Ascending Auctions and Linear Programming*

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Abstract

Based on the relationship between dual variables and sealed-bid Vickrey auction payments (established by Bikhchandani and Ostroy), we consider simpler, specific formulations for various environments. For some of those formulations, we interpret primal-dual algorithms as ascending auctions implementing the Vickrey payments. We focus on three classic optimization problems: assignment, matroid, and shortest-path problems.

Keywords Vickrey auctions, multi-item auctions, combinatorial auctions, duality, primal-dual algorithm

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1 Introduction

For situations in which a seller is interested in socially efficient allocation, the Vickrey sealed-bid auction has received considerable attention (Vickrey (1961), Clarke (1971), Groves (1973)). In such an auction, bidders submit information about their preferences to the auctioneer, who then uses it to decide how to choose an efficient allocation, and how to charge the bidders. Since payments are computed in such a way as to align (i) the bidder's objective to maximize profits and (ii) the seller's objective to maximize social surplus, it is always in each bidder's best interest to report his preference information truthfully. Specifically, a bidder's payment equals the net effect his presence has on everyone else.

To formalize this in an abstract private-goods setting, consider a set of agents N, each of whom has a (monetary) value $v_j(a_j)$ for any bundle a_j received under some "allocation" of goods $a \in A$. The seller's objective described earlier is to find the "efficient" allocation a^* which solves $V(N) \equiv \max_{a\in A} \sum_{j\in N} v_j(a_j)$.

Consider a situation in which bidder $k \in N$ were not present. The seller's objective would be to find $V(N \setminus k) \equiv \max_{a\in A} \sum_{j\in N\setminus k} v_j(a_j)$. In this sense, the net effect that k's presence has on the other bidders equals

$$V(N \setminus k) - \sum_{j\in N\setminus k} v_j(a_j^*)$$

which is precisely bidder k's *Vickrey payment*.

Bidder k's net payoff in a Vickrey auction is therefore

$$v_k(a_k^*) - \left[V(N \setminus k) - \sum_{j\in N\setminus k} v_j(a_j^*)\right] = V(N) - V(N \setminus k). \qquad (1)$$

That is, his net payoff equals his net *contribution* to attainable social surplus, which is why this amount is also called bidder k's *marginal product*.

The computation of payments in a Vickrey sealed-bid auction can be found by solving $n+1$ optimization problems: one to find $V(N)$ (and a^*), and n more to find each $V(N \setminus j)$. However, in many environments the problem of finding $V(N)$ is a linear program. Furthermore, an agent's presence can be seen in the choice of constraints (rows) or variables (columns) of the linear program. Therefore, it is tempting to think that the *effect* of his presence, i.e. his marginal product, might be encoded in the optimal dual variables of the linear program—these variables inform us of the effect of changing the right hand side of a constraint. Whenever such a connection would exist, payments for sealed-bid Vickrey auctions could be computed with a single linear program (producing $V(N)$ and a^*) and its dual (producing marginal products). Since each $v_j(a_j^*)$ can be computed from the program, the amounts of payments follow immediately from (1).

A byproduct of this connection between linear programming variables and Vickrey payoffs/payments applies to ascending auctions. While the sealed-bid version of the Vickrey auction has the appeal of the properties discussed above, there may, in some environments, be practical reasons to prefer a dynamic, ascending implementation of this auction outcome.[1] For example, the ascending English auction for a single object duplicates the Vickrey outcome.

The byproduct of which we speak involves the fact that auctions can be interpreted as decentralized algorithms. Therefore, it is also tempting to think that an algorithm which solves the above linear programs may be interpreted as an auction for that environment. Since such dynamic auctions are based on the Vickrey auction, which has the dominant-strategy incentive property alluded to earlier, it is not surprising that the dynamic versions also provide nice incentives. In particular, in the Vickrey auctions we discuss below, "truthful bidding" is an ex-post Nash Equilibrium: even if a bidder had perfect knowledge about exactly how all other bidders planned to (truthfully) behave in the auction, that bidder could do no better than by bidding truthfully.[2]

In this paper, we examine this connection between linear programming variables and Vickrey payoffs for the three polynomially solvable problems: assignment, matroid optimization, and shortest path problems.

In Section 2 we examine various generalizations of the classic assignment problem, and show how the choice of a linear programming formulation affects our interpretation of prices in various previous works. In particular, depending on the model, certain "naive" linear programming formulations may not provide Vickrey payoff information. At the other extreme, Bikhchandani and Ostroy (2000a) provide a general formulation that applies whenever the connection exists with Vickrey payoffs, and provide a necessary and suffi-

[1]Such reasons may include the auctioneer's credibility, perceptions of fairness, etc; see Ausubel (2002a).

[2]For the sake of brevity, we omit a more formal description of this property, since the formal definition of *truthful bidding* is dependent on the context of the specific auction. We believe the idea is clear; one proof of this type of result is given by Bikhchandani and Ostroy (2000b).

cient condition for the connection to exist. Other, simpler formulations for restricted models are provided by Demange, Gale, and Sotomayor (1986) and by Gul and Stacchetti (1999,2000). The latter formulation does not produce Vickrey prices. This fact, and its connection to the choice of formulation, are the emphasis of Section 2, though we also emphasize this point in the following sections.

In Section 3 we show that the connection between Vickrey payoffs and linear programming holds for matroid optimization problems. While this means that one can therefore use Bikhchandani and Ostroy's formulation above, we provide a simpler formulation. Furthermore, the primal-dual algorithm for our formulation has a natural auction interpretation, which is related to Ausubel's (2002a) auction for homogeneous goods.

Finally, we discuss shortest path problems in Section 4. In general, the connection between Vickrey payoffs and linear programming may not hold in that setting (as opposed to the matroid problem). However, there are some restricted environments where it does. We discuss one which admits a very simple formulation.

2 A Choice of Formulation

In this section, we use a brief review of the ascending auction literature for certain matching models to illustrate the way in which the choice of a formulation impacts the resulting interpretation of variables, and the corresponding construction of an auction.

Consider the case of an auctioneer who must sell a set of distinct, heterogeneous, indivisible objects, M, to a set of bidders, N. For any set of objects $S \subseteq M$, let $v_j(S) \geq 0$ be the value that bidder $j \in N$ assigns to S. We assume that valuations are monotonic: for all $j \in N$, $S \subset T \subseteq M$ implies $0 \leq v_j(S) \leq v_j(T)$.

An *allocation* is an assignment of objects to the agents. An *efficient* allocation maximizes the sum of the agents' valuations.

The problem of finding an *efficient* allocation may be formulated as follows. Let $y(S, j) = 1$ if bidder j is to be allocated the bundle $S \subseteq M$, and

$y(S,j) = 0$ otherwise. The optimization problem is

$$\textbf{(P1)} \qquad \begin{aligned} V(N) = \max & \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S,j) \\ \text{s.t.} & \sum_{S \ni i} \sum_{j \in N} y(S,j) \leq 1 \quad \forall i \in M \\ & \sum_{S \subseteq M} y(S,j) \leq 1 \quad \forall j \in N \\ & y(S,j) = 0,1 \quad \forall S \subseteq M, \forall j \in N \end{aligned}$$

The first constraint of (P1) ensures that an object is assigned at most once, while the second ensures that a bidder receives at most one subset. The size of this formulation is exponential in the number of objects.

2.1 The Assignment Problem

As a special case of the above problem, consider a model in which bidders obtain value from at most one object at a time. In this model, it is without loss of generality to assume that each bidder consumes at most one object, so we have the classic the assignment problem, where valuation functions v_j are such that $v_j(S) = \max_{i \in S} v_j(i)$.

