Zero-sum Dynamic Games and a Stochastic Variation of Ramsey Theorem

Eran Shmaya* and Eilon Solan†‡

July 11, 2003

Abstract

We show how a stochastic variation of Ramsey theorem can be used to prove the existence of the value, and to construct ε-optimal strategies, in two-player zero-sum dynamic games that have certain properties.

Keywords: Dynamic games, Ramsey theorem, value, optimal strategies.

*The School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. e-mail: gawain@post.tau.ac.il

†Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208-2001 *and* the School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. e-mail: eilons@post.tau.ac.il, e-solan@kellogg.northwestern.edu

‡The research of the second author was supported by the Israel Science Foundation (grant No. 69/01-1).

1 Introduction

Competitive interactions between two players are quite common, and it is desirable to know whether such a competition has a value. That is, whether there is some quantity such that for every $\varepsilon > 0$ the maximizing player can guarantee to receive, on average, at least this quantity (up to ε), and the minimizing player can guarantee to pay, on average, no more than this quantity (up to ε). Once the value exists, finding ε-optimal strategies for the two players (which guarantee that they receive at least the value, or pay no more than the value, up to ε) is also desirable.

When the interaction lasts for a single stage, or for a bounded number of stages, existence of the value is usually proven using a fixed-point argument, and hinges on the continuity of the payoff in the strategies of the players.

When the duration of the interaction is long but not known in advance, it is convenient to assume that the interaction lasts for countably many stages (see Aumann and Maschler (1995, p.143) and Neyman and Sorin (2002) for justifications). However, in this formulation, the payoff is often not continuous in the strategies of the players, and therefore one cannot use standard fixed-point theorems to prove the existence of the value. Various techniques where employed in the literature to handle this problem (see, e.g., Mertens and Neyman, 1981, Maitra and Sudderth, 1993, 1998, and Nowak, 1985), but it seems that each technique can be applied only in some models, under special conditions, or is not constructive.

Here we present another tool for proving existence of the value in infinite-stage competitive interactions, or two-player zero-sum dynamic games. We show how a stochastic variation of Ramsey theorem[1] can be used to reduce the analysis of the infinite-stage interaction to the analysis of finite-stage interactions.

To exhibit the new technique, we apply it to the following generalization of stopping games (see Dynkin, 1969). At the outset of the game, the state of the world is chosen according to some known probability distribution, but is not told to the players. At every stage of the game, the players learn some information concerning the state of the world; both receive the same information. Then each player chooses an action. The pair of actions, together with the state of the world, determine a probability of termination,

[1]Ramsey theorem (Ramsey, 1930) states that for every coloring of the complete infinite graph by finitely many colors there is a complete infinite monochromatic sub-graph.

and a terminal payoff if the game terminates at that stage. If the game never terminates, the payoff to both players is 0.

The goal of the maximizing player is to maximize the expected payoff, and the goal of the minimizing player is to minimize this quantity.

The paper is arranged as follows. In section 2 we formally present the model and the main result, stating that in our model the value exists. In section 3 we state the stochastic variation of Ramsey theorem that we use, and we apply it to our model to exhibit the new technique. Further discussion is relegated to section 4.

2 The Model and the Main Result

We consider infinite-stage dynamic games in discrete time that are given by

- A probability space $(\Omega, \mathcal{F}, \mathbf{P})$, that captures the uncertainty about the state of the world. We denote by $\mathbf{E}$ the expectation w.r.t. $\mathbf{P}$.
- A filtration $(\mathcal{F}_n)_{n \in \mathbf{N}}$, that describes the information available to both players at stage n.
- Two measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ of *actions* for the two players.
- For each $n \in \mathbf{N}$, $\mathcal{F}_n \otimes \mathcal{A} \otimes \mathcal{B}$-measurable functions $p_n : \Omega \times A \times B \to [0,1]$ and $g_n : \Omega \times A \times B \to [-1,1]$. p_n indicates the probability of termination, while g_n indicates the terminal payoff.

The game is played as follows. At the outset of the game, a state of the world $\omega \in \Omega$ is chosen according to the probability measure $\mathbf{P}$. At every stage n, the players choose independently and simultaneously actions $a_n \in A$ and $b_n \in B$. Players' choices must be measurable with regard to their information, namely, $\mathcal{F}_n$ and previously played actions. With probability $p_n(a_n, b_n)$ the game terminates, and the terminal payoff is $g_n(a_n, b_n)$. With probability $1 - p_n(\omega)$ the game continues to stage $n+1$.

Our model is a generalization of stopping games (see, e.g., Dynkin, 1969, Rosenberg et al., 2001, or Touzi and Vieille, 2002). It is also closely related to general stochastic games (see, e.g., Nowak, 1985, or Maitra and Sudderth, 1998).

