# Undescribable Contingencies*

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**Abstract.** We develop a model of complex undescribable contingencies. These are contingencies that are understood by economic agents — their consequences and probabilities are known — but are such that every description of such events necessarily leaves out relevant features that have a non-negligible impact on the parties' expected utilities. Using a simple co-insurance problem as backdrop, we introduce a model where states are described in terms of objective features, and the description of an event specifies a finite number of such features. In this setting, undescribable contingencies are present when the first-best risk-sharing contract varies with the states of nature in a complex way that makes it highly sensitive to the component features of the states.

We also show that two key ingredients of our model — probabilities that are finitely additive but fail countable additivity, and a state space that is small (discrete in our model) in a measure-theoretic sense — are *necessary* ingredients of any model of undescribable contingencies that delivers our results.

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## 1. Introduction

### *1.1. Motivation*

In many circumstances, even though contracting parties understand the consequences and probabilities associated with a particular contingency, they fail to take the contingency into account in ex-ante contracts, even if doing so may be mutually advantageous in principle.[1] Starting with Grossman and Hart (1986), the resulting contracts have been referred to as *incomplete* in many influential contributions to the literature.[2]

Our point of departure in this paper is the observation that there are, broadly speaking, two types of circumstances in which this failure to condition may arise; one of which is well understood and widely used in the literature, and a second which, although often cited informally, is far from settled at least from a formal modelling point of view. The latter is the focus of this paper.

First, it is well understood that contracting parties may fail to condition on contingencies that are "observable but not verifiable."[3] In this framework, whether the relevant contingency occurs or not is *observed* by (and is common knowledge among) the contracting parties. The problem is that whether the contingency has occurred or not cannot be observed by a third party. In particular, it cannot be *verified* by any third party that is charged with *enforcing* the terms of the contract (the enforcement agent, usually a court).

The second type of environment in which such failure to condition on a particular contingency arises naturally is one in which the contingency is so "complex" that the

[1] These contingencies have also been called "unforeseen" in the literature (Tirole 1999, p. 743). Sometimes, we will use this term in this sense in what follows.

Of course, the fact that a contingency is not included in an ex-ante agreement, does not, in general, imply that the *outcome* of the contractual situation cannot depend on such a contingency. This is because of the possible role of ex-post implementation mechanisms (Maskin and Tirole 1999). We return to this issue in Section 3 below.

[2] See Section 3 below for a discussion of some related papers.

[3] An exhaustive list of references here would be enormous and hence out of place. See, for instance, Holmström (1982) in which to our knowledge the term was first used in its current sense, the seminal paper by Hart and Moore (1988), and the survey by Tirole (1999).

contracting parties find it "[...] prohibitively difficult to [...] describe unambiguously in advance" (Grossman and Hart 1986, p. 696). A formal model of this type of environment is our goal in this paper.

One way to look at the two approaches simultaneously is as follows. In order for complete ex-ante contracting to take place, two key ingredients are necessary. The parties need to describe at an ex-ante stage their will to the court with full precision, and the court needs to be able to verify ex-post in which category specified by the contract the actual state of the world falls. The observable but not verifiable approach takes away the court's ability to verify ex-post what really took place. In this paper, we model the difficulties (impossibility) that the contracting parties face in describing their will to the court, leaving intact its ability to verify the realized state of the world ex-post.

It is not difficult to imagine real-world examples of the phenomenon we have in mind. In the well known case of Jacobellis v. Ohio,[4] Supreme Court Justice Potter Stewart argued that only "hard-core" pornography could be banned, but conceded:

> *"I shall not today attempt to further define the kind of materials I understand to be embraced within the shorthand definition; and perhaps I could never succeed in doing so," Stewart had said. "But I know it when I see it."* (Woodward and Armstrong 1979, p. 94)

We conclude our introduction by noticing that there is a further reason why modelling undescribable contingencies as we do here addresses more than a theoretical curiosity. In a fully specified model of what courts do, it surely would have to be the case that their information structure is (at least to some extent) *endogenous*. If the ability to verify "finer and finer" events yields large potential "gains from trade," then the appropriate resources will be invested to endow the court with the ability to do so. The model we develop here tells us that, even in the limit case in which the court can verify *all* contingencies, the possibility is still open that the parties will lack the ability to describe them fully in their contractual agreement.

[4] *Jacobellis v. Ohio*, 378 U. S. 184 (1964).

### *1.2. Overview*

The plan of the rest of the paper is as follows. Section 2 outlines the requirements that, in our view, should be met by a model of undescribable contingencies. In this section we also claim that some critical features of our model are in fact necessary features of a model that meets these requirements. We then review some related literature in Section 3. In Section 4 we set up the co-insurance problem we use as a backdrop and derive the benchmark efficient allocation that the parties can achieve in the absence of any constraint. We then define the state space and the associated probability measure in Section 5. In Section 6 we proceed to give a formal definition of the notion of a finite contract. In Section 7 we piece together all these elements and proceed to evaluate the parties' expected utilities associated with any finite contract. Section 8 presents our first batch of results: we show that for some instances of our basic co-insurance problem the only transfers that the parties would like to specify are contingent on undescribable contingencies. As a consequence, the optimal finite contract is to specify no transfers at all: the no-contract outcome obtains.

Section 9 briefly reviews our desiderata again, and verifies that they have been achieved. In Section 10 we present our results that embody the claim that finitely additive probabilities and a "small" state space are necessary ingredients of a model that meets the requirements that we set out. Section 11 outlines some extensions of our model and concludes the paper. For ease of exposition, all proofs have been relegated to the Appendix.[5]

## 2. Desiderata and Necessity

In this section we lay out the requirements that we think a model of undescribable contingencies should satisfy. We start by outlining the features we believe our model should have, and then move on to "desired results." We conclude with a claim that some critical features of our model are in fact *necessary* features of any model that meets the desiderata we have set out.

[5] In the numbering of equations, definitions, remarks and so on, a prefix of "A" indicates that the relevant item is to be found in the Appendix.

### *2.1. Model Desiderata*

*1. Expected Utility.* Recall first that we seek a model in which the consequences and probabilities of the relevant events are understood by the parties, and hence all appropriate expected utility calculations can be carried out. We call this the "expected utility requirement."

*2. Language Based.* We also want to be able to take seriously the notion that we can distinguish between physical states and their description in ex-ante agreements.[6] For want of a better term we refer to this requirement as the fact that we would like our model to be "language-based."[7]

To capture this requirement, we work with a model in which physical states of nature can be described by means of a *language* in which a countable infinity of elementary statements are possible. Each elementary statement represents a particular feature that can be either present or not in a given state of nature (the sky can be either "blue" or "not blue").

So, with little loss of generality, we take each physical state of nature $s \in \mathcal{S}$ to be fully described by (at most) an infinite list of elementary statements $\{s^1, \ldots, s^i, \ldots\}$ that determine which features are present in the state. Each feature $s^i$ can either be present ($s^i = 1$) or not ($s^i = 0$) in each state.[8]

We identify the set of statements about future states of nature that the contracting parties are able to specify in an ex-ante contract with the language that describes the

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[6]Of course this does not preclude, as will be the case in our model below, that a "full description" of a state of nature will identify the actual state uniquely.

[7]The objects of the (co-insurance) contracting problem that we use as a back-drop are states of nature (see Section 4 below). As we clarify there, depending on the context, the objects of contracting could also be some actions undertaken by the contracting parties, or the result of both these actions and the random realization of a state of nature. Here and throughout most of the paper we simply refer to the object of contractual interest as a state of nature.

[8]We limit the set of elementary statements to be at most countably infinite, in keeping with the view that in any logical endeavor a "statement" must be a finite string of symbols drawn from an alphabet that is itself at most countably infinite. Of course, depending on the cardinality of $\mathcal{S}$ a finite set of elementary statements could suffice to pin down a state uniquely. In this case $\mathcal{S}$ would have to be a finite set. The actual assumption embodied in our statement above is that a countable infinity of elementary statements is in fact always sufficient to uniquely identify a state $s$. This implies that the cardinality of $\mathcal{S}$ is at most $2^{\aleph_0}$.

states. There are two "competing" requirements that we wish to accommodate here. These are model desiderata 3 and 4 that follow.

*3. Rich Language.* We want to ensure that our model delivers contingencies that are undescribable because they are too "complex," and *not* because the contracting parties are endowed with a language that is simply "too coarse" relative to the environment they face. Since we want to rule out coarse languages, as a minimal requirement we will insist that the parties can write ex-ante contracts that vary across any two states $s'$ and $s''$. We refer to this as the requirement of a "rich language."

*4. Finitely Describable Events.* The set of statements that can be included in an ex-ante contract must embody the notion that there are in fact some contingencies that are "[...] prohibitively difficult to [...] describe unambiguously in advance" (Grossman and Hart 1986, p. 696). Given that we require that our model be language-based in the sense above, there is a completely natural way to model this notion. We will assume that only *finite* statements about the constituent features of a set of states can be included in the contract that the parties draw up. We call this the requirement of "finitely describable" events.

Notice that the requirement of finitely describable events is also appealingly weak in the following sense. It does not require us to specify a cost function for the inclusion of more and more features in a contract.[9] Clearly, any cost function that becomes "sufficiently large" in the limit would deliver ex-ante contracts that are coarser than the ones we obtain in our model.[10] This cuts at once through the problem of specifying a cost function; an exercise which, at least to some degree, is necessarily arbitrary.

It is also worth remarking at this point that since we are only restricting our descriptions of events to be *finite*, our results below are immune to changes in the elementary statements in the language that, for instance, re-code feature "1" and feature "14" into a single one. A finite statement in one language will correspond

[9] Anderlini and Felli (1999) and Battigalli and Maggi (2002) are two contributions to the literature that make explicit use of "writing costs" in a contracting problem.

[10] For instance any cost function that becomes larger than the available contractual surplus in a "first best scenario."

to a finite statement in the new one and vice-versa. This immunity to re-coding is relevant in a world in which languages obviously evolve to capture more efficiently concepts that may once have been considered complex or difficult.

### *2.2. Results Desiderata*

What are then the desiderata for our model in terms of results that embody the notion of complex undescribable events?

*1. No Approximation.* Notice that our specification of goals on model features so far does not preclude the fact that any contingency that cannot be finitely described, may be *approximated* more and more finely by events that *can* be finitely described. In any model in which utilities are sufficiently well behaved (continuous in consequences) our restriction to finitely describable events would then have a negligible impact on the parties' expected utilities.[11] This we want to rule out. Using the same terminology as in Anderlini and Felli (1994), we describe this as the requirement that the "approximation result" must not hold.

In fact, we seek the strongest possible result in this sense. We want a model that displays undescribable contingencies that cannot be approximated at all by finitely describable events.

*2. Finite Invariance and Fine Variability.* Loosely speaking, we are after a model in which for some event $\mathcal{Z}$ the following two properties hold. First of all, neither $\mathcal{Z}$ nor its complement are certain to happen. So, if we denote by $\mu(\mathcal{Z})$ the probability that $\mathcal{Z}$ takes place, then $\mu(\mathcal{Z})$ must be strictly between 0 and 1. More importantly, we require that conditioning on *any* finitely describable event $A$ does not help at all in "predicting" $\mathcal{Z}$. In other words, we require that the conditional probability $\mu(\mathcal{Z}|A)$ satisfies $\mu(\mathcal{Z}|A) = \mu(\mathcal{Z})$ for *every* such $A$. Clearly, if these conditions hold the contracting parties will not be able to gain by conditioning their ex-ante contract on any event $A$, regardless of how mutually advantageous conditioning on $\mathcal{Z}$ might be in principle. Knowing that a state belongs to a set $A$ does not help at all to

[11] See our discussion of the results in Anderlini and Felli (1994) in Section 3 below.

predict whether the state is in $\mathcal{Z}$ or not. In the words of our title, the event $\mathcal{Z}$ is an undescribable contingency.

For reasons that will become clear in the sequel, an event like $\mathcal{Z}$ described informally above will be referred to as displaying both "finite invariance" and "fine variability."

### *2.3. Necessity*

Our model meets all the desiderata we have set out, in terms of both model features and results. We verify this claim in detail in Section 9 below.

Our model also has two critical non-standard features that we discuss extensively in Section 10 below. The first is the use of a probability distribution over states of nature that is finitely additive, but *fails* countable additivity. The second is a state space that is a "small" (in fact countable) subset of the set of all possible potential states (the set of all possible infinite strings of 0s and 1s).

One of the goals of this paper is to show that these non-standard features are *necessary* ingredients of any model that obtains the desiderata we set out. Section 10 below formally argues that this is indeed the case.

Therefore, this paper can be read in two ways. The first is to conclude that it is indeed possible to model formally the notion of an undescribable contingency. The second is that the model we use below to this end, complete with its non-standard features, is what it takes to get a formal hold of this notion. There is a sense in which a rejection of the non-standard ingredients that we use here is equivalent to saying that the formal notion of a contingency that is undescribable because it is "too complex" rather than because of the parties do not have a sufficiently "rich language" is unattainable.

## 3. Related Literature

The intuitive notion of a contingency that is impossible to include in an ex-ante contract, either because it is observable but not verifiable, or because it is "too complex" has been extensively used in the contracting literature. In short, if we take *as given* that some contingencies cannot be included in an ex-ante agreement (although their consequences and probabilities are understood by the agents), and therefore that contracts are incomplete, we can then focus on the institutional arrangements that may reduce the inevitable inefficiencies that are associated with this lack of detail of the ex-ante contracts that the parties draw up.

