Continuous-time Games of Timing*

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February 21, 2003

JEL Classification: C72, C73

Abstract

We address the question of existence of equilibrium in general timing games of complete information. Under weak assumptions, any two-player timing game has a subgame perfect ϵ-equilibrium, for each $\epsilon > 0$. This result is tight. For some classes of games (symmetric games, games with cumulative payoffs), stronger existence results are established.

*We thank Elchanan Ben-Porath and Itzhak Gilboa for useful comments.

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Introduction

Many economic and political interactions revolve around timing. A well-known example is the class of war of attrition games, in which the decision of each player is when to quit, and the game ends in the victory of the player who held on longer. These games were introduced by Maynard Smith (1974), and later analyzed by a number of authors. Hendricks et al. (1988) provide a characterization of equilibrium payoffs for complete information, continuous time wars of attrition played over a compact time interval. Several models that resemble wars of attrition were studied in the literature. Ghemawat and Nalebuff (1985) analyze the exit decision of two competing firms in a declining market, and assume that the market will eventually not be profitable if none of the two firms ever drops from the market, see also Fine and Li (1989). Fudenberg and Tirole (1986) look at an incomplete information setup, in which there is a small probability that either firm will find it dominant to stay in forever. More recently Bilodeua and Slivinski (1996) studied a model where a volunteer for a public service is needed, and Bulow and Klemperer (1999) consider multi-player auctions as generalized wars of attrition.

Another important class of timing games are preemption games, in which each player prefers to stop first. The analysis is then sensitive to the specification of the payoff, were the two players to stop simultaneously, see Fudenberg and Tirole (1985, 1991 p.126-128).

Yet another class of timing games consists of duel games. These are two-player zero-sum games. In the simplest version, both players are endowed with one bullet, and have to choose when to fire. As time goes, the two players get closer and the accuracy of their shooting improves. These games are similar to preemption games in that a player who decides to act may be viewed as preempting her opponent. However, as opposed to preemption games, in duel games a player has no guarantee that firing first would result in a victory. We refer the reader to Karlin (1959) for a detailed presentation of duel games, and to Radzik and Raghavan (1994) for an updated survey.

There are many timing games that do not fall neatly into any of these known categories. Consider for instance the standard case of a declining market, with two initially present firms. If the monopoly profits in that market are not decreasing – e.g. if the market has a cyclical component – or if the monopoly profits remain consistently above the outside option, the game fails to be a war of attrition (see Fudenberg and Tirole (1991), p. 122). In another setup, when two firms compete on the patenting or the introduction of new technology, their interaction has a flavor of a preemption game. But each such firm also has an incentive to wait, since the probability of higher payoffs increases with time (and, presumably, with product quality). LaCasse et al. (2002) studied a model where volunteers for several jobs are needed. When only one volunteer is needed, the model reduces to a standard war of attrition, but when there are several jobs, the strategic considerations are more complex.

The present paper addresses the question of existence of equilibrium in general timing games. It provides a framework that unifies the specific classes of timing games discussed in the literature. Moreover, it deals with the question of equilibrium existence in many timing games that have not been studied before.

A continuous-time game of timing is described by a set I of players, and, for each non-empty $S \subseteq I$, a function $u_S : [0,\infty) \to \mathbf{R}^I$, with the interpretation that $u_S(t)$ is the payoff vector if the players in S – called the *leaders* – are the first to act, and they do so at time t. In addition, player i's time-preferences are described by a discount rate δ_i.[1]

Our first result is a general existence result for *two-player* games: assuming u_S is continuous and bounded for each S, the game has a subgame-perfect ε-equilibrium, for each $\varepsilon > 0$. This general result is tight in two respects. First, we provide an example of a two-player zero-sum game where a Nash 0-equilibrium does not exist. Second, we provide an example of a three-player zero-sum game where a Nash ε-equilibrium does not exist, for every ε sufficiently small. In these two examples, payoffs are constant over time.

For some classes of economic interest, we obtain stronger existence results. For *symmetric* games, our existence result is valid irrespective of the number of players, and the corresponding strategy profile is pure - but a symmetric ε-equilibrium need not exist.

In some applications, the payoff $u_S^i(t)$, for $i \in S$, is the sum of a payoff incurred up to t and of an outside opportunity - and therefore is independent of the identity of the other leaders (i.e., the set $S \setminus \{i\}$).[2] We call these games games with *cumulative payoffs*. For such games, our existence result is valid for any number of players.

We also address the issue of the existence of a Markov subgame-perfect ε-equilibrium (see Maskin and Tirole (2001)). We provide a positive answer for two-player games, for symmetric games, and for cumulative-payoff games with non-constant payoff, but exhibit a cumulative-payoff game with constant payoff with no Markov subgame-perfect ε-equilibria, provided ϵ is sufficiently small.

In most cases, the proofs we provide are constructive.

Finally, we provide a restrictive condition under which existence of a Nash ε-equilibrium for every $\varepsilon > 0$ implies the existence of a Nash equilibrium. The condition is that the function u_S is constant for each $S \subset I$,and that players are not discounting payoffs (but does not impose any restriction on the number of players). Incidentally, this establishes the existence of a Nash equilibrium for the corresponding class of two-player games – a class of games for which none of the

[1]The model that is described here is of a game with complete information. We shall argue that some of our results extend to games with symmetric incomplete information.

[2]On the other hand, for $i \notin S$, $u_S^i(t)$ can be interpreted as the sum of the payoff incurred up to t, and of an equilibrium payoff to i in the smaller game obtained once S is gone. Hence $u_S^i(t)$ depends on S.

known sufficient conditions for equilibrium existence hold, see, e.g., Reny (1999).

We conclude this introduction by a conceptual point. Fudenberg and Tirole (1985) discuss the relevance of continuous-time models of timing games, on the following ground. Games in continuous time are best seen as idealized models for games in discrete time, with very short time periods. As Fudenberg and Tirole point out, in certain cases a limit of discrete-time equilibria has no equivalent in the continuous-time model. This is best seen on the grab-the-dollar game.[3] At each time t, each of two players (with the same discount rate) can grab a dollar that lies between them. The game terminates once at least one of the players grabs the dollar. If at that time only one player grabbed the dollar, he receives 1, and his opponent receives 0. If both grabbed the dollar, both lose 1. In the discrete-time version of this game, the players are only allowed to act at exogenously given times (t_n), where the sequence (t_n) is increasing. The unique symmetric equilibrium has both players grab the dollar with probability 1/2 at every time t_n (if the game still goes on at that stage) – yielding a payoff of zero to both players.[4] When the stage length decreases to zero, the symmetric equilibrium strategies do not converge to any strategy profile of the continuous-time version, since such a limit strategy would have to stop with probability 1/2 at any time. Hence, the unique candidate would be the strategy profile in which both players stop with probability 1 at time zero – but the payoff associated with this profile differs from the limit of the discrete-time payoffs. Fudenberg and Tirole define an enlarged strategy space, that may be viewed as a compactification of the set of discrete-time strategy profiles.

This approach has been developed further by Simon and Stinchcombe (1989), Bergin (1992), Stinchcombe (1992) and Bergin and MacLeod (1993) for repeated games played in continuous time. In such games, a "naive" definition of a strategy profile need not yield a well-defined outcome – a problem which does not arise in timing games. These authors provide various restrictions that deal with this problem, but lose the natural simplicity of the continuous-time framework.

In our view, the problem which arises in the grab-the-dollar game is best seen as a lack of upper semi-continuity, as the stage length decreases to zero. However, as also pointed out in Fudenberg and Levine (1986) in a different context, some kind of lower semi-continuity holds: given any $\varepsilon' > \varepsilon > 0$, any ε-equilibrium profile for the continuous-time model is still, when discretized, an ε'-equilibrium in the discrete-time versions of the game, provided the time period is short enough.

We here stick with the standard interpretation of continuous-time models, viewed as an idealized framework allowing for the use of the powerful tools of mathematical analysis. This enables us to provide a simple and general equilibrium analysis of timing games. Moreover, our equilibrium recommendation in the

[3]Our discussion follows closely the discussion in Fudenberg and Tirole (1985).

[4]In addition, there are non-symmetric equilibria: any pair of strategies in which at every stage one of the players grabs the dollar and his opponent does not grab it is a subgame-perfect equilibrium.

continuous-time model approximately yields an equilibrium in all discrete-time models, with sufficiently short time periods.[5]

The paper is organized as follows. In Section 1 we state our assumptions and our results. All examples are collected in Section 2. Section 3 contains the proof of the general existence result for two-player games while the discussion of specific issues is postponed to Section 4. Finally, Section 6 concludes with few extensions.

1 The Model and the Main Results

The set of non-negative reals $[0, \infty)$ is also denoted by $\mathbf{R}^+$, and for every $t \in \mathbf{R}^+$ we identify $[t, \infty] = [t, \infty) \cup \{\infty\}$.

1.1 The model

A *game of timing* Γ is given by:

- A finite set of players I, and a discount rate $\delta_i \in \mathbf{R}^+$ for each player $i \in I$.
- For every non-empty subset $\emptyset \subset S \subseteq I$, a continuous and bounded function $u_S : [0, \infty) \to \mathbf{R}^I$.

A pure strategy t_i of player i is simply a time to act, namely an element of $[0, \infty]$, where the alternative $t_i = \infty$ corresponds to never acting.

Given a pure strategy profile $(t_i)_{i \in I}$, we let $\theta := \min_{i \in I} t_i$ denote the terminal time, and $S_* := \{i \in I \mid t_i = \theta\}$ be the coalition of leaders. We define the payoff $g^i((t_j)_j)$ to player i to be $e^{-\delta_i \theta} u^i_{S_*}(\theta)$ if $\theta < \infty$ – *i.e.*, if the game terminates in finite time – and 0 otherwise.

In most timing games of economic interest, the players incur costs, or receive profits prior to the end of the game, and the discounted sum of profits/costs up to t is bounded as a function of t. This case reduces to the case under study here by deducting/adding the total cost/profit up to time t from the discounted $u_S(t)$. Hence, our standing assumption that $g^i = 0$ if $\theta = \infty$ is a normalization convention, and entails no loss of generality.

1.2 Strategies and payoffs

A *mixed strategy* for player i is a probability distribution σ^i over the set $[0, \infty]$. The *expected payoff* given a strategy profile $\sigma = (\sigma^i)_{i \in I}$ is:

$$\gamma^i_0(\sigma) = \mathbf{E}_{\otimes_{i \in I} \sigma^i}[g^i(t_1, \ldots, t_I)]. \tag{1}$$

[5] This is also true when the time periods are not known in advance, but follow a stochastic process with small increments; see Section 6.

The subscript reminds that payoffs are discounted back to time zero. We denote by $\gamma^i_t(\sigma) = e^{\delta_i t}\gamma^i_0(\sigma)$, the expected payoff discounted to time t.

Equivalently, a mixed strategy σ^i can be described by its c.d.f. (cumulative distribution function), *i.e.,* by the function $F^i : \mathbf{R}^+ \to [0,1]$ defined by $F^i_t = \sigma^i([0,t])$. Plainly, F^i is right-continuous and non-decreasing. Note also that $1 - \lim_{t \nearrow \infty} F^i_t$ is the probability under σ^i that player i never acts, and that F^i_0 is the probability that player i acts immediately. We let $\mathcal{F}$ denote the set of all such functions F^i.

