal{I}} \mathbf{e}^i(s)$, for all $s \in \mathcal{S}$. Since there are finitely many shocks, this allows us to describe the economy by finitely many spot utility functions, production sets, endowment vectors and asset payoffs.

3.2 Close-by Economies

As explained in the introduction we are interested in analyzing equilibria of economies $\mathcal{E}'$ which are close-by to an original Markovian economy $\mathcal{E}$ in the sense that all individuals' endowments and preferences are close-by. In order to formalize this idea, we index economies by preferences and endowments, i.e. we write $\mathcal{E} = (P^{\mathcal{I}}, e^{\mathcal{I}})$, where $P^{\mathcal{I}}$ denotes the profile of preferences across agents and $e^{\mathcal{I}}$ denotes the profile of individual endowments. We also parameterize economies by node-dependent perturbations $o(\sigma) \in \mathcal{O} \subset \mathbb{R}^N$ and write $\mathcal{E}((o(\sigma))_{\sigma \in \Sigma})$ for a given (possibly non-stationary) perturbed economy. In the original economy $o(\sigma) = 0$ for all $\sigma \in \Sigma$. The vector $o(\sigma)$ may contain perturbations of endowments or preferences or both.

We need to define a metric on economies, i.e. distances for preferences and for endowments. Throughout the paper, for a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the sup-norm,

$$\|x\| = \max\{|x_1|, \ldots, |x_n|\}.$$

For an element of the commodity space $x \in \ell_\infty$ we define

$$\|x\| = \sup_{(\sigma,l) \in \Sigma \times \mathcal{L}} \|x_l(\sigma)\|.$$

In many applications we are interested in examining close-by economies with identical preferences. In these cases $o(\sigma)$ are additive perturbations of endowments of individuals which are active in markets at node σ. Individual endowments are then called 'close-by' if the sup-norm of their difference is small.

While small differences in individual endowments are easy to interpret, differences in preferences are much harder to quantify. However, in some cases (e.g. when endowments are specified to lie on the boundary and we do not want to consider interior endowments) we need to perturb preferences in order to provide an economically meaningful interpretation for an approximate equilibrium.

Following Postlewaite and Schmeidler (1981) and Debreu (1969) we use the Hausdorff distance to define closeness of two preferences P and P'. However, in the models we consider, aggregate endowments are always bounded. While it is conceivable that an agent may contemplate consumption bundles that exceed aggregate endowments, it simplifies the analysis considerably to call preferences close-by if they are close over fixed bounded sets. For this purpose, we define

$$\bar{c} = \max_{s \in \mathcal{S}} \|\mathbf{e}(s)\| \quad \text{and} \quad \bar{C} = \{(x, y) \in X \times X : \|(x, y)\| \leq \bar{c}\}.$$

The distance between two preferences is then

$$d^H(P, P') = \max\left\{ \sup_{(x,y) \in P \cap \bar{C}} \left(\inf_{(x',y') \in P'} \|(x, y) - (x', y')\| \right), \sup_{(x',y') \in P' \cap \bar{C}} \left(\inf_{(x,y) \in P} \|(x, y), -(x', y')\| \right) \right\}.$$

We define a distance between the economies,

$$d(\mathcal{E}, \mathcal{E}') = \max_{i \in \mathcal{I}} \left(\max\left\{ \|e^i - e^{i\prime}\|, d^H(P^i, P^{i\prime}) \right\} \right).$$

3.2.1 Admissible Perturbations of Preferences

We assume throughout the paper that preferences can be represented by a time-separable expected utility function. We consider linear additive perturbations to Bernoulli utilities (as is often done in general equilibrium analysis, see e.g. Mas-Colell (1985)). For simplicity the following discussion focuses on the case where agents are infinitely lived and their consumption sets are infinite dimensional. The case of finitely lived agents follows immediately from this case.

Given common beliefs and discount factors (we discuss possible perturbations in beliefs and discounting below), Π and β, for an infinitely lived agent i there exists a Bernoulli function $u^i : \mathbb{R}^L_+ \times \mathcal{S} \to \mathbb{R}$ such that with

$$U^i(x) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u^i(x(s^t), s^t),$$

we have

$$(x, y) \in P^i \text{ if and only if } U^i(x) \geq U^i(y).$$

In the original unperturbed economy, Bernoulli utilities only depend on the current shock, i.e. $u^i(x, s^t) = u^i(x, s_t)$. We assume that each u^i is continuously differentiable, strictly increasing and concave for $x \in \mathbb{R}_{++}$. As a result there exists $\bar{m} > 0$ such that $D_{x_l} u^i(x, s) \geq \bar{m}$ for all $l \in \mathcal{L}$ and all $x \in \mathbb{R}^L_+$ with $\|x\| < 2\bar{c}$, and all $s \in \mathcal{S}$.

Given $u^i(x, s_t)$ and a utility perturbation $\omega^i(s^t) \in \mathbb{R}^L$, the perturbed Bernoulli utility is

$$\tilde{u}^i(x, s^t) = u^i(x, s^t) - \omega^i(s^t) \cdot x$$

For the perturbed utility function to remain strictly increasing we need to restrict attention to sufficiently small perturbations and thus require that $\|\omega^i(s^t)\| < \frac{\bar{m}}{2}$ for all $s^t \in \Sigma$. If P^i denotes the original preferences, we denote the implied perturbed preferences by $\tilde{P}^i$ which we represent by the utility function

$$\tilde{U}^i(x) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) \left(u^i(x(s^t), s^t) + \omega^i(s^t) \cdot x(s^t) \right).$$

The following lemma giving bounds on $d^H(P^i, \tilde{P}^i)$ is proven in the Appendix.

LEMMA 1 *Given perturbations* $(\omega^i(\sigma))_{\sigma \in \Sigma}$, *define* $\bar{\omega}^i = \sup_\sigma \|\omega^i(\sigma)\|$. *Then a bound on the distance between original and perturbed preferences is as follows.*

$$d^H(P^i, \tilde{P}^i) \leq L\ \bar{c}\ \frac{\bar{\omega}^i}{\bar{m}}.$$

It is clear that as $\|(\omega^i(\sigma))_{\sigma \in \Sigma}\| \to 0$ we have that $d^H(P^i, \tilde{P}^i) \to 0$. Moreover, the bound in the lemma is invariant to affine transformations in $\tilde{u}^i$. Note, however, that the bound does depend on the lower bound $\bar{m}$ on marginal utilities. As a result, multiplying all endowments by some factor does affect the bound on the preference distance. This comes as no surprise, since a fixed perturbation $\omega^i(s^t)$ of marginal utility will be much more significant when the original marginal utility was rather small. Also note that because of the relationship $\bar{\omega}^i < \frac{\bar{m}}{2}$ we have an upper bound $d^H(P^i, \tilde{P}^i) \leq L\ \bar{c}\ \frac{\bar{\omega}^i}{\bar{m}} < \frac{L\bar{c}}{2}$. Clearly this bound is too crude: we always want $\bar{\omega}^i$ to be orders of magnitude smaller than $\bar{m}$.

3.2.2 Possible Other Perturbations ?

While we can show that the considered linear perturbations in Bernoulli functions lead to close-by preferences, it is obviously not true that given a preference, P, all close-by preferences can be represented by utility functions with linear perturbations. It will become clear in Section 6 below that in general most additional perturbations in preferences do not facilitate the error analysis. However, it may appear tempting to perturb conditional probabilities and node-dependent discount factors. It is therefore useful to point out that such perturbations may lead to preferences which are very far away from the original ones. Perturbations in resulting unconditional probabilities may get arbitrarily large for events far along the event-tree and therefore marginal rates of substitution for the perturbed preferences will be far from those of the original preferences. The preferences will be far in the Hausdorff distance.

3.3 Equilibrium

A competitive equilibrium for the economy $\mathcal{E}(o(\sigma))_{\sigma\in\Sigma})$ is a process of endogenous variables $(z(\sigma))_{\sigma\in\Sigma}$ with $z(\sigma) \in \mathcal{Z} \subset \mathbb{R}^M$, which solve agents' optimization problems and clear markets. We refer to the collection of the economy and the endogenous variables, $(\mathcal{E}((o(\sigma))_{\sigma\in\Sigma}), (z(\sigma))_{\sigma\in\Sigma})$, as an 'economy in equilibrium'.

3.3.1 The Expectations Correspondence

For the computation of competitive equilibria it is important that equilibrium conditions can be summarized in a set of inequalities which relate current period exogenous and endogenous variables to endogenous and exogenous variables one period ahead. Duffie et al. (1994) describe this relation via an *expectations correspondence*. We use their terminology but slightly alter the concept for our specific purposes.

We restrict attention to economies where a time invariant expectations correspondence can encompass all conditions for agents' optimality and market clearing. A competitive equilibrium can then be characterized by an expectations correspondence which maps endogenous variables today to possible (i.e. consistent with individuals' Euler equations and market clearing) endogenous variables and perturbations of the fundamentals at the S possible shocks next period. That is, we want to be able to define a correspondence

$$H : \mathcal{S} \times \mathcal{Z} \rightrightarrows \bigotimes_{s\in\mathcal{S}} (\mathcal{O} \times \mathcal{Z}),$$

where $(z(\sigma))_{\sigma\in\Sigma}$ is an equilibrium for $\mathcal{E}((o(\sigma))_{\sigma\in\Sigma})$ if for all $s^t \in \Sigma$,

$$(o(s^t 1), z(s^t 1), \ldots, o(s^t S), z(s^t S)) \in H(s_t, z(s^t)).$$

We assume furthermore that elements in the graph of the expectations correspondence can be characterized as (part of) a solution to a system of equations, i.e. we assume that there exists a set $\mathcal{K} \subset \mathbb{R}^K$, and a function

$$h : \mathcal{S} \times \mathcal{Z} \times \mathcal{K} \times (\bigotimes_{s \in \mathcal{S}} (\mathcal{O} \times \mathcal{Z})) \to I\!R^L$$

such that $(o_1, z_1, \ldots, o_S, z_S) \in H(\hat{s}, \hat{z})$ if and only if there exists $\kappa \in \mathcal{K}$ such that

$$h(\hat{s}, \hat{z}, \kappa, o_1, z_1, \ldots, o_S, z_S) = 0.$$

In this formulation the variables $\kappa \in \mathcal{K}$ should be thought of representing slack variables in inequalities or Kuhn-Tucker multipliers. In the applications below the functions h consist of individuals' intertemporal Euler equations, market clearing equations and first order conditions for spot optimality. We refer to h as the equilibrium equations. In Section 2, we described these equations for a simple two-period model.