For every measurable space M, denote by $\mathcal{P}(M)$ the space of probability distributions over M.

The space of infinite plays is $(A\times B)^{\mathbf{N}}\times\Omega$. We equip it with the product σ-algebra $(\mathcal{A}\otimes\mathcal{B})^{\mathbf{N}}\otimes\mathcal{F}$. We denote by $\mathcal{G}_n=(\mathcal{A}\times\mathcal{B})^{n-1}\otimes\mathcal{F}_n$ the σ-algebra that represents the information available for the players at stage n. It is convenient to consider $\mathcal{F}_n$ as sub-σ-algebra of $\mathcal{G}_n$.

A *strategy* $\sigma=(\sigma_n)_{n\in\mathbf{N}}$ of player 1 is a collection of functions such that $\sigma_n:(A\times B)^{\mathbf{N}}\times\Omega\to\mathcal{P}(A)$ is $\mathcal{G}_n$-measurable, for every $n\in\mathbf{N}$. Strategies τ of player 2 are defined analogously.

Every pair (σ,τ) of strategies, together with $\mathbf{P}$, naturally defines a probability distribution over $(A\times B)^{\mathbf{N}}\times\Omega$. The corresponding expectation operator is denoted by $\mathbf{E}_{\sigma,\tau}$.

Denote by θ the stage of termination, so that $\theta=+\infty$ if termination never occurs. For every pair (σ,τ) of strategies, the expected payoff is

$$\gamma(\sigma,\tau)=\mathbf{E}_{\sigma,\tau}[\mathbf{1}_{\{\theta<+\infty\}}g_\theta(a_\theta,b_\theta)],$$

where $\mathbf{1}$ is the indicator function.

The goal of player 1 is to maximize the expected payoff, while the goal of player 2 is to minimize this quantity.

Definition 1 *If the equality*

$$\sup_\sigma\inf_\tau\gamma(\sigma,\tau)=\inf_\tau\sup_\sigma\gamma(\sigma,\tau) \tag{1}$$

holds, then the common value is the value *of the game. Given $\varepsilon\geq 0$, every strategy σ of player 1 that attains the supremum on the left-hand side of (1) up to ε is* ε-optimal *for player 1. Every strategy τ of player 2 that attains the infimum on the right-hand side of (1) up to ε is* ε-optimal *for player 2.*

Our main result is the following

Theorem 1 *If A,B are compact metric spaces, and the functions $g_n(\omega,\cdot,\cdot)$ and $p_n(\omega,\cdot,\cdot)$ are continuous for every $\omega\in\Omega$ and every $n\in\mathbf{N}$, the game has a value.*

Remark 1 *We have chosen to study a simple model, where the action sets are compact and independent of the stage or the information available for the players. The latter assumption is without loss of generality, as one can always add "fictitious" actions for player 1, that yield extremely low payoff, and "fictitious" actions for player 2, that yield extremely high payoff. The* *former assumption can be relaxed; all that we need is to be able to apply a Minimax theorem. Thus, our technique extends to the case where A and B are Borel spaces, and one of them is compact, as well as to other setups, see section 4.*

Our main contribution is not in the technical result, but in the new technique that we use for the proof.

We now briefly compare our result to existing literature. Under some regularity assumptions on $(\Omega, \mathcal{F}, \mathbf{P})$, the model we consider is a class of general stochastic games. Maitra and Sudderth (1993, 1998) proved the existence of the value in a most general setup of stochastic games.

Maitra and Sudderth (1993), using the operator approach and transfinite induction, proved that every stochastic game admits a value, and both players have universally measurable ε-optimal strategies, for every $\varepsilon > 0$. Relative to this result, our contribution is that we give a constructive argument for the existence of ε-optimal strategies, which are also uniformly ε-optimal in a sense defined below (see section 4).

Maitra and Sudderth (1998), using the fact that every Borel game is solvable, proved that every stochastic game admits a value, and both players have finitely-additive ε-optimal strategies, for every $\varepsilon > 0$ (see also Martin, 1998). Thus, relative to this result, our contribution is that both players have σ-additive uniformly ε-optimal strategies for every $\varepsilon > 0$, rather than finitely additive.

Rosenberg et al. (2001), using the technique of vanishing discount factors, proved that the value exists when the sets A and B are finite.

3 The Proof

3.1 A stochastic variation of Ramsey Theorem

Ramsey theorem (Ramsey, 1930) can be stated as follows. For every function c that attaches to every two non-negative integers $k < l$ an element $c(k, l) \in C$, where C is a finite set, there is an increasing sequence of integers $k_1 < k_2 < \ldots$ such that $c(k_1, k_2) = c(k_i, k_j)$ for every $i < j$.

