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PRE-LEONTIEF FUNCTIONS AND LEAST ELEMENTS

by

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Pre-Leontief functions and least elements

Introduction

In a previous paper [6] we introduced the Z-functions and showed that a nonempty feasible region defined by a continuous Z-function always contains a least element which is also a complementary solution. An iterative procedure to find the least element was also given in [6]. In this work we generalize the Z-property to introduce the pre-Leontief functions. It is then shown that these functions define feasible regions containing least elements.

Following the generalization of the linear complementarity problem presented in [2], we demonstrate a complementarity property associated with the least elements.

A modification of the algorithm presented in [6] for Z-functions, is then shown to be applicable for finding least elements of regions corresponding to pre-Leontief functions.

**Definition:** Let \( f : \mathbb{R}_+^n \to \mathbb{R}^m \) be a function from \( \mathbb{R}_+^n = \{ x | x \in \mathbb{R}^n, x \geq 0 \} \) to \( \mathbb{R}^m \), whose components are \( f_1, \ldots, f_m \). \( f \) is a pre-Leontief function on \( \mathbb{R}_+^n \) if for each \( i, j = 1, \ldots, m \), there exists an integer \( k(i), 1 \leq k(i) \leq n \), such that for all \( x \in \mathbb{R}_+^n \) and \( t \geq 0 \)

\[
f_i(x) \geq f_j(x + t e_j) \quad \forall \ j \neq k(i),
\]

where \( e_j \) is the \( j \)-th unit vector in \( \mathbb{R}^n \).

If \( m = n \) and \( k(i) = i \) \( f \) is said to be a Z-function.

Note that the pre-Leontief property ensures that each component of \( f \) is nonincreasing with respect to at least \( (n-1) \) of its arguments.

If \( f \) is linear and characterized by an \( m \times n \) matrix then it is pre-Leontief if each of its \( n \) rows contains at most one positive element. If the matrix is square and all its off-diagonal elements are nonpositive \( f \) is a Z-function.
As indicated in [6], the Z-functions constitute a natural extension of the simple linear Leontief Interindustry Model, corresponding to \( n \) products and \( n \) industries, each with one type of output. Following the exposition of [6], we immediately observe that the pre-Leontief function can be used to describe the case where several industries may produce the same type of product.

In [6] we provided a constructive proof of the following theorem, dealing with the existence of a least element.

**Theorem 1**: Let \( f : \mathbb{R}_+^n \to \mathbb{R}^n \) be a continuous Z-function and let \( q \) be in \( \mathbb{R}^n \). If \( X_q^+ = \{ x | f(x) + q \geq 0, \ x \geq 0 \} \) is nonempty then it contains a least element \( \bar{x} \) (i.e. \( \bar{x} \in X_q^+ \) and \( \bar{x} \leq y \) for all \( y \in X_q^+ \)), and \( \bar{x} \) satisfies

\[
\bar{x}'(f(\bar{x})+q) = 0.
\]

The proof given in [6] is based on a modification of the well known Gauss-Seidel and Jacobi iterative procedures.

Note that the main significance of the least element is that it (simultaneously) minimizes any real isotope objective function defined on \( X_q^+ \).

We now generalize Theorem 1 to pre-Leontief functions.

**Existence of Least Elements**

**Theorem 2**: Let \( f : \mathbb{R}_+^n \to \mathbb{R}^n \) be a continuous pre-Leontief function.

Given \( a \geq 0 \) in \( \mathbb{R}_+^n \) and \( q \in \mathbb{R}^n \), if \( X_{q,a}^+ = \{ x | f(x)+q \geq 0, \ x \geq a \} \) is nonempty it contains a least element.
Proof:

If $z \in X_{q,a}^+$ it is clearly the least element. Hence, assume that $a$ is not contained in $X_{q,a}^+$. The continuity of $f$ implies that $X_{q,a}^+$ is closed. Let $x$ be the element of $X_{q,a}^+$ which is closest to $a$, with respect to the Euclidean norm. We show that $x$ is the least element.

Suppose that $y \in X_{q,a}^+$ and $y_j < x_j$ for some $j$, $1 \leq j \leq n$. Define $z$ in $R^n_+$ by $z_i = \sin(x_i,y_i)$ for $i = 1, \ldots, n$. To complete the proof it is then sufficient to show that $z \in X_{q,a}^+$.