Given $F \in \mathcal{F}$ and $t \in [0,\infty]$, we let $F_{t-} = \lim_{s \nearrow t} F_s$ denote the left-limit of F at t (with $F_{0-} := 0$ and $F_{\infty-} = \lim_{t\to\infty} F_t$) and we denote by $\Delta F_t := F_t - F_{t-}$ the jump of F at t.

When expressed in terms of c.d.f's, formula (1) reduces to

$$\begin{aligned}\gamma^l_0(F^1,\ldots,F^I) \;&=\; \sum_{i\in I}\int_{[0,\infty)} e^{-\delta_l t} u^l_{\{i\}}(t)\prod_{j\neq i}(1-F^j_t)dF^i_t \\ &\quad + \sum_{S\subseteq I, |S|\geq 2}\sum_{t=0}^{\infty} u^l_S(t)\prod_{i\in S}\Delta F^i_t \prod_{i\notin S}(1-F^i_t),\end{aligned}$$

where the integral is a Stieltjes integral w.r.t. F^i (the notation $\int_{[0,\infty)}$ stresses that the jump of F^i at zero is explicitly taken into account in the value of the integral).

The notions of pure and mixed strategies do not suffice when studying subgame-perfect equilibria. Indeed, pure and mixed strategies indicate when the player acts for the *first* time. However, they do not indicate how the player plays if the game starts at some time $t > 0$ which is beyond his acting time.

For every $t \geq 0$, the subgame that starts at time t is the game of timing Γ_t with player set I, where the payoff function when coalition S terminates is $u'_S(s) = u_S(t+s)$. Thus, payoffs are evaluated at time t.

Definition 1.1 *A* super strategy *of player 1 is a function $\widehat{\sigma}^i : t \mapsto \sigma^i_t$ that assigns to each $t \geq 0$ a mixed strategy σ^i_t that satisfies*

- *Properness: σ^i_t assigns probability one to $[t,\infty]$.*
- *Consistency: for every $0 \leq t < s$ and every Borel set $A \subseteq [s,\infty]$, one has*

$$\sigma^i_t(A) = (1-\sigma^i_t([t,s)))\sigma^i_s(A).$$

The properness condition asserts that σ^i_t is a mixed strategy in the subgame that starts at time t: the probability that player i acts before time t is 0. The consistency condition asserts that as long as a strategy does not act with probability 1, later strategies can be calculated by Bayes' rule.

Given a super-strategy profile $\widehat{\sigma}$, a player $i \in I$ and $t \in \mathbf{R}^+$, we denote by $\gamma^i_t(\widehat{\sigma}) := \gamma^i_t(\sigma_t)$ the payoff induced by $\widehat{\sigma}$ in the subgame starting at time t.

1.3 Results and outline

Let $\varepsilon > 0$ be given. A profile of mixed strategies is a *Nash ε-equilibrium* if no player can profit more than ε by deviating to any other mixed strategy. This condition is equivalent to saying that no player can profit more than ε by deviating to any pure strategy.

A profile of super strategies $\widehat{\sigma} = (\sigma_t)_{t\geq 0}$ is a *subgame-perfect ε-equilibrium* if for every $t \geq 0$, the profile of mixed strategies σ_t is a Nash ε-equilibrium in the subgame that starts at time t (when payoffs are discounted to time t).

In Section 2, we provide few examples, that show that our existence results are tight. Section 3 contains the proof of our main existence result, stated below.

THEOREM 1.2 *Every two-player discounted game of timing in continuous time admits a subgame-perfect ε-equilibrium, for every $\varepsilon > 0$. If $\delta_i = 0$ for some i, the game admits an ε-equilibrium, for each $\varepsilon > 0$.*

The proof is essentially constructive. In many cases of interest, a *pure* subgame-perfect ε-equilibrium exists.

In Section 4, we take a brief look at some classes of timing games of specific interest. We first analyze *games with cumulative payoffs*, defined by the property that for $i \in S$, the payoff $u^i_S(t)$ does not depend on which other player(s) happen to act at that time. Formally, $u^i_S(t) = u^i_{\{i\}}(t)$ whenever $i \in S$. This class includes games in which each player receives a stream of payoffs until he/she exits from the game (and the game proceeds with the remaining players). In particular, it includes models of shrinking markets, (see, e.g., Fudenberg and Tirole (1986) and Ghemawat and Nalebuff (1985)). It can also accommodate the case in which there is a collection of winning coalitions $\mathcal{S}$, and the game terminates at the first time t in which the coalition of remaining players S_t is a winning coalition. One model of this sort is the model of multi-object auctions studied in Bulow and Klemperer (1999).

THEOREM 1.3 *Every game with cumulative payoffs has a subgame-perfect ε-equilibrium, for each $\varepsilon > 0$. Moreover, there is a subgame-perfect ε-equilibrium in which symmetric players play the same super strategy.*[6]

In many cases of economic interest, the players enjoy symmetric roles, in the sense that the payoff $u^i_S(t)$ to player i if S acts depends only on t, on the size of S, and on whether i belongs or not to S. Formally, a symmetric I-player game of timing is described by functions $\alpha_k : \mathbf{R}^+ \to \mathbf{R}$, $\beta_k : \mathbf{R}^+ \to \mathbf{R}$, $k \in \{1, \ldots, |I|\}$, with the interpretation that, for $|S| = k$, one has $u^i_S(t) = \alpha_k(t)$ if $i \in S$, and $u^i_S(t) = \beta_k(t)$ otherwise. For symmetric games, our existence result is surprisingly strong.

[6]Players i and j are symmetric if (i) $u^i_S = u^j_S$, for every S that either contains both i and j, or none of them, and (ii) $u^i_{S\cup\{i\}} = u^j_{S\cup\{j\}}$ for every S that contains neither i nor j.

Theorem 1.4 *Every symmetric discounted game of timing admits a pure subgame-perfect ε-equilibrium, for each $\varepsilon > 0$.*

The *grab-the-dollar* game is an example of a symmetric game that does not have a symmetric ε-equilibrium, provided ε is sufficiently small. To prove this claim formally, one can use similar arguments to those we use in Section 2.2.

In Section 4.3 we prove that two-player games, as well as symmetric games and games with non-constant cumulative payoffs, have a Markov subgame-perfect ε-equilibrium, but that games with constant cumulative payoffs, need not have one.

Finally, in Section 1.5, we prove that under somewhat restrictive assumptions, the existence of an ε-equilibrium implies the existence of an equilibrium.

Theorem 1.5 *Let I be a finite set of players, let $u_S(\cdot)$ be a constant function for each $\emptyset \neq S \subseteq I$, and let $\delta_i = 0$ for each $i \in I$. If the game of timing $(I, (u_S)_S)$ has an ε-equilibrium for each $\varepsilon > 0$, then it also has a zero equilibrium.*

In particular, combined with Theorem 1.2, Theorem 1.5 implies that every two-player, constant-payoff, undiscounted game of timing has a (mixed) Nash equilibrium. This equilibrium existence result is not standard. It is worth noting that it does *not* follow from the most general existence result due to Reny (1999). Indeed, Theorem 3.1 in Reny assumes that both strategy spaces are compact Hausdorff spaces, and that the game is so-called better-reply secure. In the context of timing games, one is tempted to endow the mixed strategy spaces with the topology of weak convergence.[7] Consider the constant-payoff timing game defined by $u_{\{1\}} = (3, 1)$, $u_{\{2\}} = (0, 0)$ and $u_{\{1,2\}} = (2, 3/2)$, and any strategy profile σ where player 1 acts at time zero, but player 2 does not act at time zero: $\sigma^1(\{0\}) = 1$ and $\sigma^2(\{0\}) = 0$. Plainly, σ yields the payoff $(3, 1)$ but is not an equilibrium. Since 3 is the highest payoff player 1 may possibly get in the game, player 1 can not secure at σ a higher payoff, in the sense of Reny. On the other hand, any strategy $\tilde{\sigma}^2$ of player 2 that secures at σ a payoff strictly above one must act with some positive probability η at time zero. Let now σ^1_n be a sequence of strategies that weakly converges to σ^1 and with no atom at time zero. Plainly, $\lim_{n\to\infty} \gamma^2(\sigma^1_n, \tilde{\sigma}^2) = (1 - \eta) < 1$ – hence Reny's condition does not hold.

2 Examples

In the present section we study several examples, which show that the results we present in the paper are sharp. We first present a two-player zero-sum game that has no Nash equilibrium. We then present a three-player zero-sum game with no ϵ-equilibrium, provided ϵ is sufficiently small. As mentioned in the Introduction,

[7] As in the two-player zero-sum timing game of Example 5.1 in Reny.

the grab-the-dollar game is a symmetric game with no symmetric ϵ-equilibrium, but it does admit a pure (non-symmetric) equilibrium. Our third example is an example of a two-player symmetric game with no pure equilibrium. We then provide two examples of games with no Markov equilibrium (see Section 4.3 for a definition of Markov equilibria in our context.) One game is a two-player game with non-constant payoffs, and the other is a three-player game with cumulative payoffs.

2.1 A two-player zero-sum game with no equilibrium

Consider the two-player zero-sum game defined by $u_S^1(t) = 1$ if $|S| = 1$ and $u_{\{1,2\}}^1(t) = 0$, with $\delta_1 > 0$.

We first argue that player 1 can guarantee a payoff $1 - \varepsilon$, for every $\varepsilon > 0$. Indeed, consider the mixed strategy σ^1 that acts at a random time in the interval $[0, \eta]$, where $\eta > 0$ satisfies $e^{-\delta_1 \eta} \geq 1 - \varepsilon$. Formally, the corresponding c.d.f. F^1 is defined by $F_t^1 = \min\{t/\eta, 1\}$. Since player 1 acts at a random time, the probability that both players act simultaneously is 0, whatever be the strategy used by player 2. Since the game terminates by time η, player 1's payoff is 1 with probability 1, and taking the discount rate into account, his expected payoff is at least $e^{-\delta_1 \eta} \geq 1 - \varepsilon$. Since the highest payoff he can get in the game is 1, this means that an ε-equilibrium exists for every $\varepsilon > 0$.

We now claim that player 1 cannot guarantee 1. Indeed, the discounted payoff of player 1 is 1 only if, with probability one, the game terminates at time 0, and only one player acts at that time. This can happen only if one player acts with probability one at time 0, while the other does not act. However, if player 1 acts with probability 1 at time 0, it is optimal for player 2 to act at time 0 as well, whereas if player 1 does not act at time 0, it is optimal for player 2 not to act at time 0 as well.

2.2 A three-player zero-sum game with no ε-equilibrium

We here analyze the three-player zero-sum game of timing with constant payoffs that is defined by[8] $u_{\{i\}}^i(t) = 1$, $u_{\{i\}}^{i+1}(t) = 0$, $u_{\{i\}}^{i+2}(t) = -1$, $u_{\{i,i+1\}}^i(t) = 0$, $u_{\{i,i+1\}}^{i+1}(t) = -1$, $u_{\{i,i+1\}}^{i+2}(t) = 1$ and $u_{\{1,2,3\}}^i(t) = 0$ for every $i \in I$ and every $t \in \mathbf{R}^+$. The game is described by the following matrix

Don't Act

	Don't Act	Act
Don't Act		$-1, 1, 0$
Act	$1, 0, -1$	$0, -1, 1$

Act

	Don't Act	Act
Don't Act	$0, -1, 1$	$1, 0, -1$
Act	$-1, 1, 0$	$0, 0, 0$

Figure 1

[8]Here addition is understood modulo 3.

in which players 1, 2 and 3 choose respectively a row, a column and a matrix. We assume that the three players have the same discount rate $\delta \geq 0$. The value of δ plays no role in the analysis. In particular, we allow for the possibility that $\delta = 0$, allowing in effect for the case of an undiscounted game.