In this case, formulation (P1) can be rewritten as follows, where $x_{ij} = 1$ signifies that bidder j consumes object i. This standard formulation of the assignment problem has integral extreme points, so we can use a linear formulation.

$$\begin{aligned} \max & \sum_{j \in N} \sum_{i \in M} v_j(i) x_{ij} \\ \text{s.t.} & \sum_{j \in N} x_{ij} \leq 1 \quad \forall i \in M \\ & \sum_{i \in M} x_{ij} \leq 1 \quad \forall j \in N \\ & x \geqq 0 \end{aligned}$$

Its dual is:

$$\begin{aligned}
\min & \sum_{j\in N} u_j + \sum_{i\in M} p_i \\
\text{s.t. } & u_j + p_i \geq v_j(i) \quad \forall j \in N, \forall i \in M \\
& u, p \geqq 0
\end{aligned}$$

Leonard (1983) shows that among all optimal dual solutions, (u^*, p^*), the one that maximizes $\sum_{j\in N} u_j$ yields the bidders' Vickrey payments: Specifically u_j^* is the marginal product of agent j. Since the value of a dual variable represents the change in objective function value from a small change in the right hand side of the relevant constraint, then this should come as no surprise in the unit demand setting. Indeed, changing the right hand side of the constraint $\sum_{i\in M} x_{ij} \leq 1$ to zero effectively removes bidder j from the auction. As we shall see in Section 2.2, this need not be true in more general applications of (P1).

Another feature of this model is that if we interpret p_i as a price for object i, we see that Vickrey payments can be supported by *Walrasian* (or *anonymous*) object prices: each agent consumes a demanded object (and makes payments) with respect to prices that are indexed only by the identity of the object. This is another result that does not extend to more general models (Section 2.2).

Of particular interest at present is that a particular implementation of the primal-dual algorithm for solving the assignment problem—one which finds a dual solution that maximizes $\sum_{j\in N} u_j$—produces an ascending auction that duplicates the outcome of the sealed-bid Vickrey auction. In this auction, it is an ex-post Nash equilibrium for all players to bid truthfully in each round.

Such an auction was proposed by Demange, Gale, and Sotomayor (1986) (building on the work of Crawford and Knoer (1981)). To relate it to a primal-dual algorithm, consider an initial, feasible (but typically non-optimal) dual solution in which $p_i = 0$ for each object $i \in M$, and $u_j = \max_i(v_{ij} - p_i)$ for each bidder $j \in N$. We interpret p_i as the price of object i and u_j as the potential surplus that bidder j could achieve when facing prices p. To determine whether (u, p) is optimal, we search for a primal solution x with which it satisfies the complementary slackness conditions.

There are three types of complementary slackness conditions, which have interpretations as follows. First (for u_j), a bidder who has positive surplus $u_j > 0$ must be assigned an object. Second (for p_i), an object with positive price must be assigned. Third (for x_{ij}), if object i is assigned to bidder j, then it must be surplus-maximizing: $v_j(i) - p_i = u_j$.

Given our initial dual solution, it turns out that the only "important" complementary slackness conditions are the first and third, together stating that a bidder with positive surplus must be assigned a surplus-maximizing object. One restricted primal for the algorithm attempts to find an assignment of objects that maximizes the number of positive-surplus agents who consume a surplus-maximizing object. Either an assignment exists which satisfies each such agent, or there exists an *overdemanded* set of objects, $S \subset M$: a set whose cardinality is less than the number of agents whose surplus-maximizing objects are all in S.[3] The primal-dual algorithm adjusts (u, p) by increasing the prices of objects in overdemanded sets (and decreases u accordingly).

There is one subtlety remaining. Vickrey prices are obtained by finding an optimal (u, p) which maximizes $\sum u_j$, or equivalently minimizes $\sum p_i$. This choice can be imposed by perturbing the original dual problem, above. The effect this has on the restricted primal is that prices are adjusted only on *minimal* (with respect to set inclusion) overdemanded sets.

With this caveat, we have the auction of Demange, Gale, and Sotomayor. Denote the round-t price of object i by p_i^t (initially, $p_i^0 \equiv 0$). In each round, bidders are asked to name their surplus-maximizing (*demanded*) objects. If a demand-satisfying assignment exists, the auction ends at current prices. Otherwise, the price of each object in a (or all) minimally overdemanded set(s) is incrementally increased for the next round. When bidders report their demanded objects truthfully in each round, the auction mimics the primal-dual algorithm described above, yielding the Vickrey outcome.

In summary, for the assignment problem,

- Dual variables for formulation (P1) (or a simplified version) yield Vickrey payments.

- A primal-dual algorithm yields the ascending auction of Demange, Gale, and Sotomayor (1986).

- Object prices can be expressed anonymously, i.e. they are not bidder-specific.

[3]The existence of such a set is proven via Hall's Marriage Theorem. It can also be derived from the dual of the restricted primal.

2.2 Consuming Multiple Objects

Now we consider a more general application of formulation (P1), when agents may receive value from multiple objects. Auctions were proposed for this setting by Kelso and Crawford (1982) and by Gul and Stacchetti (2000). While these auctions can be interpreted as primal-dual algorithms, they do not implement Vickrey pricing; hence it is not surprising that they do not share the incentives properties of the assignment auction discussed above.

Kelso and Crawford (1982) and Gul and Stacchetti (2000) both impose an assumption on bidders valuation functions v_j, which ensures that the linear relaxation of (P1) has an optimal integer solution. To describe this property let $p \in \mathbb{R}^m$ be a list of individual object prices. Denote bidder j's *demand set* (with respect to p) as follows.

$$D_j(p) = \{S \subseteq M : v_j(S) - \sum_{i \in S} p_i \geq v_j(T) - \sum_{i \in T} p_i,\ \forall T \subseteq M\}.$$

The condition introduced by Kelso and Crawford (1982) is the *gross substitutes* condition, which, loosely speaking, states that a price increase on a set of objects should not *decrease* a bidder's demand for the other objects. If a set of objects A is demanded, and the prices of some of those objects increase, then the remaining objects should still be part of some demanded set B.

Definition 1 *Bidder j's value function satisfies* ***gross substitutes*** *if for any two price vectors $p' \geq p$, and all $A \in D_j(p)$, there exists $B \in D_j(p')$ such that $\{i \in A : p_i = p'_i\} \subseteq B$.*

With our monotonicity condition assumed earlier, gross substitutes implies that v_j is submodular.

Gul and Stacchetti (2000) and Kelso and Crawford (1982) show that under gross substitutes, the linear relaxation of (P1) has an optimal integer solution.[4]

The dual to the linear relaxation of (P1) is

$$\textbf{(D1)} \qquad \min \sum_{j \in N} u_j + \sum_{i \in M} p_i$$

$$\text{s.t. } u_j + \sum_{i \in S} p_i \geq v_j(S) \quad \forall j \in N,\ \forall S \subset M$$

$$u, p \geqq 0$$

[4] A direct proof of this is provided in Theorem 4 of Bikhchandani et al. (2002).

Analogous to the interpretations in the assignment problem, p_i represents a (bidder-independent) price for object i, and u_j represents the profit of bidder j. The complementary slackness conditions are also analogous to those of the assignment problem.

Gul and Stacchetti's auction can be compared to a primal-dual algorithm which finds an optimal solution to (D1), and which is analogous to the one discussed above for the assignment problem. For any current object-prices (initialized with $p_i \equiv 0$), determine each bidder's *demand set*,

$$D_j(p) = \{S \subseteq M : v_j(S) - \sum_{i \in S} p_i \geq v_j(T) - \sum_{i \in T} p_i \quad \forall T \subseteq M\}.$$

which is the list of surplus-maximizing *subsets* of objects. Define the *minimum demand requirement* of a bidder from a subset of objects, T, as

$$r_j(T, p) = \min_{S \in D_j(p)} |S \cap T|.$$

Gul and Stacchetti (2000) show that this function is the dual rank function of a matroid.