We are going to attach for every non-negative integer n and every stopping time τ a $\mathcal{F}_n$-measurable function $c_{n,\tau}$, whose range is some finite set C. We also impose a consistency requirement: if $\tau_1 = \tau_2$ on a $\mathcal{F}_n$-measurable set F, then $c_{n,\tau_1} = c_{n,\tau_2}$ on F. Under these conditions, a weaker conclusion than that of Ramsey theorem can be derived: for every $\varepsilon > 0$ there exists an increasing sequence of stopping times $\nu_1 < \nu_2 < \dots$ such that $\mathbf{P}(c_{\nu_1,\nu_2} = c_{\nu_2,\nu_3} = c_{\nu_3,\nu_4} = \cdots) > 1 - \varepsilon$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and $(\mathcal{F}_n)$ a filtration. A stopping time ν is *adapted* (to the filtration $(\mathcal{F}_n)_{n\in\mathbf{N}}$) if for every $n \in \mathbf{N}$, the set $\{\nu = n\}$ is $\mathcal{F}_n$-measurable. In the sequel, all stopping times are adapted. For every $A, B \in \mathcal{F}$, *A holds on B* if and only if $\mathbf{P}(A^c \cap B) = 0$.

Definition 2 *A* NT-function *is a function that assigns to every integer $n \geq 0$ and every bounded stopping time ν a $\mathcal{F}_n$-measurable r.v. that is defined over the set $\{\nu > n\}$. We say that a NT-function f is C-*valued*, for some set C, if the r.v. $f_{n,\nu}$ is C-valued, for every $n \geq 0$ and every ν.*

Definition 3 *A NT-function f is* consistent *if for every $n \geq 0$, every $\mathcal{F}_n$-measurable set F, and every two bounded stopping times ν_1, ν_2, we have*

$$\nu_1 = \nu_2 > n \text{ on } F \text{ implies } f_{n,\nu_1} = f_{n,\nu_2} \text{ on } F.$$

When f is a NT-function, and $\nu_1 < \nu_2$ are two bounded stopping times, we denote $f_{\nu_1,\nu_2}(\omega) = f_{\nu_1(\omega),\nu_2}(\omega)$. Thus f_{ν_1,ν_2} is a $\mathcal{F}_{\nu_1}$-measurable r.v.

The following stochastic variation of Ramsey Theorem was proved by Shmaya and Solan (2002, Theorem 4.3)

Theorem 2 *For every finite set C, every C-valued consistent NT-function f, and every $\epsilon > 0$, there exists a sequence of bounded stopping times $1 \leq \nu_1 < \nu_2 < \nu_3 < \dots$ such that*

$$\mathbf{P}(f_{\nu_1,\nu_2} = f_{\nu_2,\nu_3} = f_{\nu_3,\nu_4} = \dots) > 1 - \epsilon.$$

3.2 Application to games

For every two bounded stopping times $\nu_1 < \nu_2$, and every $\mathcal{F}_{\nu_2}$-measurable function h, Let $\Gamma(\nu_1, \nu_2, h)$ be the two-player zero-sum game that starts at stage ν_1 and, if not terminated earlier, terminates at stage ν_2 with terminal payoff h. We do not introduce a new concept of a strategy in $\Gamma(\nu_1, \nu_2, h)$. Rather, we take the strategy space in $\Gamma(\nu_1, \nu_2, h)$ to coincide with that of Γ, and we will condition on the event $\{\theta \geq \nu_1\}$.

The following standard lemma states that $\Gamma(\nu_1, \nu_2, h)$ admits a value (see, e.g., Nowak, 1985, Theorem 5.2). It follows using backward induction from

Sion's (1958) Minimax theorem and a measurable selection theorem (e.g., Kuratowski and Ryll-Nardzewski, 1965).

Lemma 1 *Let $\nu_1 < \nu_2$ be bounded stopping times, and h a $\mathcal{F}_{\nu_2}$-measurable function such that $\|h\|_\infty \leq 1$. Under the assumptions of Theorem 1, there exists a $\mathcal{F}_{\nu_1}$-measurable function $v(\Gamma(\nu_1, \nu_2, h))$, and a pair (σ^*, τ^*) of strategies such that for every pair (σ', τ') of strategies,*

$$\begin{aligned}&\mathbf{E}_{\sigma^*,\tau'}[\mathbf{1}_{\{\nu_1 \leq \theta < \nu_2\}} g_\theta(a_\theta, b_\theta) + \mathbf{1}_{\{\theta_2 \leq \theta\}} h \mid \mathcal{G}_{\nu_1}] \geq v(\Gamma(\nu_1, \nu_2, h))\mathbf{1}_{\{\nu_1 \leq \theta\}}, \textit{and} \\ &\mathbf{E}_{\sigma',\tau^*}[\mathbf{1}_{\{\nu_1 \leq \theta < \nu_2\}} g_\theta(a_\theta, b_\theta) + \mathbf{1}_{\{\theta_2 \leq \theta\}} h \mid \mathcal{G}_{\nu_1}] \leq v(\Gamma(\nu_1, \nu_2, h))\mathbf{1}_{\{\nu_1 \leq \theta\}}.\end{aligned} \tag{2}$$