This line of research has proved extremely fertile. Among other things, it has afforded important insights concerning the boundaries of a firm (Grossman and Hart 1986), the allocation of ownership rights over physical assets (Hart and Moore 1990), the allocation of authority (Aghion and Tirole 1997) and power (Rajan and Zingales 1998) in organizations and the judicial role of courts (Anderlini, Felli, and Postlewaite 2003).

Perhaps precisely because of its prominence and usefulness in modelling a wide range of economic phenomena, the plain assumption that contracting agents may face some contingencies that are unforeseen (undescribable) has itself been the subject of intense scrutiny in a number of recent papers.[12]

As we mentioned above, this paper puts forth a model of undescribable contingencies that are impossible to include in an ex-ante agreement because they are too complex for this to be feasible at all.

Anderlini and Felli (1994) and Al-Najjar (1999) are two existing contributions that are closely related to the results presented here.

In Anderlini and Felli (1994), the contracting parties are restricted to ex-ante agreements that are finite in a sense that is analogous to the one we postulate in this paper. However, crucially, in Anderlini and Felli (1994), there is a *continuum* of states of nature. One of the results reported there is the so-called *approximation*

[12] It should be noted at this point that the term "unforeseen contingencies" has also been used in a number of decision-theoretic and epistemic models (see for instance Kreps (1992), and more recently Dekel, Lipman, and Rustichini (2001) and the survey in Dekel, Lipman, and Rustichini (1998)). Once again (see footnote 1 above), here we are using the term unforeseen contingency in a different sense — as a synonym of undescribable. Our contracting parties understand (have common knowledge of) the consequences and probabilities of unforeseen contingencies. They are simply unable to describe them in advance and hence to incorporate them in any ex-ante agreement.

*result*: in a model with a continuum of states, under general conditions of continuity, the restriction that only finitely many of the constituent features of a state of nature can be included in any ex-ante agreement has a *negligible impact* on the parties' expected utilities.

The restriction to finite agreements clearly precludes the agents from writing some possible ex-ante contracts.[13] Intuitively, the reason why the impact of this restriction is in fact negligible lies in the requirement that the parties must be able to compute the *expected utilities* that an ex-ante agreement generates. In short, if an ex-ante agreement yields well defined expected utilities to the contracting parties, then it must yield them utility levels that are "integrable" as a function of the state of nature. Since a function that is integrable can always be approximated by a sequence of step functions, it is now enough to notice that (a "sufficiently rich" set of) step functions can be viewed as *finite* ex-ante agreements. In the terminology of our Subsection 2 above, in the model studied in Anderlini and Felli (1994), a "rich language" is sufficient to generate the "approximation result" that instead fails to hold in this paper.

Intuitively the difference between the two environments can be traced to the cardinality of the state space (countable versus continuous) and the nature of the associated probability measure (finitely additive "frequencies" in this paper, "standard" probability measures over the interval $[0, 1]$ in Anderlini and Felli (1994)).[14]

In Al-Najjar (1999) the state space is akin to the one used here: it is discrete and is equipped with finitely additive "frequencies," as in the analysis below. Using this apparatus, in a very different set-up from the one analyzed below, Al-Najjar (1999) addresses the question of whether competitive differences between agents get washed out by imitation. In a model with a continuum of states it is possible to show that the performance of a successful agent can be replicated asymptotically as more and

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[13]A simple counting argument suffices to prove this point. It is easy to see that in the world of Anderlini and Felli (1994) there are countably many possible finite ex-ante contracts, while there are uncountably many possible ex-ante agreements.

[14]As we mentioned already, we discuss the role of these two features of our model at length in Section 10 below.

more data become available: a version of the approximation result described above holds in this case. However, in a complex environment, imitation does not eliminate all competitive advantages, even in the limit when an arbitrarily large amount of data becomes available.

Two further papers have investigated contractual environments in which the approximation result described above fails. The analysis in both Anderlini and Felli (1998) and Krasa and Williams (1999) centers on the observation that the approximation result in Anderlini and Felli (1994) requires the parties utilities to be *continuous* in outcomes. The focus of Anderlini and Felli (1998) is to characterize the effects of discontinuities in the parties' utilities in a principal-agent model in which only finite agreements are allowed. Krasa and Williams (1999) focus on a condition that they label "asymptotic decreasing importance" which, in their model, is necessary and sufficient for the required continuity conditions, and hence for the approximation result, to hold. By contrast, in this paper the parties' utilities *are assumed* to be continuous in outcomes.

The results in Machina (2003) are also related to our work. His paper is motivated by a search for events that embody "objective uncertainty" in a standard decision-theoretic model. He works with a continuous state space and standard countably additive measures (with an additional "smoothness" condition). He constructs sequences of events whose probabilities converge to the *same* value, *regardless* of the overall probability measure placed on the state space. Thus, near the limit these are "almost-objective events" in the sense that all decision makers (regardless of their information and priors) will (almost) agree on their likelihood. In a sense, an "almost-objective event" that has probability neither zero nor one, behaves similarly to our undescribable contingencies that display finite invariance and fine variability. The key difference between the two is that in one case (Machina's) the focus is on what happens "near the limit," while in the other (ours) the finite invariance and fine variability hold in the limit world, which is actually well defined. As we show in Section 10 what allows us to look directly at the limit world is our departure from the countably additive measures and continuum of states used in Machina's set-up.

These non-standard features are necessary for our results.

Finally, we view this paper as orthogonal to the debate on the role of message games in models in which complete ex-ante contracting cannot be achieved (Tirole 1999, Maskin and Tirole 1999, Segal 1999, Hart and Moore 1999, Reiche 2001, Maskin 2002, among others). In particular, a number of authors have argued that message games can in fact *substitute* for complete ex-ante contracting. The contracting parties play an ex-post message game in which their private information is revealed in equilibrium. This enables them to make the contractual outcome depend on contingencies that the ex-ante contract neglects. As we have stressed already, our contribution here is to model undescribable contingencies that cannot feature in an ex-ante contract. If these are present, then the type of message game that is appropriate to the environment at hand will be the parties only hope to condition on the contingencies that they cannot specify directly in their ex-ante agreement.

## 4. The Contracting problem

For the sake of concreteness, throughout the paper we work using a standard co-insurance problem as backdrop. Two risk-averse agents, labelled $i = 1, 2$ face a risk-sharing problem. The uncertainty in the environment is captured by the realization of a state of nature, denoted by $s$; the set of all possible states of nature is denoted by $\mathcal{S}$. The preferences of agent $i$ are represented by the state contingent utility function $U_i : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$. The agents' utilities depend on $s$ according to whether or not $s$ falls in a subset $\mathcal{Z}$ of the state space $\mathcal{S}$.

The two agents can agree to a state-contingent monetary transfer $t \in \mathbb{R}$, which by convention represents a payment from 2 to 1. We write the utility of 1 in state $s$, if the transfer is $t$ as

$$U_1(t, s) = \begin{cases} V(1 + t) & \text{if } s \in \mathcal{Z} \\ V(t) & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{1}$$

where $\overline{\mathcal{Z}}$ denotes the complement of $\mathcal{Z}$ in $\mathcal{S}$. Party 2's utility in state $s$ is instead written as

$$U_2(t,s) = \begin{cases} V(-t) & \text{if } s \in \mathcal{Z} \\ V(1-t) & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{2}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a twice differentiable, increasing and strictly concave function satisfying the Inada conditions

$$\lim_{y \to -1} V'(y) = +\infty, \quad \lim_{y \to +1} V'(y) = 0.$$

Ex-ante, 1 makes a take-it-or-leave-it offer of a contract $t : \mathcal{S} \to \mathbb{R}$ to 2, where $t(s)$ is the monetary transfer from 2 to 1 if state $s$ is realized. Of course, 1's take-it-or-leave-it offer to 2 will have to satisfy a participation constraint for 2 which will be specified shortly.

The co-insurance problem we have just described is a completely standard one. Since in (1) and (2) we have specified the agents utilities so that complete insurance is in fact feasible, in the absence of any additional restrictions, the optimal contract $t^*$ will involve only two levels of transfers $t_{\mathcal{Z}}$ and $t_{\overline{\mathcal{Z}}}$ with

$$t^*(s) = \begin{cases} t_{\mathcal{Z}} & \text{if } s \in \mathcal{Z} \\ t_{\overline{\mathcal{Z}}} & \text{if } s \in \overline{\mathcal{Z}} \end{cases} \tag{3}$$

and $1 + t_{\mathcal{Z}} = t_{\overline{\mathcal{Z}}}$ so that

$$U_1(t(s), s) = V(1 + t_{\mathcal{Z}}) = V(t_{\overline{\mathcal{Z}}}) \quad \forall\, s \in \mathcal{S} \tag{4}$$

and

$$U_2(t(s), s) = V(-t_{\mathcal{Z}}) = V(1 - t_{\overline{\mathcal{Z}}}) \quad \forall\, s \in \mathcal{S} \tag{5}$$

Agent 2's participation constraint can be easily specified if we define the probability $p = \Pr\{s \in \mathcal{Z}\}$ that $s$ falls in $\mathcal{Z}$. In the absence of any agreed transfers 2's expected utility is $pV(0) + (1-p)V(1)$. Since 2 is the recipient of a take-it-or-leave-it offer, his participation constraint will bind. Therefore, in addition to (4) and (5) the optimal contract $t^*$ is characterized by

$$pV(-t_{\mathcal{Z}}) + (1-p)V(1-t_{\overline{\mathcal{Z}}}) = pV(0) + (1-p)V(1) \tag{6}$$

Clearly, equations (4), (5) and (6) uniquely pin down the values of $t_{\mathcal{Z}}$ and $t_{\overline{\mathcal{Z}}}$, so that the characterization of the solution to our co-insurance problem in the standard case is complete.

Before we move on to a detailed description of our state space and the probability measure that we place on it, it is worth emphasizing here that the co-insurance problem that we use to exemplify our results is adopted mostly for the sake of simplicity. In fact our results in this paper can be easily translated to apply to other contracting problems.

Starting with Hart and Moore (1988) a class of models that fall within the following broad sketch has become somewhat canonical in the incomplete contracting literature.[15] Two contracting parties, a buyer and a seller, have the opportunity to undertake an ex-ante unobservable relationship-specific investment that affects the cost and/or value of the object (a "widget") of the potential exchange. Subsequently, the cost and value of the widget are realized, typically as a function of the realization of a state of nature as well as of the levels of relationship-specific investment. The presence of non-contractible variables in this set-up then gives rise to a hold-up problem, which in turn determines inefficient levels of ex-ante investments. In particular the ex-ante investments, the actual cost and value of the widget and the state of nature cannot be directly contracted on, even though it would be advantageous in principle to the parties to write an ex-ante contract that conditions the sale price of the widget (and possibly whether the exchange is to take place or not) on (a combination of) these variables.

Our results below could be applied, virtually unchanged, to yield a model of the

[15] What follows is not meant to be a summary description of the actual model analyzed in Hart and Moore (1988), but merely a description of the main ingredients common to many contributions to this area of the literature. We also refer the reader to our earlier discussion of related literature in Section 3 above.

type we have just outlined in which one or more of the relevant variables cannot be profitably included in an ex-ante contract because the relevant contingencies are too complex.

## 5. States and Probabilities

We are now ready to proceed with a formal description of our state space $\mathcal{S}$ and the associated probability measure $\mu$.

As we mentioned above, both of these ingredients of our model are not of a standard form. They are building blocks of a world in which *details*, no matter how small, can matter a lot. The inability to capture these details in any finite ex-ante agreement is at the center of our model of complex undescribable contingencies.

A discussion of our modelling choices is postponed until Section 10 below.

### *5.1. The State Space*

We think of there being a countable infinity of *physical states* of the world $\mathcal{S} = \{s_1, \ldots, s_n, \ldots\}$.

The parties have a common language to *describe* each state $s_n$. The language consists of a countable infinity of *elementary statements* (characteristics) that can be true or false about each state of nature $s_n$. Hence the *complete* description of a state of nature $s_n$ can be thought of as an infinite sequence $\{s_n^1, \ldots, s_n^i, \ldots\}$ of 0's and 1's. Each element of the sequence is simply interpreted as reporting whether the $i$-th elementary statement is true ($s_n^i = 1$) or false ($s_n^i = 0$) about state $s_n$.

The formal definition of our state space simply encapsulates what we have stated so far about $\mathcal{S}$.

**Definition 1.** *State Space: The state space $\mathcal{S}$ is a countably infinite set $\{s_1, s_2, \ldots s_n, \ldots\}$. Each $s_n$ is in turn an infinite sequence of the type $\{s_n^1, \ldots, s_n^i, \ldots\}$ with $s_n^i \in \{0, 1\}$ for every $i$ and $n$.*

### *5.2. Probabilities*

As we mentioned already, the probability measure $\mu$ that we place over $\mathcal{S}$ is non-standard in the sense that it fails countable additivity. Again, we postpone a discussion of this and other features of our model until Section 10 below.

Our first step is to define the *density* of a set of states.