One primal-dual algorithm for (P1) attempts to assign objects in a demand-satisfying way. If no such assignment exists, then there must exist an overdemanded set, T:[5]

$$\sum_{j \in N} r_j(T, p) > |T|$$

The algorithm adjusts the current dual solution by incrementally increasing the price of each object in a (minimal) overdemanded set.

Assuming sincere bidding at each round, the auction concludes with a list of object prices p which equals the minimal Walrasian price[6] vector. As noted by Gul and Stacchetti, the auction is not guaranteed to produce Vickrey prices. Typically, bidders' payoffs at (any) Walrasian prices are higher than their Vickrey payoffs, implying that Vickrey payoffs cannot be implemented with anonymous, single-object (additive) prices.

In summary, for the heterogeneous objects model under gross substitutes,

- Dual variables for formulation (P1) yield Walrasian (anonymous) prices, which typically do not coincide with Vickrey payments.

[5] See Gul and Stacchetti (2000).

[6] Prices are Walrasian for an allocation of objects if, given those prices, each agent's consumption maximizes his payoff (net of prices).

- A primal-dual algorithm for this type of formulation yields the ascending auction of Gul and Stacchetti (2000) and Kelso and Crawford (1982).

2.3 The Bikhchandani and Ostroy formulation

Since the variables of (D1) give anonymous object prices, a primal-dual algorithm for this formulation cannot yield Vickrey payments except in special cases such as the assignment game. To obtain Vickrey payments in the general case, a richer pricing language (i.e. a stronger formulation) is needed. One way to do this is by using auxiliary variables. Below, we illustrate this with the (exponentially sized) formulation of Bikhchandani and Ostroy (2000a),[7] who identify a necessary and sufficient condition for the dual variables to correspond to the agents' marginal products.

Bikhchandani and Ostroy (2000a) introduce a variable for every feasible integer solution to (P1). Let μ denote both a partition of the set of objects *and* an assignment of the elements of the partition to bidders. Thus μ and μ' can give rise to the same partition, but to different assignments of the parts to bidders. Let Γ denote the set of all such partition-assignment pairs. We will write $S^j \in \mu$ to mean that under μ, agent j receives the set S. Let $\delta_\mu = 1$ if the partition-assignment pair $\mu \in \Gamma$ is selected, and zero otherwise. Using these new variables the efficient allocation can be found by solving the following formulation.

$$\textbf{(P2)} \qquad V(N) = \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S,j)$$

$$\text{s.t. } y(S,j) \leq \sum_{\mu \ni S^j} \delta_\mu \quad \forall j \in N, \forall S \subseteq M$$

$$\sum_{S \subseteq M} y(S,j) \leq 1 \quad \forall j \in N$$

$$\sum_{\mu \in \Gamma} \delta_\mu \leq 1$$

$$y(S,j), \delta_\mu = 0,1 \quad \forall S \subseteq M, \forall j \in N, \forall \mu \in \Gamma$$

A straightforward argument shows that the linear relaxation of this formulation has the integrality property: To each partition-assignment pair μ

[7] For a formulation with fewer variables—but still exponentially sized—see Section 2.2 of Bikhchandani et al. (2002).

there is an associated integer solution: $y^\mu(S,j)$ equals one if S is assigned to agent j under μ and equals zero otherwise, and $\delta_\mu = 1$. If (y^*, δ^*) is an optimal, fractional solution with value $V_{LP}(N)$, then consider a solution that, with probability δ^*_μ, selects μ as the partition assignment pair. The expected cost of this solution is

$$\begin{aligned}
&\sum_{\mu} \delta^*_\mu \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y^\mu(S,j) \\
&= \sum_{j \in N} \sum_{S \subseteq M} v_j(S) \sum_{\mu \ni S^j} \delta^*_\mu \\
&\geq \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y^*(S,j) = V_{LP}(N)
\end{aligned}$$

where the inequality follows from the first constraint. Since the randomly generated integer solution is at least as good as (y^*, δ^*), at least one of these integer solutions must have a value at least as large as the optimal LP solution.

The significance of (P2) is that the dual variable associated with the second constraint can be interpreted as agent j's marginal product. For a bidder $j \in N$, reducing the right hand side of the corresponding constraint to zero has the effect of removing him from the auction.

The dual is

$$\textbf{(D2)} \qquad \begin{aligned}
\min & \sum_{j \in N} \pi_j + \pi^s \\
\text{s.t. } & p_j(S) + \pi_j \geq v_j(S) \quad \forall j \in N, \forall S \subseteq M \\
& -\sum_{S^j \in \mu} p_j(S) + \pi^s \geq 0 \quad \forall \mu \in \Gamma \\
& \pi_j, \pi_S, p_j(S) \geq 0 \quad \forall j \in N, \forall S \subseteq M
\end{aligned}$$

Each variable $p_j(S)$ can be interpreted as bidder j's price for subset S, while π_j can be interpreted as j's surplus. The variable π^s can be interpreted as the auctioneer's surplus.

The difficulty is to ensure that a dual solution exists such that, *simultaneously*, each π_j equals bidder j's marginal product. Bikhchandani and Ostroy (2000a) derive the following necessary and sufficient condition for this to be realized.

Definition 2 *The **agents are substitutes** condition holds if*

$$V(N) - V(N \setminus K) \geq \sum_{j \in K} [V(N) - V(N \setminus j)] \quad \forall K \subseteq N. \tag{2}$$

Bikhchandani et al. (2002, Section 4.1) prove that this condition is implied by the gross substitutes condition. It is clearly implied by submodularity of V. This condition formalizes the notion that the contribution (i.e. marginal product) of a group of agents is more than the sum of the contributions (marginal products) of the individual members of the group. Such a condition applies, for example, to situations in which workers are better-off forming a union rather than bargaining separately with management.

Theorem 1 (Bikhchandani and Ostroy, 2000a) *The "agents are substitutes" condition (2) holds if and only if there exists an optimal solution (p, π) to (D2) such that $\pi_j = V(N) - V(N \setminus j)$ for all $j \in N$, i.e. the dual provides each agent's marginal product.*

To summarize,

- Richer formulations such as (P2) can provide agents' Vickrey payments via dual variables.

To implement the Vickrey outcome, Ausubel (2002b) proposes using a collection of ascending auctions that together implement the Vickrey outcome (under the gross substitutes condition). This auction can be interpreted as $n + 1$ simultaneous applications of the Gul–Stacchetti algorithm, where one application determines the assignment of objects, while the other n help to determine payments. For more general environments, Ausubel and Milgrom (2002) and Parkes and Ungar (2000) have proposed similar ascending auctions that are subgradient algorithms for (P2). This allows them to drop the gross substitutes assumption and determine (non-linear, non-anonymous) non-Vickrey prices that support an efficient allocation. For a discussion of the connections between these auctions and algorithms for solving (P2) see de Vries, Schummer, and Vohra (2002).

3 Matroids

Here we consider an abstract setting involving the auction of various combinatorial entities. A special case is Ausubel's (2002a) ascending auction for homogenous goods with diminishing marginal utilities.

The setting considered involves a ground set E, and for each agent $j \in N$ a set $E_j \subseteq E$ that j has the possibility of acquiring. We assume that $(E_j)_{j \in N}$ is a partition for E, though this can be relaxed. For each $e \in E$, let $v_e \in \mathbb{Z}_{++}$ be the integer value of e. The value of a set $S \subseteq E_j$ to agent j is $\sum_{e \in S} v_e$. Let I be a family of independent subsets of E and suppose $\mathcal{M} = (E, I)$ forms a matroid with rank function r. The optimization problem we consider is that of finding a maximum weight basis of $\mathcal{M}$. We derive a primal-dual algorithm for this problem which will implement the Vickrey outcome.