We prove that this game has no ε-equilibrium, provided $\varepsilon > 0$ is small enough. It is interesting to recall that three-player games of timing in *discrete* time that have constant payoff functions do have a subgame-perfect ε-equilibrium (see Solan (1999)). Thus, this example stands in sharp contrast with known results in discrete time.

We first verify that this game has no equilibrium. Let σ be a strategy profile. If σ is an equilibrium, the probability that the game terminates at time 0 is below one. Otherwise, at least one player, say player 1, would act with probability one at time 0. By the equilibrium condition, player 2 would act with probability 0: given that player 1 acts, *act* is a strictly dominated action for player 2. Hence, player 3 would act with probability one at time 0, and player 1 would find it optimal not to act at time 0 – a contradiction. Next, given that the game does not terminate at time 0, each player i can get a payoff arbitrarily close to one, by acting immediately after time 0, that is, by acting at time $t > 0$, where t is sufficiently small so that the probability that σ^{i+1} or σ^{i+2} act in the time interval $(0, t]$ is arbitrarily small. Thus, the continuation equilibrium payoff of each player must be at least one – a contradiction to the zero-sum property. Hence σ is not an equilibrium.

We now prove that the game has no ε-equilibrium. For every $w \in [-1, 1]^3$ let $G(w)$ be the one-shot game with payoff matrix as in Figure 1, where the payoff if no player acts is w. We actually proved the following claim: for every $w \in [-1, 1]^3$ with $\sum_{i=1}^{3} w^i = 0$, the probability that the game terminates at time 0, under any Nash equilibrium in $G(w)$, is strictly less than 1. Since the correspondence that assigns to each $w \in [-1, 1]^3$ and every $\varepsilon > 0$ the set of ε-equilibria of the game $G(w)$ has a closed graph, there is $\varepsilon > 0$ such that for every $w \in [-1, 1]^3$ with $\sum_{i=1}^{3} w^i = 0$, the probability that the game terminates at time 0, under any ε-equilibrium in $G(w)$, is strictly less than $1 - 2\varepsilon$.

Let σ be an ε-equilibrium of the timing game. In particular, the probabilities $\sigma^i(\{0\})$ assigned to *act* at time zero form an ε-equilibrium of the game $G(w)$, taking for w the continuation payoff vector in the game. Since the game is zero-sum, the continuation payoff at time 0 of at least one player is non-positive. As argued above, by acting right after time 0, this player can improve his payoff by almost 1 if the game is not terminated at time 0. By the previous paragraph, this event has probability at least 2ε, hence the deviation improves by more than ε – a contradiction.

2.3 A symmetric game with no pure equilibrium

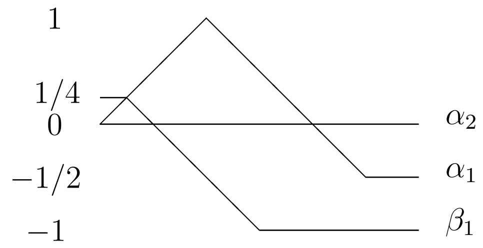
We here provide a symmetric two-player game with no pure equilibrium. It is defined by

$\alpha_2(t) = 0$ for every t : if both players act simultaneously, no-one gets anything;

$\alpha_1(t) = t\mathbf{1}_{t\le 1} + (2-t)\mathbf{1}_{1<t<5/2} - \frac{1}{2}\mathbf{1}_{t\ge 5/2}$: if only one player is to act, he will do it at time 1;

$$\beta_1(t) = \frac{1}{4}\mathbf{1}_{t\le 1/4} + (\frac{1}{2} - t)\mathbf{1}_{1/4<t<3/2} - \mathbf{1}_{t\ge 3/2}.$$

Graphically, the payoff functions look as follows.



We assume $\delta_1 = \delta_2 = 0$, but our arguments remain valid as long as the discount rates are sufficiently small.

Observe that the strategy profile in which both players act at a random time uniformly chosen from the interval $[1/4, 1/4+\varepsilon]$ is a symmetric ε-equilibrium. Indeed, the corresponding payoff to both players is 1/4, whereas the best payoff a player can get by deviating is at most $1/4+\varepsilon$. It is also easy to verify that the strategy profile in which player 1 acts at time 1/4 and player 2 acts at time $1/4+\epsilon$ is an ϵ-equilibrium.

Assume that there is a Nash equilibrium in pure strategies.

If both players act simultaneously at time $t_* \in \mathbf{R}^+ \cup \{\infty\}$, the equilibrium payoff is 0. Since $\beta_1(t) > \alpha_2(t)$ for $t < 1/2$, we must have $t_* \ge 1/2$. Each player would then rather act alone at some time $0 < t < \min\{2, t_*\}$.

Assume now that players 1 acts at time t_*, and player 2 acts at time $t_{**} > t_*$, possibly infinity. Since $\alpha_2(t) > \beta_1(t)$ for $t > 1/2$, we must have $t_* \le 1/2$. Since the function α_1 increases until $t = 1$, player 1 is better off by acting at any time $t \in (t_*, \min\{t_{**}, 1\})$.

2.4 A game with cumulative payoffs and no Markov equilibrium

Consider the following three-player game with constant cumulative payoffs.

	Don't Act: Don't Act	Don't Act: Act
Don't Act		$2,1,-\frac{5}{2}$
Act	$1,-\frac{5}{2},2$	$1,1,2$

	Act: Don't Act	Act: Act
Don't Act	$-\frac{5}{2},2,1$	$2,1,1$
Act	$1,2,1$	$1,1,1$

As we argue in Section 4.3, when payoffs are constant, the only Markov super-strategies $\widehat{\sigma}^i = (\sigma^i_t)_t$ are (i) σ^i_t acts at time t, for every t, (ii) σ^i_t assigns probability 1 to ∞, for each t and (iii) σ^i_t is an exponential distribution over $[t,+\infty)$.

Fix ϵ sufficiently small.

One can verify that the only equilibrium of the corresponding game in *discrete* time is to have each player act with probability 1/2 at each stage. It follows that there is no ϵ-equilibrium in which all players play Markov strategies, and at least one player plays the super strategy of type (i).

If the game terminates by a single player the sum of payoffs to the three players is 1/2. In particular, if all players follow a super strategy of type (ii) or (iii), the expected payoff of at least one player is below 1/2, but that player can receive 1 by acting at time 0.

Consequently, the game admits no Markov ϵ-equilibrium. Observe that the super-strategy profile that indicates each player to act with probability 1/2 whenever t is an integer, and not to act otherwise, is a non-Markovian equilibrium.

3 Subgame-perfect equilibria in two-player games

This section is devoted to the proof of Theorem 1.2. The proof combines a backward induction argument with a compactness, or diagonal extraction, principle. We provide here a brief outline.

We start with few definitions, that will be in use throughout the section. We let a two-player game of timing $(u_S(\cdot))_{\emptyset\neq S\subseteq\{1,2\}}$ be given, together with the discount rates $\delta_1,\delta_2>0$ of the two players. For ease of presentation, we denote by $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ the three functions $u_{\{1\}}(\cdot)$, $u_{\{2\}}(\cdot)$ and $u_{\{1,2\}}(\cdot)$ respectively.

Note that for every continuous function $f:\mathbf{R}^+\to\mathbf{R}^N$, and every $\eta,\delta>0$, there is a strictly increasing sequence $(t_k)_k$, with limit ∞, such that for every k and every $t_k\le s<t\le t_{k+1}$, $\|e^{-\delta(s-t)}f(s)-f(t)\|<\eta$.

Given $\varepsilon>0$, we let $\eta>0$ be small enough. We apply the previous paragraph to the $\mathbf{R}^6$-valued function $f=(a,b,c)$, to η and to $\delta=\min\{\delta_1,\delta_2\}$, and obtain a sequence $(t_k)_k$ that strictly increases to ∞.

The proof is divided into two parts. Given $n\in\mathbf{N}$, we consider the version of the timing game that terminates at time t_n with a payoff of zero if no player acted before. In this game with finite horizon, we define inductively, for $0\le k<n$, a super-strategy profile $\widehat{\sigma}_k(n)$ over the time interval $[t_k,t_{k+1})$. We prove that the profile obtained by concatenating the profiles $\widehat{\sigma}_k(n)$ is a subgame-perfect ε-equilibrium in the game with finite horizon.

Next, we let n go to ∞. We observe that, for fixed k, the sequence $(\widehat{\sigma}_k(n))_n$ takes only finitely many values, so that by a diagonal extraction argument a limit $\widehat{\sigma}$ of $\widehat{\sigma}(n)$ exists. This limit is our candidate for a subgame-perfect ϵ-equilibrium.

3.1 An auxiliary class of games

The induction step mentioned above takes as given a timing game played between times t_k and t_{k+1} and with a terminal payoff that may differ from zero. We deal in this section with such games.

Given $0 \leq \tau < \theta < \infty$ and $v \in \mathbf{R}^2$, we define the *induction game* $G([\tau,\theta);v)$ to be the game that starts at time τ and ends at time θ, with a payoff of v if no player acted in between. In this game, each player is allowed to act at any time in $[\tau,\theta)$, and the payoff is v if no one ever acts. Since the interval $[\tau,\theta)$ is homeomorphic to $\mathbf{R}^+$, the induction game is formally equivalent to a game of timing, as introduced in Section 1, except that the terminal payoff may differ from zero, and that discounting is not exponential. The definitions of a pure, mixed and super strategy, as well as of a subgame-perfect ε-equilibrium, are analogous to those given for infinite horizon games. Hence, a *pure* strategy in the induction game is an element in $[\tau,\theta) \cup \{\infty\}$, and a super strategy of player i is a map $\widehat{\sigma}^i$ that assigns to each $t \in [\tau,\theta)$ a probability distribution over $[\tau,\theta) \cup \{\infty\}$, and satisfies the analogs of the Properness and Consistency requirements of Definition 1.1.

We shall later obtain super-strategy profiles in the infinite-horizon game by concatenating profiles of successive induction games. For clarity, we use the letter g for the payoff function in $G([\tau,\theta);v)$: given a super-strategy profile $\widehat{\sigma}$ in $G([\tau,\theta);v)$ and $t \in [\tau,\theta)$, $g_t(\widehat{\sigma})$ is the payoff induced by $\widehat{\sigma}$ in the subgame starting from t, and evaluated at time t.