We first note that $z \geq a$. Let $i$, $1 = 1, \ldots, m$, and consider $f_i$.

Using the pre-Leontief property and supposing that $x_k(i) = x_k(i)$ we get $f_i(z) + q \geq f_i(x) + q \geq 0$. (If we had assumed that $x_k(i) = y_k(i)$ we would have obtained $f_i(z) + q \geq f_i(y) + q \geq 0$). Hence $f(z) + q \geq 0$ and $z \in X_{q,a}^+$.

We note that the preceding theorem generalizes a result due to Cottle and Veinott [3], who dealt with pre-Leontief matrices, i.e. matrices for which each row contains at most one positive element.

In fact, when $f(x)$ is linear the existence of a least element for $X_{q,a}^+$ for all $a \in R^n_+$ and $q \in R^m$, provided $X_{q,a}^+$ is not empty, implies that the matrix defining $f$ is pre-Leontief. The following example illustrates that this is not always true when nonlinear functions are considered.

Example 1: Let $f : R^2_+ \rightarrow R^1$ be defined by

$$f(x_1,x_2) = \begin{cases} \frac{x_1+1}{x_2+1} & x_1 \leq x_2 \\ \frac{x_2+1}{x_1+1} & x_1 \geq x_2 \end{cases}$$
It is easily verified that for each \( a \in \mathbb{R}^n_+ \) and scaler \( q \) \( X_{q,a}^+ \) contains a least element, provided it is not empty, but \( f \) is clearly not pre-Leontief.

Although the pre-Leontief property is not satisfied globally the following result can be interpreted as a local pre-Leontief property.

**Theorem 3**: Let \( f : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) be such that for each \( a \in \mathbb{R}^n_+ \) and \( q \in \mathbb{R}^n_+ \), \( X_{q,a}^+ \neq \emptyset \) implies the existence of a least element in \( X_{q,a}^+ \). Then for each \( x \in \mathbb{R}^n_+ \), and \( i, i=1,\ldots,m \) there exists \( k(i,x), i \leq k(i,x) \leq n \) such that \( \ell_i(x+s e_j) \leq \ell_i(x) \) for \( 0 \leq t \) and \( j \neq k(i,x) \).

**Proof**: Suppose on the contrary that for some \( x \in \mathbb{R}^n_+ \) and \( i, i=1,\ldots,m \), there exist \( r \) and \( p \leq r, 1 \leq r, p \leq n \) such that \( \ell_i(x+s e_r) > \ell_i(x) \), and \( \ell_i(x+s e_p) > \ell_i(x) \) where \( s > 0, s_1 > 0 \). For any \( j, j=1,\ldots,m, j \neq i \) let \( m_j = \min \{ \ell_j(x), \ell_j(x+s_1 e_p), \ell_j(x+s_1 e_r) \} \) and \( m_r = \min \{ \ell_r(x+s_0 e_r), \ell_r(x+s_1 e_p) \} \). Consider \( X_{q,x}^+ \) where \( q = -(m_1, m_2, \ldots, m_m) \). \( x + s_1 e_p \) and \( x + s_1 e_r \) belong to \( X_{q,x}^+ \), which in turn implies that \( x \) is the least element of \( X_{q,x}^+ \). A contradiction to the definition of \( m_r \).

As shown by the next theorem, the result of Theorem 3 can be strengthened to achieve the (global) pre-Leontief property, if separability is assumed.

**Theorem 4**: Let \( f : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) be given by \( f(x_1,\ldots,x_n) = f^1(x_1) + f^2(x_2) + \ldots + f^n(x_n) \) and suppose that for each \( a \in \mathbb{R}^n_+ \) and \( q \in \mathbb{R}^n_+ \), \( X_{q,a}^+ \neq \emptyset \) imply the existence of a least element in \( X_{q,a}^+ \). Then \( f(x) \) is pre-Leontief.
Proof:

We show that for each \( i, i=1, \ldots, m \), there exists \( k(i), 1 \leq k(i) \leq n \), such that for any \( j \neq k(i) \) \( j=1, \ldots, n \) the scaler function \( f^j_i(x_j) \) is nonincreasing.

