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OPTIMAL TARGET DATES AND PENALTIES
FOR CONTRACT WORK*

by

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1. Introduction

Construction and procurement contracts often attempt to insure the party letting the contract (the "owner") against late completion by the party performing the work (the "contractor"). This paper analyses these contracts from the owner's point of view.

A number of different forms of contract are possible [5]. However, the present analysis is primarily concerned with the following situation. It is assumed that the owner chooses a target date, τ , for completion of the work and a financial penalty of p dollars per time period if the work is not completed by time, τ . The chosen values of p and τ form part of the tender documents together with other conditions of the contract and the plans for the finished work. The tender documents are made available to various contractors who then submit lump-sum bids for completing the project. The time, $T(p, \tau)$, at which the project is finished by the contractor who is awarded the contract will be a random variable with a probability distribution depending on the incentive provided by the owner's choice of p and τ . When the project is finished, the winning contractor will receive the amount of his bid minus a penalty, $(T(p, \tau) - \tau)p$ if $T(p, \tau) > \tau$. The owner's problem is to choose p and τ to minimize the total expected cost of the project. This has three components: the expected value of the winning bid; the expected value of the opportunity cost incurred while the project remains unfinished; and the expected value of the penalty incurred by the contractors.

state of the system at time t , is a scalar variable representing the amount of work remaining to be completed. For example, in a procurement contract, x_t , might be measured in terms of units of product to be supplied while in a construction project it might be measured in terms of cubic yards of earth to be moved or square feet of pavement to be laid and so on. The decision at time t , $a_t \in R^m$, specifies the levels of the m resources which are to be used during the next time period. The technology of the activity is described by a sequence of cost functions, $c_t(a_t)$, and production functions, $f_t(a_t)$, $t = 0, 1, 2, \dots$. In general, uncertainty will exist concerning these functions. For example, uncertainty about future wages and prices will prevent exact specification of the function, c_t , and uncertainty with respect to such factors as the quality of the work force, quality of material inputs, and future weather conditions will prevent exact specification of the production function, f_t . These uncertainties are modelled by including additive random disturbance terms, γ_t and ξ_t , in the cost and production functions as shown in (1) below. The functions c_t and f_t are themselves assumed to be deterministic and continuous. Uncertainty will also exist with respect to the total quantity of work, x^0 , involved in the task. This occurs for example, because estimates of the quantity of work involved are obtained from blueprints which may be based on only approximate data concerning actual topological and geological conditions. It is assumed that x^0 is a random variable with mean, μ_0 , and that $x^0, \xi_0, \xi_1, \dots, \gamma_0, \gamma_1, \dots$ are independent. Although in general the states x_t cannot be observed exactly, an assumption of perfect observation will be made throughout this paper. For the class of project models considered here it was shown in [10] that this assumption is not of great importance in that the expected value of perfect information will usually be small.

In (1) the expectation is taken with respect to $x^0, \xi_0, \xi_1, \dots, \gamma_0, \gamma_1, \dots$. The time of the last decision, $T-1$, is a random variable. It is assumed that the output $f_t(a_t) + \xi_t$, and cost, $c_t(a_t) + \gamma_t$, occur uniformly over time. The random variable defined by the ratio, $\frac{x_{T-1}}{f_{T-1}(a_{T-1}) + \xi_{T-1}}$, in the objective function is therefore the fraction of the last time period in which work takes place and the term, $\frac{(c_{T-1}(a_{T-1}) + \gamma_{T-1})}{f_{T-1}(a_{T-1}) + \xi_{T-1}} x_{T-1}$, in (1) is the cost incurred in the last time period.

In [10], the activity model (1) was specialized to the time-invariant case where $c_t = c$, $f_t = f$, $A_t = A$, $t = 0, 1, 2, \dots$, and where $\{\xi_t, t = 0, 1, 2, \dots\}$ and $\{\gamma_t, t = 0, 1, 2, \dots\}$ were each assumed to be identically distributed sequences of random variables. Note that $p = 0$ and so neither p nor τ enter the analysis. Because of the assumptions concerning c_t , f_t and A_t this problem will always have a solution. In most cases part of the costs included in the cost functions will be 'fixed' in the sense that some costs will be incurred in each period even at zero levels of activity. These expenses can be thought of as being incurred in order to maintain a capability to perform the work. High fixed (or 'period') costs provide an incentive to finish the task earlier. This is accomplished by increasing a_t ; however, the convexity of the cost function or diminishing returns to scale in the production function (or a combination of these factors) will tend to decrease the optimal level of a_t . The optimal policy will strike a balance between these two factors.