3.1.1 Classification

We will say that the induction game $G([\tau,\theta);v)$ is of:
Type **C** if $c^1(\tau) \geq b^1(\tau)$ and $c^2(\tau) \geq a^2(\tau)$;
Type **V** if $e^{-\delta_1(\theta-\tau)}v^1 + \eta \geq a^1(\tau)$ and $e^{-\delta_2(\theta-\tau)}v^2 + \eta \geq b^2(\tau)$;
Type **A1** if $a^1(\tau) \geq e^{-\delta_1(\theta-\tau)}v^1 + \eta$ and $a^2(\tau) \geq c^2(\tau)$;
Type **B1** if $b^2(\tau) \geq e^{-\delta_2(\theta-\tau)}v^2 + \eta$ and $b^1(\tau) \geq c^1(\tau)$;
Type **A2** if $a^1(\tau) \geq e^{-\delta_1(\theta-\tau)}v^1 + \eta$ and $a^2(\tau) \geq b^2(\tau)$;
Type **B2** if $b^2(\tau) \geq e^{-\delta_2(\theta-\tau)}v^2 + \eta$ and $b^1(\tau) \geq a^1(\tau)$;
Type **A3** if $a^1(\tau) \geq b^1(\tau)$ and $a^2(\tau) \geq c^2(\tau)$;
Type **B3** if $b^2(\tau) \geq a^2(\tau)$ and $b^1(\tau) \geq c^1(\tau)$.

Each of these types may easily be interpreted. In a game of type **C**, the players will agree to act simultaneously. In a game of type **V**, the players will agree not to act on $[\tau,\theta)$.

Each induction game has at least one type, and possibly several. Indeed, assume that $G([\tau,\theta);v)$ has no type. If $a^1(\tau) \geq e^{-\delta_1(\theta-\tau)}v^1 + \eta$, one must have $a^2(\tau) < b^2(\tau)$ by **A2**, $b^1(\tau) < c^1(\tau)$ by **B3**, $a^2(\tau) > c^2(\tau)$ by **C** and $a^1(\tau) < e^{-\delta_1(\theta-\tau)}v^1 + \eta$ by **A1** – a contradiction. If $a^1(\tau) < e^{-\delta_1(\theta-\tau)}v^1 + \eta$ then one must have $b^2(\tau) \geq e^{-\delta_2(\theta-\tau)}v^2 + \eta$ by **V**, so that by the previous chain of implications, applied to player 2, one reaches a contradiction.

Plainly if (v_n) is a convergent sequence in $\mathbf{R}^2$, with limit v, and if the induction game $G([\tau,\theta);v_n)$ is of type T for every n, then $G([\tau,\theta);v)$ is also of type T.

3.1.2 Definition of the super-strategy profile

We next proceed to define a super-strategy profile $\widehat{\sigma}$ in the game $G([\tau,\theta);v)$. The payoff that will correspond to (σ_t^1,σ_t^2) is $c(t)$ (resp. v discounted to time t) if the type is **C** (resp. **V**), and is approximately $a(t)$ (resp. $b(t)$) if the type is **A1**, **A2** or **A3** (resp. **B1**, **B2** or **B3**).

If the game is of

- type **C**, we let σ_t^i act with probability one at time t, for each $t \in [t,\theta)$, and $i = 1,2$; hence $\gamma_t(\sigma_t) = c(t)$.
- type **V**, we let σ_t^i act with probability zero over the time interval $[t,\theta)$, for each t and $i = 1,2$; hence $\gamma_t^i(\sigma_t) = e^{-\delta_i(\theta-t)}v^i$.
- type **A1**, we let σ_t^1 act with probability one at time t, and σ_t^2 assign probability zero to $[t,t_{k+1})$; hence $\gamma_t(\sigma) = a(t)$.
- type **A2**, we let σ_t^1 be the uniform distribution over $[t,\theta)$, and σ_t^2 act with probability zero over the time interval $[t,\theta)$; hence $\gamma_t(\sigma_t) \approx a(t)$ provided the maximal variation of a over the interval $[\tau,\theta)$ is small.
- type **A3**, we let σ_t^1 act with probability one at time t, and σ_t^2 be the uniform distribution over $[t,\theta)$; hence $\gamma_t(\sigma_t) = a(t)$.

Finally, the types **B1**, **B2** and **B3** correspond respectively to types **A1**, **A2** and **A3**, when exchanging the roles of the two players, and the definition of σ_t^i for those types is to be deduced from the definitions for their symmetric counterpart.

It is clear that $\widehat{\sigma}$ satisfies the Properness requirement, and one can verify that it also satisfies the Consistency requirement.

As explained earlier, the inductive proof will apply this construction to time intervals $[\tau,\theta)$ over which the maximal variation of $u_S(\cdot)$ is close to zero, for each S. We now prove that, under such assumptions, the profile $\widehat{\sigma}$ is a subgame-perfect ε-equilibrium of the game $G([\tau,\theta);v)$.

Proposition 3.1 *Let $\tau,\theta \in \mathbf{R}^+$ and $v \in \mathbf{R}^2$ be given. Assume that, for every $f \in \{a,b,c\}$, and for $\delta = \min\{\delta_1,\delta_2\}$, and $\tau \leq s < t < \theta$ one has $\|e^{-\delta(s-t)}f(s) -$*

$f(t)\| < \eta$ and moreover that $(1 - e^{-\delta(\theta-\tau)})\|v\| < \eta$. Then, for each $t \in [\tau, \theta)$, the profile (σ_t^1, σ_t^2) is a 4η-equilibrium of the game $G([t, \theta); v)$. Moreover, if σ_t^2 assigns probability one to ∞, then player 1 does not profit by not acting, and the same holds when exchanging the roles of the two players.

Proof. Let $t \in [\tau, \theta)$ be arbitrary. We prove that no pure strategy of player 1 improves upon σ_t by more than 4η. The argument for player 2 is symmetric.

Assume that under σ_t^2 player 2 does not act in the interval $[t, \theta)$ (types **V**, **A1**, **A2**). Any deviation of player 1 yields at most

$$\max\{e^{-\delta_1(\theta-t)}v^1, \sup_{s\in[t,\theta]} e^{-\delta_1(s-t)}a^1(s)\} \leq \max\{e^{-\delta_1(\theta-t)}v^1, a^1(t)\} + \eta, \qquad (2)$$

whereas the payoff to player 1 under (σ_t^1, σ_t^2) is $e^{-\delta_1(\theta-t)}v^1$ if the type is **V**, $a^1(t)$ if the type is **A1**, and at least $\inf_{s\in[t,\theta]} e^{-\delta_1(s-t)}a^1(s) \geq a^1(t) - \eta$ if the type is **A2**. In each case, by the definition of the types, this payoff is higher than the quantity in (2) minus 2η.

Observe that by not acting player 1 receives $e^{-\delta_1(\theta-t)}v^1$ which is at most what he receives in each of these cases. This establishes the second assertion of the Proposition.

Assume next that under σ_t^2 player 2 acts at time t (types **C**, **B1**, **B3**). Any pure deviation of player 1 yields either $b^1(t)$ or $c^1(t)$. However, the payoff to player 1 under (σ_t^1, σ_t^2) is $c^1(t)$ (resp. $b^1(t)$) if the type is **C** (resp. **B1** or **B3**), which, by the definition of the types, is equal in both cases to $\max\{b^1(t), c^1(t)\}$.

Assume finally that σ_t^2 is the uniform distribution over $[t, \theta)$ (types **A3**, **B2**). Any deviation of player 1 yields at most $\max\{a^1(t), b^1(t)\} + \eta$. However, the payoff to player 1 under (σ_t^1, σ_t^2) is at least $a^1(t) - \eta$ (resp. $b^1(t) - \eta$) if the type is **A3** (resp. **B2**), which, by the definition of the types, is equal in both cases to $\max\{a^1(t), b^1(t)\} - \eta$. In particular, player 1 cannot gain more than 2η by deviating. ■

3.2 The proof

We here explicit the induction and the limit argument that were sketched in the introduction to this section.

Given $n \in \mathbf{N}$, we associate to each $k \in \{0, \dots, n\}$ a payoff $v_k(n) \in \mathbf{R}^2$ and a type $j_k(n)$, as follows:

- we set $v_n(n) := (0, 0)$;
- for $k < n$, we let $j_k(n)$ be a type of the induction game $G([t_k, t_{k+1}); v_{k+1}(n))$, and we let $v_k(n)$ be the payoff induced by the 4η-equilibrium that was defined in Section 3.1.2: $v_k(n) = g_{t_k}(\sigma_{t_k})$.

We now let n go to infinity. Since there are finitely many types, and since payoffs are bounded, a diagonal extraction argument implies that there is an increasing sequence of indices $(n_m)_{m\in\mathbf{N}}$ such that the sequences $(v_k(n_m))_{m\in\mathbf{N}}$ and $(j_k(n_m))_{m\in\mathbf{N}}$ converge for every $k \geq 0$. Denote for every $k \geq 0$ $v_k = \lim_{m\to\infty} v_k(n_m)$ and $j_k = \lim_{m\to\infty} j_k(n_m)$. By the remark at the end of Section 3.1.1, j_k is a type of $G([t_k, t_{k+1}); v_{k+1})$.

We next proceed to the definition of a super-strategy profile $(\widehat{\sigma}^1, \widehat{\sigma}^2)$ in the timing game (with infinite horizon). Given $k \in \mathbf{N}$, we denote by $(\widehat{\sigma}^{1,k}, \widehat{\sigma}^{2,k})$ the super-strategy profile in the game $G([t_k, t_{k+1}); v_{k+1})$ corresponding to type j_k, as defined in Section 3.1.2. Note that, for $i \in I$ and $t \in [t_k, t_{k+1})$, $\sigma_t^{i,k}$ is a probability distribution over $[0, \infty]$ which gives probability 1 to $[t, t_{k+1}) \cup \{\infty\}$.

By Proposition 3.1, for each $t \in [t_k, t_{k+1})$, the profile $(\sigma_t^{1,k}, \sigma_t^{2,k})$ is a 4η-equilibrium of the game $G([t, t_{k+1}); v_{k+1})$.

Intuitively, we shall define $\widehat{\sigma}_t^i$, $t \in \mathbf{R}^+$, as the concatenation of the different super strategies $(\widehat{\sigma}^{i,k})_{k\in\mathbf{N}}$. Formally, this is achieved via the following construction.

Given a mixed strategy σ^i in an induction game $G([t, t'); v)$ and a mixed strategy σ'^i in an induction game $G([t', t''); v')$, we define their concatenation $\sigma^i \circ \sigma'^i$ to be the strategy in $G([t, t''); v')$ that assigns probability $\sigma^i(A)$ to every Borel set $A \subseteq [t, t')$, and probability $(1 - \sigma^i([t, t')))\sigma'^i(A)$ to every Borel set $A \subseteq [t', t'') \cup \{\infty\}$. For every k and every $t \in [t_k, t_{k+1})$ define

$$\sigma_t^i = \sigma_t^{i,k} \circ \sigma_{t_{k+1}}^{i,k+1} \circ \sigma_{t_{k+2}}^{i,k+2} \circ \ldots .$$

One can verify that $\widehat{\sigma}^i = (\sigma_t^i)_{t\in\mathbf{R}^+}$ satisfies both the Properness and the Consistency requirement in Definition 1.1. We omit this verification.