4 Approximate Equilibria and Their Interpretation

As mentioned in the introduction, we want to give conditions which allow us to interpret the results of algorithms used in practice. It is therefore useful to define a notion of ϵ-equilibrium which is general enough that it exists in most interesting specifications of the model and that is tractable in the sense that actual approximations in the literature can be interpreted as such ϵ-equilibria (or at least that ϵ-equilibria can be constructed fairly easily from the output of commonly used algorithms).

DEFINITION 1 *An ϵ-equilibrium is a finite set $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_S$, $\mathcal{F}_s \subset \mathcal{Z}$ for all $s = 1, \ldots, S$, such that for all $\hat{s} \in \mathcal{S}$ and all $\hat{z} \in \mathcal{F}_{\hat{s}}$ there exist $(z_1, \ldots, z_S) \in \mathcal{F}$ such that*

$$\min_{\kappa \in \mathcal{K}} \| h(\hat{s}, \hat{z}, \kappa, \mathbf{0}, z_1, \ldots, \mathbf{0}, z_S) \| < \epsilon. \tag{1}$$

In most interesting models one can show existence of ϵ-equilibria for all $\epsilon > 0$ (the existence of a competitive equilibrium is often a sufficient but not a necessary condition for the existence of ϵ-equilibria). Obviously ϵ-equilibria are only computationally feasible if they have some simple Markovian structure, i.e. if the sets $\mathcal{F}_s$ are 'small' relative to $\mathcal{Z}$. In the applications below this is the case.

We define an ϵ-equilibrium as a finite collection of points because we want to be able to verify whether a candidate solution constitutes an ϵ-equilibrium, and with our definition this verification involves checking only finitely many inequalities.

4.1 Recursive Methods

The applied computational literature often refers to recursive equilibria. These equilibria are characterized by policy functions which map the current 'state' of the economy into choices and prices and by transition functions which map the state today into a probability distribution over the next period's state. While in dynamic GEI models, recursive equilibria do not always exist and no non-trivial assumptions are known which guarantee the existence of recursive equilibria (for counterexamples to existence see e.g. Hellwig (1982), Kubler and Schmedders (2002) and Kubler and Polemarchakis (2003)) recursive methods are useful for computational purposes. In most models, recursive ϵ-equilibria exist whenever ϵ-equilibria exist and formulating these ϵ-equilibria recursively facilitates the notation and the error analysis. In the following we always assume that a given ϵ-equilibrium also has a recursive representation. We therefore now define a recursive ϵ-equilibrium formally.

The relevant endogenous state space $\Psi \subset \mathbb{R}^D$ depends on the underlying model - it is determined by the payoff-relevant pre-determined endogenous variables; that is, by variables sufficient for the optimization of individuals at every date-event, given the prices. If Ψ is the 'endogenous state space' there must exist sets $\mathcal{Z}_1^*, \ldots, \mathcal{Z}_S^* \subset \mathbb{R}^{M-D}$ such that for all $s \in \mathcal{S}$, $\mathcal{F}_s = \Psi \times \mathcal{Z}_s^*$. The value of the state variables $(s_0, v_0) \in \mathcal{S} \times \Psi$ in period 0 is called 'initial condition' and is part of the description of the economy. It will be often be useful to make this explicit and to write $\mathcal{E}_{s_0, v_0}$. A recursive ϵ-equilibrium is defined as follows.

DEFINITION 2 *Given an ϵ-equilibrium $\mathcal{F}$ and a state space Ψ, a recursive ϵ-equilibrium consists of a policy function $\rho : \mathcal{S} \times \Psi \to \mathbb{R}^{M-D}$ such that*

$$\mathcal{F}_s = \mathit{graph}(\rho_s) \text{ for all } s \in \mathcal{S},$$

as well as transition functions $\tau_{ss'} : \Psi \to \Psi$, for all $s, s' \in \mathcal{S}$ such that for all $\bar{s} \in \mathcal{S}$ and all $\bar{z} = (\bar{v}, \bar{z}^*) \in \mathcal{F}_{\bar{s}}$,

$$(z_1, \ldots, z_S) = ((\tau_{\bar{s}1}(\bar{v}), \rho_1(\tau_{\bar{s}1}(\bar{v}))), \ldots, (\tau_{\bar{s}S}(\bar{v}), \rho_S(\tau_{\bar{s}S}(\bar{v}))))$$

satisfies

$$\min_{\kappa \in \mathcal{K}} \|h(\bar{s}, \bar{z}, \kappa, \mathbf{0}, z_1, \ldots, \mathbf{0}, z_S)\| < \epsilon.$$

This definition shows that recursive methods enable us to approximate an infinite dimensional equilibrium by a finite set. Given an initial value of the shock, s_0 and initial values for the endogenous state, v_0, a recursive ϵ-equilibrium assigns a value of endogenous variables to any node in the infinite event tree: For any node s^t, the value of the endogenous state is given by $v(s^t) = \tau_{s_{t-1}s_t}(v(s^{t-1}))$, the value of the other endogenous variables is given by $\rho_{s^t}(v(s^t))$. We call the resulting stochastic process an ϵ-equilibrium process and write $(z^\epsilon(\sigma))_{\sigma \in \Sigma}$.

In most contexts it will be straightforward to derive the transition function from the policy function. For example, in a finance economy, the beginning-of-period portfolio holdings constitute the endogenous state. The policy function assigns new portfolio holdings which then form the endogenous state next period.

4.2 Construction of ϵ-equilibria

This paper does not develop explicit algorithms to compute ϵ-equilibria. In fact, it is usually not feasible to compute them directly since their discrete nature does not allow directly for the application of standard methods in numerical analysis (which usually assume smoothness). For the purpose of the present paper we assume that from some algorithm the output $z(s^t)$ for any finite sequence of shocks can be computed. We want to construct a recursive ϵ-equilibrium from this output.

For this purpose, fix a small $\delta > 0$. Starting from the root node s_0 collect all pairs of shocks and (output of) endogenous variables $(s_t, z(s^t))$ in a set $\mathfrak{Y}_t$. Define the set of states rounded to within δ by

$$\mathfrak{Y}_t^{\delta} = \{(s, z^{\delta}) : \frac{z_m^{\delta}}{\delta} \in \mathbb{N} \text{ for } m = 1, \ldots, M \text{ and there exists } (s, z) \in \mathfrak{Y}_t \text{ with } \|z - z^{\delta}\| < \delta\}.$$

In other words, we restrict the endogenous variables to lie on a grid with grid size δ. The set $\mathfrak{Y}_t^{\delta}$ contains all combinations of exogenous states s and endogenous variable values on the grid that appear in the computed solution for time t. Next we collect all rounded states which have occurred up to time t in $\mathfrak{Y}^{\delta t}$, so we define $\mathfrak{Y}^{\delta t} = \cup_{t'=0}^{t} \mathfrak{Y}_{t'}^{\delta}$. Obviously, for fixed grid size $\delta > 0$ the number of elements of $\mathfrak{Y}^{\delta t}$ is finite. If for some t^*, $\mathfrak{Y}_{t^*}^{\delta} \subset \mathfrak{Y}^{\delta(t^*-1)}$ then the set $\mathfrak{Y}^{\delta(t^*-1)}$ contains all states that will ever be visited in the constructed approximate equilibrium! In other words, all states that may ever occur along the event tree have been reached at least once. By construction the set $\mathfrak{Y}^{\delta(t^*-1)}$ yields an ϵ-equilibrium for some $\epsilon > 0$. The actual ϵ can be computed by evaluating the error in the expectations correspondence at all s, $z \in \mathcal{F}_s$[3]. It is evident that this procedure has two problems that may potentially render it useless. First, the procedure may be hopelessly inefficient since for sufficiently small δ the potential number of elements in $\mathcal{F}$ is huge. Secondly, the resulting value for ϵ may not be as small as desired. Surprisingly, we found that in our economic applications these problems are not severe. The output from our algorithm computing smooth approximations for policy and transition functions is good enough in order to make ϵ sufficiently small. Moreover, after a surprisingly small number of periods t^* all states start

[3] In many applications researchers verify errors only along one randomly determined path. The implicit assumption is that simulating the economy along one sample path suffices to verify the accuracy of computations. However, an example in Kubler and Schmedders (2003b) shows that this is often not sufficient. In particular, for a given finite sample path, it is obviously impossible to infer the maximal error from the error along the path.

to be revisited, even when the dimension of the endogenous state space is fairly large. In summary, the outlined procedure for computing an ϵ-equilibrium from smooth equilibrium approximations appears to work well for interesting economic applications.

4.3 Error Analysis

In the context of recursive equilibria den Haan and Marcet (1994) and Judd (1998) suggest to evaluate the quality of a candidate solution using Euler-equation residuals. In these methods relative maximal errors in Euler equations of ϵ usually imply that the solution describes an ϵ-equilibrium. Unfortunately ϵ-equilibria defined by condition (1) are very difficult to interpret. What does an ϵ-equilibrium describe for, say $\epsilon = 0.001$? Should this be regarded as a good approximation or as a bad one? The example in Section 2 shows that the computed ϵ-equilibrium may be far away from an exact equilibrium for the economy, no matter how small ϵ.

Several authors (e.g. Judd (1998)) justify ϵ-equilibria (or maximal relative errors in Euler equations) as a measure of quality of a solution via a bounded rationality argument. They argue that economic agents have bounded computational capacity and can only find approximately optimal choices. Any improvement over their choices results in an extra gain of at most ϵ. However, in dynamic general equilibrium models with rational expectations there is a tension between assuming that agents have rational expectations about future prices and assuming that they make errors in choosing their consumptions given the correct forecasts for prices. Furthermore, since markets clear exactly, future prices must already reflect the agents' optimization errors. We therefore want to move away from a bounded rationality justification and interpret ϵ-equilibria as approximating exact equilibria of a close-by economy.

Ideally a recursive ϵ-equilibrium would generate an ϵ-equilibrium process that is close-by to a competitive equilibrium for a close-by economy at all date-events. If this were the case one could find small perturbations of endowments and preferences of the original economy such that the perturbed economy has a competitive equilibrium which is well approximated by the ϵ-equilibrium process at *each* node of the event tree. This observation leads to the following definition of approximate equilibrium.