Let a^* be a solution to:

$$(2) \quad c^* = \frac{c(a^*) + E[\gamma_0]}{f(a^*) + E[\xi_0]} = \min_{a \in A} \left\{ \frac{c(a) + E[\gamma_0]}{f(a) + E[\xi_0]} \right\}$$

and define the policy α^* by $\alpha_t^*(x_t) = a^*$, $t = 0, 1, 2, \dots$. Also let a' be the solution to

Because of (i) the policy, α^* , can be considered to be a "certainty-equivalence" policy. The cost using this policy gives the upper bound in (4) and can be computed by evaluation of the criterion (1) with $a_t = a^*$, $t = 0, 1, 2, \dots$. For any given assumption concerning the probability distribution of the disturbances, ξ_t , $t = 0, 1, 2, \dots$ this calculation can be performed using a formula given in [10] or, alternatively, by using a simulation technique.

The time invariance assumption on which Theorem 1 was based is now relaxed to allow for the target date and penalty by assuming a cost function:

$$(7) \quad c_t(a_t) = \begin{cases} c(a_t) & ; 0 \leq t \leq \tau \\ c(a_t) + p & ; t > \tau \end{cases}$$

Upper and lower bounds will now be stated for the expected cost, optimal actions and expected completion time for the contract activity problem with (7) as the cost function. As before, let c^* , α^* , a^* and a' be defined by (2) and (3) for the time invariant problem with no target date or penalty i.e., $c_t \equiv c$, $t = 0, 1, 2, \dots$. Similarly, for the time-invariant problem with $c_t \equiv c + p$, $t = 0, 1, 2, \dots$. Let:

$$(8) \quad c_p^* = \frac{c(a_p^*) + p + E[\gamma_0]}{f(a_p^*) + E[\xi_0]} = \min_{a \in A} \left\{ \frac{c(a) + p + E[\gamma_0]}{f(a) + E[\xi_0]} \right\}$$

and let α_p^* be the constant policy with $a_t = a_p^*$, $t = 0, 1, 2, \dots$. Also, let a'_p be the solution to

$$(9) \quad E \left[\frac{c(a'_p) + p + E[\gamma_0]}{f(a'_p) + \xi_0} \right] = \min_{a \in A} E \left[\frac{c(a) + p + E[\gamma_0]}{f(a) + \xi_0} \right]$$

Finally, let $T(p, \tau)$ be the random variable representing the time at which the contractor will finish the work using an optimal policy under a contract with penalty, p , and target date, τ . In the following it

can be seen from Theorem 1 the solution to the certainty equivalent problem will be within the bounds given by (10). Also, the optimal cost for the certainty equivalent problem will be lower than the expected cost of the optimal policy for the original problem and the optimal completion time will be longer than that for the original problem. The notation will be simplified by re-defining the c and f functions to include $E[\gamma_0]$ and $E[\xi_0]$ respectively.

Equations (2) and (8) can then be written as:

$$(2') \quad c^* = \frac{c(a^*)}{f(a^*)} = \min_{a \in A} \left\{ \frac{c(a)}{f(a)} \right\}$$

$$(8') \quad c_p^* = \frac{c(a_p^*) + p}{f(a_p^*)} = \min_{a \in A} \left\{ \frac{c(a) + p}{f(a)} \right\}$$

From Theorem 1, a^* and a_p^* are optimal policies for the two time invariant problems and the minimal costs and optimal completion times are given by:

$$(11) \quad W(\alpha^*) = c^* x^0 \quad ; \quad T(0, \tau) = \frac{x^0}{f(a^*)} = T^*, \quad \tau > 0$$

$$(12) \quad W(\alpha_p^*) = c_p^* x^0 \quad ; \quad T(p, 0) = \frac{x^0}{f(a_p^*)}$$

Lemma 2:

If $\tau = T(p, \tau)$ then $T(q, \tau) = T(p, \tau)$ for $q \geq p$.