Actually, there are optimal strategies $\sigma^* = (\sigma^*_n)$ and $\tau^* = (\tau^*_n)$ such that σ^*_n and τ^*_n are $\mathcal{F}_n$-measurable, rather than $\mathcal{G}_n$-measurable (that is, the actions chosen at each stage n do not depend on previously chosen actions). Moreover, one can verify that (2) still holds if we replace in Lemma 1 σ^* by any strategy σ such that, for every $n \in \mathbf{N}$, $\sigma_n = \sigma^*_n$ on $\{\nu_1 \leq n < \nu_2\}$.

The following Lemma summarizes simple monotonicity and continuity properties of the value operator.

Lemma 2 *(a) If $\|h\|_\infty < 1$ then $\|v(\Gamma(\nu_1, \nu_2, h))\|_\infty \leq 1$*

(b) If $\nu_1 < \nu_2 < \nu_3$, then $v(\Gamma(\nu_1, \nu_2, v(\Gamma(\nu_2, \nu_3, h)))) = v(\Gamma(\nu_1, \nu_3, h))$.

(c) If $F \in \mathcal{G}_{\nu_1}$ and $h \leq h'$ on F, then $v(\nu_1, \nu_2, h) \leq v(\nu_1, \nu_2, h')$ on F.

(d) If $F \in \mathcal{G}_{\nu_1}$ and $lim_{n\to\infty} h_n = h$ on F, then $\lim_{n\to\infty} v(\nu_1, \nu_2, h_n) = v(\nu_1, \nu_2, h)$ on F.

Set $C = \{\text{`+'}, \text{`}-\text{'}\}$, and define a C-valued NT-function c as follows. For every $n \in \mathbf{N}$ and every stopping time ν,

$$c(n, \nu) = \begin{cases} \text{`+'} & \text{if } v(\Gamma(n, \nu, 0)) > 0, \\ \text{`+'} & \text{if } v(\Gamma(n, \nu, 0)) \leq 0. \end{cases}$$

Lemma 2(c) implies that c is a consistent NT-function.

Fix, once and for all $\varepsilon > 0$. By Theorem 2 there exists an increasing sequence (ν_k) of adapted stopping times such that

$$\mathbf{P}(c(\nu_1, \nu_2) = c(\nu_2, \nu_3) = c(\nu_3, \nu_4) = \cdots) > 1 - \varepsilon. \tag{3}$$

3.3 An auxiliary game

For every $k \in \mathbf{N}$ define

$$\begin{aligned} E_k^+ &= \{c(\nu_1,\nu_2) = \text{`+'} \text{ and } c(\nu_k,\nu_{k+1}) = \text{`}-\text{'}\}, \text{ and} \\ E_k^- &= \{c(\nu_1,\nu_2) = \text{`}-\text{'} \text{ and } c(\nu_k,\nu_{k+1}) = \text{`+'}\}. \end{aligned}$$

By (3), $\mathbf{P}(\cup_{k\in\mathbf{N}}(E_k^+ \cup E_k^-)) < \varepsilon$. Moreover, E_k^+ and E_k^- are in $\mathcal{F}_{\nu_k}$.

We now define an auxiliary game Γ'_ε. Γ'_ε is similar to Γ, except that it has a different payoff function $(g'_n)_{n\in\mathbf{N}}$, that is defined as follows.

$$g'_n(\omega) = \begin{cases} 0 & \omega \in E_k^+ \cup E_k^- \text{ and } n \geq k, \\ g_n(\omega) & \text{otherwise.} \end{cases}$$

Thus, whenever $c(\nu_k,\nu_{k+1}) \neq c(\nu_1,\nu_2)$, we set the payoff to be 0 from stage ν_k and onwards.

Denote by $\gamma'(\sigma,\tau)$ the expected payoff under the pair of strategies (σ,τ) in Γ'_ε. Since $\mathbf{P}(g_n \neq g'_n) \leq \mathbf{P}(\cup_{k\in\mathbf{N}}(E_k^+ \cup E_k^-)) < \varepsilon$, and since payoffs are bounded by 1, for every pair of strategies (σ,τ) one has $|\gamma(\sigma,\tau)-\gamma'(\sigma,\tau)| < \varepsilon$.