Before proceeding, it is useful to consider a special case in order to develop some intuition. Let $G = (V, E)$ be a complete graph with vertex set V and edge set E. Each edge may be owned by a given agent (and an agent has the right to own only a single predetermined edge). Therefore we may use the words *edge* and *agent* interchangeably. Let v_e be the weight of edge e. Our goal is to derive an ascending Vickrey auction to sell off a maximum weight spanning tree.

Though we shall speak in terms of "selling" edges, one interpretation for this problem involves a procurement setting, where the auctioneer wants to *purchase* the right to use an edge and the bidder incurs some cost $(-v_e)$ when it is used (e.g. constructing a complete communications network at minimal total social cost). In order to be consistent with the rest of the paper, we avoid procurement examples and say that the auctioneer is selling to bidders the right to use an edge, incurring a gain of $v_e \geq 0$.[8]

An important observation to make is that, instead of selling an edge, the auctioneer is actually selling the right to "cover" a cut in the graph. A bidder is competing with all other bidders that can cover the same collection of cuts that he can. This can be seen when we compute the marginal product of an edge.

Let T be a maximum weight spanning tree and suppose $e \in T$. To determine agent (or edge) e's marginal product we must identify the reduction in weight of the spanning tree when we remove agent e and replace her with a (next best) edge. If $f \notin T$ is the largest weight edge such that $T \cup f$ contains a cycle through e, then the maximum weight spanning tree that excludes e is $T \setminus \{e\} \cup f$. Thus agent e's marginal product is $v_e - v_f$.

There are a number of algorithms for finding a maximum weight spanning

[8] If instead bidders incur costs ($v_e < 0$), then we can suppose that bidders bid on the right to supply their edges for some fixed payment M. If M can be chosen sufficiently high to guarantee $M > v_e$ for each e, then this setting is equivalent to the one we describe.

tree, but not all lend themselves to an auction interpretation. Furthermore, not all of them terminate in Vickrey prices. The "greedy out" algorithm does: starting with the complete set of edges, begin delete edges in order of increasing weight. An edge is deleted only if the remaining graph is connected. An edge is skipped for deletion when all smaller weight edges that could cover the same cut have already been deleted.

This algorithm can be interpreted as an auction which begins with a price $p = 0$ on each edge. Throughout the auction, this price is increased. At each point in time, each agent announces whether he is willing to purchase his edge at the current price.

As the price increases, agents drop out of the auction as the price exceeds their value v_e for the edge, reducing the connectivity of the graph. At some point, an agent will become *critical*: removing the agent from the auction would mean that no spanning tree could be formed from the remaining edges of the other agents. At this point, the auctioneer immediately sells the edge to the critical agent at the current price. This edge is to be part of the final (maximum weight) spanning tree and does not drop out.

The auction then continues, with other agents dropping out or becoming critical. The auction ends when the last critical agent is awarded an edge, and the tree is formed.

Notice that a critical agent acquires his edge at the price where another bidder dropped out of the auction. That price is the second-largest weighted edge that could have covered the same cut as the critical agent. This is the price a Vickrey auction dictates he should pay.

In what follows we show how this auction is an instance of a primal-dual algorithm for an appropriate formulation of the underlying optimization problem. The analysis will involve some fine points that arise because of tie breaking issues, but the main thrust is as described above.

3.1 Substitutes Property

Let L denote the weight of a maximum weight basis. For any subset $E' \subseteq E$, let $L^{-E'}$ be the weight of the maximum weight basis that does not use any element of E'. Set $L^{-E'} = -\infty$ if the bases of $E \setminus E'$ have fewer elements than those of E. We assume a no-monopoly condition that ensures finite Vickrey payments for the agents: $r(E) = r(E \setminus E_j)$ for all $j \in N$.

Theorem 2 *In the (no-monopoly) matroid problem, the "agents are substitutes" condition (2) holds: For all $K \subseteq N$,*

$$L - L^{-E_K} \geq \sum_{i \in K} [L - L^{-E_i}].$$

Proof For $A \subseteq E$ let L^A denote the value of a maximum weight basis in A if it has the same cardinality as a basis of E and $-\infty$ otherwise. The claim holds, if $L^{A \cup B} + L^{A \cap B} \leq L^A + L^B$. This follows immediately from a greedy construction. Construct a maximum weight basis T_K for $K = A$, B, $A \cup B$ and $A \cap B$ simultaneously. By submodularity of the matroid rank function, $|T_{A \cup B}| + |T_{A \cap B}| \leq |T_A| + |T_B|$. So the j^{th} element added to the left has weight smaller than or equal to the j^{th} element added to the right hand side.[9] ∎

This result combined with Theorem 1 implies that (D2) contains the Vickrey prices for this setting. One could then apply a primal dual algorithm to that formulation to produce an ascending auction that implements the Vickrey outcome. Such an auction would result in prices that are both non-linear and non-anonymous.

Below, however, we provide a more parsimonious formulation that supports the Vickrey outcome by price functions that are anonymous and "almost" linear.

3.2 A Formulation

The standard formulation for identifying a maximum weight basis does not suffice because the variables in the dual price the wrong entities. By analogy with the spanning tree case, we need a formulation whose dual will price the "right" to cover a cut. Before describing this formulation we review some facts about matroids.

The matroid polytope of a matroid with rank-function r on ground set E is defined by

$$P(r) = \{x \in \mathbb{R}^n_+ : \sum_{e \in S} x_e \leq r(S) \quad \forall S \subseteq E\}$$

We use the following theorem from Section III of Nemhauser and Wolsey (1988) concerning matroid polytopes.

[9]We thank a referee for simplifying our previous proof to this argument.

Theorem 3 *If r is the rank function of a matroid then $P(r)$ is integral.*

Thus every extreme point of $P(r)$ corresponds to an independent set and vice versa.

The *closure* of a set $S \subseteq E$ is defined as $\mathrm{cl}(S) = \{e \in E\colon r(S) = r(S \cup \{e\})$. It is called *spanning* if $\mathrm{cl}(S) = E$ (i.e. $r(S) = r(E)$).

The matroid $\mathcal{M}^*$ dual to (E, I) is defined by the collection of bases $\{E \backslash S : S$ a basis of $(E, I)\}$. The rank-function r^* of the matroid dual to (E, I) is given by $r^*(S) = |S| + r(E \setminus S) - r(E)$. If S is spanning in $\mathcal{M}$ and x is its characteristic vector, then using the dual matroid, x must satisfy $\sum_{e \in T}(1 - x_e) \leq r^*(T)$ for all $T \subseteq E$, because S^c is independent in $\mathcal{M}^*$. Since $P(r^*)$ is integral, it follows that this polytope is integral. We can rewrite the inequalities that describe the polytope of spanning sets to read $r(E) - r(E \setminus T) \leq \sum_{e \in T} x_e$ for all T.

Theorem 4 *Suppose each $e \in E$ has a value v_e. Then a maximum weight basis of the matroid can be found by solving the linear program*

$$
\begin{aligned}
\textbf{(M)} \qquad & \max \sum_{e \in E} v_e x_e \\
& \text{s.t.} \sum_{e \in T} x_e \geq r(E) - r(E \setminus T) \quad \forall T \subseteq E \\
& \sum_{e \in E} x_e \leq r(E) \\
& 0 \leq x_e \leq 1 \quad \forall e \in E.
\end{aligned}
$$

All of its extreme points are integral.

Proof Since the polytope of spanning sets is integral and the inequality $r(E) - r(\emptyset) \leq \sum_{e \in E} x_e$ $(T = E)$ induces a face, it follows that the intersection of the polytope of spanning sets with the inequality $r(E) - r(\emptyset) \geq \sum_{e \in E} x_e$ is integral. In fact this polytope is the polytope of bases of the matroid. ■

For each $T \subseteq E$ define

$$r^T(S) = r(S \cup (E \setminus T)) - r(E \setminus T) \quad \forall S \subseteq E.$$

The function r^T is the rank function of the matroid with ground set T, obtained by contracting the elements of $E \setminus T$. To interpret it, consider the tree case, where T is a set of edges. When we remove the set of edges T and shrink each remaining connected components of the graph into pseudo-vertex, r^T is the rank function of the matroid defined by the tree defined over T and the set of pseudo-vertices.