Proposition 3.2 *The super-strategy profile $\widehat{\sigma}$ is a subgame-perfect ε-equilibrium of the timing game.*

Proof. We first claim that $\gamma_{t_k}(\widehat{\sigma}) = v_k$ for each $k \in \mathbf{N}$. Indeed, since $\widehat{\sigma}$ is defined as the concatenation of the profiles $\widehat{\sigma}^k$, the equation that links $\gamma_{t_k}(\widehat{\sigma})$ to $\gamma_{t_{k+1}}(\widehat{\sigma})$ is the same as the relation between v_k and v_{k+1}: if at least one player acts with probability one on the interval $[t_k, t_{k+1})$, both v_k and $\gamma_{t_k}(\widehat{\sigma})$ coincide with the corresponding payoff. On the other hand, if both players act with probability zero on $[t_k, t_{k+1})$, then $\gamma_{t_k}^i(\widehat{\sigma}) = e^{-\delta_i(t_{k+1}-t_k)}\gamma_{t_{k+1}}^i(\widehat{\sigma})$ and $v_k^i = e^{-\delta_i(t_{k+1}-t_k)}v_{k+1}^i$. Therefore, for a given k, either (i) there is $k_* > k$ such that at least one player acts with probability one on the interval $[t_{k_*}, t_{k_*+1})$, in which case reasoning backwards from k_* yields $\gamma_{t_k}(\widehat{\sigma}) = v_k$, or (ii) no such k_* exists, in which case the equality $v_k^i = e^{-\delta_i(t_l-t_k)}v_l^i$ holds for each $l > k$. Since payoffs are bounded, by letting l go to infinity we obtain $v_k = 0$ for each k, so that as above $\gamma_{t_k}(\widehat{\sigma}) = 0$.

Let $k \in \mathbf{N}$ and $t \in [t_k, t_{k+1})$ be given. We shall prove that, for each pure strategy ${\sigma'}_t^1$ in the timing game starting at t, one has

$$\gamma_t^1({\sigma'}_t^1, \sigma_t^2) \leq \gamma_t^1(\sigma_t^1, \sigma_t^2) + \varepsilon. \tag{3}$$

Since the roles of the two players are symmetric, this will imply that (σ_t^1, σ_t^2) is an ε-equilibrium of the game starting at time t. Since t is arbitrary, the Proposition will follow.

Since it is a pure strategy, ${\sigma'}_t^1$ assigns probability one to some element $t_* \in [t, \infty) \cup \{\infty\}$. We first deal with the case $t_* < \infty$.

Let $k_* \in \mathbf{N}$ be the unique integer such that $t_* \in [t_{k_*}, t_{k_*+1})$. Let $k_{**} \geq k$ be the first integer such that the type of the game $G([t_{k_{**}}, t_{k_{**}+1}); v_{k_{**}+1})$ is either **C**, **B1**, **B2**, **A3** or **B3** (with $k_{**} = \infty$ if no such integer exists). By the definition of the strategy of player 2, the game terminates before time $t_{k_{**}+1}$ with probability one, whatever player 1 plays. Set $\widehat{k} = \min\{k_*, k_{**}\}$.

We prove that for every $k < k' \leq \widehat{k}$, the expected payoff of player 1 if player 2 follows $\sigma^2_{t_{k'}}$ and player 1 acts at time t_*, discounted to $t_{k'}$, is at most $v^1_{k'} + 4\eta$.

For $k' = \widehat{k}$ this follows since (σ_t^1, σ_t^2) is a 4η-equilibrium of the induction game $G([t_{\widehat{k}}, t_{\widehat{k}+1}); v_{\widehat{k}+1})$.[9] Assume we proved the claim for $k'+1$. Since player 2 does not act before time $t_{k'+1}$, the type $j_{k'}$ of the game $G([t_{k'}, t_{k'+1}); v_{k'+1})$ must be **V**, **A1** or **A2**. By the induction hypothesis, the expected payoff of player 1 if player 2 follows $\sigma^2_{t_{k'}}$ and player 1 acts at time t_*, discounted to $t_{k'}$, is at most $e^{-\delta_1(t_{k'+1}-t_{k'})}(v^1_{k'+1} + 4\eta) \leq e^{-\delta_1(t_{k'+1}-t_{k'})}v^1_{k'+1} + 4\eta$. By the second assertion of Proposition 3.1 this amount is at most $v^1_{k'} + 4\eta$, as desired. The same argument, applied to the induction game $G([t, t_{k+1}); v_{k+1})$, delivers now (3).

For every $t \in [0, \infty]$ denote by $\delta(t)$ the pure strategy that acts at time t with probability 1.

If $t_* = \infty$, then, since $\delta_1 > 0$ and by the first part,

$$\gamma_t^1(\delta(\infty), \sigma_t^2) = \lim_{\tilde{t} \to \infty} \gamma_t^1(\delta(\tilde{t}), \sigma_t^2) \leq \gamma_t^1(\sigma_t) + 4\eta. \tag{4}$$

■

Comment. We now argue that if $\delta_1 = 0$ (or $\delta_2 = 0$), that is, if at least one of the players does not discount, then a Nash ϵ-equilibrium exists.

For every n and k, let $(\widehat{\sigma}^{1,k}(n), \widehat{\sigma}^{2,k}(n))$ be the super strategies defined in Section 3.1 for type $j_k(n)$ in the game $G([t_k, t_{k+1}); v_k(n))$. Denote $\sigma_0^i(n) = \sigma_{t_1}^{i,1}(n) \circ \sigma_{t_2}^{i,2}(n) \circ \ldots \circ \sigma_{t_{n-1}}^{i,n-1}(n)$. If under $(\sigma_0^1(n), \sigma_0^2(n))$ both players act with probability 1 before time t_n, the arguments we presented in the proof of Proposition 3.2 imply that $(\sigma_0^1(n), \sigma_0^2(n))$ is an ϵ-equilibrium.

Assume, then, that under $\sigma_0^2(n)$ player 2 never acts, for every n. Then $j_k(n)$ is **V**, **A1** or **A2** for every k and every n. The construction in Section 3.1.2 implies that $v_k^1(n) \geq 0$ for every k and every n. In particular, the strategy $\delta(\infty)$ that never acts cannot be a profitable deviation of player 1. Let n be sufficiently large such that for some $t < t_n$ one has $a^1(t) \geq \sup_{s \in [0,\infty)} a^1(s) - \eta$ and for some $t' < t_n$ one has $b^2(t') \geq \sup_{s \in [0,\infty)} b^2(s) - \eta$. In words, the best payoff by acting alone occurs before time t_n. One can verify that $(\sigma_0^1(n), \sigma_0^2(n))$ is a 5η-equilibrium. ■

[9] Strictly speaking, σ_t^i need not be an admissible strategy in $G([t_{\widehat{k}}, t_{\widehat{k}+1}); v_{\widehat{k}+1})$, but it induces one when collapsing $[t_{\widehat{k}+1}, \infty]$ to ∞.

Corollary 3.3 *Assume that, for every t one has either (i) $b^1(t) \geq c^1(t)$ and $a^2(t) \geq c^2(t)$, or (ii) $b^1(t) \leq c^1(t)$ and $a^2(t) \leq c^2(t)$. Then for every $\epsilon > 0$,*

- *if $min\{\delta_1, \delta_2\} > 0$, there exists a pure subgame-perfect ε-equilibrium.*
- *if $min\{\delta_1, \delta_2\} = 0$, there exists a pure ε-equilibrium.*

Observe that in wars of attrition, condition (i) holds for every t.

Proof. It suffices to show that all the induction games $G([t_k, t_{k+1}), v_{k+1}(n))$ that appear in the proof are of type **C**, **V**, **A1** or **B1**. This is a matter of straightforward verification. ∎

4 Special classes of games

Many proofs in this section are minor variations upon the proof of Theorem 1.2. Hence few details will be omitted.

4.1 Games with cumulative payoff

We here prove Theorem 1.3. Let Γ be a game with cumulative payoffs. Fix a strictly increasing sequence (s_n) with $s_0 = 0$ and $\lim_{n\to\infty} s_n = \infty$ such that $\sup_n \sup_{s_n \leq s < t \leq s_{n+1}} |e^{-\delta(s-s_n)} u^i_S(s) - u^i_S(t)| < \varepsilon$ for every S and every i. Define an auxiliary game Γ^* in which players can act *only* at times $\{s_n, n \geq 0\}$ and *must* continue in all other times. The auxiliary game Γ^* is equivalent to a discounted[10] game Γ^{**} in discrete time with countably many states s_n. The stochastic game Γ^{**} has quite a specific structure: at state s_n, each player can either act or not. If at least one player acts, the game reaches an absorbing state. If no one acts, the game moves to state s_{n+1}.

Every strategy profile τ_{**} in the game Γ^{**} naturally induces a super-strategy profile in the game Γ^*, and therefore it induces a super-strategy profile $\widehat{\tau}$ in the game Γ. Observe that for every n, the expected payoff under τ_{**} starting from state s_n is equal to the expected payoff induced by $\widehat{\tau}$ in Γ, starting from time s_n.

By Fink (1964) the discounted stochastic game Γ^{**} has a subgame-perfect 0-equilibrium $\tau_{**} = (\tau^i_{**})_{i\in I}$. Moreover, there is such a subgame-perfect 0-equilibrium in which symmetric players play the same strategy.

Denote by $\widehat{\sigma}$ the profile of super strategies in Γ induced by τ_{**}. Then $\widehat{\sigma}^i = \widehat{\sigma}^j$ for every pair of symmetric players $i \neq j$. Moreover, under $\widehat{\sigma}$ players act only at times $(s_n)_{n\geq 0}$, that is, the probability distribution σ^i_t gives weight one to the set $\{s_n, n \geq 0\}$, for each $t \in \mathbf{R}^+$.

We will prove that $\widehat{\sigma}$ is a subgame-perfect ε-equilibrium. Let $t \in \mathbf{R}^+$ be given, and let τ^i be a pure strategy of player i in the subgame starting at time t, which acts at time $t_i \in [t, \infty]$.

[10] with a state-dependent discount factor

We denote by $\tilde{\tau}^i$ the auxiliary pure strategy that acts at time s_k, where $k \in \mathbf{N} \cup \{\infty\}$ is the minimal integer such $s_k \geq t_i$. By construction, under both (σ_t^{-i}, τ^i) and $(\sigma_t^{-i}, \tilde{\tau}^i)$ no player in $S \setminus \{i\}$ acts in the time interval (t_i, s_k). Therefore,

$$|\gamma_t^i(\sigma_t^{-i}, \tilde{\tau}^i) - \gamma_t^i(\sigma_t^{-i}, \tau^i)| < |e^{-\delta_i(s_k - t_i)} u_{\{i\}}^i(t_i) - u_{\{i\}}^i(s_k)| \leq \varepsilon. \quad (5)$$

The pure strategy $\tilde{\tau}^i$ is a valid strategy in Γ^*, and therefore naturally induces a pure strategy $\tilde{\tau}_{**}^i$ in Γ^{**}. Since τ_{**} is a subgame-perfect 0-equilibrium, the payoff induced by $(\tilde{\tau}_{**}^i, \tau_{**}^{-i})$ in the stochastic game Γ^{**}, starting from state s_k, does not improve upon the payoff induced by τ_{**} in that game. Since these payoffs coincide with $\gamma_t^i(\sigma_t^{-i}, \tilde{\tau}^i)$ and $\gamma_t^i(\sigma_t)$ respectively, and by (5), one gets

$$\gamma_t^i(\sigma_t^{-i}, \tau^i) \leq \gamma_t^i(\sigma_t) + \varepsilon,$$

as desired.