Suppose on the contrary, that there exist \( r \) and \( p \) \( 1 \leq r, p \leq n \) \( r \neq p \) such that \( f^r_i(x_r) \) and \( f^p_i(x_p) \) are not nonincreasing. Hence, there exist \( \bar{x}_r \geq 0, \bar{x}_p \geq 0, s_0 > 0 \) and \( s_1 > 0 \) satisfying \( f^p_i(\bar{x}_p + s_1) > f^p_i(\bar{x}_p) \) and \( f^r_i(\bar{x}_r + s_0) > f^r_i(\bar{x}_r) \).

For any \( u, u=1, \ldots, m \), let \( q_u = -\min\{f_u((\bar{x}_p + s_1)e_p + \bar{x}_r e_r); f_u(\bar{x}_p e_p + (\bar{x}_r + s_0)e_r)\} \).

Then it is easily seen that \((\bar{x}_p + s_1)e_p + \bar{x}_r e_r\) as well as \(\bar{x}_p e_p + (\bar{x}_r + s_0)e_r\) belong to \(X^+_q,a\) as defined above and \(a = \bar{x}_p e_p + \bar{x}_r e_r\). This in turn implies that \(a\) is the least element of \(X^+_q,a\) a contradiction to the definition of \(q_u\).

As shown in [9], if \(f\) is affine the (global) pre-Leontief property is equivalent to the following condition.

For each \(q \in \mathbb{R}^m\), \(X^+_q,0 \neq \emptyset\) implies the existence of a least element in \(X^+_q,0\).

The next example shows that the latter condition, (i.e. when the existence of least elements for \(X^+_q,a\) for all \(q\) and \(a\) is replaced by the existence of least elements for \(X^+_q,0\) for all \(q\)), is not even sufficient to yield the (local) pre-Leontief property stated in Theorem 3.
Example 2: Let $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}^1$ be defined by $f(x_1, x_2) = h(x_1) + g(x_2)$ where

$$h(x_1) = \begin{cases} x_1 & 0 \leq x_1 \leq 1 \\ 2x_1 - 1 & 1 \leq x_1 \leq 2 \\ 0 & x_1 \geq 2 \end{cases} \quad \text{and} \quad g(x_2) = \begin{cases} 1 - x_2 & 0 \leq x_2 \leq 1 \\ x_2 & 1 \leq x_2 \leq 2 \\ 1 & x_2 \geq 2 \end{cases}$$

When some differentiability and regularity assumptions are imposed on the function $f$, one can derive additional necessary conditions for the existence of a least element. Specifically, we will assume an arbitrary qualification for constrained optimization (see [4]).

Theorem 5: Let $f : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function and let $\bar{x}$ be a least element of $X^+_q, a^*$. Denote $I = \{i | f_i(\bar{x}) + q_i = 0\}$ and $J = \{i | \bar{x}_i = a_i \}$ and suppose that a constraint qualification is satisfied at $\bar{x}$.

Then $|I| + |J| \geq n$ and there exists an $n \times n$, $(|I| + |J|)$ nonnegative matrix $A$ such that

$$A = \begin{bmatrix} \frac{\nabla f_i(\bar{x})}{\bar{I}_J} \\ \bar{I}_I \end{bmatrix} = I_{n \times n}$$  \hspace{1cm} (1)

where $I_{n \times n}$ is the identity matrix of order $n$, $\nabla f_i(\bar{x})$ is the Jacobian of the functions $f_i(x)$, $i \in I$, and $\bar{I}_J$ is the $|J| \times n$ Jacobian matrix of the functions $g_i(x) = x_i$, $i \in J$. 
Proof:

It is assumed that some constraint qualification which is required by the Kuhn-Tucker conditions is satisfied. (See [4]).

\( \bar{x} \) being a least element of \( X^+_{q,a} \) implies that for each (row) unit vector \( e_j \) in \( \mathbb{R}^n \)

\[ e_j \bar{x} \leq e_j x \quad \forall \ x \in X^+_{q,a} \]

Hence for any \( j, j=1, \ldots, n \), there exists (row) vectors of multipliers \( u^j \geq 0, u^j \in \mathbb{R}^{|I|} \) and \( v^j \geq 0, v^j \in \mathbb{R}^{|J|} \) such that

\[ e_j - u^j \frac{\nabla f_i(x)}{\nabla f_j(x)} - v^j = 0 \quad , \quad j=1,2,\ldots,n. \]

Let \( A \) be the \( n \times (|I| + |J|) \) matrix having \( (u^j, v^j) \) as its \( j \)th row, then

\[ A \begin{bmatrix} \frac{\nabla f_i(x)}{\nabla f_j(x)} \\ \vdots \\ \frac{\nabla f_i(x)}{\nabla f_j(x)} \end{bmatrix} = I_{n \times n}. \]

In particular we obtain that \(|I| + |J| \geq n\), and the proof is complete.