DEFINITION 3 *An ϵ-equilibrium process $(z^\epsilon(\sigma))_{\sigma\in\Sigma}$ for an economy $\mathcal{E}_{s_0,v_0}$ path-approximates an economy in equilibrium with precision δ if there exists a close-by economy in equilibrium, $(\mathcal{E}'_{s_0,v_0}, (\tilde{z}(\sigma))_{\sigma\in\Sigma})$, $\mathcal{E}' = \mathcal{E}((o(\sigma))_{\sigma\in\Sigma})$, with*

$$d(\mathcal{E},\mathcal{E}') < \delta \text{ and } \sup_{\sigma\in\Sigma} \|z^\epsilon(\sigma) - \tilde{z}(\sigma)\| < \delta.$$

In models where agents are infinitely lived, it is not to be expected that a recursive ϵ-equilibrium actually gives rise to a process that path-approximates a close-by economy in equilibrium. If agents make small errors in their choices each periods, these are likely to propagate over time and after sufficiently many periods the ϵ-equilibrium allocation will be far away from the exact equilibrium allocation. The following simple example illustrates that it is easy to construct ϵ-equilibria which do not path-approximate an economy in equilibrium for any reasonable precision δ, no matter how small ϵ.

EXAMPLE 1 *Consider an infinite horizon exchange economy with two infinitely lived agents, a single commodity and no uncertainty. Suppose that agents have identical initial endowments $e^i > 0$ for all t and identical preferences with $u_i(c_t) = \log(c_t)$ and with a common discount factor $\beta \in (0, 1)$. There is a consol in unit net supply which pays 1 unit of the consumption good each period. The price of the consol is q_t, portfolios are θ^i_t. Each agent i, $i = 1, 2$, faces a short sale constraint $\theta^i_t \geq 0$ for all t.*

Even though the example is simple, it is useful to explicitly spell out the equilibrium equations. Let the endogenous variables be $z = ((\theta^i_-, \theta^i, c^i, m^i)_{i=1,2}, q)$. Admissible perturbations are $o = (o^i_P, o^i_e)_{i=1,2} \in \mathbb{R}^4$, i.e. we allow for perturbations in endowments as well as in preferences. The expectations correspondence is characterized by $h(\bar{z}, \kappa, o, z) = 0$ where $h = (h^1, \ldots, h^6)$ with

$$\begin{aligned}
h^1 &= -1 + \beta\frac{(1+q)m^i}{q\bar{m}^i} + \kappa^i && i = 1, 2\\
h^2 &= \kappa^i\bar{\theta}^i && i = 1, 2\\
h^3 &= c^i - \theta^i_-(q+1) + \theta^i q - (e^i + o^i_e) && i = 1, 2\\
h^4 &= \theta^i_- - \bar{\theta}^i && i = 1, 2\\
h^5 &= m^i - (u'_i(c^i) - o^i_P) && i = 1, 2\\
h^6 &= \theta^1 + \theta^2 - 1
\end{aligned}$$

The natural endogenous state space for this economy consists of beginning-of-period consol holdings. We build market clearing into the state space and only consider θ^1_-, θ^2_- with $\theta^1_- + \theta^2_- = 1$. We write $\vartheta = \theta^1_-$ to represent a typical state of the economy, implicitly assuming market clearing.

Obviously, for any initial condition $\vartheta_0 \in (0, 1)$, the unique exact equilibrium is no trade in the consol with each agent consuming $\theta^i_{0-} + e^i$, $\theta^1_{0-} = \vartheta_0 = 1 - \theta^2_{0-}$, each period and in the absence of bubbles the price of the consol being $q_t = \frac{\beta}{1-\beta}$ for all $t \geq 0$. However, it is also an ϵ-equilibrium if each period agent 1 sells a small amount of the tree to agent 2. In this case, the consumption of agent 1 converges to e^1 while the consumption of agent 2 converges to $e^2 + 1$. Unfortunately, there is no economy with close-by endowments for which this allocation is an approximate equilibrium allocation.

To analyze this case in more detail we construct a recursive ϵ-equilibrium as follows. Define

$$\tau(\vartheta) = \begin{cases} \vartheta - \delta & \text{if } \vartheta > \delta \\ \vartheta & \text{otherwise.} \end{cases}$$

Define $q = \rho_q(\omega) = \frac{\beta}{1-\beta}$, $\theta^1 = \rho_{\theta^1}(\omega) = \tau(\omega)$ and $c^1(\omega) = \omega(q-1) - \theta q$, $c^2(\theta) = 1 + e^1 + e^2 - c^1(\theta)$. Given any initial condition $\omega_0 \in (0,1)$, these functions describe a recursive ϵ-equilibrium as long as $0 \leq \delta < \frac{\epsilon(1-\beta)}{\beta e^i}$. Except for the Euler equations h^1 all equilibrium equations hold with equality. For $\theta_- = \omega > 2\delta$, the error in Euler equations for agent 1 is given by

$$|h^1| = \left| -1 + \frac{e^1 - (\theta+\delta)(q+1) - \theta q}{e^1 + \theta(q+1) - \theta q + \delta q}\right| < \frac{\delta(q+1)}{e^i}$$

For $2\delta \geq \omega > \delta$, we have

$$|h^1| = \left| -1 + \frac{e^1 + (\theta+\delta)(q+1) - \theta q}{e^1 + \theta(q+1) - \theta q}\right| \leq \frac{\delta(q+1)}{e^i}$$

and finally for $\omega < \delta$ we have $h^1 = 0$. The argument for agent 2 is analogous.

For the initial condition $\omega_0 = 0.5$, the constructed recursive equilibrium obviously implies an ϵ-equilibrium process which, in the sup-norm, is far from any exact equilibrium of a close-by economy. Note that this is a general problem that does not only occur in economies with incomplete markets, the same phenomenon can even arise for an approximate solution to a single-agent decision problem. In the applied literature this problem is commonly solved by ignoring it[4]: A computed solution is considered a good approximation if the computed policy function is close-by the true policy function. Usually no reference is made about how the stochastic process of choices arising from the approximate policy function relates to the stochastic process of optimal choices. For some purposes this may be good enough. At least for the first few periods, given any initial conditions, the constructed recursive ϵ-equilibrium is a good approximation to the exact equilibrium if ϵ is sufficiently small.

We generalize this idea and apply it to our general framework. Instead of requiring that the exact equilibrium process is well approximated by the ϵ-equilibrium process we merely require that for each node $s^t \in \Sigma$ and value of the exact equilibrium $z(s^t)$ there is some $\tilde{z} \in \mathcal{F}_{s_t}$ which is close to $z(s^t)$. Of course this would be a vacuous condition if there are many $\tilde{z} \in \mathcal{F}_{s_t}$ which do not approximate any equilibrium values. These considerations lead us to the following definition.

Definition 4 *An ϵ-equilibrium $\mathcal{F}$ for the economy $\mathcal{E}$ weakly approximates an economy in equilibrium with precision δ if there exists a close-by economy in equilibrium, $(\mathcal{E}', (\tilde{z}(\sigma))_{\sigma\in\Sigma})$, $\mathcal{E}' = \mathcal{E}((\rho(\sigma))_{\sigma\in\Sigma})$, with $d(\mathcal{E}, \mathcal{E}') < \delta$, such that*

$$\min_{z\in\mathcal{F}_{s_t}} \|z - \tilde{z}(s^t)\| < \delta \text{ for all } s^t$$

[4] A notable exception is Peralta-Alva and Santos (2002) who derive sufficient conditions for sample-path stability in a representative agent model.

and such that for all s, $z \in \mathcal{F}_s$ there exist initial conditions (s_0, c_0) and an equilibrium for the economy $\mathcal{E}'_{s_0, c_0}$, $(z(\sigma))_{\sigma \in \Sigma}$, such that

$$\inf_{\sigma \in \Sigma} \|z(\sigma) - z^{\epsilon}\| < \delta.$$

Intuitively, the definition merely requires that for the recursive ϵ-equilibrium the policy function is close to the policy function of an exact recursive equilibrium. In the models we consider in this paper, existence of exact recursive equilibria cannot be established. We therefore need to state the definition in terms of competitive equilibria.

This condition is much weaker than requiring that the ϵ-equilibrium process strongly approximates an economy in equilibrium. This is to be expected since closeness in policy functions generally do not imply anything about how close equilibrium allocations are, even in models where recursive equilibria do exist. The definition only requires that there exists some process with values in $\mathcal{F}$ which approximates the exact equilibrium but does not explicitly state how to construct this process. However, given an ϵ-equilibrium that satisfies the definition and given any T one can construct a process which is within δ of the exact equilibrium process for up to T periods. This will be made explicit in Section 6 below.

It is easy to see that the ϵ-equilibrium in Example 1 approximates the exact equilibrium very well, and that Definition 4 is satisfied. However, without knowing the exact equilibrium it can be difficult to verify the definition from the ϵ-equilibrium alone. In order to derive a sufficient condition which does not involve the (generally unknown) exact equilibrium we need to use the ϵ-equilibrium $\mathcal{F}$ to construct a process which is an exact equilibrium for a close-by economy. Given initial conditions in Example 1, suppose endogenous variables in the first period are equal to z_1. How can we find values for the next period which are close to an element of $\mathcal{F}$ and for which the equilibrium equations hold exactly? One possible way to proceed is to search over $\mathcal{F}$ for consumption values which make the error in h^1 identical across agents. Evidently this is obtained if next period's individual consumption is set equal to this period's consumption. This will lead to an exact equilibrium in this simple example if we perturb endowments.

In general, of course, verifying that an ϵ-equilibrium satisfies Definition 4 will not be as straightforward as in this example. In particular, it will generally be impossible to find elements of $\mathcal{F}$ which ensure that Euler equations hold with equality. We now describe how to generalize the idea of the example to more interesting models.

4.4 A Sufficient Condition

In this section we describe a general method to construct an exact equilibrium process for a close-by economy from a given ϵ-equilibrium. In Section 6 we apply this method to a concrete example.

We assume throughout that for all $\bar{s} \in \mathcal{S}$, $\bar{z} \in \mathcal{F}_{\bar{s}}$ we can find values of endogenous variables which are close-by to elements in $\mathcal{F}$ and small perturbations at all direct successor nodes such that the equilibrium equations are satisfied exactly. The problem is that for a given constructed equilibrium process, endogenous variables at some node s^t will not be exactly equal to any element $z \in \mathcal{F}_{s_t}$. Therefore we require that for all initial endogenous variables in a small neighborhood (within some $\delta_0 \leq \delta$) of $\bar{z}$ perturbations and values of endogenous variables can be found for the direct successor nodes such that the endogenous variables again lie in the same δ_0 neighborhood of some value in $\mathcal{F}$. If this is the case, we say that the ϵ-equilibrium is *balanced* and there exists an infinite process which approximates an economy in equilibrium.