3.4 Sufficiency of the analysis of the auxiliary game

The following lemma asserts that if for every $\varepsilon > 0$ there are 3ε-optimal strategies in Γ'_ε, then the original game admits a value.

Lemma 3 *If for every $\varepsilon > 0$ there exist $V_\varepsilon \in [-1,1]$ and a pair $(\sigma_\varepsilon,\tau_\varepsilon)$ of strategies that satisfy* $\inf_{\tau'} \gamma'(\sigma_\varepsilon,\tau') \geq V_\varepsilon - 3\varepsilon$ *and* $\sup_{\sigma'} \gamma'(\sigma',\tau_\varepsilon) \leq V_\varepsilon + 3\varepsilon$, *then $V := \lim_{\varepsilon\to 0} V_\varepsilon$ exists, and is the value of* Γ.

Observe that we do not require that V_ε is the value of Γ'_ε, or, for that matter, that the games $(\Gamma'_\varepsilon)_\varepsilon$ have values.

Proof. Let V be any accumulation point of the sequence $(V_\varepsilon)_{\varepsilon>0}$ as ε goes to 0. Since $|\gamma(\sigma,\tau)-\gamma'(\sigma,\tau)| < \varepsilon$, the assumptions imply that $\inf_{\tau'} \gamma(\sigma_\varepsilon,\tau') \geq V_\varepsilon - 4\varepsilon$ and $\sup_{\sigma'} \gamma(\sigma',\tau_\varepsilon) \leq V_\varepsilon + 4\varepsilon$.

Therefore, for every δ there is $\varepsilon > 0$ sufficiently small such that $\inf_{\tau'} \gamma(\sigma_\varepsilon,\tau') \geq V_\varepsilon - 4\varepsilon \geq V - \delta$ and $\sup_{\sigma'} \gamma(\sigma',\tau_\varepsilon) \leq V_\varepsilon + 4\varepsilon \leq V + \delta$. In particular, V is the value of Γ. ■

Thus, our goal is to find $V_\varepsilon \in \mathbf{R}$ and to construct a pair (σ,τ) of strategies such that $\inf_{\tau'} \gamma'(\sigma,\tau') \geq V_\varepsilon - 3\varepsilon$ and $\sup_{\sigma'} \gamma'(\sigma',\tau) \leq V_\varepsilon + 3\varepsilon$.

In section 3.5 we define V_ε. In section 3.6 we define σ, and in section 3.7 we prove that $\inf_{\tau'} \gamma'(\sigma,\tau') \geq V_\varepsilon - 3\varepsilon$. The construction of τ, and the proof that $\sup_{\sigma'} \gamma'(\sigma',\tau) \leq V_\varepsilon + 3\varepsilon$, is analogous to that of σ, hence omitted.

3.5 Properties of the coloring

By construction of Γ'_ε, If $v(\Gamma'_\varepsilon(\nu_1,\nu_2,0)) > 0$ then $v(\Gamma'_\varepsilon(\nu_k,\nu_{k+1},0)) \geq 0$ for every $k \in \mathbf{N}$, whereas if $v(\Gamma'_\varepsilon(\nu_1,\nu_2,0)) \leq 0$ then $v(\Gamma'_\varepsilon(\nu_k,\nu_{k+1},0)) \leq 0$ for every $k \in \mathbf{N}$.

Let $D_+ = \{v(\Gamma'_\varepsilon(\nu_1,\nu_2,0)) > 0\} \in \mathcal{F}_{\nu_1}$, and $D_- = \{v(\Gamma'_\varepsilon(\nu_1,\nu_2,0)) \leq 0\} \in \mathcal{F}_{\nu_1}$. Plainly, (D_+, D_-) is a partition of Ω.

On D_+, $v(\Gamma'_\varepsilon(\nu_k,\nu_{k+1},0)) \geq 0$ for every $k \in \mathbf{N}$. By Lemma 2(b,c),

$$v(\Gamma'_\varepsilon(\nu_k,\nu_{l+1},0)) = v(\Gamma'_\varepsilon(\nu_k,\nu_l,v(\Gamma'_\varepsilon(\nu_l,\nu_{l+1},0)))) \geq v(\Gamma'_\varepsilon(\nu_k,\nu_l,0)) \quad \text{on } D_+.$$

Similarly,

$$v(\Gamma'_\varepsilon(\nu_k,\nu_{l+1})) \leq v(\Gamma'_\varepsilon(\nu_k,\nu_l,0)) \quad \text{on } D_-.$$

In particular, for every fixed $k \in \mathbf{N}$, the sequence $(v(\Gamma'_\varepsilon(\nu_k,\nu_l,0))_{l>k}$ is a sequence of $\mathcal{F}_{\nu_k}$-measurable functions, monotonic non-decreasing on D_+, and monotonic non-increasing on D_-. In particular, this sequence has a limit h^*_k, which is $\mathcal{F}_{\nu_k}$-measurable.