**Definition 2.** *Density: Given any $Q \subseteq \mathcal{S}$, let $\chi_Q$ denote the characteristic function of $Q$ so that $\chi_Q(s_n) = 1$ if $s_n \in Q$ and $\chi_Q(s_n) = 0$ if $s_n \notin Q$. We define the* density *of $Q$ to be*

$$\mu(Q) \quad = \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_Q(s_n) \tag{7}$$

*when the limit in (7) exists. The density is otherwise left undefined. We denote by $\mathcal{D}$ the collection of subsets of $\mathcal{S}$ that have a well defined density.*

Two points should be noted. First, the density of a set $\mu(Q)$ is its "frequency" in the standard sense of the word. Thus, for instance, every finite set of states has a density of zero and the set of all "even numbered" states $\{s_2, s_4, s_6, \ldots\}$ has a density of $1/2$. Second, the density of a set (and whether or not it is well defined) depends on the *ordering* of the states $\{s_1, \ldots, s_n, \ldots\}$. This ordering is taken as given and fixed throughout the paper.[16]

We conclude this subsection with two observations that will become useful below.

First, given two sets $Q'$ and $Q''$ that have well defined densities and such that $\mu(Q') > 0$ and $\mu(Q' \cap Q'')$ is also well defined, we can define the *conditional density* $\mu(Q'' \mid Q')$ as $\mu(Q' \cap Q'')/\mu(Q')$.

Secondly, if we let $\Sigma$ be the set of all subsets of $\mathcal{S}$. Then there exists an extension to $\Sigma$ of the density $\mu$ in Definition 2 above which is a finitely additive probability measure. In other words

[16]The class of permutations of the states of nature that leave our results unaffected includes all finite permutations. We do not attempt a general characterization of such permutations in this paper.

**Remark 1.** *Finitely Additive Probability Measure: There exists a finitely additive probability measure $\tilde{\mu}$ over $(\mathcal{S}, \Sigma)$ that for every set of states $B \subset \mathcal{S}$ satisfies $\tilde{\mu}(B) = \mu(B)$, whenever $\mu(B)$ is defined.*[17]

## 6. Finitely Definable Sets and Finite Contracts

The set of ex-ante contracts that our agents can draw up intuitively coincides with those agreements that are *finite* in a sense to be defined shortly in a formal way.

It is convenient to start our description of what a finite contract is by introducing the notion of a *finitely definable set.* Intuitively, these are subsets of $\mathcal{S}$ that can be defined referring only to a *finite* set of their features.

For each state of nature $s_n$, let $s_n^i \in \{0, 1\}$ indicate the value of the $i$-th feature of $s_n$. Define also

$$A(i, j) \quad = \quad \{s_n \in \mathcal{S} \quad \text{such that} \quad s_n^i \; = \; j\} \tag{8}$$

so that $A(i, j)$ is the set of states that have the $i$-th feature equal to $j \in \{0, 1\}$. These are the elementary statements of the underlying language to which we referred informally in Subsection 2 above.

We are now ready to define the finitely definable subsets of $\mathcal{S}$. These are the sets that can be described in the language of our contracting parties.

**Definition 3.** *Finitely Definable Sets: Consider the* algebra *of subsets of $\mathcal{S}$ generated by the collection of sets of the type $A(i, j)$ defined in (8). Let this algebra be denoted by $\mathcal{A}$. We refer to any $A \in \mathcal{A}$ as a finitely definable set.*

Elements of $\mathcal{A}$ can be obtained by complements and/or finite intersections and/or finite unions of the sets $A(i, j)$. Hence every element of $\mathcal{A}$ can be defined by finitely many elementary statements about the features of the states of nature that it contains.

[17]See, for example, Rao and Rao (1983, p. 41) for a proof.

A suitable definition of a finite contract is now easy to get. The key feature of a finite contract is that it should specify a set of transfers that is conditional only on finitely definable sets. For simplicity we also restrict attention to contracts that specify a finite set of values for the actual transfer $t$. This is clearly without loss of generality in our simple co-insurance problem described in Section 4 above.

**Definition 4.** *Finite Contracts: A contract is* finite *if and only if the transfer rule $t(\cdot)$ that it prescribes is measurable with respect to $\mathcal{A}$, and takes finitely many values $\{t_1, \ldots, t_M\}$. The set of finite contracts is denoted by $\mathcal{F}$.*

While it is possible, as we do here, to take Definition 4 as a primitive that embodies the notion of a contract as a finite object, it is important to point out that this requirement can be supported in a different way (than just taking Definition 4 at face value).

Anderlini and Felli (1994) put forward the idea that it is natural to consider contracts that yield a value for a sharing rule that is *computable* by a Turing machine as a function of the state of nature. The justification for this requirement is a claim that if a function is computable in a finite number of steps by any imaginable finite device then it must be computable by a Turing machine.[18] Obviously, any finite contract must be computable. It is also possible to show that the converse holds: requiring that contracts be finite exhausts the set of all computable contracts. For reasons of space, we omit any formal analysis of this topic.

## 7. Computing Expected Utilities

We now have set out all the ingredients of our model. In essence we want to characterize what the agents can achieve using finite contracts when the state space and associated probability measure are as in Section 5.

As we mentioned already, we want to restrict attention to those cases in which the agents can base their choices on the *expected utility* that an ex-ante contract yields.

[18] This claim is known in the literature on computable functions as *Church's thesis*. See for instance Cutland (1980), or Rogers (1967).

Since we want the agents to be able to contemplate *all* possible finite contracts, we need to ensure that all such contracts can be evaluated in this way. So far, there is nothing in our framework that guarantees that this is the case. This is because our Definition 2 above does not, by itself, guarantee that all finitely definable sets have a well defined density. The proposition that follows guarantees that this can indeed be done.

**Proposition 1.** *Existence: There exists a state space $\mathcal{S}$ as in Definition 1 such that every $A \in \mathcal{A}$ has a well defined density $\mu(A)$. In other words, there exists an $\mathcal{S}$ such that $\mathcal{A} \subseteq \mathcal{D}$.*

The proof of Proposition 1 is a simple consequence of the law of large numbers. Think of $\mathcal{S}$ as a realization of countably many i.i.d. draws from, say, a (countably additive) density $\hat{\mu}$ over $\{0,1\}^{\mathbb{N}}$. It is then sufficient to observe that the law of large numbers guarantees that, with probability one, the limit frequency of draws that falls into any finitely definable set $A$ is in fact well defined and equal to its density $\hat{\mu}(A)$. The set of realizations of these i.i.d. draws that have the properties required by the statement of the proposition has probability one in the space of realizations of this process. It then follows that it must be not empty. Hence, setting $\mathcal{S}$ to be equal to a "typical" realization of these i.i.d. draws as described is sufficient to prove the claim.

To evaluate the expected utility accruing to each party from any finite contract we will also need to refer to the *conditional* densities of certain events. This is an easy task if we restrict attention to finitely definable sets. The following remark is stated without proof since it is a direct consequence of the fact that, by assumption, since $\mathcal{A}$ is an algebra, the intersection of two finitely definable sets is itself finitely definable.

**Remark 2.** *Well Defined Conditional Densities: Let $\mathcal{S}$ be as in Proposition 1 and let $A'$ and $A''$ be two finitely definable sets with $\mu(A') > 0$. Then the conditional density $\mu(A'' \mid A')$ is well defined.*

Of course, to compute the expected utility of a finite contract, the parties must be able to compute more than the frequencies of finitely definable sets. They need to compute the density of the intersection of $\mathcal{Z}$ with any finitely definable set.

Our next definition makes precise what it means for a set of states to meet this requirement.

**Definition 5.** *Well-Defined Frequencies: A set $\mathcal{Z}$ has* well defined frequencies *if*

$$\mathcal{Z} \cap A \in \mathcal{D} \quad \forall\, A \in \mathcal{A}$$

*in other words $\mathcal{Z}$ has a well defined density, conditional on any finitely definable set $A$ (provided of course that $\mu(A) > 0$).*

We have now introduced all the elements that will allow us to study a class of co-insurance problems in which undescribable contingencies can arise, and in which the expected utilities for both agents from any finite contract are well defined and can be computed in a simple way.

The fact that undescribable contingencies can arise in this model is the subject of our next section. For the time being, we remark that the expected utilities from any finite contract are well defined.

Our next statement takes the shape of a *definition* (rather than a proposition) since we are in fact defining what the natural meaning of expected utilities is in a world in which probabilities are equated with the densities of Definition 2 above.

**Definition 6.** *Expected Utilities: Consider the co-insurance problem described in Section 4. Let a density $\mu$ as in Definition 2 be given and let $\mathcal{S}$ be as in Proposition 1. Assume further $\mathcal{Z}$ has well defined frequencies in the sense of Definition 5. Let also any finite contract $t : \mathcal{S} \to \{t_1, \ldots, t_M\}$ be given. Then the expected utility to agent 1 from contract $t$ is defined as*

$$EU_1(t) \;=\; \sum_{i=1}^{M} V(1+t_i)\, \mu[t^{-1}(t_i) \cap \mathcal{Z}] + \sum_{i=1}^{M} V(t_i)\, \mu[t^{-1}(t_i) \cap \overline{\mathcal{Z}}\,] \tag{9}$$

*while 2's expected utility is*

$$EU_2(t) = \sum_{i=1}^{M} V(-t_i)\,\mu[t^{-1}(t_i) \cap \mathcal{Z}] + \sum_{i=1}^{M} V(1-t_i)\,\mu[t^{-1}(t_i) \cap \overline{\mathcal{Z}}\,] \tag{10}$$

We conclude this section with an observation. Using the finitely additive probability measure $\tilde{\mu}$ of Remark 1 that extends $\mu$ to all subsets of $\mathcal{S}$ it is possible to compute the density of every set $D \in \Sigma$. This in turn would allow us to compute the expected utility of a much broader class of contracts that are not necessarily finite, allowing also for a much broader class of state-dependent utilities. Of course, to do this we would need a way to integrate a much broader class of functions than we effectively do in (9) and (10) above. Fortunately, there is an elaborate theory of integration with respect to finitely additive probabilities, which for the most part is analogous to the usual theory of integration.[19]

In this paper, we restrict attention to contracts that are measurable with respect to $\mathcal{A}$ and to contracting problems in which $\mathcal{Z}$ has well defined frequencies in the sense of Definition 5. Of course, when we restrict attention to this case, the more general type of integration that we are referring to gives exactly the expected utilities that we have defined above.[20]

To simplify matters further, we also restrict attention (without any loss of generality in our co-insurance setup) to contracts that take a finite number of values. It should be noted, however, that the restriction to finitely-valued functions, is introduced only for expository simplicity; our analysis is applicable more generally (although this would require some additional machinery).

[19]Dunford and Schwartz (1958) is a classic textbook which provides a unified treatment of integration for both finite and countably additive measures. A more specialized treatment can be found in Rao and Rao (1983).

[20]We proceed as we do here instead than using the more general machinery that we have referred to because this makes our results more transparent in at least two ways. First, all equations (9) and (10) of Definition 6 allow the reader to look "directly inside" what would otherwise be buried in the definition of an "exotic" integral sign. Second, and more important, this way of proceeding clarifies the fact that our results below depend only of the rather intuitive (frequencies) properties of the density $\mu$ rather than on its non-unique extension to the power set $\Sigma$.

## 8. Complex Undescribable Contingencies

### *8.1. Finite Invariance and Fine Variability*

In contrast to the cases of a continuous state space and of a countable state space with a countably additive probability measure, finite contracts *cannot* always *approximate* the first best in the model we have set-up here. The idea is that the allocation $t^*$ that the agents may be trying to attain could exhibit fine variability as a function of the state of nature. Any finite contract is bound not to capture part (or all) of this variability. It is important to stress again that this is in fact possible when the state-dependence of the agents' preferences is such that the expected utility of *any* finite contract (Definition 6) is well defined.

We begin with two abstract definitions that capture the idea that in the model we have set up it is possible that a set $\mathcal{Z}$ may "look the same" if we look at its restriction over any finitely definable set, but at the same time may have a characteristic function that varies "finely" with the state of nature. It will be precisely this type of fine variability that finite contracts cannot capture and hence give rise to undescribable contingencies below.

**Definition 7.** *Finite Invariance: We say that $\mathcal{Z} \subseteq \mathcal{S}$ displays* finite invariance *if for every $A \in \mathcal{A}$ with $\mu(A) > 0$,*

$$\mu(\mathcal{Z}|\, A) \;=\; \mu(\mathcal{Z}) \tag{11}$$

*So, $\mathcal{Z}$ displays finite invariance if its density is the same conditional on all finitely definable sets that have positive measure under $\mu$.*

In other words, if $\mathcal{Z}$ displays finite invariance, knowing that $s$ belongs to any finitely definable subset of $\mathcal{S}$ does not help us to "predict" better whether it belongs to $\mathcal{Z}$ or to its complement. It should be noted at this point that the possibility that Definition 7 may have a non-trivial content is a feature of the model we have set up, which *does not* hold in say a standard model with a continuum of states with $\mathcal{Z}$ a measurable set. In fact, it is clear that in this case if $\mathcal{Z}$ displays finite invariance then it must have measure either 0 or 1. This is not the case in our model, as we will demonstrate shortly in Proposition 2 below.

The second abstract definition that we state is a property that we have labelled fine variability.[21]

**Definition 8.** *Fine Variability: Let $\mathcal{Z}$ be a set that displays finite invariance.*

*We say that $\mathcal{Z}$ displays* fine variability *if and only if*

$$0 \; < \; \mu(\mathcal{Z}) \; < \; 1$$

The properties that we have just defined may *simultaneously* hold for a set $\mathcal{Z}$ that also has well defined frequencies as in Definition 5. Our next proposition asserts that, for some state spaces $\mathcal{S}$ as in Proposition 1 even though a set $\mathcal{Z}$ may have well defined frequencies and display finite invariance, its characteristic function may be far from being constant over $\mathcal{S}$.