The problem of finding the maximum weight bases can be formulated as follows (where variables in parenthesis represent dual variables). For each $F \subseteq E_j$, set $v_j(F) = \sum_{e \in F} v_e$. For each $j \in N$ and $F \subseteq E_j$, interpret $y_j(F) = 1$ to mean F is selected to be in a maximum weight basis.

$$
\textbf{(MP)} \qquad \begin{aligned}
&\max \sum_{j \in N} \sum_{F \subseteq E_j} v_j(F) y_j(F) \\
&\text{s.t.} \sum_{j \in N} \sum_{F \subseteq E_j} r^T(F) y_j(F) \geq r(E) - r(E \setminus T) \quad \forall T \subseteq E \qquad (\mu_T) \\
&\sum_{j \in N} \sum_{F \subseteq E_j} |F| y_j(F) \leq r(E) \qquad (\mu) \\
&\sum_{F \subseteq E_j} y_j(F) \leq 1 \quad \forall j \in N \qquad (\lambda_j) \\
&y_j(F) \geq 0 \quad \forall j \in N, \forall F \subseteq E_j.
\end{aligned}
$$

Theorem 5 *Formulation (MP) has an optimal integral solution.*

Proof Let y denote an optimal extreme solution (possibly fractional) to (MP). For each agent j and element $e \in E_j$, set $z_e = \sum_{F \ni e} y_j(F)$. We show that z is a feasible solution to (M).

Furthermore, since we prove below that each vertex of (M) has an integral preimage in (MP), we conclude that (M) is a projection of (MP). Given an integral solution x^* to (M), we can construct an integral solution y^* to (MP). Let $M_j = \{e \in E_j : x^*_e = 1\}$. For all $j \in N$, set $y^*_j(M_j) = 1$ and $y^*_j(F) = 0$ for $F \neq M_j$. Observe that $\sum_{e \in E} v_e x^*_e = \sum_{j \in N} v_j(M_j) y_j(M_j)$, i.e. both solutions have the same objective function value.

Returning to z_e, notice that $z_e \geq 0$. Also,

$$
\sum_{e \in E} z_e = \sum_{j \in N} \sum_{e \in E_j} \sum_{F \subseteq E_j : e \in F} y_j(F) = \sum_{j \in N} \sum_{F \subseteq E_j} |F| y_j(F) \leq r(E).
$$

Next, for $T \subseteq E$

$$\sum_{e \in T} z_e = \sum_{j \in N} \sum_{e \in E_j \cap T} \sum_{F \subseteq E_j : e \in F} y_j(F)$$
$$= \sum_{j \in N} \sum_{F \subseteq E_j} |F \cap T| y_j(F) \geq \sum_{j \in N} \sum_{F \subseteq E_j} r^T(F) y_j(F) \geq r(E) - r(E \setminus T).$$

Notice also that $\sum_{j \in N} \sum_{F \subseteq E_j} v_j(F) y_j(F) = \sum_e v_e z_e$. Hence, if z_e is integral then we are done. If not, we can express z_e as a convex combination of integral points of (M). Each of these, however, corresponds to an integral point of (MP). Thus, y can be expressed as a convex combination of integral points of (MP), which is a contradiction. ■

The dual to (MP) is

$$\textbf{(DMP)} \qquad \min - \sum_{T \subseteq E} (r(E) - r(E \setminus T)) \mu_T + r(E)\mu + \sum_{j \in N} \lambda_j$$
$$\text{s.t.} - \sum_{T \subseteq E} r^T(F) \mu_T + |F| \mu + \lambda_j \geq v_j(F) \quad \forall j \in N, \forall F \subseteq E_j$$
$$\mu, \mu_T \geq 0 \quad \forall T \subseteq E$$
$$\lambda_j \geq 0 \quad \forall j \in N.$$

Here, λ_j can be interpreted as agent j's surplus; μ represents a price of any element $e \in E$; the interpretation of μ_T may not be obvious, a priori. However, the description of the primal-dual algorithm in Section 3.3 leads to an interpretation where μ_T represents a discount to agents who are sold elements that cover T. This leads to an interpretation of prices analogous to one regarding (D2): the total price that agent j faces for $F \subset E_j$ is $-\sum_{T \subseteq E} r^T(F)\mu_T + |F|\mu$ (see the first constraint).

If B is a maximum weight basis, let $B^j = E_j \cap B$ for each $j \in N$. Observe that each B^j is an independent set. For each k let G^k be the largest weight set in $\cup_{j \neq k} E_j$ such that $B^{-k} = (B \setminus B^k) \cup G^k$ is a basis. By the property of matroids this is well defined. Furthermore, B^{-k} is the maximum weight basis that excludes all elements of E_k. Thus, agent k's marginal product is $v_k(B^k) - \sum_{j \neq k} v_j(E_j \cap G^k)$.

Theorem 6 *There is an optimal solution to (DMP) such that*

$$\lambda_k = v_k(B^k) - \sum_{j \neq k} v_j(E_j \cap G^k) \quad \forall k \in N.$$

Proof In (DMP) we set $\lambda_k = v_k(B^k) - \sum_{j\neq k} v_j(E_j \cap G^k)$ for all $k \in N$ and show how to choose μ and $\{\mu_T\}_{T\subseteq E}$ to maintain dual feasibility.

First, for each $e \in B^j$ we can find $e' \in G^j$ such that $B \cup e'$ contains a unique circuit through e and $v_{e'} \leq v_e$. In this way we can, for each $e \in B^j$ pair it with a unique element e' of G^j such that $v_e \geq v_{e'}$. Hence $\lambda_j \geq 0$. Modify the weight of all elements by decreasing the weight of each v_e to $v_{e'}$ for all $e \in B$. The weights of elements outside B remain unchanged. Call the new weight vector w. Hence $w_j(F) = v_j(F \setminus B^j) + \sum_{e\in F\cap B^j} v_j(e')$. The set B is still a maximum weight basis with respect to the weight vector w.

Denote by **(wMP)** and **(wDMP)** the primal and dual formulations respectively of the matroid optimization problem with weight vector w. Let $(\mu^*, \mu^*_T, \lambda^*)$ be an optimal solution to (wDMP).

We know that under weight vector w, B is an optimal weight basis. Therefore there is an optimal solution to (wMP), such that $y_j(F) = 0$ for all j and $F \neq B^j$. In addition, B^{-k} is also an optimal basis under weight vector w. That is, there is an optimal solution to (wMP) where $y_k(B^k) = 0$. Invoking complementary slackness we deduce that $\lambda^*_k = 0$ for all $k \in N$. Thus $-\sum_{T\subseteq E} r^T(F)\mu^*_T + |F|\mu^* \geq w_j(F)$ for all $j \in N$, $F \subseteq E_j$.