4.2 Symmetric games

We here prove Theorem 1.4. Let an I-player symmetric timing game be given. We set

$$T_I = \{t \in [0, \infty) \mid \alpha_I(t) \geq \beta_{I-1}(t)\},$$

and

$$T_k = \{t \in [0, \infty) \mid \alpha_k(t) \geq \beta_{k-1}(t) \text{ and } \alpha_{k+1}(t) \leq \beta_k(t)\}, \text{ for } k = 2, 3, \ldots, I-1.$$

If $t \in T_I$ then the strategy profile in which all players act at time t is a 0-equilibrium in Γ_t. Indeed, under this profile the payoff for all players is $\alpha_I(t)$, while any deviator who will not act at time t will receive $\beta_{I-1}(t) \leq \alpha_I(t)$.

Similarly, if $t \in T_k$, for $k = 2, \ldots, I-1$, any strategy profile in which exactly k players act at time t is a 0-equilibrium in the game starting from time t. Indeed, any one of the k players who acts at time t receives $\alpha_k(t)$, while if such a player deviates and does not act at time t he will receive $\beta_{k-1}(t) \leq \alpha_k(t)$. Any one of the $I-k$ players who does not act at time t receives $\beta_k(t)$, while if such a player deviates and acts at time t he will receive $\alpha_{k+1}(t) \leq \beta_k(t)$.

For $k = 2, 3, \ldots, I$, we let T_k^* be the closure of the interior of T_k. Then each T_k^* is the union of at most countably many disjoint closed intervals: $T_k^* = \cup_{n=1}^\infty [c_n^k, d_n^k]$. Set $\widehat{T}_k = \cup_{n=1}^\infty [c_n^k, d_n^k)$.

We set $T_0 = [0, \infty) \setminus \cup_{k=2}^I \widehat{T}_k$. Observe that $T_0 = \cup_{n=1}^\infty [c_n^0, d_n^0)$ is a union of disjoint half-closed half-open intervals.

Given $t \in \mathbf{R}^+$, one has $t \in \cup_{k \geq 2} T_k$ as soon as $\alpha_2(t) \geq \beta_1(t)$. Therefore, $\alpha_2(t) \leq \beta_1(t)$ for every $t \in T_0$.

We already defined a pure 0-equilibrium for initial times $t \in \cup_k \widehat{T}_k$. To complete the proof, it is now sufficient to prove that a subgame-perfect ε-equilibrium exists in each game $G([c_n^0, d_n^0); v)$, where v is the equilibrium payoff we defined starting from time d_n^0. If $d_n^0 = \infty$, we set this terminal payoff to zero. To prove this claim, we shall mimic the proof of Theorem 1.2. We shall only sketch the main steps of the proof. We let the game $G([c_n^0, d_n^0); v)$ and $\varepsilon > 0$ be given. Choose $\eta > 0$ to be very small. Consider an increasing sequence $(t_k)_k$ that converges to d_n^0 and such that $\sup_{s,t \in [t_k, t_{k+1}]} |e^{-\delta(s-t_k)}\alpha_1(s) - \alpha_1(t)| < \eta$. If $d_n^0 < \infty$, we define the sequence so that it contains only finitely many terms $(t_k)_{k \leq K}$, with $t_K = d_n^0$. In that case, the profile is constructed by backward induction, starting with the game $G([t_{K-1}, d_n^0); v)$. If $d_n^0 = \infty$, the sequence (t_k) contains infinitely many terms, and the induction proceeds as in the proof of Theorem 1.2, as explained below.

Fix $k \in \mathbf{N}$, and look at the game $G([t_k, t_{k+1}); v_k(n))$ that appears in the induction step. We use the symmetry of the game to simplify the classification into types. Specifically, we say that $G([t_k, t_{k+1}); v_k(n))$ is of

Type **V** if $e^{-\delta(t_{k+1}-t_k)} \min_{i \in I} v_k^i(n) + \eta \geq \alpha_1(t_k)$.

Type $\mathbf{1}i$ if $e^{-\delta(t_{k+1}-t_k)} v_k^i(n) + \eta < \alpha_1(t_k)$.

Following the proof of Theorem 1.2, we define a *pure* super-strategy profile in the game $G([t_k, t_{k+1}); v_k(n))$, depending on the type of that game. If it is of type **V**, we let σ_t^i act with probability zero on the time interval $[t, t_{k+1})$, for each $t \in [t_k, t_{k+1})$. If it is of type $\mathbf{1}i$ for some i, we let σ_t^i act with probability one at t, and σ_t^j act with probability zero on the time interval $[t, t_{k+1})$, for each $j \neq i$ and $t \in [t_k, t_{k+1})$. The rest of the proof follows the proof of Theorem 1.2.

4.3 Markov equilibrium

We here discuss the existence of a Markov subgame-perfect ε-equilibrium in timing games. According to a Markov strategy, the behavior at time t depends only on payoff relevant past events, see Maskin and Tirole (2001). In the context of timing games, this requirement is expressed as follows. A real number $T \in \mathbf{R}^+$ is a *period* of the game if $u_S(t+T) = u_S(t)$, for each $t \in \mathbf{R}^+$ and $S \subseteq I$. A super-strategy profile σ is Markovian if, for every $t \in \mathbf{R}^+$ and every $i \in I$, the mixed strategy σ_{t+T}^i is obtained from σ_t^i by translation: for each Borel set $A \subseteq \mathbf{R}^+$, one has $\sigma_t^i(A) = \sigma_{t+T}^i(A+T)$. In this section, we provide a partial answer to the existence problem of a Markov subgame perfect ε-equilibrium.

When payoffs are constant, one can provide an explicit characterization for the set of Markov strategies. Let $\widehat{\sigma}^i$ be a Markov super-strategy of player i. If $\sigma_0^i(0) = 1$ then $\sigma_t^i(0) = 1$ for every $t \in \mathbf{R}^+$: under $\widehat{\sigma}^i$ the player acts at every time t.

If $\sigma_0^i(0) < 1$ then $\sigma_0^i(\eta) < 1$ for some $\eta > 0$ sufficiently small. By the Markov requirement, this implies that $\sigma_0^i(s) < 1$ for every $s \in \mathbf{R}^+$; indeed, by induction over k, $\sigma_0^i((k+1)\eta) = \sigma_0^i(k\eta) + (1 - \sigma_0^i(k\eta))\sigma_0^i(\eta) < 1$. Moreover, the Markov requirement implies that $(1 - \sigma_0^i(t))(1 - \sigma_0^i(s)) = 1 - \sigma_0^i(t+s)$, so that by the characterization of the exponential distribution (see, e.g., Billingsley, 1995, p.189) σ_0 is an exponential distribution over $\mathbf{R}^+$, and for $t > 0$ σ_t is obtained by translation. To summarize, if a super-strategy $\widehat{\sigma}$ is Markov, then σ_t is obtained from σ_0 by translation. Moreover, σ_0 is either a unit mass located at 0 or ∞, or is an exponential distribution over $[0, \infty)$. Conversely, any such super-strategy has the Markov property.

PROPOSITION 4.1 *Every two player game function has a Markov subgame-perfect ε-equilibrium, for each $\varepsilon > 0$.*

Proof. We shall use the notations of section 3. We first assume that $a(\cdot), b(\cdot)$ and $c(\cdot)$ are constant, and we adapt the proof of Theorem 1.2. Since payoffs are constant, it is sufficient for our proof to consider only one induction game $G([0, \infty); \vec{0})$. In most cases (i.e., **C**, **V**, **A1**, **B1**, **A2** and **B3** for player 2, **A3** and **B2** for player 1) the super strategies we defined are either never to act, or always to act, which are Markov. In the other four cases replace the current definition of σ_t^i by an exponential distribution over $[t, \infty)$ with sufficiently high parameter α. Given $\varepsilon > 0$, if α is sufficiently high, then under the new definition the game terminates before time $t + \varepsilon$ with probability at least $1 - \varepsilon$; since the payoff functions are constant this implies that no player can profit in discounted terms more than 3ε by deviating, provided ε is sufficiently small.

Next, we assume that the functions $a(\cdot), b(\cdot)$ and $c(\cdot)$ have a common period $T < \infty$. We shall discuss two cases. Up to symmetries, these cases exhaust all possible cases.

Case 1: $a^1(t) \leq b^1(t)$ and $a^2(t) \geq b^2(t)$ for each $t \in \mathbf{R}^+$.

In a sense, each player would rather see his opponent stop. We adapt the proof of Theorem 1.3, see section 4.1. We shall only sketch the proof, without providing all the details. Given $\epsilon > 0$, we let $\eta > 0$ be small enough, and let $0 = t_0 < t_1 < \cdots < t_n = T$ be a finite subdivision of $[0, T]$, such that a, b and c do not vary by more than η on each subinterval $[t_k, t_{k+1}]$, $k = 0, 1, \ldots, n-1$.

Consider the stochastic game Γ^{**} with finitely many states labelled $t_0, \ldots, t_{n-1}$ where (i) the game moves cyclically from one state to the next one in the sequence (and from t_{n-1} to t_0) as long as no player ever acts, (ii) player 1 (resp. player 2) can only act in states with *odd* index (resp. with *even* index), and (iii) the payoff by acting at state t_k is $a(t_k)$ or $b(t_k)$ depending on k. The game Γ^{**} has a subgame-perfect equilibrium $\widehat{\sigma}$ in stationary strategies – strategies that depend only on the current state. When reverting to the interpretation of t_k as a time rather than a state, this profile corresponds to a periodic profile – still denoted $\widehat{\sigma}$ – in the timing game. We derive a modified, periodic super-strategy profile $\widehat{\tau}$ as follows. Loosely, if player i stops with probability p at time t_k under $\widehat{\sigma}$, we will have him act under $\widehat{\tau}$ with probability p over the whole time-interval $[t_k, t_{k+1})$. Specifically, for $k < n$, the mixed strategy $\tau_{t_k}^i$ has no atoms, assigns to the interval $[t_k, t_{k+1})$ the probability $\sigma_{t_k}^i(\{t_k\})$ with which $\sigma_{t_k}^i$ acts at time t_k, and can be calculated using Bayes' rule from $\tau^i_{t_{k+1}}$ on the interval $[t_{k+1}, \infty]$. For $t \neq t_k$, τ_t is defined via Bayes rule. Note that, for each $t \in \mathbf{R}^+$, the payoffs $\gamma_t(\widehat{\sigma})$ and $\gamma_t(\widehat{\tau})$ differ by at most η.

We claim that $\widehat{\tau}$ is a subgame-perfect ϵ-equilibrium, provided η is small enough. Plainly, it is enough to prove that player 1 can not deviate profitably in the game that starts at time 0. This claim is supported by the following arguments.

Let $\tilde{\tau}^i_0$ be a pure strategy of player 1 in the timing game. If it never acts, it is payoff equivalent – up to η to the strategy in Γ^{**} that never acts.[11] If it acts at time $t \in [t_k, t_{k+1})$ for some odd k, it is payoff-equivalent to the strategy in Γ^{**} that acts at state t_k. Finally, if it acts at time $t \in [t_k, t_{k+1})$ for some even k, it yields a *lower* payoff than the strategy that acts at time t_{k+1}, by the assumption on payoffs.

Case 2: $a^2(t_*) < b^2(t_*)$ for some $t_* \in \mathbf{R}^+$.