More formally, in order to show that $\mathcal{F}$ weakly approximates an economy in equilibrium with precision δ is suffices to show that there are non-negative numbers $\delta_1, \ldots, \delta_M < \delta$ such that for all $\bar{s}$, $\bar{z} \in \mathcal{F}_{\bar{s}} \subset \mathbb{R}^M$, all $\tilde{z}$ with $|\tilde{z}_m - \bar{z}_m| \leq \delta_m$ for all $m = 1, \ldots, M$, there exist, for all $s \in \mathcal{S}$, $o(s)$ with $\|o(s)\| < \delta$, as well as $z(s)$ such that there is a $z \in \mathcal{F}_s$ with $|z_m - z_m(s)| < \delta_m$ for all $m = 1, \ldots, M$, which satisfy

$$\min_{\kappa \in \mathcal{K}} \| h(\bar{s}, \tilde{z}, \kappa, o(1), z(1), \ldots, o(S), z(S)) \| = 0.$$

Evidently, in the simple example above, only marginal utilities need to be considered. For all marginal utilities which lie in a δ neighborhood of marginal utilities in $\mathcal{F}$ we can find next period marginal utilities in the same neighborhood. We discuss a more complicated (and more interesting) example of balancedness in Section 6 where we consider the Lucas model with heterogeneous agents.

5 A Model with Overlapping Generations and Production

As the first application of our methods, we consider a model of a production economy with overlapping generations and several commodities. This is a generalization of models frequently used in macroeconomics and public finance (see e.g. Rios-Rull (1996)) and of the overlapping generations model analyzed in Duffie et al. (1994).

We show that in this model the ϵ-equilibrium process actually path approximates an economy in equilibrium and we derive bounds on the distance between the close-by economy in equilibrium and the specified economy. These bounds are constructed from the ϵ-equilibrium $\mathcal{F}$ using linear algebra. In Section 5.2 we present a detailed error analysis which is at times very tedious because we consider a model with production where individual endowments can lie on the boundary. Nevertheless, we present all necessary steps to show how to implement our ideas in practice. A reader who is mainly interested in the computational or in the conceptual aspects of this paper may want to skip this subsection.

5.1 The economy

While the olg model is fairly standard, we describe it in some detail to fix notation. At each date-event a single individual commences his economic life; he lives for N dates. An individual is identified by the date event of his birth, (s^t). He consumes at the date-event $s^t, \ldots, s^{t+N-1}$; the age of an individual is $a = 1, \ldots, N$.

There are L physical commodities, $l \in \mathcal{L}$, and one representative firm at each date-event, s^t. The firm produces in spot markets using a constant returns to scale technologies which depends on the current shock alone. In order to simplify the error analysis we assume that commodities $1, \ldots, K$, $K < L$, are always used as inputs to spot production and commodities $K+1, \ldots, L$ are always outputs. We assume that the technology can be described by a function $f(., s) : \mathbb{R}^K_+ \to \mathbb{R}^{L-K}_+$. A production plan $y \in \mathbb{R}^L$ is feasible for the firm at shock s if and only if $(y_{K+1}, \ldots, y_L) - f((-y_1, \ldots, -y_K), s) \leq 0$.

Households have access to a risky intertemporal technology which, for simplicity, is assumed to be linear. For each shock s we define an $L \times L$ matrix, $D(s)$, where the element $d_{l'l}$ denotes how much of commodity l' is produced from one unit of commodity l used as input in the previous period. We denote by the vector $\alpha^\sigma(s^t) \in \mathbb{R}^L_+$ the bundle of commodities invested by individual σ into the technology at date event s^t, the output at s^{t+1} is given by $D(s_{t+1})\alpha^\sigma(s^t) \in \mathbb{R}^L_+$. In order to distinguish between spot production of the firm and intertemporal household production, we refer to the latter as storage.

An agent born at date event s^t has individual endowments at nodes $s^t, \ldots, s^{t+N-1}$ which are a function of the shock and his age alone, i.e. for all $a = 1, \ldots, N$, $e^{s^t}(s^{t+a-1}) = \mathbf{e}^a(s_{t+a-1})$ for some function $\mathbf{e}^a : \mathcal{S} \to \mathbb{R}^L_+$. For an agent $\sigma = s^T$ we denote his consumption choices over his lifetime by $x^\sigma = (x^\sigma(s^t))_{t=T,\ldots,T+N-1, s^t \succeq \sigma}$, and his investment choices by $\alpha^\sigma = (\alpha^\sigma(s^t))_{t=T,\ldots,T+N-1, s^t \succeq \sigma}$. To simplify notation we define $\alpha^{s^t}(s^{t-1}) = 0$ for all s^t. The agent has an intertemporal, von Neumann-Morgenstern utility function

$$U^{s^t}(x) = \mathrm{E}_{s^t} \sum_{a=1}^{N} u_a(x(s^{t+a-1}), s_{t+a-1}).$$

The Bernoulli utility u depends on the age and the current shock alone. At the root node, s_0, there are individuals of all ages $s^{-1}, \ldots, s^{-N+1}$ with initial holdings $\alpha^{s^{-a}}(s^{-1})$. These determine the 'initial condition' of the economy.

A competitive equilibrium is a collection of prices, choices of individuals and choices of the firm $(p(\sigma), (\alpha^i(\sigma), x^i(\sigma))_{i \in \mathcal{I}(\sigma)}, y(\sigma))_{\sigma \in \Sigma}$ such that markets clear and agents optimize, i.e. for all nodes $s^t \in \Sigma$ we have

- Market clearing:

$$\sum_{a=1}^{N} \left(x^{s^{t-a+1}}(s^t) - \mathbf{e}^a(s_t) - D(s_t)\alpha^{s^{t-a+1}}(s^{t-1}) + \alpha^{s^{t-a+1}}(s^t) \right) = y(s^t).$$

- Individual's maximize utility:

$$(x^{s^t}, \phi^{s^t}) \in \arg\max_{(x,\phi)\geq 0} U^{s^t}(x) \text{ s.t.}$$

$$p(s^{t+a})\cdot\left(x^{s^t}(s^{t+a}) - \mathbf{e}^{a-1}(s_{t+a}) + \phi^{s^t}(s^{t+a}) - D(s_{t+a})\phi^{s^t}(s^{t+a-1})\right) \leq 0,\ a = 0,\ldots,N-1$$

Optimality conditions for initially alive agents, $s^{-1},\ldots,s^{-N+1}$ are analogous.

- The firm maximizes profits,

$$(y_1(s^t),\ldots,y_K(s^t)) \in \arg\max_{(y_1,\ldots,y_K)\leq 0} \sum_{l=K+1}^{L} p_l(s^t) f((-y_1,\ldots,-y_K),s_t) + \sum_{l=1}^{K} p_l(s^t) y_l(s^t)$$

$$(y_{K+1}(s^t),\ldots,y_L(s^t)) = f((-y_1(s^t),\ldots,-y_K(s^t)),s_t)$$

The Expectations Correspondence

We want to characterize competitive equilibria by an expectations correspondence. We define the endogenous variables at some node σ to consist of investments from the previous period, $\phi_- = (\phi_-^1,\ldots,\phi_-^N)$, new investments, $\phi = (\phi^1,\ldots,\phi^N)$, consumptions, $x = (x^1,\ldots,x^N)$ as well as excess demands, $\xi = (\xi^1,\ldots,\xi^N) \in \mathbb{R}^{NL}$, and Lagrange multipliers, $\lambda = (\lambda^1,\ldots,\lambda^N) \in \mathbb{R}_+^N$, for all individuals alive; of the firm's choice, y, and spot prices, p, so

$$z = (\phi_-,\phi,x,\xi,\lambda,y,p).$$

We build bounds and normalizations into the admissible endogenous variables, i.e. we only consider z for which $\phi_-^1 = 0$, $\phi^N = 0$, $\phi \geq 0$, $\phi_- \geq 0$, $c \geq 0$ and $p_1 = 1$.

We consider perturbations in individual endowments and preferences, i.e. define $o(\sigma) = (o_e(\sigma), o_P(\sigma)) \in \mathbb{R}^{2LN}$ to be perturbations in endowments and preferences across all agents alive at a current node σ. As explained in Section 3 preferences are perturbed by perturbing Bernoulli utility functions node by node. We write for agent s^t's perturbed Bernoulli function at node s^{t+a}, $u^a(x,s_t,o_P(s^t)) = u^a(x,s_t) + o_P(s^t)\cdot x$.

We characterize the expectations correspondence H via the equilibrium equations. So, $(o_1^{\mathcal{I}}, z_1,\ldots,o_S^{\mathcal{I}}, z_S) \in H(\bar{s},\bar{z})$ if and only if there exist $\kappa \in \mathbb{R}_+^{(N-1)L}$ such that

$$h(\bar{s},\bar{z},\kappa,o_1^{\mathcal{I}},z_1,\ldots,o_S^{\mathcal{I}},z_S) = 0.$$

We define $h = (h^1, \ldots, h^9)$ with

$$
\begin{array}{lll}
h^1 = & \omega^a_-(s) - \bar{\omega}^{a-1} & a = 2, \ldots, N, s \in \mathcal{S} \\
h^2 = & p(s) \cdot \xi^a(s) & a = 1, \ldots, N, s \in \mathcal{S} \\
h^3 = & x^a(s) - (\mathbf{e}^a(s) - \omega^a_\epsilon(s)) - \xi^a(s) + \omega^a(s) - D(s)\omega^a_-(s) & a = 1, \ldots, N, s \in \mathcal{S} \\
h^4 = & -\bar{p}\bar{\lambda}^{a-1} + \beta \mathrm{E}_{s,\bar{s}} p(s) D(s) \lambda^a(s) + \kappa^a & a = 2, \ldots, N \\
h^5 = & \kappa^a_l \bar{\omega}^{a-1}_l & a = 2, \ldots, N, l \in \mathcal{L} \\
h^6 = & D_x u^a(x^a(s), s) + \omega_P(s) - \lambda^a(s) p(s) & a = 1, \ldots, N, s \in \mathcal{S} \\
h^7 = & \sum_{l=K+1}^{L} p_l(s) \frac{\partial f_l((-y_1(s), \ldots, -y_K(s)), s)}{\partial y_{l'}} - p_{l'}(s) & s \in \mathcal{S}, l' = 1, \ldots, K \\
h^8 = & (y_{K+1}(s), \ldots, y_L(s)) - f(-y_1(s), \ldots, -y_K(s), s) & s \in \mathcal{S} \\
h^9 = & \sum_{a=1}^{N} \xi^a(s) - y(s) & s \in \mathcal{S}
\end{array}
$$

We assume throughout that in h^6 derivatives are only taken with respect to commodities which enter the utility function.