We now show that

$$-\sum_{T\subseteq E} r^T(F)\mu^*_T + |F|\mu^* + \lambda_j \geq v_j(F) \quad \forall j \in N, \forall F \subseteq E_j,$$

by proving that $w_j(F) \geq v_j(F) - \lambda_j$. This shows that $(\mu^*, \mu^*_T, \lambda)$ is feasible for (DMP). Now

$$\begin{aligned} v_k(F) - \lambda_k &= v_k(F \setminus B^k) + v_k(F \cap B^k) - v_k(B^k) + \sum_{j\neq k} v_j(E_j \cap G^k) \\ &= w_k(F \setminus B^k) + w_k(B^k) + v_k(F \cap B^k) - v_k(B^k) \leq w_k(F). \end{aligned}$$

To conclude we show that $(\mu^*, \mu^*_T, \lambda)$ is optimal for (DMP). Observe that

$$\begin{aligned} &-\sum_{T\subseteq E}(r(E) - r(E \setminus T))\mu^*_T + r(E)\mu^* + \sum_{j\in N}\lambda_j \\ &\qquad = \sum_{k\in N} w_k(B^k) + \sum_{k\in N}\left(v_k(B^k) - \sum_{j\neq k} v_j(E_j \cap G^k)\right) = \sum_{k\in N} v_k(B^k). \end{aligned}$$

completing the proof. ∎

3.3 Primal-Dual Algorithm

An initial dual feasible solution can be had by setting $\mu = 0$, $\mu_T = 0$ for all $T \subseteq E$, and $\lambda_j = \max_{F \subseteq E_j} v_j(F)$ for all $j \in N$. This step can be interpreted as the auctioneer initially setting a price of zero for each subset of E. For subsequent iterations of the algorithm,

$$\lambda_j = \max\{0, \max_{F \subseteq E_j} v_j(F) + \sum_{T \subseteq E} r^T(F)\mu_T - |F|\mu\}. \tag{3}$$

Denote agent j's demand correspondence, i.e. collection of surplus maximizing subsets of elements, as

$$\Delta^j = \{F \subseteq E_j \colon -\sum_{T \subseteq E} r^T(F)\mu_T + |F|\mu + \lambda_j = v_j(F)\}.$$

Notice that $\lambda_j = 0$ implies $\emptyset \in \Delta^j$.

Given a current feasible dual solution we seek a primal solution with which it satisfies complementary slackness. One complementary slackness condition is that if $F \in E_j \setminus \Delta^j$, then $y_j(F) = 0$.[10] Therefore, the remaining conditions (based on primal constraints) can be written as follows.

1. If $\lambda_j > 0$ then $\sum_{F \in \Delta^j} y_j(F) = 1$.
2. If $\mu > 0$ then $\sum_{j \in N} \sum_{F \in \Delta^j} |F| y_j(F) = r(E)$.
3. If $\mu_T > 0$ then $\sum_{j \in N} \sum_{F \in \Delta^j} r^T(F) y_j(F) = r(E) - r(E \setminus T)$.

A restricted primal is formed by appending these four conditions to the constraints of (MP).

For the rest of the section, we make an assumption on the feasibility of a system related to this restricted primal. This assumption corresponds to situations in which prices are "lower" than the final Vickrey prices, i.e. to situations in which prices are such that agents demand "too much."

[10]This is interpreted as saying that a non-demanded set F cannot be used.

Consider the following system.

$$\textbf{(OD)}\quad \sum_{j\in N}\sum_{F\in\Delta^j} r^T(F)y_j(F) \geq r(E) - r(E\setminus T) \quad \forall T\subseteq E$$

$$\sum_{j\in N}\sum_{F\in\Delta^j} r^T(F)y_j(F) = r(E) - r(E\setminus T) \quad \forall T\subseteq E \text{ with } \hat{\mu}_T > 0$$

$$\sum_{j\in N}\sum_{F\in\Delta^j} |F|y_j(F) \geq r(E)$$

$$\sum_{F\in\Delta^j} y_j(F) = 1 \quad \forall j\in N$$

$$y_j(F) = 0 \quad \forall F\notin\Delta^j$$

$$y_j(F) \geq 0 \quad \forall j\in N, \forall F\in\Delta^j.$$

We say that the *overdemand condition* holds if (OD) has a feasible solution, and every such solution satisfies the third constraint with strict inequality: $\sum_{j\in N}\sum_{F\in\Delta^j}|F|y_j(F) > r(E)$.

Under the initial choice of dual variables (μ's equal zero, $\lambda_j = \max v_j(F)$), the overdemand condition holds because $\Delta^j = \{E_j\}$ for all $j\in N$.

Below (Theorem 7) we show that the overdemand condition holds at each subsequent iteration of the algorithm except at termination. The economic interpretation of this condition is that at current prices, the demand from the agents in $\{j\in N : \lambda_j > 0\}$ exceeds the supply of available elements to form a basis of E. Given the overdemand condition, we can reformulate the restricted primal as follows.

$$
\begin{aligned}
\textbf{(RP)} \quad & \min \sum_{j \in N : \lambda_j > 0} |E_j| z_j + z \\
& \text{s.t.} \sum_{j \in N} \sum_{F \in \Delta^j} r^T(F) y_j(F) \geq r(E) - r(E \setminus T) \quad \forall T \subseteq E \\
& \sum_{j \in N} \sum_{F \in \Delta^j} |F| y_j(F) - z \leq r(E) \\
& \sum_{F \in \Delta^j} y_j(F) + z_j = 1 \quad \forall j \in N : \lambda_j > 0 \\
& \sum_{F \in \Delta^j} y_j(F) \leq 1 \quad \forall j \in N : \lambda_j = 0 \\
& y_j(F) \geq 0 \quad \forall j \in N, \forall F \in \Delta^j \\
& z, z_j \geq 0 \quad \forall j \in N : \lambda_j > 0
\end{aligned}
$$

Observe that (RP) is integral.

The dual to the restricted primal is

$$
\begin{aligned}
\textbf{(DRP)} \quad & \min - \sum_{T \subseteq E} (r(E) - r(E \setminus T)) \hat{\mu}_T + r(E) \hat{\mu} + \sum_{j \in N} \hat{\lambda}_j \\
& \text{s.t.} - \sum_{T \subseteq E} r^T(F) \hat{\mu}_T + |F| \hat{\mu} + \hat{\lambda}_j \geq 0 \quad \forall j \in N, \forall F \subseteq \Delta^j \\
& \hat{\mu}_T \geq 0 \quad \forall T \subseteq E \\
& \hat{\lambda}_j \geq -|E_j| \quad \forall j \in N : \lambda_j > 0 \\
& \hat{\lambda}_j \geq 0 \quad \forall j \in N : \lambda_j = 0 \\
& 0 \leq \hat{\mu} \leq 1
\end{aligned}
$$

We describe one iteration of the primal-dual algorithm. So as not to confound the main idea with tie breaking issues we suppose that no agent is exactly indifferent between purchasing a single element and purchasing nothing, i.e. that for all $j \in N$,

$$
Z^j \equiv \{e \in E_j : v_e + \sum_{T \subseteq E_j} r^T(\{e\}) \mu_T - |\{e\}| \mu = 0\} = \emptyset.
$$

We also assume that $U^j \equiv E_j \setminus \cup_{F \in \Delta^j} = \emptyset$. Subadditivity of r^T implies that $v_e + \sum_{T \subseteq E_j} r^T(\{e\}) \mu_T - |\{e\}| \mu < 0$ if and only if $e \in U^j$. Subsequently, we amend the description to account for $Z^j, U^j \neq \emptyset$.

If, in every optimal solution to the restricted primal, $z_k = 0$, it is because some element of Δ^k is required to satisfy one of the constraints

$$\sum_{j \in N} \sum_{F \in \Delta^j} r^T(F) y_j(F) \geq r(E) - r(E \setminus T).$$

For every agent $j \in N$, consider the matroid obtained by contracting the elements of $\bigcup_{k \neq j} \bigcup_{F \in \Delta^k} F \setminus E_j$ and removing all of its loops;[11] call this matroid $\mathcal{M}_j$, its ground set T^j, and its rank function r^j. If $T^j \neq \emptyset$, we call agent j and/or T^j **critical**.

We make two observations about this choice of T^j. The first is that for all $F \in \Delta^k$, $k \neq j$ we have $r^{T_j}(F) = r(F \cup (E \setminus T^j)) - r(E \setminus T^j) = 0$, as $F \cap T_j = \emptyset$. Second, there exists $F \in \Delta^j$ with $r^{T_j}(F) = r(E) - r(E \setminus T^j)$. If not, the set T^j would not be covered, contrary to the overdemand requirement.