Proof:

This result follows since the optimal policy for the contract (p, τ) is still optimal if the target date is $T(p, \tau)$ and the penalty is $q \geq p$.

To analyse the case where the costs are given by (7) let $v_t(x_t; p, \tau)$ be the cost of completing the contract given that state, x_t , has been reached at stage t . For $t \geq \tau$ the problem is time invariant and from (12):

The solutions to the contract activity problem for a given value of $p > 0$ and different values of $\tau \geq 0$ are now stated. This will facilitate the solution to the owner's problem discussed in Section 4. For given $p \geq 0$ there are three regions in which τ can lie:

(a) $\tau \leq T(p,0)$: From Lemma 3 it is clear that the contractor's optimal policy is to finish the project at time $T(p,0)$ and so from (15):

$$(16) \quad v_0(x^0; p, \tau) = c_p^* x^0 - \tau p \quad ; \quad T(p, \tau) = T(p, 0)$$

and the optimal policy, $\alpha_t^0(x_t; p, \tau) = a_p^*$, $t = 0, 1, 2, \dots$

(b) $T(p,0) < \tau < T^*$: For this case the optimal policy will involve the maximum completion time consistent with $x_\tau = 0$. This follows since, from the continuity of c and f and Lemma 1, there exists a penalty, $q \leq p$, corresponding to the target date τ such that $T(q,0) = \tau$. From Lemma 2 the optimal policy for the contract (p, τ) is the same as that for the contract (q, τ) . From Theorem 1 it can be seen that:

$$(17) \quad v_0(x^0; p, \tau) = \tau c(f^{-1}(\frac{x^0}{\tau})) \quad ; \quad T(p, \tau) = \tau$$

$$\text{and } \alpha_t^0(x_t; p, \tau) = f^{-1}(\frac{x_t}{\tau}), \quad t = 0, 1, 2, \dots$$

(c) $T^* < \tau$: Since τ is greater than or equal to the optimal completion time with no penalty:

$$(18) \quad v_0(x^0; p, \tau) = c^* x^0 \quad ; \quad T(p, \tau) = T^*$$

$$\text{and } \alpha_t^0(x_t; p, \tau) = a^*, \quad t = 0, 1, 2, \dots$$

the contractor toward risk, and the perceived competition from other contractors in the bidding process. The problem of choosing an optimal bidding strategy has received much attention and a number of different solution procedures have been proposed. These involve both non-game-theoretic approaches (for example, [2] and [6]) and game theoretic approaches (for example, [4]).

4. The Owner's Problem

It can be seen that an exact solution to the owner's problem would require complete knowledge of every contractor's expected cost flows from all sources, utility function, subjective probability distribution over future events, and so on. Clearly, this is an impossible requirement. An approximate procedure for selecting p and τ which greatly reduces the informational requirements is now described.

The basic idea is that the owner should try to estimate the cost function, c , the production function, f , and the profit mark-up, m , of the contractor who will be awarded the contract. If the contractor is known beforehand, this task could be relatively simple in many cases. If there are a number of possible contractors, the owner might use industry cost and performance figures to help in forming the estimates of the contractor's costs.

In many cases the owner's unique knowledge of the work required will be an advantage in estimating the expected costs. To estimate the value of the profit mark-up on which the winning bid is based the owner can use any of the methods referenced in the preceding section. In some industries the estimation of the profit mark-up will be relatively easy because the mark-ups used by different firms are remarkably similar (see for example [3]).

$$(22) \quad D(p, \tau) = -p \int_{\tau}^{\infty} (s - \tau) dF(s; p, \tau)$$

Note that the expected penalty cost is also imbedded in the bid price $B(p, \tau)$. In effect, the contractor partially insures himself against the penalty when calculating the bid $B(p, \tau)$. The latter becomes a contractual cost for the owner. However, if the project is not completed before the penalty date, the owner will receive penalty payments and the expected cost of the project to the owner must be reduced accordingly.

The owner's problem can be stated in general terms as:

$$(23) \quad \min_{p, \tau} \{A(p, \tau) = B(p, \tau) + C(p, \tau) + D(p, \tau)\}$$

where $B(p, \tau)$, $C(p, \tau)$ and $D(p, \tau)$ are defined by (19), (21) and (22) respectively.