Applying Lemma 2(b,d), we get

$$\begin{aligned} h^*_k &= \lim_{l\to\infty} v(\Gamma'_\varepsilon(\nu_k,\nu_l,0)) \\ &= \lim_{l\to\infty} v(\Gamma'_\varepsilon(\nu_k,\nu_{k+1},v(\Gamma'_\varepsilon(\nu_{k+1},\nu_l,0))) \\ &= v(\Gamma'_\varepsilon(\nu_k,\nu_{k+1},h^*_{k+1})). \end{aligned} \tag{4}$$

Set $V_\varepsilon = \mathbf{E}[v(\Gamma'_\varepsilon(1,\nu_1,h^*_1))]$.

3.6 Definition of a strategy σ

Choose $l \in \mathbf{N}$ sufficiently large such that

$$\mathbf{P}(D_+ \cap \{v(\Gamma'_\varepsilon(\nu_1,\nu_l,0)) \geq h^*_1 - \varepsilon\}) > \mathbf{P}(D_+) - \varepsilon. \tag{5}$$

For every $k \in \mathbf{N}$ choose for player 1 an optimal strategy σ_k in the game $\Gamma'_\varepsilon(\nu_k,\nu_{k+1},0)$, and an optimal strategy σ^*_k in the game $\Gamma'_\varepsilon(\nu_k,\nu_{k+1},h^*_{k+1})$.

Choose for player 1 an optimal strategy $\sigma_{1,l}$ in the game $\Gamma'_\varepsilon(\nu_1,\nu_l,0)$. Finally, choose for player 1 an optimal strategy σ_0 in the game $\Gamma'_\varepsilon(1,\nu_1,h^*_1)$.

Recall that D_+ and D_- are $\mathcal{F}_{\nu_1}$-measurable. Define a strategy σ for player 1 as follows.

- σ follows σ_0 up to stage ν_1.
- If $\omega \in D_-$, σ follows σ_k^* between stages ν_k and ν_{k+1}, for every $k \in \mathbf{N}$.
- If $\omega \in D_+$, σ follows $\sigma_{1,l}$ between stages ν_1 and ν_l. Then, for every $k \geq l$, it follows σ_k between stages ν_k and ν_{k+1}.

3.7 The strategy σ is 3ε-optimal

Let τ' be an arbitrary strategy of player 2. We prove that $\gamma'(\sigma, \tau') \geq V_\varepsilon - 3\varepsilon$. For convenience, set $r_\theta = g'_\theta(a_\theta, b_\theta)$ if $\theta < +\infty$ and $r_\theta = 0$ if $\theta = +\infty$. This is the terminal payoff in the game.

Since $\sigma_{1,l}$ is optimal in $\Gamma'_\varepsilon(\nu_1, \nu_l, 0)$,

$$\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{\{\nu_1 \leq \theta < \nu_l\}} r_\theta \mid \mathcal{G}_{\nu_1}] \geq v(\Gamma'_\varepsilon(\nu_1, \nu_l, 0))\mathbf{1}_{\{\nu_1 \leq \theta\}} \quad \text{on } D_+. \tag{6}$$

Since for every $k \in \mathbf{N}$, σ_k is optimal in $\Gamma'_\varepsilon(\nu_k, \nu_{k+1}, 0)$,

$$\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{\{\nu_k \leq \theta < \nu_{k+1}\}} r_\theta \mid \mathcal{G}_{\nu_k}] \geq v(\Gamma'_\varepsilon(\nu_k, \nu_{k+1}, 0))\mathbf{1}_{\{\nu_k \leq \theta\}} \geq 0 \quad \text{on } D_+.$$

Taking conditional expectation w.r.t. $\mathcal{G}_{\nu_1}$, and summing over $k \geq l$, gives us

$$\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{\{\nu_l \leq \theta\}} r_\theta \mid \mathcal{G}_{\nu_1}] \geq 0 \text{ on } D_+ \tag{7}$$

From (6) and (7) we have

$$\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{\{\nu_1 \leq \theta\}} r_\theta \mid \mathcal{G}_{\nu_1}] \geq v(\Gamma'_\varepsilon(\nu_1, \nu_l, 0))\mathbf{1}_{\{\nu_1 \leq \theta\}} \text{ on } D_+.$$