Under standard assumptions on preferences and the production function which guarantee that first order conditions are necessary and sufficient, a competitive equilibrium can be characterized by these equations. Kubler and Polemarchakis (2003) prove the existence of ϵ-equilibria.

5.2 Error Analysis

Throughout this section we use the following well known fact from linear algebra. For an under-determined system $Ax = b$ with a matrix A that has linearly independent rows, denote by $A^+ = A^T(AA^T)^{-1}$ the pseudo inverse of A (where A^T denotes the transpose of the matrix A). The unique solution of the system that minimizes the Euclidean norm $\|x\|_2$ is then given by $x_{LS} = A^+ b$. We assume that $A^+ b$ can be computed without error. While this is obviously incorrect the error analysis for this problem is well understood and explicit bounds on the errors are usually very small (see e.g. Higham (1996), Chapter 20). We use the pseudo inverse below without explicitly assuming that AA^T is invertible. If AA^T is singular in our analysis below, there is no bound on errors. While we are interested in maximum errors, we use the Euclidean (or two-) norm here since it is well understood how to compute $A^+ b$ accurately. Evidently, for an $x \in \mathbb{R}^n$, we have that

$$\|x\|_\infty \leq \|x\|_2$$

and so this approach will immediately yield an upper bound on the minimal sup-norm of a solution x.

For the error analysis, we assume that Bernoulli utility is strictly increasing in all commodities which enter the utility function and that there exists at least on commodity l^* such that agents' choices always satisfy $x_{l^*} > 0$. We restrict the consumption set of each agent to be bounded above in each component by twice the maximal aggregate consumption occurring in the ϵ-equilibrium

$$\bar{c} = 2 \max_{s \in \mathcal{S}, z \in \mathcal{F}_s} \| \sum_{a=1}^{N} c^a \|$$

and define

$$\bar{m} = \min_{s \in \mathcal{S}, a=1,\ldots,N} \left(\min_{x \geq 0, \|x\| \leq \bar{c}} \| D_x u^a(x, s) \| \right).$$

where again we only take derivatives with respect to those commodities which enter the utility function. It holds that $\bar{m} > 0$ since we assume that the utility function is strictly increasing.

Let $\mathcal{G}^1 \subset \mathcal{L}$ consist of all those commodities which are inputs of intertemporal production (storable commodities).

$$\mathcal{G}^1 = \{l \in \mathcal{L} : \exists l' \in \mathcal{L}, s \in \mathcal{S} : d_{l'l}(s) \neq 0\}.$$

and let $\mathcal{G}^2 \subset \mathcal{L}$ consist of all those commodities which are output of intertemporal production (stored commodities).

$$\mathcal{G}^2 = \{l \in \mathcal{L} : \exists l' \in \mathcal{L}, s \in \mathcal{S} : d_{ll'}(s) \neq 0\}.$$

We want to distinguish between inputs and outputs of spot production which have previously been stored and inputs and outputs which can not be produced through the storage technology. In order to do so with as little notation as possible, we assume[5] that there are integers j and m, $0 \leq j \leq K \leq m \leq L$ such that

$$\mathcal{G}^2 = \{l \in \mathcal{L} : l \leq j \quad \text{or} \quad K+1 \leq l \leq m\}.$$

Given any $\bar{s}, \bar{z} \in \mathcal{F}_s$, let

$$(z(1), \ldots, z(S)) = \arg \min_{(z_1, \ldots, z_S) \in \mathcal{F}} \left(\min_{\kappa \in \mathcal{K}} \| h(\bar{s}, \bar{z}, \kappa, \mathbf{0}, z_1, \ldots, \mathbf{0}, z_S) \|_\infty \right).$$

Without loss of generality we can restrict attention to ϵ-equilibria for which Equations h^1, h^5, h^8 and h^9 hold with equality at $(\bar{s}, \bar{z}, z(1), \ldots, z(S))$ since an error in these equations can be easily put to zero by increasing the error in other equations. The other equations will generally only hold with some error. In order to facilitate the error analysis, it is useful to state explicitly the variables these functions depend on.

- $h^2 = (h^2_{a,s})_{a=1,\ldots,N, s \in \mathcal{S}}$, $h^2_{a,s}(p(s), \xi^a(s)) = \epsilon^2_{a,s} \in \mathbb{R}$

[5]This assumption implicitly states that all commodities are either inputs or outputs. This simplifies the notation but the analysis can also be conducted if some commodities are not part of production.

- $h^3 = (h^3_{a,l,s})_{a=1,\ldots,N, l\in\mathcal{L}, s\in\mathcal{S}},\ h^3_{a,s,l}(x^a_l(s), \xi^a_l(s), \alpha^a_l(s), \alpha^a_{l-}(s)) = \epsilon^3_{a,s,l} \in \mathbb{R}$
  We assume w.l.o.g. that $h^3_{a,s,l}$ holds with equality whenever the individual a has no endowments in commodity l and that commodity does not enter his utility function.

- $h^4 = (h^4_{a,l})_{a=2,\ldots,N, l\in\mathcal{G}^1},\ h^4_{a,l}(\bar{p}_l, \bar{\lambda}^{a-1}, (p_n(s))^{s\in\mathcal{S}}_{n\in\mathcal{G}^2}, (\lambda^a(s))_{s\in\mathcal{S}}, \hat{\kappa}) = \epsilon^4_{a,l} \in \mathbb{R}$ where

$$\hat{\kappa} = \arg\min_{\kappa>0} \left| h^4_{a,l}(\bar{p}_l, \bar{\lambda}^{a-1}, (p_n(s))^{s\in\mathcal{S}}_{n\in\mathcal{G}^2}, (\lambda^a(s))_{s\in\mathcal{S}}, \kappa) \right| \text{ s.t. } \kappa\bar{\alpha}^{a-1}_l = 0$$

  Note that for all commodities which cannot be stored, i.e. $l \notin \mathcal{G}^1$, there is no equilibrium equation.

- $h^6 = (h^6_{a,s,l})_{a=1,\ldots,N, l\in\mathcal{L}, s\in\mathcal{S}},\ h^6_{a,s,l}(x^a(s), \lambda^a(s), p_l(s)) = \epsilon^6_{a,s,l}$

- $h^7 = (h^7_s)_{s\in\mathcal{S}},\ h^7_s(y(s), p(s)) = \epsilon^7_s \in \mathbb{R}^K$.

The general strategy to derive error bounds will be as follows. We identify a set of commodities whose prices we can perturb at any node in order to ensure that $h^7(y(s), \tilde{p}) = 0$ for the perturbed $\tilde{p}$. These prices do not appear in h^4. We then give bounds on the errors caused in h^4 by perturbations in previous periods of $\bar{\lambda}$ and $\bar{p}$. The perturbations in λ and p then determine the perturbations necessary in individual endowments to satisfy the budget constraints and in Bernoulli utility functions to satisfy $h^6 = 0$.

Errors in h^7

For a given $s = 1, \ldots, S$ and $z(s)$, let the $(L-K)\times K$ matrix J denote the Jacobian of $f(., s)$ with respect to all inputs at $y(s)$. For our analysis it is helpful to divide J into 4 submatrices. Denoting the row index by l and the column index by l' we write

$$J = \left(\frac{\partial f_l(-y_1(s), \ldots, -y_K(s), s)}{\partial y_{l'}}\right)^{l=K+1,\ldots,L}_{l'=1,\ldots,K} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \quad \text{with}$$

$$J_{11} = \left(\frac{\partial f_l}{\partial y_{l'}}\right)^{l=K+1,\ldots,m}_{l'=1,\ldots,j}, \qquad J_{12} = \left(\frac{\partial f_l}{\partial y_{l'}}\right)^{l=K+1,\ldots,m}_{l'=j+1,\ldots,K},$$

$$J_{21} = \left(\frac{\partial f_l}{\partial y_{l'}}\right)^{l=m+1,\ldots,L}_{l'=1,\ldots,j}, \qquad J_{22} = \left(\frac{\partial f_l}{\partial y_{l'}}\right)^{l=m+1,\ldots,L}_{l'=j+1,\ldots,K}.$$

Define a $(K-j)+(L-m)$ vector

$$\epsilon r^1(s) = \begin{pmatrix} 0 & J_{21}^T \\ -I_{K-j} & J_{22}^T \end{pmatrix}^{-1} \epsilon^7_s.$$

Recall that $\bar{s}, \bar{z}$ denote the shock and the endogenous variables from the previous period and let

$$\Delta_1(\bar{s}, \bar{z}) = \max_{s\in\mathcal{S}} \max\{ \max_{i=1,\ldots,K-j} \left|\frac{\epsilon r^1_i(s)}{p_{j+i}(s)}\right|, \max_{i=1,\ldots,L-m} \left|\frac{\epsilon r^1_{K-j+i}(s)}{p_{m+i}(s)}\right| \}.$$

The proof of Lemma 2 in the Appendix shows that this bound denotes the maximal relative perturbation necessary in prices to obtain equality in Equation h^7 for z. Finally let $\overline{\Delta}_1 = \max_{s,z \in \mathcal{F}_s} \Delta_1(s, z)$ denote the upper bound on this perturbation across all points in the ϵ-equilibrium. We can now ensure that there exist relative perturbations in the prices of commodities $l = j+1, \ldots, K$ and $l = m+1, \ldots, L$ (that is, for $l \in \mathcal{L} - \mathcal{G}^2$) which are uniformly bounded by $\overline{\Delta}_1$ and which guarantee that for all s, $z \in \mathcal{F}_s$, $h^7_s(\bar{z}) = 0$. The following lemma states this formally (see the Appendix for the proof).

LEMMA 2 *For each s, $z \in \mathcal{F}_s$, there exist $\tilde{p}_l$, $l \in \mathcal{L} - \mathcal{G}^2$, with $\left|\frac{p_l - \tilde{p}_l}{p_l}\right| < \overline{\Delta}_1$ such that*

$$h^7_s(y, (p_1, \ldots, p_j, \tilde{p}_{j+1}, \ldots, \tilde{p}_K, p_{K+1}, \ldots, p_m, \tilde{p}_{m+1}, \ldots, \tilde{p}_L)) = 0.$$

Note that prices of commodities which are output of the intertemporal storage technology, i.e. p_l for $l \in \mathcal{G}^2$, are not perturbed. Therefore, the performed perturbations do not affect the error in Equation h^4. Below we need some notation for the perturbed vector of all prices: we write $\tilde{p}_l(s) = p_l(s) + v_l(s)$ where $v_l(s) = \epsilon r^1(s)$ for $l \in \mathcal{L} - \mathcal{G}^2$ and $v_l(s) = 0$ for $l \in \mathcal{G}^2$. In vector notation, $\tilde{p}(s) = p(s) - v(s)$.