Given an optimal solution to (RP) we construct a solution to the (DRP) as follows, where θ is a step-size to be specified later.

1. If $z > 0$ set $\hat{\mu} = \theta$.
2. If $z = 0$ but at least one $z_k = 1$, set $\hat{\mu} = \theta$.
3. If $z_k = 0$ in every optimal solution, find T^k and set $\hat{\mu}_{T^k} = \hat{\mu}$.
4. Set $\hat{\lambda}_k = \max_{F \in \Delta^k} \{-|F|\hat{\mu} + \sum_{T \subseteq E} r^T(F)\hat{\mu}_T\}$.

It is easy to check that this solution is feasible in (DRP).

To show that it represents a valid direction along which to change the current dual solution we must verify that it has a negative objective function value. For each $k \in N$ let $F^k \in \arg\max_{F \in \Delta^k} (-|F|\hat{\mu} + \sum_{T \subseteq E} r^T(F)\hat{\mu}_T)$.

By complementary slackness these F^k's can be chosen to span E. Then

$$\hat{\lambda}_k = -|F^k|\hat{\mu} + \sum_{T \subseteq E} r^T(F^k)\hat{\mu}_T = -|F^k|\hat{\mu} + r^{T^k}(F^k)\hat{\mu}_{T^k}.$$

Thus

$$\begin{aligned}
&- \sum_{T \subseteq E} (r(E) - r(E \setminus T))\hat{\mu}_T + r(E)\hat{\mu} + \sum_{j \in N} \hat{\lambda}_j \\
&= - \sum_{j \in N} (r(E) - r(E \setminus T^j))\hat{\mu}_{T^j} + r(E)\hat{\mu} - \hat{\mu} \sum_{j \in N} |F^j| + \sum_{j \in N} \hat{\mu}_{T^j} r^{T^j}(F^j) \\
&\leq r(E)\hat{\mu} - \hat{\mu} \sum_{j \in N} |F^j| < 0,
\end{aligned}$$

[11] Loops are elements that do not belong to any independent set.

where the last inequality follows from the overdemand hypothesis.

To adjust the current dual feasible solution $(\mu, \{\mu_T\}_{T\subseteq E}, (\lambda_j)_{j\in N})$, increase μ by $\hat{\mu}$, increase μ_{T^j} by $\hat{\mu}_{T^j}$, and recalculate λ_j according to eqn. 3. For any critical agent j, the price on $F \subseteq E_j$ rises by $(|F| - r^{T^j}(F))\theta$. For all non-critical agents, the price of F rises by $|F|\theta$.

To deal with the case when $Z^j, U^j \neq \{\emptyset\}$ for one or more $j \in N$, a modification is needed. At each iteration we delete all elements in $\bigcup_{j\in N} Z^j \cup \bigcup_{j\in N} U^j$ and amend the restricted primal to find a basis not of $\mathcal{M}$ but of $\mathcal{M}\setminus (\bigcup_{j\in N} Z^j \cup \bigcup_{j\in N} U^j)$. Essentially, we freeze prices on elements of $(\bigcup_{j\in N} Z^j \cup \bigcup_{j\in N} U^j)$ but do not assign them until the end of the algorithm. The absence of these elements does not affect the surplus of the agents. To see why, observe that at any iteration, we cannot have $F \in \Delta^j$, $e \in Z^j \cap F$ and $F \setminus e \notin \Delta^j$ (a similar argument applies with Z^j replaced with U^j). This is because

$$
\begin{aligned}
&v_j(F) - |F|\mu + \sum_{T\subseteq E} r^T(F)\mu_T = v_j(F) - |F|\mu + \sum_{T\subseteq E} r^T((F\setminus e)\cup e)\mu_T \\
&\leq v_j(F\setminus e) - (|F|-1)\mu + \sum_{T\subseteq E} r^T(F\setminus e)\mu_T + v_e - \mu + \sum_{T\subseteq E} r^T(e)\mu_T \\
&= v_j(F\setminus e) - (|F|-1)\mu + \sum_{T\subseteq E} r^T(F\setminus e)\mu_T.
\end{aligned}
$$

When the algorithm terminates, we have a maximum weight basis, B', of a subset E' of E. However, a subset of the elements in $E \setminus E'$ can be added to B' to form a basis B of E. These elements all have zero surplus and are added (provided they do not form a circuit) to B' in the reverse order in which they are deleted.

Theorem 7 *For sufficiently small step size θ, if overdemand held before an iteration then either it holds after the iteration, or the algorithm has terminated.*

Proof Denote agent j's demand correspondence at the start of iteration t by $\Delta^j(t)$. Let $s^j_t(F)$ be the surplus that agent j enjoys on set F at the start of iteration t. By complementary slackness it suffices to prove that

$$\arg\max_{F\in\Delta^j(t)} (-|F|\hat{\mu} + \sum_{T\subseteq E} r^T(F)\hat{\mu}_T) \subseteq \Delta^j(t+1).$$

Let $A \in \arg\max_{F \in \Delta^j(t)}(-|F|\hat{\mu} + \sum_{T \subseteq E} r^T(F)\hat{\mu}_T)$ and $B \in \Delta^j(t)$. Then

$$s_t^j(A) - |A|\hat{\mu} + \sum_{T \subseteq E} r^T(A)\hat{\mu}_T \geq s_j^t(B) - |B|\hat{\mu} + \sum_{T \subseteq E} r^T(B)\hat{\mu}_T.$$

Now suppose $B \notin \Delta^j(t)$. Then $s_t^j(A) > s_t^j(B)$. By choosing the step size, θ sufficiently small we can ensure that

$$s_t^j(A) - |A|\hat{\mu} + \sum_{T \subseteq E} r^T(A)\hat{\mu}_T \geq s_j^t(B) - |B|\hat{\mu} + \sum_{T \subseteq E} r^T(B)\hat{\mu}_T.$$

completing the proof. ∎

A simple but tedious argument shows that for integer valuations, a step size of 1 suffices. To conclude our description of the primal-dual algorithm, we must show that it terminates in the optimal dual variables that correspond to the Vickrey prices.

Theorem 8 *The algorithm terminates in the optimal dual solution that maximizes* $\sum_{j \in N} \lambda_j$.

Proof If we choose $\epsilon > 0$ sufficiently small, the optimal solution to (DMP) that maximizes $\sum_{j \in N} \lambda_j$ is the optimal solution to a perturbed version of (DMP): Let **(DMP$_\epsilon$)** denote the program with objective

$$\min\ -\sum_{T \subseteq E}(r(E) - r(E \setminus T))\mu_T + r(E)\mu + (1-\epsilon)\sum_{j \in N} \lambda_j$$

and with the same constraints as (DMP). A primal-dual algorithm applied to (DMP$_\epsilon$) would terminate in the desired optimal dual solution. Call the corresponding primal problem (MP$_\epsilon$).