A similar version of the above theorem was also proved by Bod [1].

Following Mangasarian [5], we note that (1) is equivalent to the inverse isotonicity of \( \begin{bmatrix} \frac{\nabla f_i(x)}{\nabla f_j(x)} \\ \vdots \\ \frac{\nabla f_i(x)}{\nabla f_j(x)} \end{bmatrix} \). (A matrix \( B \) is inverse isotone if for all \( x \neq 0 \) implies \( x \geq 0 \).)
As a consequence of the theorem it follows that if
\[
E = \left( \frac{\partial f_i(x)}{\partial x_j} \right)_{i \in I, j \in J},
\]
then there exists a nonnegative matrix A of order \((n-|J|) \times |I|\) such that \(AB = \mathbf{I} (n-|J|, (n-|J|))\). The latter implies that the partial Jacobians of the binding functions, \(f_i, i \in I\), with respect to the nonbinding variables, \(x_j, j \notin J\) is inverse isotone.

Finally we observe that the regularity assumptions in Theorem 3 cannot be omitted as illustrated by the function \(f(x_1, x_2) = -(x_1-1)^2 - (x_2-1)^2\) and \(X^+_{Q,a}\) where \(q = 0\) and \(a = (0,0)\).

An Algorithm Finding the Least Element:

Focussing on a constructive approach to find a least element we next show that the algorithm suggested in [6] for \(Z\)-functions can be modified to be applied to pre-Leontief functions. (Note that by Theorem 2 the least element can be found by solving
\[
\min_{i=1}^n x_i \quad \text{s.t. } x \in X^+_{Q,a}.
\]

Let \(f : X^0 \to \mathbb{R}^n\) be a pre-Leontief function and \(q \in \mathbb{R}^m\). For any \(i, i = 1, \ldots, n\), let
\[
I(i) = \{ j \mid f_j \text{ is not monotonically nonincreasing in } x_i \}.
\]

Further, if \(I(i) \neq \emptyset\) we define
\[
g_i(x) = \min_{j \in I(i)} \{ f_j(x) + q_j \}.
\]
We observe that the pre-Leontief property ensures that $I(i)$, $i = 1, \ldots, n$, are well defined, i.e. $i \neq j \Rightarrow I(i) \cap I(j) = \emptyset$. We also verify that $g_i^*(x) : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is pre-Leontief for each $i$ such that $I(i) \neq \emptyset$. In fact, $g_i^*(x)$ is monotonically nonincreasing in $x_j$ for all $j \neq i$. We define the following sets of indices

$$J = \{ j | j \notin I(i) \} \text{ and } I(i) = \{ j | I(i) \neq \emptyset \} \quad (4)$$

We will further assume without loss of generality that

$$I = \{ i(1), i(2), \ldots, i(t) \} \text{ where } i(k) < i(k+1), 1 \leq k < t \quad (5)$$

Having associated the above notation with a pre-Leontief function $f$, we prove the following result, which will be found useful in the application of the algorithm of [6].

**Lemma 6:** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be pre-Leontief, and let $a \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$. If $\bar{x}$ is a least element of $X_{q,a}^+$, then $x_i = a_i$ for each $i \notin I$, where $I$ is given by (4).

**Proof:**

If $I$ implies that for each $j$, $j = 1, \ldots, m$, $f_j(x)$ is monotonically nonincreasing in $x_i$, i.e. for any $x \in \mathbb{R}^n$ and $t \geq 0$

$$f_j(x) \leq f_j(x + te_i).$$

If $\bar{x}$ is the least element in $X_{q,a}^+$, then

$$0 \leq q_j + f_j(\bar{x}) \leq q_j + f_j(\bar{x} - (x_i - a_i)e_i),$$

for all $i$, $j = 1, \ldots, n$, which in turn proves that $x_i = a_i$. 
As a consequence of the above lemma we can set \( x_i^+ = a_i \) for all \( i \in I \), as a start in our effort to find the least element of \( X_{q,a}^+ \). Our next step is to introduce a Z-function which will be shown to be equivalent to \( f(x) \) in the sense of finding least elements.