By taking expectation and using (5), we obtain

$$\begin{aligned} \mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{D_+ \cap \{\nu_1 \leq \theta\}} r_\theta] &\geq \mathbf{E}[\mathbf{1}_{D_+ \cap \{\nu_1 \leq \theta\}} v(\Gamma'_\varepsilon(\nu_1, \nu_l, 0))] \\ &\geq \mathbf{E}[\mathbf{1}_{D_+ \cap \{\nu_1 \leq \theta\}} h_1^*] - 3\epsilon. \end{aligned} \tag{8}$$

Since for every $k \in \mathbf{N}$ σ_k^* is optimal in $\Gamma'_\varepsilon(\nu_k, \nu_{k+1}, h_{k+1}^*)$, and by the recursive relation (4),

$$\begin{aligned} &\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{\{\nu_k \leq \theta < \nu_{k+1}\}} r_\theta + \mathbf{1}_{\{\theta \geq \nu_{k+1}\}} h_{k+1}^* \mid \mathcal{G}_{\nu_k}] \geq \\ &\qquad \mathbf{1}_{\{\nu_k \leq \theta\}} v(\Gamma'_\varepsilon(\nu_k, \nu_{k+1}, h_{k+1}^*)) = \mathbf{1}_{\{\nu_k \leq \theta\}} h_k^* \quad \text{on } D_-. \end{aligned} \tag{9}$$

By (9) and an iterative use of Lemma 2(c), for every $m > 1$ one has

$$\mathbf{E}[\mathbf{1}_{\{\nu_1 \leq \theta < \nu_m\}} r_\theta + \mathbf{1}_{\{\nu_m \leq \theta\}} h_m^* \mid \mathcal{G}_{\nu_1}] \geq \mathbf{1}_{\{\nu_1 \leq \theta\}} h_1^* \text{ on } D_-.$$

Since on D_- $h^*_m \leq 0$ for every m, it follows by taking expectation that,

$$\begin{aligned}\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{D_-\cap\{\nu_1\leq\theta<\nu_m\}}r_\theta] &\geq \mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{D_-\cap\{\nu_1\leq\theta<\nu_m\}}r_\theta + \mathbf{1}_{D_-\cap\{\theta\geq\nu_m\}}h^*_m] \\ &\geq \mathbf{E}[\mathbf{1}_{D_-\cap\{\nu_1\leq\theta\}}h^*_1].\end{aligned}$$

By the bounded convergence theorem, we deduce that

$$\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{D_-\cap\{\nu_1\leq\theta\}}r_\theta] \geq \mathbf{E}[\mathbf{1}_{D_-\cap\{\nu_1\leq\theta\}}h^*_1]. \tag{10}$$

By (8) and (10),

$$\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{\{\nu_1\leq\theta\}}r_\theta] \geq \mathbf{E}[\mathbf{1}_{\{\nu_1\leq\theta\}}h^*_1] - 3\epsilon. \tag{11}$$

Since σ_0 is optimal in the game $\Gamma'_\varepsilon(1,\nu_1,h^*_1)$,

$$\mathbf{E}_{\sigma,\tau'}[\mathbf{1}_{\{\theta<\nu_1\}}r_\theta + \mathbf{1}_{\{\nu_1\leq\theta\}}h^*_1] \geq \mathbf{E}[v(\Gamma'_\varepsilon(1,\nu_1,h^*_1))] = V_\varepsilon. \tag{12}$$

By (11) and (12),

$$\gamma'(\sigma,\tau') = \mathbf{E}_{\sigma,\tau'}[r_\theta] \geq V_\varepsilon - 3\varepsilon. \tag{13}$$

4 Further Discussion

Here we discuss the assumptions that our proof hinges on, as well as further topics.

Our argument hinges on the assumption that the evolution of the game is independent of the actions chosen by the players. That is, the players cannot affect the information that they receive along the game. All they control is the probability of termination and the terminal payoff. It is most desirable to extend our technique to the case that players do affect their information.

Another aspect that we critically need is the fact that the information is symmetric: both players have the same information at every stage. If this is not the case, then the value need not exist (see Laraki, 2000, for an example.) It is interesting to see under which informational structure the value still exists.

The strategy σ that we constructed in section 3.6 is uniform in the following sense. There is $n \geq 0$ such that

$$\inf_\tau \mathbf{E}_{\sigma,\tau}[\mathbf{1}_{\{\theta\leq n\}}g_\theta(a_\theta,b_\theta)] \geq V_\varepsilon - 5\varepsilon.$$

That is, the strategy σ is $5\varepsilon+|V-V_\varepsilon|$-optimal in every finite-stage interaction, provided the interaction is sufficiently long.

The proof relies on the observation that if for some bounded stopping time ν, where $\nu \geq \nu_k$ for k sufficiently large, the expected payoff under (σ, τ) up to stage ν is significantly different from V_ε, then the probability of termination between stages ν_k and ν must be bounded away from 0. Therefore, such an event can occur only finitely many times. Details are standard and omitted.