Errors in h^2

Given $\bar{s}, \bar{z}, z(1), \ldots, z(S)$ from above and given perturbed prices, $\tilde{p}(s)$, to ensure that h^7 holds with equality, we need to perturb $\xi^a(s)$ for all $a = 1, \ldots, N$ to ensure that $\tilde{p}(s)\xi^a(s) = 0$. Since we assume constant returns to scale, $\tilde{p}(s) \cdot y(s) = 0$ and since $h^9 = 0$ we have that $\tilde{p}(s) \sum_{a=1}^{N} \xi^a(s) = 0$. Since l^* is a commodity that is desired by all agents, we have that $\tilde{p}_{l^*}(s) > 0$, therefore a sufficient perturbation would be

$$\tilde{\xi}^a_{l^*}(s) = -\frac{1}{\tilde{p}_{l^*}(s)} \sum_{l \neq l^*} \xi^a_l(s)\tilde{p}_l(s).$$

Now note that $\tilde{p}_l(s)\xi^a_l(s) = (p_l(s)+v_l(s))\,\xi^a_l(s)$ and thus $\tilde{p}(s)\xi^a(s) = \sum_l (p_l(s)+v_l(s))\,\xi^a_l(s) = e^2_{a,s} + \sum_l v_l(s)\xi^a_l(s)$. Hence, an upper bound for $|\tilde{\xi}^a_{l^*} - \xi^a_{l^*}|$ is given by

$$\Delta_2(\bar{s}, \bar{z}) = \max_{s \in \mathcal{S}, a=1,\ldots,N} \frac{1}{p_{l^*}(s)(1 - \Delta_1(\bar{s}, \bar{z}))} \left(|\epsilon^2_{a,s}| + \sum_{l \in \mathcal{L}} |v_l(s)\xi^a_l(s)| \right).$$

Define

$$\overline{\Delta}_2 = \max_{s,z \in \mathcal{F}_s} \Delta_2(s, z).$$

Errors in h^3

Given $\bar{s}, \bar{z}, z(1), \ldots, z(S)$ and $\Delta_2(\bar{s}, \bar{z})$ from above, recall that h^3 is assumed to hold with equality for all commodities which do not enter the utility function **and** in which the agent has zero endowments.

For commodities which do not enter the utility function, but in which the agent has positive endowments, adjust $o^a_\ell(s)$ to ensure equality. These adjustments all lie within $|\epsilon^3|$ since by construction, the excess demands ξ in these commodities were not perturbed in the previous step. As we mentioned above, we assume that endowments are sufficiently large to ensure non-negativity of perturbed endowments.

For commodities which enter the utility function, adjust $x^a(s)$ to ensure equality. These commodities include l^* for which ξ_{l^*} has been perturbed. Define

$$\nu^a_s = \max\{\Delta_2(\bar{s}, \bar{z}) + \epsilon^3_{l^*}, \max_{l \neq l^*} \epsilon^3_l\}.$$

An upper bound on the necessary perturbations in consumptions is therefore given by

$$\Delta_3(\bar{s}, \bar{z}) = \max_{a=1,\ldots,N, s \in \mathcal{S}} \nu^a_s.$$

Define $\overline{\Delta}_3 = \max_{s \in \mathcal{S}, z \in \mathcal{F}_s} \Delta_3(s, z)$.

For commodities that do enter the utility function, this perturbation may increase the error in h^6. In order to capture this effect, define

$$\Delta_6(\bar{s}, \bar{z}) = \max_{a=1,\ldots,N, s \in \mathcal{S}} \left(|\epsilon^6_{as}| + |D_x u^a(x^a(s) + \nu_s \mathbf{1}, s) - D_x u^a(x^a(s), s)| \right).$$

Define $\overline{\Delta}_6 = \max_{s \in \mathcal{S}, z \in \mathcal{F}_s} \Delta_6(s, z)$. We must take this error into account when we examine Equation h^6.

Errors in h^4

Given $\bar{s}, \bar{z}, z(1), \ldots, z(S)$ from above define a payoff matrix A by

$$A = \left(\pi(s, \bar{s}) \sum_{l \in \mathcal{G}^2} d_{ll'}(s) p_l(s) \right)_{l's}, \quad l' \in \mathcal{G}^1, s \in \mathcal{S}.$$

Define $er^2(a) = A^+(\epsilon^4_{al})_{l \in \mathcal{G}^1}$ and let $\Delta_4(\bar{s}, \bar{z}) = \max_{a=2,\ldots,N} \max_{s \in \mathcal{S}} |er^2_s(a)/\lambda^a_s|$. Define

$$\overline{\Delta}_4 = \left((1 + \max_{s, z \in \mathcal{F}_s} \Delta_4(s, z))(1 + \overline{\Delta}_1) \right)^N.$$

It is straightforward to see that this imposes an upper bound on necessary relative perturbations in λ^a which ensure that h^4 holds with equality, given the perturbations in prices for h^7 and given the errors in h^4.

Errors in h^6: Necessary Perturbations in Bernoulli Utilities

Finally we need to perturb Bernoulli utilities in order to impose equality on the first order conditions for individuals' spot optimality. A bound on the necessary (linear) perturbations is given by

$$\Delta^P = \overline{\Delta}_6 + \overline{\Delta}_4 \max_{s, (o, \ldots, o, x, \xi, \lambda, y, p) \in \mathcal{F}_s} \left(\max_{a=1,\ldots,N} p_l \lambda^a \right).$$

The following theorem summarizes the above discussion and uses Lemma 1 to give bounds on the overall perturbations in endowments and preferences necessary.

THEOREM 1 *Given an ϵ equilibrium $\mathcal{F}$ for an economy $\mathcal{E}$, with $\overline{\Delta}_j$, $j = 1, \ldots, 4$ and Δ^P as defined above, there exists an economy $\mathcal{E}'$ with $d(\mathcal{E}, \mathcal{E}') < \Delta^P L \frac{\bar{\epsilon}}{m}$ and with a competitive equilibrium $(z(\sigma))_{\sigma \in \Sigma}$ such that the ϵ equilibrium process $z^\epsilon = (\phi^\epsilon_-, \phi^\epsilon, x^\epsilon, \xi^\epsilon, \lambda^\epsilon, y^\epsilon, p^\epsilon)$ satisfies for all $\sigma \in \Sigma$,*

$$
\begin{aligned}
\phi^\epsilon_-(\sigma) - \phi_-(\sigma) &= 0 \\
\phi^\epsilon(\sigma) - \phi(\sigma) &= 0 \\
y^\epsilon(\sigma) - y(\sigma) &= 0 \\
\|x^\epsilon(\sigma) - x(\sigma)\| &\leq \overline{\Delta}_3 \\
\|\lambda^\epsilon(\sigma) - \lambda(\sigma)\| &\leq \overline{\Delta}_4 \|\lambda^\epsilon(\sigma)\| \\
\|p^\epsilon(\sigma) - p(\sigma)\| &\leq \overline{\Delta}_1 \|p^\epsilon(\sigma)\| \\
\|\xi^\epsilon(\sigma) - \xi(\sigma)\| &\leq \overline{\Delta}_2
\end{aligned}
$$

Note that the portfolios $(\phi^\epsilon_-, \phi^\epsilon)$ and the firm's output were not perturbed.

5.3 Parametric Examples

We illustrate by means of an example that the bounds in Theorem 1 are fairly tight and that methods which lead to low maximal errors in Euler equations usually approximate an economy in equilibrium very well in our model. Suppose that there are only three commodities: Capital, k, labor, l, and a consumption good, c. The numéraire commodity is taken to be capital, its price is always 1. Agents have access to a risk-less storing technology, which transforms one unit of the consumption good at node s^t into 1 unit of capital at each node $s^{t+1} \succ s^t$. The risky spot production function is Cobb-Douglas

$$f(k, l, s) = \eta(s) k^\alpha l^{1-\alpha} + (1 - \delta(s)) k^\alpha$$

for shocks η, δ. Agents live for 9 periods and only derive utility from the consumption good. An agent born at shock s^t has utility function

$$U^{s^t} = \mathrm{E}_{s^t} \sum_{a=1}^{N} \beta^{a-1} u(c(s^{t+a-1})).$$

We assume that there are 4 shocks which are iid with $\pi_s = 0.25$ for $s = 1, \ldots, 4$. Bernoulli utilities are of the CRRA form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{2}$$

with a coefficient of relative risk aversion $\gamma = 3$. Suppose that $\beta = 0.8$ and that individual endowments are deterministic and given by

$$(\mathbf{e}^1, \ldots, \mathbf{e}^N) = (1, 1, 1, 1, 1, 1, 0.5, 0, 0).$$

We consider 4 different specifications for the shocks to production.

Table 1: Specifications for Shocks				
	State 1	State 2	State 3	State 4
Case 1: η	0.95	1.05	0.95	1.05
Case 1: δ	0.7	0.7	0.7	0.7
Case 2: η	0.85	1.15	0.85	1.15
Case 2: δ	0.7	0.7	0.7	0.7
Case 3: η	0.95	1.05	0.95	1.05
Case 3: δ	0.5	0.5	0.9	0.9
Case 4: η	0.85	1.15	0.85	1.15
Case 4: δ	0.5	0.5	0.9	0.9

We do *not* impose that individual investment has to be non-negative. This is done to simplify computations. It is easy to see that the above error analysis remains valid even without the non-negativity restriction.

5.3.1 Computation

Due to the finite nature of recursive ϵ-equilibria it should be possible to derive a globally convergent algorithm which computes an ϵ-equilibrium for any given $\epsilon > 0$ and any given specification of preferences and endowments. However, since agents live for 9 periods the endogenous state space Θ is of dimension 7 and any discrete algorithm will be hopelessly inefficient. Krueger and Kubler (2003) develop an algorithm to approximate equilibria in OLG models where agents live for several periods using polynomial approximations. The algorithm assumes that pricing and policy functions which describe a recursive ϵ-equilibrium are defined over a compact set Θ and that these functions exhibit a high degree of smoothness. They approximate them by polynomials, using Smolyak's method to avoid a curse of dimensionality. The unknown polynomial coefficients are solved for through a time iteration algorithm.