We start with a simple observation. Assume that, for some $t \in \mathbf{R}^+$ and $\eta > 0$, there is a super-profile $\widehat{\sigma}$ such that (i) $\widehat{\sigma}$ is a subgame-perfect ϵ-equilibrium in $G([t, t+\eta); v)$, irrespective of v and (ii) for each $s \in [t, t+\eta)$, under σ_s, at least one player will act before $t + \eta$. Then there is a Markov ϵ-equilibrium.

Indeed, by translation we can assume that $t \geq T$. By the backward-induction argument presented in section 3.2 we construct a pure ϵ-equilibrium in the period $[t + \eta - T, t + \eta]$. By (2), the super-strategy profile in the original game that is defined by repeating periodically this ϵ-equilibrium is a subgame-perfect ϵ-equilibrium in the original game.

Given this fact, we shall mimic the proof of Theorem 1.2, see section 3.2, where we choose the sequence (t_k) so that $t_* = t_{k_*}$ for some $k_* \in \mathbf{N}$. If, for some $n \in \mathbf{N}$, the induction game $G([t_{k_*}, t_{k_*+1}); v_{k_*}(n))$ is either of type **A3**, **B3** or **C**, we may apply the above observation with $[t, t+\eta) = [t_{k_*}, t_{k_*+1})$ and the result follows. Otherwise, it must be that $a^1(t_*) < b^1(t_*)$. Indeed, since $a^2(t_*) < b^2(t_*)$, one first has $b^1(t_*) < c^1(t_*)$ by **B3**, next $a^2(t_*) > c^2(t_*)$ by **C**, and finally $a^1(t_*) < b^1(t_*)$ by **A3**.

To conclude, we let $[t, t+\eta) = [t_{k_*}, t_{k_*+1})$, and define a super-profile $\widehat{\sigma}$ in $G([t, t+\eta); v)$ by having both players acting time be distributed according to an exponential distribution[12] over $[t, t+\eta)$. The parameter of player 2's distribution is chosen to be much larger than the parameter of player 1's distribution. We then apply the basic observation. ■

Next, we show that in symmetric games and in games with non-constant cumulative payoff a Markov ε-equilibrium always exists, irrespective of the number

[11] To be precise: faced with τ_0^{-i} in the timing game, it yields approximately the same payoff as the strategy *never act* in Γ^{**}, faced with $\widehat{\sigma}^{-i}$

[12] To be precise, it is the image of an exponential distribution over $\mathbf{R}^+$ under an increasing homeomorphism that maps $\mathbf{R}^+$ to $[t, t+\eta)$.

of players.

PROPOSITION 4.2 *Every multi-player symmetric game of timing has a pure Markov subgame-perfect ϵ-equilibrium.*

Proof. We modify the proof given in Section 4.2. If payoffs are constant, the proof is similar to the proof of Proposition 4.1.

Assume now that the payoffs are periodic with period $T > 0$. We shall use the observation made in Case 2 of the previous proof. Observe that if $t \in T_k$ for some $k = 2, \ldots, K$, and if $\eta > 0$ is small enough, then the profile that requires k players to act and $I - k$ players to continue satisfies the two requirements of that observation. Therefore, we can assume w.l.o.g. that $T_0 = [0, \infty)$.

If $\sup \alpha_1(\cdot) \leq 0$, there is a subgame-perfect equilibrium in which no player ever acts. Thus, we may assume that $\sup \alpha_1 > 0$. We divide the proof in three cases. Since α_1 and β_1 are continuous, these exhaust all possible cases.

Case 1: $\alpha_1(t) = \beta_1(t)$ for some t.

We let η be small enough, and let (t_n) be an increasing sequence with limit $t + \eta$ and such that $t_0 = t$. We define σ as follows: player 1 (resp. player 2) acts at each time $s \in [t_n, t_{n+1})$ for even n (resp. for odd n). Players $3, 4, \ldots, I$ never act. We then use the first observation.

Case 2: $\alpha_1(t) > \beta_1(t)$ for each $t \in \mathbf{R}^+$.

We divide the time interval $[0, T]$ into a large, finite, even number of intervals, and define a periodic super-profile $\widehat{\sigma}$ as follows: player 1 (resp. player 2) acts at each time $s \in [t_n, t_{n+1})$ for even n (resp. for odd n). Players $3, 4, \ldots, I$ never act. It is straightforward to check that $\widehat{\sigma}$ is a subgame-perfect ϵ-equilibrium, provided the partition of $[0, T]$ is fine enough.

Case 2: $\alpha_1(t) < \beta_1(t)$ for each $t \in \mathbf{R}^+$.

Choose $t_* \geq T$ such that $\alpha_1(t_*) = \sup_{t \in \mathbf{R}^+} \alpha_1(t)$, and let $\eta > 0$ be small enough. We divide the period $[t_* - T + \epsilon, t_* + \epsilon)$ into finitely many small intervals $[t_k, t_{k+1})$, $k = 0, \ldots, k_*$ and apply the backward construction that appears in the proof of Theorem 1.4. We initialize the induction with player 1 acting at each $s \in [t_{k_*}, t_{k_*+1})$, while players $2, \ldots, I$ do not act on $[t_{k_*}, t_{k_*+1})$. Hence $v^1_{k_*} = \alpha_1(t_*)$, while $v^i_{k_*} = \beta_1(t_*)$ for each $i = 2, \ldots, I$. One can check inductively that $0 < v^1_k < v^i_k$ for each $k = 1, \ldots, k_*$ and $i = 2, \ldots, I$ – so that each induction game is either of type **1-1** or **C**, while the last one, $G([t_0, t_1); v_1)$ is of type **1-1**. Therefore, this construction generates a periodic profile. ■

PROPOSITION 4.3 *In every multi-player game with non-constant cumulative payoffs a Markov subgame-perfect ϵ-equilibrium exists. Moreover, there is a Markov equilibrium where symmetric players play the same super-strategy.*

Proof. The proof is essentially the same as the proof of Theorem 1.2. All one should note is that since payoffs are periodic, one can construct the stochastic game Γ^{**} in discrete time to have finitely many states, that correspond to one period of the game in continuous time. ■

5 An equilibrium existence result

We here prove Theorem 1.5. It will be helpful to explain first the gist of the argument. In a sense, it relies on a compactness principle. We shall exhibit a compact set $\mathcal{G}$ of profiles that satisfies:

a) if there is an ϵ-equilibrium, then there is an ϵ-equilibrium in $\mathcal{G}$, and

b) the payoff function $\gamma(\cdot)$ is continuous on $\mathcal{G}$.

The second property will imply that any accumulation point of ϵ-equilibria in $\mathcal{G}$, as ϵ goes to 0, is an equilibrium, while the first property, together with the compactness of $\mathcal{G}$, will imply that under the assumptions of Theorem 1.5 such an accumulation point exists.

The set $\mathcal{F}^I$ of all profiles, endowed with the weak topology, does not satisfy the second property, since the payoff function is not continuous over $\mathcal{F}^I$. Discontinuities may arise for two reasons. First, in the weak topology, several atoms may merge to a single atom at the limit. Second, a sequence of non-atomic distributions may weakly converge to an atomic distribution.

We illustrate these two phenomena with two examples. Both examples involve two players. We let $F = (F^1, F^2)$ be the profile in which both players act with probability 1 at time 0: $F^i_t = 1$ for every $t \in \mathbf{R}^+$.

Example 1: Player 1 acts with probability 1 at time 0, while player 2 acts with probability 1 at time $1/n$. Formally, for every $n \in \mathbf{N}$, $F^1(n) = F^1$ whereas $F^2_t(n) = \mathbf{1}_{t \geq 1/n}$. Plainly the sequence $(F(n))$ weakly converges to F, but $\gamma(F(n)) = u_{\{1\}}$ while $\gamma(F) = u_{\{1,2\}}$.

Example 2: Both players act uniformly in the interval $[0, 1/n]$. Formally, $F^1_t(n) = F^2_t(n) = \min\{1, nt\}$. The sequence $(F(n))$ weakly converges to F. Since for every $n \in \mathbf{N}$ the probability that under $F(n)$ both players act simultaneously is 0, $\gamma(F(n)) = \frac{1}{2}u_{\{1\}} + \frac{1}{2}u_{\{2\}}$, while $\gamma(F) = u_{\{1,2\}}$.

Roughly speaking, the auxiliary space $\mathcal{G}$ contains all profiles $G = (G^1, \ldots, G^I)$ that satisfy (A) if G^i has a jump of ΔG^i_t at t, then all G^j's are constant in the interval $(t - \Delta G^i_t, t)$, and (B) the slope of $\frac{1}{n}\sum_i G^i$ is 1 whenever this function is continuous.

The first requirement implies that as one goes to the limit, it cannot be that two atoms merge. Indeed, if for each $n \in \mathbf{N}$ $G^i(n)$ and $G^j(n)$ have discontinuities at t_n and s_n respectively, with $t_n < s_n$, then $\Delta G^j_{s_n}(n)$ is bounded by $s_n - t_n$. Therefore, if $\lim s_n = \lim t_n$ then the atom of $G^j(n)$ at s_n vanishes at the limit.

The second requirement implies that a sequence of non-atomic distributions in $\mathcal{G}$ cannot converge to an atomic distribution, since the slope of $G^i(n)$ is uniformly bounded by I.

We now turn to the formal presentation. Recall that $\mathcal{F}$ is the space of all functions $F : \mathbf{R}^+ \to [0,1]$ that are non-decreasing and right-continuous. It is in bijection with the set of probability measures μ over $[0,+\infty]$. We denote by λ the Lebesgue measure over $[0,+\infty)$. The set of atoms of μ^i (or equivalently, of discontinuities of F^i) is denoted by A_{μ^i}. Let $\mathcal{G} \subset (\mathcal{F})^I$ be the space of all $\mu = (\mu^1,\ldots,\mu^n)$ that satisfy the following conditions.

0) The support of each μ^i is an interval $[0,T_i]$, with $T_i \leq I$.

A) For each $i \in I$ and $t \in A_{\mu^i}$, one has $\mu^j_{[t-\mu^i_t,t)} = 0$ for every $j \in I$. Set $T_\mu := \mathbf{R}^+ \setminus \left(\cup_i \cup_{t\in A_{\mu^i}} [t-\mu^i_t,t]\right)$.

B) One has $\frac{1}{I}\sum_i \mu^i_A = \frac{1}{I}\sum_i \lambda_{A\cap[0,T_i]}$, for every $A \subseteq T_\mu$.

By Helly's Theorem (Billingsley, 1995, Theorem 25.9) and Theorem 25.10 in Billingsley (1995), the set $\mathcal{G}$ is compact for the topology of weak convergence.

Plainly, Theorem 1.5 follows immediately from Lemmas 5.1 and 5.2 below, using the compactness of $\mathcal{G}$.

LEMMA 5.1 *Let $\epsilon > 0$ be given. If the game has an ϵ-equilibrium, then it has an ϵ-equilibrium in $\mathcal{G}$.*

The proof of this lemma appears in Section 5.1.

We denote by Δ^i the set of pure strategies of player i.