Let \( I \) be given by (2) - (4), and \( g_i(y) \), \( i \not\in I \) be defined by (1). For each \( k = 1, \ldots, t \) define

\[
h_k(y_1, \ldots, y_t) = g_i(k) \left( \sum_{k=1}^t y_k e_i(k) + \sum_{i \not\in I} a_i e_i \right) \tag{6}
\]

Note that \( h = (h_1, \ldots, h_t) \) is a Z-function mapping \( R^t_+ \) to \( R^t \), where \( t \leq \min(m,n) \). We next show that for our purposes it is sufficient to concentrate on the Z-function \( h \).

**Theorem 7:** Let \( f : R^n_+ \rightarrow R^m_+ \) be pre-Leontief. Given \( a \in R^n_+ \) and \( q \in R^m_+ \), define \( h : R^n_+ \rightarrow R^t_+ \) by (6), and \( I, J \) by (4).

Let

\[
x_{q,a}^+ = \{ x \mid f_j(x) + q_j \geq 0, j \not\in J \text{ and } x \geq a \} \quad \text{and} \quad y_{q,a}^+ = \{ y \mid h_k(y) \geq 0, y_k \geq a_i(k), k = 1, \ldots, t \}.
\]

Then

1. \( y_{q,a}^+ = \emptyset \) if and only if \( x_{q,a}^+ = \emptyset \).
2. If \( \bar{y} \) is a least element in \( y_{q,a}^+ \) then
   \[
   \bar{x} = \sum_{k=1}^t \bar{y}_k e_i(k) + \sum_{i \not\in I} a_i e_i \quad \text{is a least element in} \quad x_{q,a}^+. \quad \text{Further, if also} \quad f_j(\bar{x}) + q_j \geq 0 \quad \text{for all}
   \]


\[ j \in J \text{ then } \exists x \text{ is also a least element of } X^+_q, a; \]
\[ \text{otherwise } X^+_q, a = \emptyset. \]

Proof:

(1) follows directly from the definition of \( h \). If \( y \in Y^+_q, a \)
then it is easily verified that
\[ x = \sum_{k=1}^{t} y_k e_i(k) + \sum_{i \in I} a_i e_i \in X^+_q, a. \]
Conversely, if \( x \in X^+_q, a \), observe that \( f_j(x), j = 1, \ldots, m \) is monotonically
nonincreasing in \( x_i \) for all \( i \in I \). In particular for \( j \notin J \)
\[ f_j(x) + q_j \geq 0 \text{ and } x \geq a \Rightarrow f_j \left( \sum_{k=1}^{t} x_i(k) e_i(k) + \sum_{i \in I} a_i e_i \right) + q_j \geq 0. \]
The latter implication then yields
\[ e_i \left( \sum_{k=1}^{t} x_i(k) e_i(k) + \sum_{i \in I} a_i e_i \right) \geq 0, \text{ for all } i \in I. \]
Hence, \( (x_i(1), \ldots, x_i(t)) \in Y^+_q, a \).

To prove (2), suppose as \( \bar{y} \) is a least element in \( Y^+_q, a \). Clearly,
\[ x = \sum_{k=1}^{t} y_k e_i(k) + \sum_{i \in I} a_i e_i \in X^+_q, a. \]
If \( x \) was not the least element,
there would exist \( x^0 \in \nexists_q, a \) such that \( x^0_i(k) < y_k \) for some \( 1 \leq k \leq t. \)
As demonstrated above while proving (1), \( x^0_i(1), \ldots, x^0_i(t) \in Y^+_q, a \),
which in turn contradicts the fact that \( \bar{y} \) is the least element in \( Y^+_q, a \).