Given the discussion above, using this algorithm to obtain recursive ϵ-equilibria might seem odd: By definition recursive ϵ-equilibria are a finite collection of points and there is no guarantee that the functions in the definition can be extended to smooth functions over a compact set. However, in practice the algorithm has been proven to converge very well and it is clear that it would be infeasible to compute a recursive ϵ-equilibrium directly for a 7-dimensional state space. Our concept of recursive ϵ-equilibrium is not meant to impose any restrictions on the actual computation of approximate equilibria. It merely provides a method to assess the quality of a candidate solution.

We use the method described in Section 4.2 to construct a recursive ϵ-equilibrium from the computed values of the algorithm. We set $\delta = 1/300000$. This value results in a set $\mathcal{F}$

with around 2 million elements. All states are visited within the first 24 periods, i.e. the algorithm terminates at $\mathcal{Y}_t^{\delta}$ for $t = 24$.

5.3.2 Error Analysis in Practice

In this example most of the steps in the proof of Theorem 1 reduce to a single calculation. The error in Euler equations does propagate over time, but this is the only source of high overall errors. Since neither labor nor capital enters individual's utility functions, the errors in h^7 are around machine-precision (10^{-15} on the machine used for this computation). The necessary perturbations in spot prices and consumptions are then smaller than 10^{-13}. Only errors in h^4 are significantly higher, around 10^{-4}, these then lead to the higher maximal errors.

The following table reports the errors along one simulated path of length 100000, simerr, the ϵ which resulted from taking $\delta = 1/300000$ in the above discretization procedure as well as the maximum perturbation necessary in individual Bernoulli utilities. Note that the computational error along a simulated path is always considerably smaller than that of our ϵ-equilibrium.

Table 2: Errors

	Case 1	Case 2	Case 3	Case 4
simerr	8.3 (-5)	2.5 (-4)	8.2 (-4)	1.3 (-3)
ϵ	9.2 (-4)	4.2 (-4)	1.3 (-3)	2.5 (-3)
Δ^P	8.2 (-3)	8.3 (-3)	9.9 (-3)	1.3 (-2)

6 The Lucas Model with Several Agents

As a second application we consider the model of Duffie et al. (1994, Section 3). This model is a version of the Lucas (1978) asset pricing model with finitely many heterogeneous agents. There are I infinitely lived agents, $i \in \mathcal{I}$, and a single commodity in a pure exchange economy. Each agent $i \in \mathcal{I}$ has endowments $e^i(\sigma) > 0$ at all nodes $\sigma \in \Sigma$ which are time-invariant functions of the shock alone, i.e. there exist functions $\mathbf{e}^i : \mathcal{S} \to \mathbb{R}_+$ such that $e^i(s^t) = \mathbf{e}^i(s_t)$. Agent i has von Neumann-Morgenstern utility over infinite consumption streams

$$U^i(c) = E_0 \sum_{t=0}^{\infty} \beta^t u_i(c_t)$$

for a differentiable, strictly increasing and concave Bernoulli function u_i which satisfies an Indada condition.

There are J infinitely lived assets in unit net supply. Each asset j pays shock dependent dividends $d_j(s)$, we denote its price at node s^t by $q_j(s^t)$. Agents trade these assets but are restricted to hold non-negative amounts of each asset. We denote portfolios by $\theta^i \geq 0$. At the root node s_0 agents hold initial shares $\theta^{\mathcal{I}}(s^{-1})$ which are assumed to be identical across agents and sum up to 1, so $\theta^i_j(s^{-1}) = 1/I$ for all $i \in \mathcal{I}, j = 1, \ldots, J$.

A competitive equilibrium is a collection $((c^i(\sigma), \theta^i(\sigma))_{i\in\mathcal{I}}, q(\sigma))_{\sigma\in\Sigma}$ such that market clear,

$$\sum_{i\in\mathcal{I}} \theta^i_j(\sigma) = 1 \text{ for all } \sigma \in \Sigma, j \in \mathcal{J}.$$

and such that agents optimize

$$\begin{aligned} c^i \in \arg\max\nolimits_{c\geq 0} U^i(c) \quad \text{s.t.} \quad & \forall s^t \in \Sigma \\ & c^i(s^t) = \mathbf{e}^i(s_t) + \theta^i(s^{t-1})(q(s^t) + d(s_t)) - \theta^i(s^t)q(s^t), \\ & \theta^i(s^t) \geq 0. \end{aligned}$$

6.1 The Expectations Correspondence

Following Kubler and Schmedders (2003a) it is useful to include as an endogenous variable individual shares of total financial wealth

$$w^i(s^t) = \frac{\theta^i(s^{t-1}) \cdot (q(s^t) + d(s_t))}{\sum_{j=1}^J q_j(s^t) + d_j(s_t)}$$

Note that $w^{\mathcal{I}} = (w^1, \ldots, w^I) \in \Delta^{I-1}$, the $(I-1)$-dimensional simplex in $\mathbb{R}^I$. We define the current endogenous variables to consist of wealth shares, asset prices, individuals' consumption and portfolios,

$$z = (w^{\mathcal{I}}, q, c^{\mathcal{I}}, \theta^{\mathcal{I}}).$$

As before, we built trivial normalizations into the state space, i.e. we assume that $\theta^i \geq 0$, $c^i \geq 0$ for all $i \in \mathcal{I}$ and that $w^{\mathcal{I}} \in \Delta^{I-1}$.

Since, we want to perturb individual endowments only (perturbing preferences does not simplify the analysis), we take perturbations to be I-vectors, $o^{\mathcal{I}} = (o^1, \ldots, o^I) \in \mathbb{R}^I$. The equilibrium equations are then $h(\bar{s}, \bar{z}, \kappa, o^{\mathcal{I}}_1, z_1, \ldots, o^{\mathcal{I}}_S, z_S) = 0$ with $h = h^1, \ldots, h^5$ and

$$\begin{aligned} h^1_i &= -\bar{q}u'_i(\bar{c}^i) + \beta E_{\bar{s}}\left[(q(s) + d(s))u'_i(c^i(s))\right] + \kappa^i \\ h^2_{is} &= w^i(s) - \frac{\bar{\theta}^i \cdot (q(s) + d(s))}{\sum_{j=1}^J q_j(s) + d_j(s)} \\ h^3_{is} &= c^i(s) - w^i(s)\sum\nolimits_{j\in\mathcal{J}}(q_j(s) + d_j(s)) + \theta^i(s)\cdot q(s) - (\mathbf{e}^i(s) + o^i(s)) \\ h^4_{ij} &= \kappa^i_j \bar{\theta}^i_j \\ h^5_{js} &= \sum\nolimits_{i\in\mathcal{I}} \theta^i_j(s) - 1 \end{aligned}$$

In order to obtain $h^1 = 0$ we need to perturb marginal utility which we achieve by perturbing consumptions. The residuals in h^3 are then set to zero by perturbing endowments.

Kubler and Schmedders (2003a) show that under standard assumptions on preferences and endowments competitive equilibria can be characterized by the expectations correspondence and that recursive ϵ-equilibria always exist. Let $\rho = (\rho_q, (\rho_{c^i}, \rho_{\theta^i})_{i\in\mathcal{I}})$ denote the policy function associated with a recursive ϵ-equilibrium.

6.2 Error Analysis

The main problem in the error analysis is that, because agents are infinitely lived, the necessary perturbations to correct for errors in h^1 may propagate without bounds. For example, a small reduction in consumption in a given period may result in a bigger reduction in the subsequent period which in turn results in a further reduction in the third period and so on. As a consequence the perturbed values may move far away from the ϵ-equilibrium. Therefore, as we explained in Section 4, we are no longer able to show that the ϵ-equilibrium path approximates an economy in equilibrium. Instead we need to show that the ϵ-equilibrium is balanced and weakly approximates an economy in equilibrium. For this, we need to perturb the distribution of wealth $w^{\mathcal{I}}$. As we move from period to period we must allow for small perturbations in the state of the economy in order to maintain closeness between all perturbed and ϵ-equilibrium values.

Given a (recursive) ϵ-equilibrium $\mathcal{F}$ and given any $\bar{s}, \bar{z} \in \mathcal{F}_{\bar{s}}$, let

$$(\tilde{z}(1), \ldots, \tilde{z}(S)) = \arg \min_{(z_1, \ldots, z_S) \in \mathcal{F}} \left(\min_{\kappa \in \mathcal{K}} \| h(\bar{s}, \bar{z}, \kappa, \mathbf{0}, z_1, \ldots, \mathbf{0}, z_S) \| \right)$$

As in Section 5, we can assume w.l.o.g. that Equations h^4 and h^5 hold with equality given $\tilde{z}_1, \ldots, \tilde{z}_S$.

In order for h^1 to hold with equality, we need to perturb marginal utilities through perturbations of $c_1, \ldots, c_S$. Define the $S \times J$ payoff matrix A by

$$A(\bar{s}, \bar{z})_{js} = (\beta \pi(s, \bar{s})(\tilde{q}_j(s) + d_j(s)))_{js}.$$

Given an agent i, define ϵ^i by

$$\epsilon^i = \min_{\kappa \geq 0} \| A^+(-\bar{q} u_i'(\bar{c}^i) + \beta E_{\bar{s}} \left[(\tilde{q}(s) + d(s)) u_i'(\tilde{c}^i(s)) \right] + \kappa) \| \quad \text{s.t.} \quad \kappa \cdot \bar{\theta}^i = 0$$

Let $er^1(\bar{s}, \bar{z}) = \max_{i \in \mathcal{I}} \epsilon^i$ and $\delta = \max_{s, z \in \mathcal{F}_s} er^1(s, z)$. This last bound denotes the maximum necessary perturbation of marginal utilities over the entire set $\mathcal{F}$ to obtain equality in h^1 given that marginal utilities last period were not perturbed. However, this only constitutes a lower bound on total necessary perturbations because errors will propagate over time. We use the concept of balancedness as defined in Section 4 to derive an upper bound. Given that this period's marginal utilities have been perturbed, we want to derive perturbations in next period's wealth distribution which ensure that perturbations in next period's marginal utilities *do not* propagate.

Using the fact that we consider a recursive ϵ-equilibrium, we can write current endogenous variables as functions of $w^{\mathcal{I}}$ alone and define

$$M(w^{\mathcal{I}}(1), \cdots, w^{\mathcal{I}}(S)) = \left(\beta \pi(s, \bar{s})(\rho_{q_j}(w^{\mathcal{I}}(s), s) + d_j(s)) \right)_{js}.$$

Note that at $\tilde{w}$, $M(\tilde{w}) = A(\bar{s}, \bar{z})$.