**Proposition 2.** *Finite Invariance and Fine Variability: There exists an $\mathcal{S}$ such that the following is true.*

*Let any $p \in (0, 1)$ be given. Then there exists a set $\mathcal{Z}$ with well defined frequencies that displays both finite invariance* and *fine variability and such that $\mu(\mathcal{Z}) = p$.*

The formal proof of Proposition 2 is in the Appendix. Here we only sketch the argument for the case $p = 1/2$. Let $\mathcal{S}$ be as in Proposition 1. We can then construct $\mathcal{Z}$ in the following way. For each given state of nature $s_n \in \mathcal{S}$ we set $s_n \in \mathcal{Z}$ and $s_n \in \overline{\mathcal{Z}}$ with equal probability, and with i.i.d draws across all the states $s_n$. The law of large numbers again guarantees that we can take $\mathcal{Z}$ to be a "typical" realization

[21] Notice that we use the word "variability" with reference to a set. This is not as odd as it may seem at first sight. In our view the best intuitive way to think about the content of Definition 8 below is that of a set that has a characteristic function that varies "very finely" with the state of nature.

of this process to prove the claim. In fact, in any such typical realization, the law of large numbers ensures that the event $\mathcal{Z}$ has a density that is well defined and is equal to 1/2 conditional on any finitely definable subset of states. This clearly guarantees that $\mathcal{Z}$ displays finite invariance and fine variability, as well as having well defined frequencies, as required. As we mentioned above, the type of fine variability that is found in Proposition 2 is at the root of our model of undescribable contingencies. Our next task is to examine its impact on the simple co-insurance model described in Section 4 above.

Before proceeding any further, it is useful to dwell further on how our two results stated so far are proved. The proofs of Propositions 1 and 2 reported in the Appendix both rely on the law of large numbers. These arguments are appealing in the sense that they also show that state spaces $\mathcal{S}$ as in Proposition 1 and a $\mathcal{Z}$ as in Proposition 2 are not "knife-edge" cases in any sense of the word. This is so because the stochastic processes that we use in the two proofs in the Appendix yield a set of probability one of realizations in which the two statements hold. However, it is legitimate at this point to ask whether there are constructive arguments that can be used to prove the two claims.

The answer to the question is affirmative.[22] In fact, the following construction proves both Proposition 1 and 2.[23] We outline it for the case in which $\mu(A(i,j))$

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[22]We are indebted to an anonymous referee for asking the question, and for suggesting the answer.

[23]It may be argued that the set $\mathcal{Z}$ in this example is not complex, and that it is (obviously) describable—after all, the construction that follows is its description. We believe this to be misleading, however. To appreciate this point, let $\mathbb{N}$ be the set of natural numbers and fix any countable state space $\mathcal{S}$. Call a function $e : \mathbb{N} \to S$ an enumeration if it is one-one and onto (thus, under $e$, we are labelling a state $s \in \mathcal{S}$ as the $e^{-1}(s)$-th state). Given any infinite subset $\mathcal{Z} \subset \mathcal{S}$, it is obviously possible to find an enumeration $e_{\mathcal{Z}}$ under which $\mathcal{Z}$ has a simple description. For instance, one can easily find an $e_{\mathcal{Z}}$ under which $\mathcal{Z}$ corresponds to (i.e. $e_{\mathcal{Z}}^{-1}(\mathcal{Z})$ is) the set of even integers. Obviously $\mathcal{Z}$ is simple to describe, but only *given the enumeration* $e_{\mathcal{Z}}$.

To use the labels in $\mathbb{N}$ given by $e$ to identify a set, the description of a set must therefore include a full specification of the enumeration needed to give it a simple representation (e.g. as the set of even integers) — an infinite object by itself. The contracting agents in our model are endowed with a given language, that corresponds to describability in terms of a fixed set of features. This language is the only available vehicle to convey their will to the court. Thus, the integer labels of the states are meaningless to our contracting agents in identifying any particular set of states in an ex-ante contract. The set $\mathcal{Z}$ in this example is not describable in the language determined by these features.

$= 1/2$ for every $i$ and every $j$, and $\mu(\mathcal{Z}) = 1/2$. Start with the states just being identified by their labels, the positive integers. Now assign all odd numbered states to $\mathcal{Z}$ and all even states to $\overline{\mathcal{Z}}$. Among those states that have been placed in $\mathcal{Z}$, assign a value of 0 for the first feature to the first, third, fifth state and so on (states 1, 5, 9 etc.), and a value of 1 to the other states. Among the states that were placed in $\overline{\mathcal{Z}}$ assign a value of 1 for the first feature to the first, third, fifth state and so on (states 2, 6, 10 etc.), and a value of 0 to the other states. So now we have four subsets of states, identified by whether the state is in $\mathcal{Z}$ or $\overline{\mathcal{Z}}$ and whether the first feature is 0 or 1.

Now we can divide each of these four subsets into two subsets as follows. Among those states that have been placed in $\mathcal{Z}$ with first feature 0, assign a value of 0 for the second feature to the first, third, fifth state and so on (states 1, 9, 17 etc.), and a value of 1 to the other states. Among those states that have been placed in $\mathcal{Z}$ with first feature 1, assign a value of 0 for the second feature to the first, third, fifth state and so on (states 3, 11, 19 etc.), and a value of 1 to the other states. Symmetrically, among the states that were placed in $\overline{\mathcal{Z}}$ with first feature 1 assign a value of 1 for the second feature to the first, third, fifth state and so on (states 2, 10, 18 etc.), and a value of 0 to the other states. Finally, among the states that were placed in $\overline{\mathcal{Z}}$ with first feature 0 assign a value of 1 for the second feature to the first, third, fifth state and so on (states 4, 12, 20 etc.), and a value of 0 to the other states.

We can then complete the construction by subdividing the 16 subsets of states into two sets each in the same fashion, and continuing ad infinitum in the same way.[24]

[24]A schematic representation of the construction we have outlined is as follows.

$\mathcal{Z}$

| $s_1$ | $s_3$ | $s_5$ | $s_7$ | $s_9$ | $s_{11}$ | $\cdots$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 | $\cdots$ |
| 0 | 0 | 1 | 1 | 0 | 0 | $\cdots$ |
| 0 | 0 | 0 | 0 | 1 | 1 | $\cdots$ |
| 0 | 0 | 0 | 0 | 0 | 0 | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

$\overline{\mathcal{Z}}$

| $s_2$ | $s_4$ | $s_6$ | $s_8$ | $s_{10}$ | $s_{12}$ | $\cdots$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | $\cdots$ |
| 1 | 1 | 0 | 0 | 1 | 1 | $\cdots$ |
| 1 | 1 | 1 | 1 | 0 | 0 | $\cdots$ |
| 1 | 1 | 1 | 1 | 1 | 1 | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

It is then easy to verify that all the requirements of Propositions 1 and 2 are satisfied with $\mu(A(i,j)) = 1/2$ for every $i$ and every $j$, and $\mu(\mathcal{Z}) = 1/2$.

### *8.2. Undescribable Contingencies and Fine Variability*

The possibility that the contract $t^*$ in the co-insurance problem described in Section 4 above may have the fine variability described in Proposition 2 has far reaching consequences on what the contracting parties can achieve by means of a finite contract.

In this section, we characterize the impact of fine variability when it is associated with finite invariance. In this case, any finite contract will be unable to capture any of the fine variability of $t^*$. As a consequence the agents will choose a trivial contract that prescribes a transfer of $t = 0$ in every possible state. This is of course the same as saying that *no contract* will be drawn up.

Consider the co-insurance problem described in Section 4. For a given $\mathcal{S}$, $\mu$ and $\mathcal{Z}$, let $t^{**}$ be the optimal *finite* co-insurance contract, if it exists. In other words, if it is well defined let $t^{**}$ be the solution to

$$\begin{array}{rl} \max\limits_{t} & EU_1(t) \\ \text{s.t.} & EU_2(t) \;\geq\; \mu(\mathcal{Z})V(0) + \mu(\overline{\mathcal{Z}})V(1) \\ & t \;\in\; \mathcal{F} \end{array} \tag{12}$$

where $EU_i(t)$ are the parties' expected utilities as in Definition 6 above.

**Proposition 3.** *Optimal Finite Contract: Consider the co-insurance problem described in Section 4. Then there exist an $\mathcal{S}$, $\mu$ and $\mathcal{Z}$ with $\mu(\mathcal{Z}) \in (0,1)$ with the following properties.*

*1. The set $\mathcal{Z}$ has well defined frequencies.*

*2. The optimal finite contract $t^{**}$ that solves problem (12) exists unique, up to a set of states of $\mu$-measure zero.*

*3. The optimal finite contract $t^{**}$ prescribes no transfer between the agents in every state of nature. In other words $t^{**}(s) = 0$ for every $s \in \mathcal{S}$, up to a set of states of $\mu$-measure zero.*

Once again the formal proof of Proposition 3 is presented in the Appendix. Intuitively, Proposition 3 is a fairly direct consequence of Propositions 1 and 2 coupled with the strict concavity (in $t$) of the agents' preferences.

Again, we start with an $\mathcal{S}$ as in Proposition 1. Recall now that in the co-insurance problem described in Section 4 above the parties are able to achieve full insurance by agreeing on a transfer contingent on the event $\mathcal{Z}$. We now choose the event $\mathcal{Z}$ to display finite invariance and fine variability as in Proposition 2. Let $p_{\mathcal{Z}}$ and $p_{\overline{\mathcal{Z}}}$ be the densities of $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ respectively, conditional on any $A \in \mathcal{A}$.

Notice that by definition of finite invariance the event $\mathcal{Z}$ has been defined so that any attempt by the parties to condition on a finite set of characteristics (the only feasible ex-ante description available to them) will leave them with a set of states of which only a fraction $p_{\mathcal{Z}}$ actually belongs to $\mathcal{Z}$. This is true whatever finitely definable subset of $\mathcal{S}$ the parties decide to condition their contract on. The fact that the parties are risk averse now implies that the optimal finite contract should specify the same transfer from 2 to 1 contingent on *any* finitely definable subset of $\mathcal{S}$. Any transfer function that varies across two finitely definable sets of states will be strictly dominated (in terms of the parties expected utility) by a constant transfer that coincides with the average of the transfer function we started from.

The optimal contract $t^{**}$ is now immediately obtained from the observation that the only constant (across all states) transfers from 2 to 1 that are compatible with 2's participation constraint are non-positive. Since 1's expected utility is monotonically increasing in the constant transfer from 2, the optimal finite contract must clearly prescribe a transfer of 0 in all states.

The allocation entailed by the optimal finite contract coincides with the no-contract outcome. Clearly the fact that the two parties to the contract are strictly risk averse implies that party 1's expected utility associated with the no-contract outcome is bounded away from the full-insurance contract $t^*$ described in Section 4.

In our terminology, the event $\mathcal{Z}$ is an undescribable (or unforeseen) contingency. The agents understand its probability $p_{\mathcal{Z}}$ and use it in their expected utility computations. However, no matter how finely they attempt to describe it in a finite ex-ante agreement, they will only be correct a fraction $p_{\mathcal{Z}}$ of the time. The extreme prediction that the parties will choose an allocation equivalent to no-contract at all of course derives from the particular event $\mathcal{Z}$ we constructed above in the sense that it displays finite invariance and fine variability.

## 9. Desiderata Re-Visited

In Section 2 above we set key characteristics that a model of undescribable contingencies should, in our view, possess. There, we also specified what results should be true in a model of undescribable contingencies. In this section we briefly review our desiderata and verify that they are indeed met by our model. We also refine our desiderata in terms of results in a way that could not be specified up-front for technical reasons.

Our Model Desiderata 1 was that the contracting parties should be able to evaluate ex-ante contracts by means of expected utility. In view of Definition 6 above, this we have clearly achieved.

Our Model Desiderata 2 was that it should be language-based, so that the notion of a finite statement could be anchored to the underlying language used to describe the states. Clearly, the model that we put forth in Sections 5 and 6 satisfies this requirement. Each state (or set of states) is identified by the constituent features that describe it.

Our set-up also clearly meets Model Desiderata 3 of a rich language. Indeed, it does so in a stronger sense of the term than the "separation" property that we spelled out in Section 2 above.[25] Recall that meeting the requirement of a rich language is key

[25]The algebra of finitely definable sets $\mathcal{A}$ of Definition 3 is capable of "approximating" any state

to our claim that our model captures contingencies that are undescribable because they are too complex, and not because the parties' language or their information is too coarse for the task.

Definition 4 above specifying what we mean by a finite contract, and the definition (3) of finitely definable sets clearly meet Model Desiderata 4 of finitely describable events that we set forth above.

From Proposition 2 it is also clear that the approximation result does not hold in our model in the strong sense that our model displays both finite invariance and fine variability. Hence Results Desiderata 1 and 2 are met.

Our model delivers a version of these results that is in some sense stronger than we specified in Section 2 above. It is worth expanding on this point as it could not be addressed fully in our introductory remarks, before the formal results were laid out.

The set $\mathcal{Z}$ of Proposition 3 above that exhibits finite invariance and fine variability *is* in some intuitive sense expressible in the language defined by the features that describe each state of nature. Simply, $\mathcal{Z}$ is not expressible by any *finite* statement in the language. The formal counterpart of the intuitive claim we have just made is the following. It is in fact the case that the undescribable contingency $\mathcal{Z}$ is in the *sigma* algebra $\sigma(\mathcal{A})$ generated by the algebra $\mathcal{A}$ of finitely definable sets.[26] So, if we were to allow (countably) infinite statements in the language, we would be able to capture $\mathcal{Z}$ exactly.