To prove the second part of (2), it will be shown that if \( X^+_q, a \)
is nonempty, then \( x \) is the least element of \( X^+_q, a \). Suppose that
\( x^1 \subseteq x^0 \), in particular \( x^1 \subseteq x^0 \) and \( x^1 \geq \bar{x} \). The proof will be
cOMPLETE when we show that \( f_j(x) + q_j \geq 0 \) for all \( j \in J \). But the latter
is implied by the monotonicity of \( f_j \), \( j \in J \) in all of its \( n \) arguments.
We are now ready to apply the algorithm of [6] to find the least element of \( X_{q,a}^+ = \{ x | f(x) + q \geq 0, x \geq a \} \), provided \( X_{q,a}^+ \neq \emptyset \), when \( f : R^n_+ \to R^n \) is a continuous pre-Leontief function.

Given a continuous Z-function \( h : R^n_+ \to R^n \), and \( q \in R^n \), the algorithm presented in [6] finds a least element of \( X_{q,a}^+ = \{ x | h(x) + q \geq 0, x \geq a \} \) or indicates that \( X_{q,a}^+ \) is empty.

To find the least element of \( \{ x | h(x) + q \geq 0, x \geq a \} \) when \( a \in R^n_+ \), one has to find the least element of \( \{ y | h(y) + q \geq 0, y \geq 0 \} \), where \( h(y) = h(y + a) \) and add the vector \( a \) to this least element.

Let \( f : R^n_+ \to R^n \) be a continuous pre-Leontief function, \( a \in R^n_+ \) and \( q \in R^n \). Using Theorem 7, one can apply the following procedure to verify the existence of a least element to \( X_{q,a}^+ \) and to find the element provided it exists.

**Step 1:** Define \( I, J \) and \( h : R^n_+ \to R^n \) by (4) and (6) respectively.

**Step 2:** Apply the algorithm of [6] to find the least element of the set \( \{ y | h(y) \geq 0, y_k \geq s_i(k), k=1, \ldots, t \} \). If the set is empty, \( X_{q,a}^+ \) is empty; terminate. Otherwise apply Step 3, where \( \bar{y} \) is the least element.

**Step 3:** Define \( \bar{x} = \sum_{k=1}^{t} \frac{y_k}{s_i(k)} e_i(k) + \sum_{I \subseteq J} a_I e_I \). If \( f_j(\bar{x}) + q_j \geq 0 \) for all \( j \in J \), \( \bar{x} \) is the least element of \( X_{q,a}^+ \). Otherwise, \( X_{q,a}^+ \) is empty.
Finally we demonstrate a complementarity property associated with pre-Leontief functions. Motivated by the generalization of the linear complementarity problem due to Cottle and Dantzig [2], we refer to the generalized nonlinear complementarity problem defined as follows.

Let \( F : \mathbb{R}^n_+ \to \mathbb{R}^m \), have components \( F_1, \ldots, F_m \), and suppose that these components are partitioned into \( n \) sets \( S_j, j=1, \ldots, n \). The generalized complementarity problem is to find \( x \in \mathbb{R}^n_+ \) such that

\[
x \geq 0, \quad F(x) \geq 0 \quad \text{and} \quad x \cdot \bigcap_{r \in S_j} F_r(x) = 0, \quad j=1, \ldots, n.
\]

(C7)

Cottle and Dantzig [2] treat the linear case i.e. when \( f \) is affine and provide conditions guaranteeing the existence of a complementary solution.

It is shown in [6] that if \( h \) is a continuous \( Z \)-function and \( y \) is a least element of \( \{ y | h(y) + q \geq 0, \ y \geq 0 \} \), then \( y' (h(y)+q) = 0 \). Consequently we can conclude that if \( f \) is a continuous pre-Leontief function and \( x \) is a least element of \( X_1^+ \), then for each \( i \in I \), there exists \( j = j(i) \in I(i) \) such that \( x_i > a_i \) implies \( f_j(x) + q_j = 0 \).

The latter observation can be interpreted in terms of the generalized complementarity property presented above. If \( f : \mathbb{R}^n_+ \to \mathbb{R}^m \) is pre-Leontief and continuous define \( S_i = I(i), i=1, \ldots, n \) where \( I(i) \) is given by (2). Assuming for simplicity that \( a = 0 \) we conclude that for any \( q \in \mathbb{R}^m, x_1^+ \neq \emptyset \) implies the existence of a least element which is also a complementary solution to the generalized complementarity problem (C7) defined by \( F(x) = f(x) + q \).
References