We now determine perturbations in next period's wealth distribution which guarantee that necessary perturbations in marginal utilities next period are within 2δ, given that perturbations of marginal utilities this period are within 2δ. For any $\bar{\mu} \in \mathbb{R}^I$ with $|\bar{\mu}^i - u_i'(\bar{c}^i)| < 2\delta$, $\forall i$, we want to find $w_1^{\mathcal{I}}, \ldots, w_S^{\mathcal{I}}$ such that there exist $\tilde{\mu}_1, \ldots, \tilde{\mu}_S \in \mathbb{R}^I$ with $|\tilde{\mu}_s^i - u_i'(\rho_{c^i}(w_s^{\mathcal{I}}, s))| < 2\delta$ and with

$$-\bar{q}\bar{\mu}^i + \beta \sum_{s \in \mathcal{S}} \pi(s|\bar{s})(\rho_q(w_s^{\mathcal{I}}, s) + d_s)\tilde{\mu}_s^i = 0 \text{ for all } i \in \mathcal{I}.$$

Since the true wealth distribution is determined through $\bar{\theta}$ which cannot be perturbed in order to achieve equality in h^2, $\bar{w}^{\mathcal{I}}$ as it appears in h^3 is fixed. However, the wealth distribution which supports the new consumptions and prices generally differ from $\bar{w}^{\mathcal{I}}$. We then need to perturb endowments in h^3 in order to be able to support $\rho_q(w)$ and $\rho_c(w)$ at the predetermined wealth distribution $\bar{w}$. We want to find $w_1^{\mathcal{I}}, \ldots, w_S^{\mathcal{I}}$ which minimize these perturbations in the endowments, i.e. which minimizes

$$\left| w_s^i \sum_{j \in \mathcal{J}} (\rho_{q_j}(w_s^{\mathcal{I}}, s) + d_j(s)) - \bar{\theta}^i \cdot (\rho_q(w_s^{\mathcal{I}}, s) + d(s)) \right|$$

at all $s \in \mathcal{S}$. In order to ensure that this is possible for all $\bar{\mu}$ which are within 2δ from $u_i'(\bar{c}^i)$ it suffices to check that at the 3^I distinct points, $(u_i'(\bar{c}^i)(1 - 2\delta n_i))_{i \in \mathcal{I}}$ for all $n \in \{-1, 0, 1\}^I$, this can be achieved with the perturbations next period lying within one times δ, i.e. with

$$|\tilde{\mu}_s^i - u_i'(\rho_{c^i}(w_s^{\mathcal{I}}, s))| < \delta.$$

Since $\mathcal{F}$ is finite a simple grid search[6] allows us to determine the following number. Given $n \in \{-1, 0, 1\}^I$ define

$$err^2(n) = \min_{w = (w_1^{\mathcal{I}}, \ldots, w_S^{\mathcal{I}})} \left(\max_{i \in \mathcal{I}, s \in \mathcal{S}} \left| w_s^i \sum_{j \in \mathcal{J}} (\rho_{q_j}(w_s^{\mathcal{I}}, s) + d_j(s)) - \bar{\theta}^i \cdot (\rho_q(w_s^{\mathcal{I}}, s) + d(s)) \right| \right)$$

subject to for all $i \in \mathcal{I}$,

$$\min_{\kappa \geq 0, \kappa\theta^i = 0} \left\| (M(w))^{-} \left[\bar{q} u_i'(\bar{c}^i)(1 - 2n_i\delta) - M(w) \begin{pmatrix} u_i'(\rho_{c^i}(w(1), 1)) \\ \vdots \\ u_i'(\rho_{c^i}(w(S), S)) \end{pmatrix} - \kappa \right] \right\| < \delta.$$

Define

$$\Delta_1(\bar{s}, \bar{z}) = \max_{n \in \{-1, 0, 1\}^I} err^2(n) \text{ and } \overline{\Delta}_1 = \max_{s, z \in \mathcal{F}_s} \Delta_1(s, z).$$

The discussion above shows that this is an upper bound on perturbations in individual endowments necessary to offset the perturbations in the wealth distribution. In addition we

[6] In the example below this issue is dealt with more sophisticatedly.

need to perturb individual endowments in order to obtain the 'correct' marginal utilities. For this purpose, we define

$$\Delta_2(\bar{s}, \bar{z}) = \max_{i \in \mathcal{I}, s \in \mathcal{S}} \max\{u_i'^{-1}\left(u_i'(\tilde{c}^i(s))(1+2\delta)\right) - \tilde{c}^i(s), |u_i'^{-1}\left(u_i'(\tilde{c}^i(s))(1-2\delta)\right) - \tilde{c}^i(s)|\}$$

and $\overline{\Delta}_2 = \max_{s,z \in \mathcal{F}_s} \Delta_s(s, z)$.

The following theorem now summarizes our discussion.

THEOREM 2 *The ϵ-equilibrium $\mathcal{F}$ weakly approximates an economy in equilibrium with precision* $\overline{\Delta}_1 + \overline{\Delta}_2$.

Note that in order to achieve balancedness, it is crucial that there are sufficiently many states compared to the number of assets. In particular, with this construction will generally not be possible when markets are complete. We now turn to two examples which illustrate that when markets are incomplete, this construction often results in reasonable error bounds.

6.3 Parametric Example

We consider two examples to illustrate the analysis above. In both examples there are two agents with identical CRRA utility and a coefficient of risk aversion of 2. There are 4 shocks which are i.i.d. and equi-probable. Dividends and endowments in two cases are as follows:

1. There is a single tree. Dividends are $d(s) = 1$ for all $s = 1, \ldots, 4$. Individual endowments are $\mathbf{e}^1 = (2, 5, 2, 5), \mathbf{e}^2 = (5, 2, 5, 2)$.
2. There are two trees. Dividends are $d_1(1) = d_1(2) = 1$, $d_1(3) = d_1(4) = 2$ and $d_2(1) = d_2(3) = 1$, $d_2(2) = d_2(4) = 2$. Individual endowments are $\mathbf{e}^1 = (1, 2, 1, 2), \mathbf{e}^2 = (2, 1, 2, 1)$.

Since there are only two agents, the endogenous state space for the recursive ϵ-equilibrium simply consists of the interval $[0, 1]$. We use the algorithm described in Kubler and Schmedders (2003a) and discretize the state space into 10^8 possible wealth levels to obtain an ϵ-equilibrium. The resulting maximal error lies around 10^{-3} in both examples. The necessary (relative) perturbations in individual endowments lie around 4.3×10^{-3} in the first specification and around 1.2×10^{-2} for the second specification. The maximal error in Euler equations along one simulated path for 20000 simulated periods lies around 10^{-5}. This is a large discrepancy, but it is mainly caused by the fact that along a simulated path many areas of the state space are not visited and so many errors are simply missed. On the other hand, a maximal perturbation of 1.2 percent might still be viewed as acceptable if one is only interested in moments of asset prices.

Appendix: Proofs of Lemmas

Proof of Lemma 1. We want to find a bound on the distance between original and perturbed preferences. For this task we take as given a point $(y, x) \in \bar{C}$ on the boundary of P, $(y, x) \in \partial P \cap \bar{C}$, and construct a z such that $(z, x) \in \tilde{P}$ is close to (y, x).

For any $x \in X^i$, define $o^i \cdot x = \sum_{s^t \in \Sigma} \beta^t \pi(s^t) o^i(s^t) x(s^t)$. Since $(y, x) \in \partial P$ and $(z, x) \in \partial \tilde{P}$ it holds by definition

$$U^i(z) - o^i \cdot z \geq U^i(x) - o^i \cdot x, \quad U(x) = U(y)$$

and so,

$$U^i(z) - U^i(y) \geq o^i \cdot (x - z).$$

(Note that we can freely rearrange terms in all series since all of them are absolutely convergent.) A sufficient condition for z to satisfy this last equation is that each element $z(s^t)$ satisfies

$$u(z(s^t)) - u(y(s^t)) \geq o^i(s^t)(x(s^t) - z(s^t))$$

for all $s^t \in \Sigma$. If $o^i(s^t)(x(s^t) - y(s^t)) \leq 0$, the condition is trivially satisfied for $z = y$. Otherwise, we can find a z such that the $u(z(s^t)) - u(y(s^t)) = o^i(s^t)(x(s^t) - z(s^t))$. (This is true since for $z(s^t) \equiv 2\bar{c}$ and $y(s^t) \equiv \bar{c}$ it holds that $u(z(s^t)) - u(y(s^t)) > \bar{m}\bar{c} > o^i(s^t)(x(s^t) - z(s^t))$.) Note that the appropriate z may not be an element of $\bar{C}$ as we discussed above.

It now follows from the mean value theorem that

$$u^i(z(s^t)) - u^i(y(s^t)) = D_x u^i(\xi(s^t)) \, (z(s^t) - y(s^t))$$

for some $\xi(s^t) \in \mathbb{R}^L$, $\|\xi(s^t)\| \leq 2\bar{c}$ for all $s^t \in \Sigma$. (Note that $\xi(s^t) = \alpha z(s^t) + (1 - \alpha) y(s^t)$ for some $\alpha \in [0, 1]$.) Therefore

$$D_x u^i(\xi(s^t)) \, (z(s^t) - y(s^t)) = o^i(s^t)(x(s^t) - z(s^t)).$$

Since for all s^t, $|o^i(s^t)(z(s^t) - x(s^t))| \leq L\bar{\omega}^i \bar{c}$, and with the bound on marginal utility, $\bar{m}$, we obtain

$$\|z - y\| \leq L \frac{\bar{\omega}^i}{\bar{m}} \bar{c},$$

which proves the lemma. □

Proof of Lemma 2. For each s, z_s, the system of Equations h^7 in linear in prices. We write it as follows

$$-(p_1 \cdots p_j \cdots p_K) + (p_{m+1} \cdots p_L) \begin{pmatrix} J_{21} & J_{22} \end{pmatrix} = \epsilon - (p_{K+1} \cdots p_m) \begin{pmatrix} J_{11} & J_{12} \end{pmatrix}$$

Equivalently

$$(p_{j+1} \cdots p_K, p_{m+1} \cdots p_L) \begin{pmatrix} 0 & -I_{K-j} \\ J_{21} & J_{22} \end{pmatrix} = \epsilon - (p_1 \cdots p_j, p_{K+1} \cdots p_m) \begin{pmatrix} -I_j & 0 \\ J_{11} & J_{12} \end{pmatrix}$$

Using the definition of the pseudo inverse implies the lemma. □

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