The fact that $\mathcal{Z}$ is in the sigma algebra $\sigma(\mathcal{A})$ is a key property of the model that increases the appeal of our results. This is so not only because it brings out the fact that it is precisely the restriction to finite statements in the language that drives our

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$s \in \mathcal{S}$ in the following obvious sense. For every given $s \in \mathcal{S}$ there exists a nested decreasing sequence of sets $\{A_n\}_{n=1}^{\infty}$ with $A_n \in \mathcal{A}$ for every $n$, and such that $A_n \downarrow s$. For want of a better term, in what follows we will refer to this feature of our model as the "zoom-in" property.

[26]Throughout the rest of the paper, we use the following notation. If $\mathcal{B}$ is any algebra of sets, then $\sigma(\mathcal{B})$ denotes the sigma algebra generated by $\mathcal{B}$.

Given the zoom-in property mentioned in footnote 25 above, the formal proof of this statement is trivial. The zoom-in property tells us that every singleton $s \in \mathcal{S}$ is in $\sigma(\mathcal{A})$. But since $\mathcal{S}$ is itself countable, it then follows that *every* subset of $\mathcal{S}$ is in fact in $\sigma(\mathcal{A})$.

results. It also rules out another possible type of phenomenon that may give rise to failures of the approximation result. This is best discussed with reference to a concrete example.

It is well known that if we let $(\Omega, \sigma(\mathcal{B}), \nu)$ be a measure space (equipped with the sigma algebra $\sigma(\mathcal{B})$, and $\nu$ a countably additive finite measure), then given any non-measurable set $B^* \notin \sigma(\mathcal{B})$, we can extend the measure $\nu$ to $B^*$ in an arbitrary way as follows.[27] The sigma algebra $\sigma(\mathcal{B})$ can be (minimally) enlarged to $\sigma(\mathcal{B})^*$ so as to include $B^*$, and the measure $\nu$ can be extended to $\nu^*$, where $\nu^*$ assigns any *arbitrary* value to $B^*$, provided it is between the *inner* and *outer* measure of $B^*$ under the original measure $\nu$. Moreover, under $\nu^*$, the event $B^*$ is *independent* of any event $B$ in the original sigma algebra $\sigma(\mathcal{B})$.

Clearly, the set $B^*$ will display finite invariance and fine variability if we take our algebra of finitely definable sets to be $\mathcal{B}$ (or even the entire sigma algebra $\sigma(\mathcal{B})$ that it generates). So, the approximation result will fail in the strongest possible way. Yet, there is a clear sense in which this construction is unsatisfactory. The non-measurable set $B^*$ has in a very real sense no relationship with the algebra $\mathcal{B}$, or even with the generated sigma algebra $\sigma(\mathcal{B})$. This is precisely the reason it can be assigned an *arbitrary* measure and taken to be independent of all events in $\sigma(\mathcal{B})$. If $\mathcal{B}$ is meant to embody the statements of a language, then the set $B^*$ has no relationship with the language at all. It cannot be interpreted as a finite or even a countably infinite statement of the language embodied in $\mathcal{B}$. The viability of this arbitrary construction is simply a by-product of the fact that there exists sets that are not $\sigma(\mathcal{B})$-measurable in the first place.

By contrast, the undescribable contingency $\mathcal{Z}$ of Proposition 3 is in the sigma algebra $\sigma(\mathcal{A})$, and its density $\mu(\mathcal{Z})$ is computed in exactly the same way as the density of any finitely definable set $A \in \mathcal{A}$ — it is its limit frequency in $\mathcal{S}$.

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[27]See for instance Billingsley (1995, Exercise 1.4.10) for the case in which $\Omega$ is the unit interval, or Royden (1988, Theorem 12.38) for the general case in which $\Omega$ is any underlying space equipped as appropriate with the algebra $\mathcal{B}$.

## 10. Necessity Re-Visited

In Section 2 above we claimed that two critical ingredients of our model are *necessary* features of a model that delivers the results that we obtained here. In this section, we substantiate this claim.

### *10.1. Finitely Additive Probabilities*

The most unusual feature of our model is undoubtedly the fact that the measure $\mu$ that we place on the state space $\mathcal{S}$ is finitely additive but fails countable additivity.

It turns out to be the case that this is a necessary feature of any model in which there is an event that cannot be approximated by events in an algebra $\mathcal{A}$, but which is in the sigma algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. To put this in reverse, if we define any algebra of events $\mathcal{A}$ over a state space $\mathcal{S}$, and let $\mu$ be a countably additive measure over $(\mathcal{S}, \mathcal{A})$, then any event $\mathcal{Z}$ in $\sigma(\mathcal{A})$ can be approximated arbitrarily closely by events in $\mathcal{A}$. So, in particular, no such $\mathcal{Z}$ could display finite invariance and fine variability.

To state this claim formally, we obviously first need to specify what we mean by approximating an event in a probability space.

**Definition 9.** *Approximation: Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also $\mu$ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive). Let $\mu^*$ be* any *extension of $\mu$ to the sigma algebra $\sigma(\mathcal{A})$.*

*We say that the approximation result holds for the space $(\mathcal{S}, \mathcal{A}, \mu)$ if and only if for every $\mathcal{Z} \in \sigma(\mathcal{A})$ and every real number $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ such that $\mu^*(\mathcal{Z} \triangle A) < \varepsilon$.*[28]

Clearly, if the approximation result holds for a space $(\mathcal{S}, \mathcal{A}, \mu)$, then *no set* $\mathcal{Z} \in \sigma(\mathcal{A})$ will display finite invariance and fine variability in the sense of Definitions 7 and 8.

[28]Throughout the rest of the paper we use the standard notation $C \triangle D$ to indicate the symmetric difference between the two sets $C$ and $D$. In other words we define $C \triangle D = [C - (C \cap D)] \cup D - (D \cap C)]$.

Our next step is to formalize the claim that a model of undescribable contingencies that delivers a set $\mathcal{Z} \in \sigma(\mathcal{A})$ that cannot be approximated in the sense of Definition 9 must involve a measure $\mu$ that fails to be countably additive.

**Proposition 4.** *Finitely Additive Measures: Let a space $(\mathcal{S}, \mathcal{A}, \mu)$ as in Definition 9 be given, and assume that $\mu$ is countably additive on $\mathcal{A}$.*

*Then the approximation result holds for the space $(\mathcal{S}, \mathcal{A}, \mu)$.*

The intuition behind Proposition 4 is not hard to outline. Roughly speaking, since $\mu$ is countably additive on the algebra $\mathcal{A}$ it has, by Carathéodory's Extension Theorem,[29] a unique countably additive extension $\mu^*$ to the sigma algebra $\sigma(\mathcal{A})$. Consider now a sequence of sets $\{A_n\}$ in $\mathcal{A}$ such that the symmetric difference $A_n \Delta \mathcal{Z} \downarrow \emptyset$. Then, by countable additivity $\mu^*(A_n \Delta \mathcal{Z})$ converges to 0 and hence the approximation result holds.

If a set $\mathcal{Z}$ displays finite invariance and fine variability, the approximation result fails strongly in the sense that $\mathcal{Z}$ cannot be approximated at all ($\mathcal{Z}$ is *independent* of all $A \in \mathcal{A}$), and it fails *uniformly* over the entire state space $\mathcal{S}$. These two features of our model determine the fact that not only $\mu$ must fail countable additivity, but it must fail to be countably additive in the *strongest* possible way.

The following is a standard result that will enable us to formalize the claim we have just made.[30]

**Remark 3.** *Decomposition Theorem: Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also $\mu$ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive).*

*Then $\mu$ can be written in the form $\mu = \mu^{CA} + \mu^{FA}$, where $\mu^{CA}$ is a countably additive measure, and $\mu^{FA}$ is purely finitely additive in the sense that there does not exist a non-zero countably additive measure $\nu$ on $(\mathcal{S}, \mathcal{A})$ such that $\nu \leq \mu^{FA}$.*

[29] See, for instance, Royden (1988, Ch. 12.2).

[30] Many of the results we quote and use in our arguments below are well known in the mathematical literature. A measure that fails countable additivity is known as a "charge." The most comprehensive reference of which we are aware in this field is Rao and Rao (1983).

*Moreover, the decomposition of $\mu$ into $\mu^{CA} + \mu^{FA}$ is unique.*[31]

Finite invariance and fine variability imply failure of countable additivity in the very strong sense that $\mu$ must be purely finitely additive in the sense of Remark 3.

**Proposition 5.** *Pure Finite Additivity: Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also $\mu$ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$.*

*Assume now that there exists a set $\mathcal{Z} \in \sigma(\mathcal{A})$ that displays finite invariance and fine variability.*

*Then the unique decomposition of $\mu$ into $\mu^{FA} + \mu^{CA}$ (as in Remark 3) is such that $\mu^{FA} = \mu$, and $\mu^{CA}$ is identically equal to zero.*[32]

Intuitively, if the countably additive component of $\mu$ is not identically equal to zero then, from Proposition 4, we can approximate, at least in part, any event in the sigma algebra $\sigma(\mathcal{A})$. This contradicts the presence of a set like $\mathcal{Z}$ that displays finite invariance and fine variability.

Once we know that $\mu$ is purely finitely additive, it is easy to see that there cannot be a state $s$ in $\mathcal{S}$ that has point mass. So, another necessary feature of a model that delivers finite invariance and fine variability is a measure $\mu$ that is "diffuse" in a well defined sense.[33]

**Proposition 6.** *Diffuse Probabilities: Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also $\mu$ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$.*

*Then if $\mu$ is purely finitely additive as in Proposition 5 there cannot be a state in $\mathcal{S}$ that has point mass in the following sense. There exists no $s \in \mathcal{S}$ and $\varepsilon > 0$ such that $s \in A$ implies $\mu(A) \geq \varepsilon$ for every $A \in \mathcal{A}$.*[34]

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[31] The proof of this claim can be found for instance in Rao and Rao (1983, Theorem 10.2.1). Notice that the standard name for a purely finitely additive measure like $\mu^{FA}$ is that of a "pure charge."

[32] In the terminology of Rao and Rao (1983), $\mu$ is a "pure charge."

[33] We refrain from using the term "non-atomic" here since a whole host of technical problems arise if one attempts to define this term in a general way for a measure $\mu$ that fails countable additivity. Rao and Rao (1983, Ch. 5) devote an entire chapter to the subject.

[34] Obviously, if $\{s\} \in \mathcal{A}$, then Proposition 6 tells us that it cannot be that $\mu(s) > 0$.

### *10.2. Smallness of the State Space*

The set $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ of all infinite strings of 0s and 1s, of course has the cardinality of the continuum. Yet, the state space $\mathcal{S}$ that we use in Propositions 2 and 3 is countable. In some obvious sense, the state space that we used above to deliver complex undescribable contingencies is "small" relative to $\mathcal{C}$. This is a significant statement since, in principle, one could attempt to use the features in the underlying language to describe *any* element of $\mathcal{C}$. Nevertheless, in the model we have developed above, only countably many elements of $\mathcal{C}$ do in fact correspond to an actual "physical state."

It turns out that the fact that $\mathcal{S}$ must be a "small" subset of $\mathcal{C}$, is also a consequence of the fact that the model admits a set $\mathcal{Z}$ that displays finite invariance and fine variability. Hence this is also a necessary feature of a model of undescribable contingencies that delivers a strong failure of the approximation result, as is the case here.

We now proceed with the formal version of the claim we just made, postponing an intuitive discussion of the assumptions and result until the statement has been made precise.

Let $\lambda$ denote the "uniform" distribution on $\mathcal{C}$. By this we mean the (unique, countably additive) probability distribution on $\mathcal{C}$ obtained as the product distribution on the features and under which $\lambda(A(i,0)) = \lambda(A(i,1)) = 1/2$ for every feature $i$. Note that this may be viewed as the translation of the Lebesgue measure on $\mathcal{C}$.[35]

**Proposition 7.** *Zero Lebesgue Measure: Let $\mathcal{S}$ be any subset of $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, and let $\mathcal{A}$ be the algebra of finitely definable sets of Definition 3.*

[35]To see this, embed the interval $[0,1]$ in the real line as a subset of $\mathcal{C}$, denoted $\mathcal{C}_1$, by identifying each point in $[0,1]$ with its binary expansion. This assignment is unique except for a countable number of points in $[0,1]$ that have two possible binary expansions. For these points, we choose a unique point in $\mathcal{C}$. Then the restriction of $\lambda$ to $\mathcal{C}$ coincides with the Lebesgue measure on $[0,1]$.

The measure $\lambda$ is defined formally in Definition A.1 in the Appendix. Remarks A.4 and A.5 formalize the relationship between $\mathcal{C}$ and the interval $[0,1]$ that we have just sketched out.

*Suppose that $\mu$ is such that the space $(\mathcal{S}, \mathcal{A}, \mu)$ admits a set $\mathcal{Z} \in \sigma(\mathcal{A})$ that displays finite invariance and fine variability. Assume also that $\mu(A) > 0$ for every $A \in \mathcal{A}$.*[36] *Then $\lambda(\mathcal{S}) = 0$.*

Broadly speaking, Proposition 7 is a consequence of the fact that $\mu$ must be purely finitely additive, which in turn of course is a consequence of finite invariance and fine variability.