A solution of (23) for the case where the contractor's problem is approximated by its 'certainty-equivalent' is now presented. It is assumed that the owner's opportunity loss function is given by (20). Under these assumptions:

$$(24) \quad B(p, \tau) = (1 + m)V(p, \tau),$$

$$C(p, \tau) = LT(p, \tau),$$

$$D(p, \tau) = \begin{cases} p(T(p, \tau) - \tau) & \text{if } T(p, \tau) > \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the analysis for the contractor's problem has shown that $D(p, \tau) = 0$, if $\tau \geq T(p, 0)$

The optimal solution to the owner's problem will now be derived by considering the three possible regions of choice for τ for given p and $T(p, 0) < T^*$ (Figure 1).

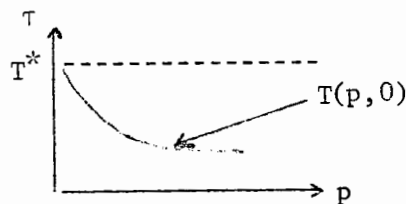


Figure 1
Possible choices for p and τ

Case (b): $T(p,0) < \tau < T^*$: From (17), (23), and (24):

$$(26) \quad A(p,\tau) = (1+m)\tau c(f^{-1}(\frac{x^0}{\tau})) + L\tau .$$

In this case, p has been set high enough to force the contractor to finish at $T(p,0)$ if $\tau = T(p,0)$. However, as shown earlier, the contractor will choose to finish at time τ instead. Substituting for τ in (26):

$$\begin{aligned} A(p,\tau) &= \left\{ \frac{L + (1+m) c(f^{-1}(\frac{x^0}{\tau}))}{f(f^{-1}(x^0/\tau))} \right\} x^0 \\ &\geq \left\{ \frac{L + (1+m) c(a_{p^*}^*)}{f(a_{p^*}^*)} \right\} x^0 \end{aligned}$$

where $p^* = \frac{L}{1+m}$ and $a_{p^*}^*$ minimizes $\frac{L/(1+m) + c(a)}{f(a)}$, $a \in A$.

Case (c): $T^* < \tau$: In this case the contractor's optimal policy will not be affected by the penalty. From (2'), (11), (18), (23) and (24):

$$\begin{aligned} (27) \quad A(p,\tau) &= (1+m) c^* x^0 + LT^* \\ &= \left\{ \frac{L + (1+m) c(a^*)}{f(a^*)} \right\} x^0 \\ &\geq \left\{ \frac{L + (1+m) c(a_{p^*}^*)}{f(a_{p^*}^*)} \right\} x^0 \end{aligned}$$

where p^* and $a_{p^*}^*$ are defined as above.

From (27) it follows that if $L > 0$ the owner should always include a penalty clause in the contract. The results of this section are summarized in the following theorem.

Theorem 2

For the owner's problem involving the certainty equivalent version of the contract activity model (1) with opportunity loss function (20), the optimal penalty, p^* , and target, τ^* are given by:

- (c) Set $\tau > T^*$ -- in which case his opportunity cost is given by $(1 + m) \{T^* c(a^*) - \tau^* c(a_{p^*}^*)\} + L(T^* - \tau^*)$ (see (27) and (28)).

5. Conclusion

The choice of optimal target dates and penalties for inclusion in contract documents is an important and practical problem which seems to have received little attention by economists and operations researchers. This paper has provided a formal statement of the problem in which it is assumed that the owner can make prior estimates of the technology (cost and production functions) and profit mark-up of the contractor who will be awarded the contract. Using this approach it should be possible for an owner to estimate the optimal values for the target date and penalty in any given situation. In the paper the particular case where the technology of the contractor is deterministic and invariant with respect to time was analysed and very simple and intuitive results were obtained. In particular it was found that the optimal penalty was independent of the technology of the contractor and dependent only on the owner's loss function and on the profit-mark-up of the winning contractor.

Much work remains to be done in this area. The assumption that the cost and production functions of the contractor do not vary over time will be reasonable in many cases where the contract involves routine or repetitive work on which the contractor has had prior experience (for example, pipe-laying or road-paving contracts might satisfy this description). However, it will also be important to investigate cases of 'learning' where the production performance improves over time. Other areas for investigation include the effects of different assumptions with respect to the owner's loss function

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