The dual to the restricted primal of (MP$_\epsilon$) would be a perturbed version of (DRP), i.e. with objective

$$\min -\sum_{T \subseteq E}(r(E) - r(E \setminus T))\hat{\mu}_T + r(E)\hat{\mu} + (1-\epsilon)\sum_{j \in N} \hat{\lambda}_j$$

It is easy to see that the solution we construct to (DRP) is feasible for this problem. The objective function value of this dual solution with respect to the perturbed objective function is bounded above by

$$-\sum_{j\in N}(r(E)-r(E\setminus T^j))\hat{\mu}_{T^j}+r(E)\hat{\mu}-(1-\epsilon)\hat{\mu}\sum_{j\in N}|F^j|+(1-\epsilon)\sum_{j\in N}\hat{\mu}_{T^j}r^{T^j}(F^j)$$
$$\leq r(E)\hat{\mu}-(1-\epsilon)\hat{\mu}\sum_{j\in N}|F^j|\leq \hat{\mu}[r(E)-(1-\epsilon)\sum_{j\in N}\min_{F\in\Delta^j}|F|]<0.$$

The last inequality follows from the overdemand assumption and the fact that $[r(E)-\sum_{j\in N}\min_{F\in\Delta^j}|F|]$ is integral. ■

3.4 An Auction Interpretation

The primal-dual algorithm reinterpreted as an auction takes the following form. Start with a price of zero for all elements. Agents announce which elements they are prepared to buy at the current price. These are called active elements. Now raise the price. As the price rises, agents withdraw some elements. Meanwhile, the auctioneer checks that for any agent, a basis is contained in the set of active elements not belonging to that agent.

Suppose this condition is first violated at current price p by agent j, and let E^a_j be the currently active set of elements belonging to agent j. Let M be any minimal subset of E^a_j whose removal from the set of all active elements would eliminate the existence of a basis. From the matroid property, all such sets have the same cardinality. The auctioneer permits each agent j to purchase any $|M|$ elements from E^a_j at a price of $p|M|$. The auction then continues.

3.5 Ausubel's "Clinching" Auction

Consider now an auction where K identical units must be auctioned off to n bidders. Denote the (marginal) value that bidder j assigns to consuming his i^{th} unit by v^i_j. Ausubel (2002a) concerns himself with the case in which bidders have decreasing marginal valuations: $v^i_j \geq v^{i+1}_j$ for each $i \leq K$. Under this assumption, the problem of finding an efficient allocation can be formulated as the problem of finding a maximum weight basis.

To do this, for each bidder j we introduce K elements, $j^1, j^2, \ldots, j^K$ with weights $v^1_j, v^2_j, \ldots, v^K_j$. Let E be the collection of these elements, and let I be the collection of all subsets of size at most K. It is easy to see that (E, I)

is a matroid. In this case the ascending auction described above coincides with Ausubel's (2002a) auction.

If bidders do not have decreasing marginal valuations, then the efficient allocation problem is not equivalent to the basic problem of finding a maximum weight basis. For example, suppose that bidder 1 has a higher marginal value for his second object than for his first: $v_1^1 < v_1^2$. In this case, (depending on the other bidders' valuations), a maximum weight basis may include the element for v_1^2, but not the one for v_1^1. However, it is not feasible to give bidder 1 his marginal valuation for a second object without giving him a first object! The allocation problem in this case requires an additional constraint; an agent cannot receive an $i+1^{\text{st}}$ object without receiving an i^{th} object. This side constraint destroys the matroid structure.

4 Shortest Paths

In this section, we consider the shortest path problem which, after the assignment and matroid problems, is the remaining member of the basic class of polynomially solvable optimization problems.

Let $G = (V, E)$ be a directed graph with vertex set V and edge set E. We denote a source node $s \in V$ and a sink node $t \in V$. Each edge $e \in E$ has a cost c_e. In what follows, we sometimes refer to an edge e as a pair of vertices (i, j) where e is directed from i to j. An efficient allocation in this setting is an s–t path of minimal total cost, i.e., the shortest path.

To be precise about the ownership structure, each agent j owns a (distinct) set of edges E^j, so $(E_j)_{j \in N}$ is a partition of E. Vickrey payments are well-defined only when no agent owns a cut that disconnects s from t; hence we assume this, analogously to the no-monopoly assumption of the previous section.

In this general model, the agents are substitutes condition (2) need not hold. This complementarity can be illustrated even when each s–t path uses at most one edge belonging to any particular agent. The example involves three agents $N = \{\alpha, \beta, \gamma\}$, each of whom own a single different edge. Each directed edge in Figure 1 is given a cost, and is labelled with the agent who owns it.

To illustrate our point, denote the shortest s–t path by P^* and its length by $L(P^*)$. Let P^{-S} be the shortest s–t path that does not use any edge belonging to an agent in $S \subseteq N$, and let $L(P^{-S})$ be its length. The substitutes

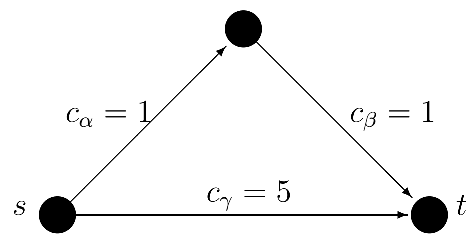

Figure 1: A shortest path problem.

condition (2) is[12]

$$L(P^{-S}) - L(P^*) \geq \sum_{j \in S} [L(P^{-j}) - L(P^*)] \quad \forall S \subseteq N.$$

In our example, the shortest path has length $L(P^*) = 2$. The marginal product of agent α (or β) is $5 - 2 = 3$, but this is the same as the marginal product of the coalition $S = \{\alpha, \beta\}$. Since $3 < 3+3$, the substitutes condition does not hold. In this example, agents α and β are complements; neither can be on an s–t path without the other.

While the substitutes condition need not hold in general in this environment, there are special cases where it does. For example, Schummer and Vohra (2001) describe a class of shortest-path problems which model the optimization problem for options-based electricity procurement with capacity constrained suppliers.

When the agents are substitutes condition does hold, there can be more than one formulation that captures Vickrey payments. We discuss two of them for the case in which each agent owns at most one edge on any s–t path.[13]

To get one formulation, one can reinterpret the variables of (P2) in order to formulate the shortest path problem. Specifically, interpret $\mu \in \Gamma$ as a feasible s–t path, and interpret S as an edge. (Values are negative lengths.) Then when the agents are substitutes condition holds, the previous results apply; the dual variables for the second constraint provide the marginal products.

[12] "Length" is the opposite of "value," which is why the terms appear to be reversed from (2).

[13] Hershberger and Suri (2001) provide an algorithm for finding Vickrey payments when valuations are given, even when the substitutes condition does not hold, assuming each agent owns a single edge.

A second, more succinct formulation eliminates the variables for paths.

$$\begin{aligned}
\min & \sum_{e \in E} c_e x_e \\
\text{s.t. } & Yx = b \\
& \sum_{e \in E_j} x_e \leq 1 \quad \forall j \in N \\
& x \geqq 0
\end{aligned}$$

Here, Y is the node-arc incidence matrix, so the first constraint is the standard one for path problems, where b is a column vector defined as $b_s = -1$, $b_t = 1$, and $b_v = 0$ for any other vertex. The second constraint is redundant.

Under our assumption that agents own at most one edge on any path, the dual variables for the second constraint correspond to marginal products. Since this formulation is more succinct than (P2), this fact is not trivial for the same reason that (P1)—a formulation more succinct than (P2)—does not provide Vickrey payments. The proof of the result, omitted for brevity, is given in Section 6.1 of Bikhchandani et al. (2002).

5 Summary

This paper surveys the connections between sealed bid Vickrey auctions and duality in linear programming. By example, we have shown how this relation can be exploited to produce iterative auctions that implement the Vickrey outcome in various scenarios. The approach can be summarized by the following steps.

1. Verify that the *agents are substitutes* condition holds.

2. Formulate a linear program where appropriate dual variables correspond to the marginal products of the agents. Under the substitutes condition, an optimal dual solution exists in which, *simultaneously*, each appropriate variable takes the value of the corresponding agent's marginal product.

3. Construct a primal-dual algorithm for the linear program. The algorithm must choose an improving direction in the dual that will cause the algorithm to terminate in the dual solution described above. This

is the one that, among all optimal dual solutions, is the one that maximizes the combined surplus of the bidders.

If the substitutes condition does not hold, it is shown in de Vries et al. (2002) that an ascending (suitably defined) auction (in which bidding sincerely is an equilibrium) yielding the Vickrey outcome cannot exist.

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