Intuitively, it is easiest to think of Proposition 7 in the following way. Suppose that we equip the set $\mathcal{C}$ with the algebra of finitely definable sets $\mathcal{A}$ and we place a *finitely* additive measure, say $\nu$, on this pair. Then, by a theorem of Kolmogorov we know that $\nu$ must necessarily be *countably additive* as well.[37] It is then clear that we could not have our state space $\mathcal{S}$ equal to $\mathcal{C}$, since to deliver finite invariance and fine variability we need a measure that is purely finitely additive, as Proposition 5 above shows.

Could it then be that $\mathcal{S}$ contains at least a subset of $\mathcal{C}$ which, conditional on say the first $m$ features being equal to a given sequence of 0s and 1s, contains all elements of $\mathcal{C}$ (a whole "cylinder")? The answer to this question is no. Roughly speaking, we could then apply the same theorem to this subset of $\mathcal{C}$ to obtain at least a "portion" of $\nu$ that is countably additive. But this is impossible if the measure is to be purely finitely additive, as Proposition 5 asserts that it must be if we are to obtain finite invariance and fine variability. It follows that the state space of a model of complex undescribable contingencies that delivers the results in Propositions 2 and 3 must have a state space that is a "small" subset of $\mathcal{C}$ as in Proposition 7.

[36]Note that we make the assumption that $\mu(A) > 0$ for every $A \in \mathcal{A}$ purely for the sake of simplicity. Without it we would need to take care separately of any possible "superfluous" portion of $\mathcal{C}$. By this we mean that, for instance, $\mu$ could assign a mass of zero to the set of all states in $\mathcal{S}$ that have, say, feature 1 equal to 1. In this case it is possible that this entire cylinder in $\mathcal{C}$ is included in $\mathcal{S}$. Since this part of $\mu$ is identically equal to 0, it would be purely finitely additive in the sense of Remark 3 since *both* its countably additive component and its purely finitely additive components are identically equal to 0.

[37]See for instance Billingsley (1995, Theorem 2.3) or Doob (1994, Theorem V.6).

## 11. Conclusions

We have shown that it is possible to construct a contracting environment in which some contingencies have the following properties. Their probabilities and consequences are understood by all concerned, and all agents involved use this information to compute expected utilities arising from any possible finite ex-ante contract. Yet these contingencies are undescribable in the sense that any attempt to describe them in a finite ex-ante agreement must fail. The contracting parties cannot describe these contingencies to any degree that will improve their expected utilities relative to an agreement that ignores them altogether. This is so notwithstanding the fact that the contracting parties' language can in fact distinguish between any two states.

In this paper we have considered an environment in which a particularly stark failure of the approximation of Anderlini and Felli (1994) takes place. In particular, we obtain undescribable contingencies, like $\mathcal{Z}$ above, that display finite invariance and fine variability. So, the approximation results fails "uniformly" in that membership of *no* finitely describable set constitutes useful information about membership of $\mathcal{Z}$.

It is clearly possible to envision intermediate cases in which, say, knowing that the first feature of a state is 0 tells us *something* about its membership of $\mathcal{Z}$, but it is still the case that $\mathcal{Z}$ cannot be approximated in the sense of Definition 9.[38]

In an earlier version of this paper (Al-Najjar, Anderlini, and Felli 2002) we develop formally a batch of results that deal with these intermediate cases. What follows is a brief sketch.

It is possible to characterize tightly what the optimal finite contract looks like in the general case in which the conditional density of $\mathcal{Z}$ is not equal across all finitely definable sets in the algebra $\mathcal{A}$. Applying again the Kolmogorov theorem that we cited in Section 10.2,[39] we can identify the unique countably additive measure on the continuum set $\mathcal{C}$ that agrees with the conditional density of $\mathcal{Z}$, $\mu(\mathcal{Z}|\cdot)$, on every $A$ in

[38]Note that in this case, from Proposition 4 we still know that the measure $\mu$ must fail to be countably additive.

[39]See footnote 37 above and Remark A.3 in the Appendix.

$\mathcal{A}$.[40] Using this measure and keeping fixed the parties' utility functions we can then define an "auxiliary" contracting problem on the state space $\mathcal{C}$.

Since the ingredients of the auxiliary contracting problem are all "standard" it can be solved using familiar techniques. It is then relatively easy to show that the solution to the auxiliary problem fully characterizes the optimal finite contract in the general case.

Hence the optimal finite contract is not "null" in the general case. It captures the variability of the conditional density of $\mathcal{Z}$ that can be embodied in its unique countably additive "translation" to $\mathcal{C}$ that we have mentioned above. All other variability in the characteristic function of $\mathcal{Z}$ cannot be captured at all by any finite contract. Hence it can be safely ignored in the characterization of the optimal finite contract that the parties will sign.

## Appendix

**Proof of Proposition 1:** Consider the set $\mathcal{C}$ of infinite sequences of 0s and 1s, $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, with typical element $c$ and let $c^i$ be the $i$-th digit of the sequence $c$. Let also

$$\tilde{A}(i,j) \;=\; \{c \in \mathcal{C} \;\; \text{such that} \;\; c^i \;=\; j\} \tag{A.1}$$

Let $H$ denote the set of all infinite sequences $\{c_1, \ldots, c_n, \ldots\}$ with $c_n \in \mathcal{C}$ for every $n$. Let $\{\tilde{c}_n\}_{n=1}^{\infty}$ be an infinite sequence of i.i.d. random variables with (countably additive) distribution $\tilde{\mu}$ over $\mathcal{C}$, and let $P$ be the (product) probability distribution that this yields for $H$.

For any $i$ and $j$ now consider the event $M(i,j) \subset H$ such that $\lim\limits_{N\to\infty} (1/N) \sum_{n=1}^{N} \chi_{\tilde{A}(i,j)}(c_n) = \tilde{\mu}(\tilde{A}(i,j))$. By the law of large numbers, $P(M(i,j)) = 1$ for every $i$ and $j$.

Now define,

$$M \;=\; \bigcap_{\substack{i\in\mathbb{N} \\ j\in\{0,1\}}} M(i,j) \tag{A.2}$$

Clearly, since $P(M(i,j)) = 1$ for every $i$ and $j$, and of course $P$ is countably additive, we must also have $P(M) = 1$, and therefore $M \neq \emptyset$.

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[40] Of course, in the case of finite invariance, this would be the uniform measure on $\mathcal{C}$.

It is now sufficient to choose $\mathcal{S}$ to be equal to any element of $M$ to prove the claim. ■

**Proof of Proposition 2:** Fix any $p \in (0,1)$ as in the statement of the proposition. Assume that $\mathcal{S}$ is as in Proposition 1, and that it has the property that any finitely definable set $A$ contains a countable infinity of elements. This is clearly possible from the construction in the proof of Proposition 1.

Define a stochastic process $\{\tilde{h}_1, \ldots, \tilde{h}_n, \ldots\}$ where each random variable $\tilde{h}_n$ takes values in $\{0,1\}$. Let $H$ denote the set of all realizations of this process, and let $P$ be the probability distribution on $H$ under which $\{\tilde{h}_1, \ldots, \tilde{h}_n, \ldots\}$ are i.i.d. random variables with distribution $(p, 1-p)$. Notice that a realization $h = \{h_1, \ldots, h_n \ldots\} \in H$ of this process can be taken to be a candidate for the characteristic function $\chi_{\mathcal{Z}} : \mathcal{S} \to \{0,1\}$. We now proceed to show that the claim can be proved by setting $\chi_{\mathcal{Z}}$ equal to any such realization of this process in a set of probability 1.

Let any $h \in H$ be given and let $A(h)$ be the set of states $s_n$ such that $s_n \in A$ and $h_n = 1$. The law of large numbers holds for any $A \in \mathcal{A}$ in the following sense. There is a set $H_A \subset H$ with $P(H_A) = 1$ such that $h \in H_A$ implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{A(h)}(s_n) = p\, \mu(A) \tag{A.3}$$

Since $P(H_A) = 1$, clearly $Q = \bigcap_{A\in\mathcal{A}} H_A$ also has probability 1. Therefore $Q \neq \emptyset$. Now select any element $h = \{h_1, \ldots, h_n, \ldots\}$ of $Q$, and set $\chi_{\mathcal{Z}}(s_n) = h_n$ for every $n$. This is our candidate $\chi_{\mathcal{Z}}$.

Since equation (A.3) holds for any $A \in \mathcal{A}$ it is obvious that $\mathcal{Z}$ displays finite invariance as in Definition 7. Again from the fact that equation (A.3) holds for any $A \in \mathcal{A}$, it is clear that $\mathcal{Z}$ has well defined frequencies as in Definition 5. Lastly, again from equation (A.3) it is immediate that for any $A \in \mathcal{A}$ with $\mu(A) > 0$ we must have that $\mu(\mathcal{Z}|A) = p$, as required. ■

**Lemma A.1**: *Consider problem (12). Let $\mathcal{Z}$ have well defined frequencies as in Definition 5 and display finite invariance, as in Definition 7.*

*Let any finite contract $t(\cdot) \in \mathcal{F}$ that is feasible in problem (12) be given, and $\{t_1, \ldots, t_M\}$ be the range of $t(\cdot)$. Finally, for every $i = 1, \ldots, M$, let $T_i$ be the inverse image of $t_i$ under $t(\cdot)$.*

*Assume now that $t(\cdot)$ has the following property. There exist an $i \in \{1, \ldots, M\}$ and a $j \in \{1, \ldots, M\}$ such that $\mu(T_i) > 0$ and $\mu(T_j) > 0$. Then there exists another finite contract $t'(\cdot) \in \mathcal{F}$ that is* constant *over $T_i \cup T_j$, which is also feasible in problem (12) and which yields a higher expected utility for agent 1.*

PROOF: Let $t'(\cdot)$ be the same as $t(\cdot)$ for every $s_n \notin T_i \cup T_j$, and set

$$t'(s_n) \;=\; \frac{\mu(T_i)t_i \;+\; \mu(T_j)t_j}{\mu(T_i) \;+\; \mu(T_j)} \qquad \forall s_n \in T_i \cup T_j \tag{A.4}$$

The claim now follows directly by concavity of $V$, defining $U_1$ and $U_2$ as in (1) and (2). The rest of the details are omitted. ■

**Lemma A.2**: *Let $\mathcal{Z}$ have well defined frequencies (as in Definition 5) and display finite invariance (as in Definition 7). Then an optimal finite contract $t^{**}$ that solves problem (12) exists unique, up to a set of states of $\mu$-measure zero. Moreover, $t^{**}(s_n) = 0$ for all $s_n \in \mathcal{S}$, up to a set of states of $\mu$-measure zero.*

PROOF: Let $\mathcal{Z}$ as in the statement of the Lemma be given. Consider now the following maximization problem.

$$\begin{array}{ll} \max\limits_{x} & V(1+x)\,\mu(\mathcal{Z}) + V(x)\,\mu(\overline{\mathcal{Z}}) \\ \text{s.t.} & V(-x)\,\mu(\mathcal{Z}) + V(1-x)\,\mu(\overline{\mathcal{Z}}) \geq V(0)\,\mu(\mathcal{Z}) + V(1)\,\mu(\overline{\mathcal{Z}}) \\ & x \in \mathbb{R} \end{array} \tag{A.5}$$

The strict concavity of $V(\cdot)$ implies that problem (A.5) has a unique solution by completely standard arguments. Let this solution be denoted by $\tilde{x}$.

The expected utility $V(-x)\,\mu(\mathcal{Z}) + V(1-x)\,\mu(\overline{\mathcal{Z}})$ is monotonically decreasing in $x$. Therefore the constraint in problem (A.5) is satisfied only when $x \leq 0$. Since the objective function in problem (A.5), $V(1+x)\,\mu(\mathcal{Z}) + V(x)\,\mu(\overline{\mathcal{Z}})$, is monotonically increasing in $x$ we conclude that the unique solution of problem (A.5) is $\tilde{x} = 0$.