We assumed that the functions $p_n(\omega, \cdot, \cdot)$ and $g_n(\omega, \cdot, \cdot)$ are continuous for every $\omega \in \Omega$. However, all that we need is that for every $\mathcal{F}_{n+1}$-measurable function f, the one-stage game $\Gamma(n, n+1, f)$ with terminal payoff f admits a value. More formally, we now present a more general version of the one-shot game.

Definition 4 *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, X and Y two measurable sets of strategies, and $u\colon \Omega \times X \times Y \to [-1, 1]$ be measurable payoff function. The game $(\Omega, \mathcal{F}, \mathbf{P}, X, Y, u)$ admits a* value *if there exist a $\mathcal{F}$-measurable function $v\colon \Omega \to [-1, 1]$, and, for every $\epsilon > 0$, there exist $\mathcal{F}$-measurable functions $x\colon \Omega \to X$ and $y\colon \Omega \to Y$, such that*

$$\sup_{x' \in X} u(\omega, x', y(\omega)) - \epsilon \leq v(\omega) \leq \inf_{y' \in Y} u(\omega, x(\omega), y') + \epsilon, \quad \mathbf{P} - a.e.$$

The proof of the following extension of Theorem 1 follows the same lines as the proof we presented.

Theorem 3 *Let $\Gamma = (\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_n), A, B, (g_n, p_n))$ be an infinite-stage dynamic game. Assume that for every $n \in \mathbf{N}$ and every $\mathcal{F}_n$-measurable function $h\colon \Omega \to [-1, 1]$ the one-shot game $(\Omega, \mathcal{F}_n, \mathbf{P}, \mathcal{P}(A), \mathcal{P}(B), u)$ admits a value, where $u(\omega, x, y) = \int_A \int_B p_n(a, b) g_n(a, b) + (1 - p_n(a, b)) h \; dx(a) dy(b)$. Then the game Γ admits a value.*

The stochastic variation of Ramsey theorem that we use can be applied to prove existence of an equilibrium in two-player non-zero-sum stopping games in discrete time (see Shmaya and Solan, 2002). However, whereas for non-zero-sum finite-stage games a lot of structure is needed to ensure existence of an equilibrium that satisfies certain desirable properties, for zero-sum games the technique works in a much more general setup.

References

[1] Aumann R.J. and Maschler M.B. (1995) Repeated Games with Incomplete Information, The MIT Press

[2] Dynkin E.B. (1969) Game Variant of a Problem on Optimal Stopping, *Soviet Math. Dokl.*, **10**, 270-274

[3] Kuratowski K. and Ryll-Nardzewski C. (1965) A General Theorem on Selectors, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, **13**, 397-403

[4] Laraki R. (2000) Jeux répétés à information incomplète: approche variationnelle, *Thèse de Doctorat de l'Univérsité Paris 6, France.*

[5] Maitra A. and Sudderth W. (1993) Borel Stochastic Games with Limsup Payoff, *Ann. Probab.*, **21**, 861-885

[6] Maitra A. and Sudderth W. (1998) Finitely Additive Stochastic Games with Borel Measurable Payoffs, *Int. J. Game Theory*, **27**, 257-267

[7] Martin D.A. (1998) The Determinacy of Blackwell Games. *J. Symbolic Logic*, **63**, 1565-1581

[8] Mertens J.F. and Neyman A. (1981) Stochastic Games, *Int. J. Game Th.*, **10**, 53-66

[9] Neyman A. and Sorin S. (2001) Zero-Sum Two-Person Repeated Games with Public Uncertain Duration Process, Discussion paper 259, Center for Rationality, The Hebrew Univeristy of Jerusalem

[10] Nowak A.S. (1985) Universally Measurable Strategies in Zero-Sum Stochastic Games, *Ann. Probab.*, **13**, 269-287

[11] Ramsey F.P. (1930) On a Problem of Formal Logic, *Proc. London Math. Society*, **30**, 264-286

[12] Rosenberg D., Solan E. and Vieille N. (2001) Stopping Games with Randomized Strategies, *Probab. Th. Related Fields*, **119**, 433-451

[13] Shmaya E. and Solan E. (2002) Two-Player Non-Zero-Sum Stopping Games in Discrete Time, Discussion paper 1347, Center for Mathematical Studies in Economics and Management Science, Kellogg School of Management, Northwestern University

[14] Sion M. (1958) On General Minimax Theorems, *Pacific J. Math.,* **8**, 171-176

[15] Touzi N. and Vieille N. (2002) Continuous-Time Dynkin Games with Mixed Strategies. *SIAM J. Cont. Opt.*, **41**, 1073-1088