From Lemma A.1 above it is immediate that a solution to problem (A.5) must yield a solution to problem (12). Therefore setting $t^{**}(s_n) = 0$ for every $s_n \in \mathcal{S}$ yields the unique (up to a set of $\mu$-measure zero) solution to problem (12). ■

**Proof of Proposition 3:** Let $\mathcal{S}$ be as in Proposition 1. Using Proposition 2 we can now choose $\mathcal{Z}$ to have well defined frequencies, display finite invariance and exhibit fine variability, with $\mu(\mathcal{Z}) \in (0,1)$. The claim now follows directly from Lemma A.2. ■

**Proof of Proposition 4:** Since $\mu$ is countably additive on $\mathcal{A}$ by Catahéodory's Extension Theorem there exists a unique extension $\mu^*$ of $\mu$ to $\sigma(\mathcal{A})$. Since $\mathcal{Z} \in \sigma(\mathcal{A})$, we must then have that $\mu^*(\mathcal{Z})$ is equal to the outer measure of $\mathcal{Z}$ induced by $\mu$. In other words it must be that

$$\mu^*(\mathcal{Z}) \;=\; \inf \sum_n \mu(O_n) \tag{A.6}$$

where the infimum extends over all finite and infinite sequences $\{O_n\}$ that satisfy

$$O_n \in \mathcal{A} \;\; \forall\, n \quad \text{and} \quad \mathcal{Z} \subseteq \bigcup_n O_n \tag{A.7}$$

Hence, for any real number $\xi > 0$ there exists a sequence $\{O_n\}$ satisfying (A.7) and

$$\sum_n \mu(O_n) \;-\; \mu^*(\mathcal{Z}) \;<\; \xi \tag{A.8}$$

Since the first term in (A.8) is a convergent series, for any real number $\eta > 0$ there exists a finite $m$ such that

$$\sum_n \mu(O_n) \;-\; \sum_{n=1}^{m} \mu(O_n) \;<\; \eta \tag{A.9}$$

Notice next that (A.8) implies that

$$\mu^*(\bigcup_n O_n\,) \;-\; \mu^*(\mathcal{Z}) \;<\; \xi \tag{A.10}$$

Since the sequence $\{O_n\}$ satisfies (A.7), the inequality in (A.10) implies that

$$\mu^*(\,\mathcal{Z} \,\triangle \bigcup_n O_n\,) \;<\; \xi \tag{A.11}$$

From (A.9) we can now deduce that

$$\sum_{n>m} \mu(O_n) \;<\; \eta \tag{A.12}$$

and hence that

$$\mu^*(\,\bigcup_{n>m} O_n\,) \;<\; \eta \tag{A.13}$$

from which it follows immediately that

$$\mu^*(\bigcup_n O_n \triangle \bigcup_{n=m}^{m} O_n) < \eta \tag{A.14}$$

It is straightforward to verify that the operator $\mu^*(\cdot \triangle \cdot)$ is in fact a pseudo-metric on the sigma algebra of sets $\sigma(\mathcal{A})$. Hence it satisfies the triangular inequality. Hence

$$\mu^*(\mathcal{Z} \triangle \bigcup_{n=1}^{m} O_n) \leq \mu^*(\mathcal{Z} \triangle \bigcup_n O_n) + \mu^*(\bigcup_n O_n \triangle \bigcup_{n=m}^{m} O_n) \tag{A.15}$$

Using (A.11) and (A.14), (A.15) immediately yields

$$\mu^*(\mathcal{Z} \triangle \bigcup_{n=1}^{m} O_n) \leq \xi + \eta \tag{A.16}$$

Finally, since $\xi$ and $\eta$ are both arbitrary, and the finite union $\bigcup_{n=1}^{m} O_n$ is clearly an element of $\mathcal{A}$, (A.16) is obviously enough to prove the claim. ■

We will use the following result in the proof of Proposition 5 below. We state it here without proof purely for the sake of completeness. For the proof see Rao and Rao (1983, Theorem 10.3.1).

**Remark A.1**: *Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also $\mu$ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive).*

*Then $\mu$ is purely finitely additive if and only if for every countably additive measure $\nu$ on $(\mathcal{S}, \mathcal{A})$, every $A \in \mathcal{A}$, and every $\eta > 0$ there exists a set $M \in \mathcal{A}$ such that $M \subseteq A$*

$$\nu(M) < \eta \quad \text{and} \quad \mu(A) - \mu(M) < \eta \tag{A.17}$$

**Remark A.2**: *Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also $\mu$ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive), and consider its (unique decomposition) into $\mu^{CA} + \mu^{FA}$ as in Remark 3.*

*Then for every $\eta > 0$ there exists a set $B \in \mathcal{A}$ such that*

$$\mu^{CA}(B) > \mu^{CA}(\mathcal{S}) - \eta \quad \text{and} \quad \mu^{FA}(B) < \eta \tag{A.18}$$

PROOF: The claim is a straightforward consequence of Remark A.1.

Since $\mu^{CA}$ is countably additive and $\mu^{FA}$ is purely finitely additive, in Remark A.1 we can set $\mu = \mu^{FA}$ and $\nu = \mu^{CA}$. Hence, setting $A = \mathcal{S}$, Remark A.1 now tells us that for every $\eta > 0$ there exists a set $M \in \mathcal{A}$ such that

$$\mu^{CA}(M) < \eta \quad \text{and} \quad \mu^{FA}(\mathcal{S}) - \mu^{FA}(M) < \eta \tag{A.19}$$

Next, set $B = \overline{M}$. We then note that $\mu^{CA}(M) = \mu^{CA}(\mathcal{S}) - \mu^{CA}(B)$ and $\mu^{FA}(M) = \mu^{FA}(\mathcal{S}) - \mu^{FA}(B)$. Substituting these equalities in (A.19) now immediately yields that for every $\eta > 0$ there exists a set $B \in \mathcal{A}$ such that

$$\mu^{CA}(\mathcal{S}) - \mu^{CA}(B) < \eta \quad \text{and} \quad \mu^{FA}(\mathcal{S}) - \mu^{FA}(\mathcal{S}) + \mu^{FA}(B) < \eta \tag{A.20}$$

Rearranging (A.20) then immediately gives the result. ■

**Proof of Proposition 5:** Use Remark 3 to write $\mu = \mu^{CA} + \mu^{FA}$. From Proposition 4 we know that $\mu^{CA}(\mathcal{S}) < 1$. Assume now that the Proposition is false. Then it must also be the case that $\mu^{CA}(\mathcal{S}) > 0$.

Using Remark A.2 we know that for every $\eta > 0$ there exists a set $B \in \mathcal{A}$ such that

$$\mu^{CA}(B) > \mu^{CA}(\mathcal{S}) - \eta \quad \text{and} \quad \mu^{FA}(B) < \eta \tag{A.21}$$

Since by assumption $\mu(\mathcal{Z}) \in (0,1)$, we can choose $\eta$ in (A.21) to satisfy

$$\eta < \mu^{CA}(\mathcal{S}) \frac{\mu(\mathcal{Z}) - \mu(\mathcal{Z})^2}{2 + \mu(\mathcal{Z}) + \mu(\mathcal{Z})^2} \tag{A.22}$$

Notice next that since we know that $\mu^{CA}(\mathcal{S}) > 0$ (the contradiction hypothesis), and by assumption $\mu(\mathcal{Z}) \in (0,1)$, the inequalities in (A.21) and (A.22) guarantee that $\mu(B) \geq \mu^{CA}(B) > 0$. Therefore, we can define the restrictions of $\mu$ and $\mu^{CA}$ to $B \in \mathcal{A}$ as $\mu_B = \mu/\mu(B)$ and $\mu_B^{CA} = \mu^{CA}/\mu^{CA}(B)$. Further, define $\mu_B^{FA}$ to be identically equal to 0 if $\mu^{FA}(B) = 0$, and $\mu_B^{FA} = \mu^{FA}/\mu^{FA}(B)$ if $\mu^{FA}(B) > 0$. Therefore, we can now write

$$\mu_B = \alpha \, \mu_B^{CA} + (1 - \alpha) \, \mu_B^{FA} \tag{A.23}$$

where $\alpha = \mu^{CA}(B)/\mu(B)$. Notice that, since $\mu^{CA}(\mathcal{S}) \geq \mu^{CA}(B)$, we can now use (A.21) and (A.22) to conclude that

$$\alpha = \frac{\mu^{CA}(B)}{\mu^{CA}(B) + \mu^{FA}(B)} > \frac{\mu^{CA}(\mathcal{S}) - \eta}{\mu^{CA}(\mathcal{S}) + \eta} > \frac{1 + \mu(\mathcal{Z})^2}{1 + \mu(\mathcal{Z})} \tag{A.24}$$

Next, define $\mathcal{Z}_B = \mathcal{Z} \cap B$, and notice that since $\mathcal{Z}$ displays finite invariance we have that $\mathcal{Z}_B$ displays finite invariance with respect to the restriction $\mu_B$. In other words whenever $A \in \mathcal{A}$ and $A \subseteq B$ we must have that $\mu_B(\mathcal{Z}_B|A) = \mu_B(\mathcal{Z}_B)$, with the latter of course also equal to $\mu(\mathcal{Z})$.

Clearly, $\mu_B^{CA}$ is countably additive. Applying Proposition 4, for every real number $\xi > 0$ there exists $Q_\xi \in \mathcal{A}$ such that

$$\mu_B^{CA}(\mathcal{Z}_B|\overline{Q}_\xi) \; < \; \xi \quad \text{and} \quad \left| \mu_B^{CA}(Q_\xi) \; - \; \mu_B^{CA}(\mathcal{Z}_B) \right| \; < \; \xi \tag{A.25}$$

Therefore

$$\begin{aligned} \mu_B(\mathcal{Z}_B|\overline{Q}_\xi) \quad &= \quad \frac{\alpha\, \mu_B^{CA}(\mathcal{Z}_B|\overline{Q}_\xi)\, \mu_B^{CA}(\overline{Q}_\xi) + (1-\alpha)\, \mu_B^{FA}(\mathcal{Z}_B|\overline{Q}_\xi)\, \mu_B^{FA}(\overline{Q}_\xi)}{\alpha\, \mu_B^{CA}(\overline{Q}_\xi) + (1-\alpha)\, \mu_B^{CA}(\overline{Q}_\xi)} \\ &< \quad \frac{\xi + (1-\alpha)\, \mu_B^{FA}(\mathcal{Z}_B|\overline{Q}_\xi)\, \mu_B^{FA}(\overline{Q}_\xi)}{\alpha\, \mu_B^{CA}(\overline{Q}_\xi) + (1-\alpha)\, \mu_B^{CA}(\overline{Q}_\xi)} \\ &< \quad \frac{\xi + 1 - \alpha}{\alpha\, \mu_B^{CA}(\overline{Q}_\xi)} \end{aligned} \tag{A.26}$$

In other words, using the fact that $\mathcal{Z}$ displays finite invariance with respect to $\mu_B$, we can now write

$$\alpha\, \mu_B(\mathcal{Z}_B)(1 \; - \; \mu_B^{CA}(Q_\xi)) \; < \; \xi \; + \; 1 \; - \; \alpha \tag{A.27}$$

Since $\mu_B(\mathcal{Z}_B) \geq \alpha\, \mu_B^{CA}(\mathcal{Z}_B)$, we can now use (A.25) to re-write (A.27) as

$$\alpha \; < \; \frac{1 + \xi + \mu_B(\mathcal{Z}_B)^2}{1 + \mu_B(\mathcal{Z}_B)\,(1 - \xi)} \tag{A.28}$$

Since $\mu_B(\mathcal{Z}_B) = \mu(\mathcal{Z})$, for $\xi$ sufficiently small (A.28) implies that

$$\alpha \; < \; \frac{1 + \mu(\mathcal{Z})^2}{1 + \mu(\mathcal{Z})} \tag{A.29}$$

However, since (A.29) directly contradicts (A.24) this is clearly enough to prove our claim. ■

**Proof of Proposition 6:** Since $\mu$ is purely finitely additive, from Remark 3 we know that $\mu^{CA}$ is identically equal to 0. Hence from Theorem 10.2.2 of Rao and Rao (1983) we can conclude directly that

$$0 \; = \; \inf \left\{ \sum_n \mu(A_n) \right\} \tag{A.30}$$

where the infimum extends over all (finite or infinite) sequences of disjoint sets $\{A_n\}$ such that $A_n \in \mathcal{A}$ for every $n$, and $\bigcup_n A_n = \mathcal{S}$.

Suppose now by way of contradiction that the statement of the Proposition is false. Then there exists an $s \in \mathcal{S}$ such that $\mu(A) \geq \varepsilon$ whenever $A$ contains $s$. Since for any sequence $\{A_n\}$ as above we must have that $s \in A_n$ for some $n$, this implies that the infimum in (A.30) is at least $\varepsilon$. This contradiction is enough to establish the result. ■

We will use the following result in the proof of Lemma A.3 below. We state it here without proof purely for the sake of completeness. For the proof see Billingsley (1995, Theorem 2.3) or Doob (1994, Theorem V.6).

**Remark A.3**: *Consider the set $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, and any subset $\mathcal{S}$ of $\mathcal{C}$. Assume that $\mathcal{S}$ is equipped with the algebra $\mathcal{A}$ of finitely definable sets, and equip $\mathcal{C}$ with the algebra $\tilde{\mathcal{A}}$ corresponding to the algebra of finitely definable sets as follows.*

*As in (A.1) of the proof of Proposition 1, for each $c \in \mathcal{C}$ let $\{c^i\}_{i\in\mathbb{N}}$ be the sequence of digits in $\{0,1\}$ that define $c$, and for every $i \in \mathbb{N}$ and $j \in \{0,1\}$ let*

$$\tilde{A}(i,j) \;=\; \{c \in \mathcal{C} \quad \text{such that} \quad c^i \;=\; j\} \tag{A.31}$$

*and let $\tilde{\mathcal{A}}$ be the algebra of subsets of $\mathcal{C}$ generated by the collection of sets of the type $\tilde{A}(i,j)$. Notice that in this way, using (8), we obviously have that for every $\tilde{A} \in \tilde{\mathcal{A}}$ it must be that $\tilde{A} \cap \mathcal{S} = A \in \mathcal{A}$.*

*Let $\mu$ be any finitely additive measure on $(\mathcal{A}, \mathcal{S})$ (not necessarily countably additive). Then there exists a unique countably additive measure $\tilde{\mu}$ on $(\sigma(\tilde{\mathcal{A}}), \mathcal{C})$ that satisfies $\tilde{\mu}(\tilde{A}) = \mu(A)$ whenever $\tilde{A} \cap \mathcal{S} = A$.*[41]

**Lemma A.3**: *Let any $\mathcal{S} \subset \mathcal{C}$ be given, and consider a purely finitely additive measure $\mu$ on $(\mathcal{S}, \mathcal{A})$. Let $\tilde{\mu}$ be the extension of $\mu$ to $(\sigma(\tilde{\mathcal{A}}), \mathcal{C})$ as in Remark A.3 above.*

*Then, for every real number $\varepsilon > 0$ there exists $\tilde{A}_\varepsilon \in \sigma(\tilde{\mathcal{A}})$ such that $\mathcal{S} \subseteq \tilde{A}_\varepsilon$ and $\tilde{\mu}(\tilde{A}_\varepsilon) < \varepsilon$.*

Proof: Since $\mu$ is purely finitely additive, appealing again to Theorem 10.2.2 of Rao and Rao (1983) we can conclude directly that

$$0 \;=\; \inf \left\{ \sum_n \mu(A_n) \right\} \tag{A.32}$$

where the infimum extends over all (finite or infinite) sequences of disjoint sets $\{A_n\}$ such that $A_n \in \mathcal{A}$ for every $n$, and $\bigcup_n A_n = \mathcal{S}$. Hence, for every $\varepsilon > 0$ there exists a sequence of disjoint sets $\{A_{n,\varepsilon}\}$ such that $A_{n,\varepsilon} \in \mathcal{A}$ for every $n$, $\bigcup_n A_{n,\varepsilon} = \mathcal{S}$ and

$$\sum_n \mu(A_{n,\epsilon}) \;<\; \varepsilon \tag{A.33}$$

[41] With a slight abuse of language we refer to $\tilde{\mu}$ as the *extension* of $\mu$ to $(\sigma(\tilde{\mathcal{A}}), \mathcal{C})$.

Consider any sequence $\{A_{n,\varepsilon}\}$ as in (A.33) and the sequence $\{\tilde{A}_{n,\varepsilon}\}$ of subsets of $\mathcal{C}$ corresponding to it in the sense of Remark A.3, so that $\tilde{A}_{n,\varepsilon} \cap \mathcal{S} = A_{n,\varepsilon}$ for every $n$. Let $\tilde{A}_\varepsilon = \bigcup_n \tilde{A}_{n,\varepsilon}$. Observe that clearly $\tilde{A}_\varepsilon \in \sigma(\tilde{\mathcal{A}})$.

Notice next that $\bigcup_n A_{n,\varepsilon} = \mathcal{S} \cap \bigcup_n \tilde{A}_{n,\varepsilon}$. Hence $\mathcal{S} = \tilde{A}_\varepsilon \cap \mathcal{S}$, and therefore $\mathcal{S} \subseteq \tilde{A}_\varepsilon$. Since $\tilde{\mu}$ is countably additive we now have that $\tilde{\mu}(\tilde{A}_\varepsilon) = \sum_n \tilde{\mu}(\tilde{A}_{n,\varepsilon})$. Since by construction we must have that $\tilde{\mu}(\tilde{A}_{n,\varepsilon}) = \mu(A_{n,\varepsilon})$ for every $n$ we also know that

$$\tilde{\mu}(\tilde{A}_\varepsilon) \;=\; \sum_n \tilde{\mu}(\tilde{A}_{n,\varepsilon}) \;=\; \sum_n \mu(A_{n,\varepsilon}) \tag{A.34}$$

Using (A.33), and (A.34) it is now immediate that $\tilde{\mu}(\tilde{A}_\varepsilon) < \varepsilon$, as required. ■

**Lemma A.4**: *Let any $\mathcal{S} \subset \mathcal{C}$ be given, and consider a purely finitely additive measure $\mu$ on $(\mathcal{S}, \mathcal{A})$. Let $\tilde{\mu}$ be the extension of $\mu$ to $(\tilde{\mathcal{A}}, \mathcal{C})$ as in Remark A.3 above.*

*Then, there exists $\tilde{\mathcal{S}} \in \sigma(\tilde{\mathcal{A}})$ such that $\mathcal{S} \subseteq \tilde{\mathcal{S}}$ and $\tilde{\mu}(\tilde{\mathcal{S}}) = 0$.*

Proof: From Lemma A.3 we know that, given any sequence $\varepsilon_m \to 0$ we can construct a corresponding sequence of sets $\{\tilde{A}_{\varepsilon_m}\}$ such that $\mathcal{S} \subseteq \tilde{A}_{\varepsilon_m}$, $\tilde{\mu}(\tilde{A}_{\varepsilon_m}) < \varepsilon_m$, and $\tilde{A}_{\varepsilon_m} \in \sigma(\tilde{\mathcal{A}})$ for every $m$. To prove the claim it is then sufficient to set $\tilde{\mathcal{S}} = \bigcap_m \tilde{A}_{\varepsilon_m}$ and to notice that it must be the case that $\tilde{\mathcal{S}} \in \sigma(\tilde{\mathcal{A}})$. ■

**Remark A.4**: *Each element $c$ of $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ can be interpreted as the binary expansion of a real number $r$ in the interval $[0,1]$ by taking the elements of the sequence $c$ to be the digits of the binary expansion of $r$ following a "0" and the "decimal" point.*

*This map assign a unique real in $[0,1]$ to each element of $\mathcal{C}$ except for those that are of the form $\{c_1, \ldots, c_m, 1, 0, \ldots, 0, \ldots\}$ and $\{c_1, \ldots, c_m, 0, 1, \ldots, 1, \ldots\}$ which obviously correspond to the same real number $r$. Notice that there are countably many such pairs of elements of $\mathcal{C}$.*

*In what follows we will denote by $\mathcal{C}_0$ the set of elements of $\mathcal{C}$ that are of the form $\{c_1, \ldots, c_m, 1, 0, \ldots, 0, \ldots\}$, excluding $\{0, \ldots, 0, \ldots\}$, and by $\mathcal{C}_1$ the remainder of $\mathcal{C}$ so that $\mathcal{C}_1 = \mathcal{C} - \mathcal{C}_0$.*

*From what we have just stated, it is clear that we can assign a unique real in $[0,1]$ to each element of $\mathcal{C}_1$ and a unique element of $\mathcal{C}_1$ to every real in $[0,1]$.*

*Finally, notice that if we define the sigma algebra $\sigma(\tilde{\mathcal{A}}_1)$ of subsets of $\mathcal{C}_1$ as consisting of the collection of sets $\tilde{A} \cap \mathcal{C}_1$ for every $\tilde{A} \in \sigma(\tilde{\mathcal{A}})$ we obtain that $\sigma(\tilde{\mathcal{A}}_1)$ contains all the half-open intervals in $[0,1]$ of the form $(a,b]$ where $a$ and $b$ are reals in $[0,1]$.*

**Remark A.5**: *Consider the sigma algebra $\sigma(\tilde{\mathcal{A}}_0)$ of subsets of $\mathcal{C}$ consisting of the collection of sets $\tilde{A} \cap \mathcal{C}_0$ for every $\tilde{A} \in \sigma(\tilde{\mathcal{A}})$. Consider also the sigma algebra $\sigma(\tilde{\mathcal{A}}_1)$ of Remark A.4.*

*Then $\sigma(\tilde{\mathcal{A}}) = \sigma(\tilde{\mathcal{A}}_0) \cup \sigma(\tilde{\mathcal{A}}_1)$.*

Proof: Since $\mathcal{C}_0$ is a countable set it is enough to notice that every singleton set is already contained in $\sigma(\tilde{\mathcal{A}})$. Hence $\sigma(\tilde{\mathcal{A}}_0)$ consists of all subsets of $\mathcal{C}_0$. The assertion is then immediate from the definition of $\sigma(\tilde{\mathcal{A}}_0)$ and $\sigma(\tilde{\mathcal{A}}_1)$. The details are omitted. ■

**Definition A.1**: *Recall that from Remark A.5 we know that $\sigma(\tilde{\mathcal{A}}) = \sigma(\tilde{\mathcal{A}}_0) \cup \sigma(\tilde{\mathcal{A}}_1)$. The Lebesgue measure $\lambda$ on $\mathcal{C}$ is then defined as follows.*

*For every $\tilde{A}$ in $\sigma(\tilde{\mathcal{A}})$, set $\lambda(\tilde{A}) = 0$ if $\tilde{A} \in \sigma(\tilde{\mathcal{A}}_0)$, and $\lambda(\tilde{A}) = \mathcal{L}(\tilde{A})$ if $\tilde{A} \in \sigma(\tilde{\mathcal{A}}_1)$ where $\mathcal{L}$ is the Lebesgue measure on the real interval $[0, 1]$ defined in the standard way.*

*Finally, as is standard, we take $\lambda$ to be the* completion *of the measure we have just defined in the sense that it is defined and is equal to zero on all subsets of all measurable sets that have zero measure.*[42]

**Lemma A.5**: *Let $\tilde{\mu}$ be the extension of $\mu$ to $(\sigma(\tilde{\mathcal{A}}), \mathcal{C})$ as in Remark A.3, and assume that $\mu$ is such that $\mu(A) > 0$ for every $A \in \mathcal{A}$.*

*Then $\mathrm{supp}(\tilde{\mu}) = \mathcal{C}$, where $\mathrm{supp}(\cdot)$ indicates the support of a given measure.*

Proof: Suppose not. Then there is a non-empty open set $O$ in $\mathcal{C}$ such that $\tilde{\mu}(O) = 0$. (We take $O$ to be open in the product topology generated by the discrete topology on each coordinate of the elements of $\{0, 1\}^{\mathbb{N}}$.)

We will show that for every open set $O$ we can find an $\tilde{A} \in \tilde{\mathcal{A}}$ that is contained in $O$. Since $\tilde{\mu}(\tilde{A}) = \mu(\tilde{A} \cap \mathcal{S})$ and the latter is, by assumption, positive this yields a contradiction and hence is sufficient to prove the claim.

Assume by way of contradiction that we can find a non-empty open $O \subseteq \mathcal{C}$ such that $\tilde{A} \not\subseteq O$ for every $\tilde{A} \in \tilde{\mathcal{A}}$.

Fix $c \in O$ and consider the nested sequence of sets $\{\tilde{A}_n\}$ where for every $n$, $\tilde{A}_n \in \tilde{\mathcal{A}}$ is the set (the "cylinder") of all those $\hat{c}$s that have the first $n$ digits equal to the first $n$ digits of $c$.

By our contradiction hypothesis it must be that $\tilde{A}_n \not\subseteq O$ for every $n$. Hence, for every $n$ we must be able to find a $\hat{c}_n \in \tilde{A}_n$ and $\hat{c}_n \notin O$.

Clearly, the sequence $\{\hat{c}_n\}$ converges to $c$. But since $\hat{c}_n \notin O$ for every $n$, and $c \in O$, this contradicts the fact that $O$ is open. ■

[42] See for instance Billingsley (1995, p. 45).

**Proof of Proposition 7:** Let $\tilde{\mu}$ be the extension of $\mu$ to $(\sigma(\tilde{\mathcal{A}}), \mathcal{C})$ as in Remark A.3 and $\lambda$ be the Lebesgue measure on $\mathcal{C}$ as in Definition A.1.

By Lemma A.4 we know that there exists a set $\tilde{\mathcal{S}} \in \sigma(\tilde{A})$ such that $\mathcal{S} \subseteq \tilde{\mathcal{S}}$ and $\tilde{\mu}(\tilde{\mathcal{S}}) = 0$, and By Lemma A.5 we know that $\text{supp}(\tilde{\mu}) = \mathcal{C}$.

Since $\lambda$ is, by definition, *complete* in the sense that it assigns measure zero to all subsets of any set in $\sigma(\tilde{\mathcal{A}})$ that have $\lambda$-measure zero, it is enough to show that $\lambda(\tilde{\mathcal{S}}) = 0$.[43] We proceed by contradiction. Hence suppose that $\lambda(\tilde{\mathcal{S}}) > 0$.

By the "Lebesgue Decomposition Theorem,"[44] we know that $\tilde{\mu}$ can be (uniquely) written as $\tilde{\mu} = \tilde{\mu}^C + \tilde{\mu}^S$ where $\tilde{\mu}^C$ is absolutely continuous with respect to $\lambda$, and $\tilde{\mu}^S$ is singular with respect to $\lambda$.

Let $Q^S = \text{supp}(\tilde{\mu}^S)$ and $Q^C = \text{supp}(\tilde{\mu}^C)$. Since $\text{supp}(\tilde{\mu}) = \mathcal{C}$, we must have that $\mathcal{C} = Q^S \cup Q^C$. Hence $\tilde{\mathcal{S}} = [\tilde{\mathcal{S}} \cap Q^S] \cup [\tilde{\mathcal{S}} \cap Q^C]$.

Notice that, since $\tilde{\mu}^S$ is singular with respect to $\lambda$, we immediately know that $\lambda(\tilde{\mathcal{S}} \cap Q^S) = 0$. Hence, by our contradiction hypothesis it must be that $\lambda(\tilde{\mathcal{S}} \cap Q^C) > 0$.

Now let $f$ be the Radon-Nikodym derivative of $\tilde{\mu}^C$ with respect to $\lambda$, which of course we know exists because $\tilde{\mu}^C$ is absolutely continuous with respect to $\lambda$. Notice that it must be the case that $f > 0$ except for a set of $\lambda$-measure zero on $\tilde{S} \cap Q^C$. Hence $\lambda(\tilde{S} \cap Q^C) > 0$ implies that

$$\tilde{\mu}^C(\tilde{\mathcal{S}} \cap Q^C) \;=\; \int_{\tilde{\mathcal{S}} \cap Q^C} f \, \mathrm{d}\, \lambda \;>\; 0 \tag{A.35}$$

However, since $\tilde{\mu} = \tilde{\mu}^C + \tilde{\mu}^S$ and $\tilde{\mu}(\tilde{\mathcal{S}}) = 0$, we must obviously have that $\tilde{\mu}^C(\tilde{\mathcal{S}} \cap Q^C) = 0$. This contradiction is sufficient to prove the claim. ■

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[43] See footnote 42 above.

[44] See for instance Royden (1988, Theorem 11.24).

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