LEMMA 5.2 *The payoff function γ is continuous over $\mathcal{G}$. Moreover, let $(G(n))_{n\in\mathbf{N}}$ be a convergent sequence in $\mathcal{G}$, with limit G, and let $\tilde{G}^i \in \Delta^i$, for some $i \in I$. Then there exists a sequence $\tilde{G}^i(n) \in \Delta^i$, such that*

$$\lim_{n\to+\infty} \gamma^i(\tilde{G}^i(n), G^{-i}(n)) = \gamma^i(\tilde{G}^i, G^{-i}).$$

The proof of this lemma appears in Section 5.2.

5.1 Time-changes

Our goal in this section is to prove Lemma 5.1. A *time-change* is a non-decreasing, right-continuous function defined over some interval of $\mathbf{R}^+$, with values in $\mathbf{R}^+$. Given an ϵ-equilibrium $(F^1,\ldots,F^I)$, we shall construct a time-change u such that the profile $(G^1,\ldots,G^I)$ defined by $G^i_t = F^i_{u(t)}$ is in $\mathcal{G}$, and is an ϵ-equilibrium.

For $s \in \mathbf{R}^+$, we define the *s-level set* of F to be the interval $F^{-1}(\{s\})$.

5.1.1 Straightening F

We here define a first time-change, relative to a given *continuous* function $F \in \mathcal{F}$. In effect, the clock will be adjusted in such a way that: (i) the duration of the level sets of F will not be affected and (ii) the increasing portions of F will be transformed into affine portions with slope one.

We first introduce a usual time-change (see, e.g., Revuz and Yor (2000), Chapter 0):

$$C_s = \inf\{t \geq 0 \mid F_t > s\}, \text{ for } s \in [0, F_{\infty-}).$$

The function C is defined on $[0, F_\infty)$, with values in $\mathbf{R}^+$. It is increasing (since F is continuous) and right-continuous. Moreover, the s-level set of F coincides with the interval $[C_{s-}, C_s)$.

Plainly, the function $s \mapsto F_{C_s}$ increases linearly from 0 to $F_{\infty-}$, at unit speed. We now proceed to introduce the non-trivial level sets of F. More precisely, we will let the value of F at time t be reached, under the time-change, at a time which is the sum of two components, the time F_{t-} that is needed to reach the level F_{t-} at unit speed, and the cumulative length of all level sets up to time t.

As mentioned above, the length of the $F_{t'}$-level set is $\Delta C_{F_{t'}}$. Therefore, the cumulative length of all level sets up to time t is

$$\sum_{t'<t} \Delta C_{F_{t'}} + t - C_{F_{t-}} :$$

the first summation is the total length of all level sets lying entirely to the left of t, while $t - C_{F_{t-}}$ is the time elapsed since the current level set was initiated.

This leads us to introduce the function v_1 defined by

$$v_1(t) := F_t + \sum_{t'<t} \Delta C_{F_{t'}} + t - C_{F_{t-}}.$$

The next lemma lists few easy properties of v_1. The proof is omitted.

Lemma 5.3 *The function v_1 is continuous and increasing. In addition, $v_1(0) = 0$, and*[13] *$v_1(\infty-)$ is infinite or finite depending on whether F is eventually constant or not.*

5.1.2 Playing with level sets

We here define a second time-change, relative to an arbitrary $F \in \mathcal{F}$. In effect, we shall adjust the length of level sets of F to the size of nearby discontinuities. Formally, the value of F at time t will be reached, according to the new clock, at time s, which is obtained from t by substracting the cumulative length of all

[13]Recall that $f(\infty-) = \lim_{t\to\infty} f(t)$.

level sets prior to t, and by adding the cumulative sum of jumps prior to time t. That is, we set

$$\begin{aligned} v_2(t) &= t + \sum_{t'<t} \Delta F_{t'} - \left(\sum_{t'<t} \Delta C_{F_{t'}} + t - C_{F_t-} \right) \\ &= C_{F_t-} + \sum_{t'<t} \Delta F_{t'} - \sum_{t'<t} \Delta C_{F_{t'}} \end{aligned}$$

The proof of the following basic properties of v_2 is left to the reader.

Lemma 5.4 *The function v_2 is non-decreasing and right-continuous.*

5.1.3 Time-changes and the equilibrium property

We let here $\epsilon > 0$ and an ϵ-equilibrium $(F^i)_{i\in I}$ be given. Loosely speaking, our goal is to show that applying the above time-changes to the profile $(F^i)_{i\in I}$ does not affect the ϵ-equilibrium property.

We will make extensive use of the following change-of-variable formula for Stieltjes integrals, which is a minor variation upon Prop. 4.10 in Revuz and Yor (2000, Chapter 0).

Lemma 5.5 *Let $u : [a,b] \to \mathbf{R}^+$ be a right-continuous, non-decreasing map. Let $F \in \mathcal{F}$ and g be a bounded, Borel measurable map. Assume that $F_{u(t)-} = F_{u(t)}$ whenever $\Delta u(t) > 0$. Then*

$$\int_{[u(a),u(b)]} g(s) dF_s = \int_{[a,b]} g(u(t)) dF_{u(t)}.$$

For $i \in I$, we let $\tilde{F}^i$ denote the continuous part of F^i: $\tilde{F}^i_t = F^i_t - \sum_{t'<t} \Delta F^i_t$ for $t \in \mathbf{R}^+$. Next, we set $\tilde{F} = \frac{1}{I} \sum_{i\in I} \tilde{F}^i$ and consider the function v_1 relative to $\tilde{F}$, as defined in section 5.1.1. Let u_1 be the inverse map of v_1.

For $i \in I$, we define G^i to be the image of F^i under the time-change u_1: $G^i_s = F^i_{u_1(s)}$ for $s < v_1(\infty-)$ and $G^i_s = F^i_{\infty-}$ for $s \geq v_1(\infty-)$. Plainly, $G^i \in \mathcal{F}$ for each $i \in I$.

Lemma 5.6 *The profile $(G^i)_{i\in N}$ is an ϵ-equilibrium.*

Proof. We fix $i \in I$, and prove that player i has no pure deviation that increases his payoff by more than ϵ. Let $\tilde{G}^i$ be a pure strategy.

Case 1: $\tilde{G}^i_s = 0$ for every $s \in \mathbf{R}^+$ (player i never acts).

Since $(F^1, \ldots, F^I)$ is an ϵ-equilibrium,

$$\gamma^i(\tilde{G}^i, G^{-i}) = \gamma^i(\tilde{G}^i, F^{-i}) \leq \gamma^i(F^i, F^{-i}) + \epsilon = \gamma^i(G^i, G^{-i}) + \epsilon,$$

where the equalities follow by the change-of-variable formula.

Case 2: $\tilde{G}^i_s = 1_{s \geq s_0}$ for some $s_0 \in \mathbf{R}^+$ (player i acts at time s_0).

If $s_0 < v_1(\infty-)$, we set $t_0 = u_1(s_0)$ and we define $\tilde{F}^i_t = 1_{t \geq t_0}$.

Since $(F^1, \ldots, F^I)$ is an ϵ-equilibrium,

$$\gamma^i(\tilde{G}^i, G^{-i}) = \gamma^i(\tilde{F}^i, F^{-i}) \leq \gamma^i(F^i, F^{-i}) + \epsilon = \gamma^i(G^i, G^{-i}) + \epsilon,$$

where the equalities follow by the change-of-variable formula.

Assume now that $s_0 \geq v_1(\infty-)$. In particular, $v_1(\infty-) < \infty$. For $\bar{s} \leq v_1(\infty-)$, define $1^{\bar{s}} \in \mathcal{F}$ by $1^{\bar{s}}_s = 1_{s \geq \bar{s}}$.

Plainly,

$$\begin{aligned}\gamma^i(\tilde{G}^i, G^{-i}) &= \gamma^i(1^{v^1(\infty-)}, G^{-i}) \\ &= \lim_{\bar{s} \nearrow v^1(\infty-)} \gamma^i(1^{\bar{s}}, G^{-i}) \leq \gamma^i(G^i, G^{-i}) + \epsilon,\end{aligned}$$

where the last inequality follows by the analysis of the case $\bar{s} < v^1(\infty-)$. ■

We now analyze the impact of the second time-change on $(G^i)_{i \in I}$. We let v_2 be the time-change relative to $\frac{1}{I}\sum_{i \in I} G^i$, as defined in section 5.1.2. We let u_2 be the generalized inverse of v_2: $u_2(s) = \inf\{t : v_2(t) > s\}$. The function u_2 is defined over $[0, v_2(\infty-))$, is right-continuous and non-decreasing. Note that a level set of u_2 with positive length corresponds to a jump in v_2. Also, a jump in u_2 corresponds to a non-trivial level set of v_2. For $i \in I$, we let $H^i_s = G^i_{u_2(s)}$ for $s < v_2(\infty-)$ and $H^i_s = G^i(\infty-)$ for $s \geq v_2(\infty-)$.

LEMMA 5.7 *The profile $(H^i)_{i \in I}$ is an ϵ-equilibrium in $\mathcal{G}$.*

Proof. We prove that player i has no pure profitable deviation. Let $\tilde{H}^i \in \Delta^i$ be arbitrary. The case $\tilde{H}^i = 0$ can be dealt with as in the previous proof. Assume now that $\tilde{H}^i = 1_{s \geq s_0}$ for some $s_0 \in \mathbf{R}^+$. As observed at the end of the previous proof, it is enough to deal with the case $s_0 < v_2(\infty-)$. Set $t_0 = u_2(s_0)$. If u_2 is continuous at s_0, the inequality $\gamma^i(\tilde{H}^i, H^{-i}) \leq \gamma^i(H^i, H^{-i}) + \epsilon$ follows by the change-of-variable formula.

If u_2 is not continuous at s_0, then the change-of-variable cannot be applied (at least for the integral w.r.t. $\tilde{H}^i$). In that case, we let (s^n) be a increasing sequence of continuity points of u_2, that converges to s_0, and we let $\tilde{H}^{i,n}_s = 1_{s \geq s^n}$. It is not difficult to check that $\lim_{n \to \infty} \gamma^i(\tilde{H}^{i,n}, H^{-i}) = \gamma^i(\tilde{H}^i, H^{-i})$. Hence, by the previous paragraph, $\gamma^i(\tilde{H}^i, H^{-i}) \leq \gamma^i(H^i, H^{-i}) + \epsilon$. Therefore, $(H^i)_{i \in I}$ is an ϵ-equilibrium. ■

5.2 Proof of Lemma 5.2

We shall only prove the first assertion of Lemma 5.2. The second one can be established using similar ideas.

Let $(F(n))$ be a sequence in $\mathcal{G}$ that weakly converges to $F \in \mathcal{G}$.

For every non-empty subset S of I we let π_S be the probability that under F the game terminates, and the terminating coalition is S. For $n \in \mathbf{N}$, we denote by $\pi_S(n)$ the analogous probability under $F(n)$.

Since $\gamma(F) = \sum_S \pi_S u_S$ and $\gamma(F(n)) = \sum_S \pi_S(n) u_S$, it is enough to prove that $\lim_{n\to\infty} \pi_S(n) = \pi_S$ for every S.

Note first that $F^i_t = \lim_{n\to\infty} F^i_t(n)$ for each $i \in I$ and for every continuity point t of F^i. In particular, the equality holds for λ-a.e. $t \in \mathbf{R}^+$, which implies

$$\lim_{n\to\infty} F^i_{t-}(n) = F^i_{t-}, \text{ for every } t \in \mathbf{R}^+ \text{ and every } i \in I. \tag{6}$$

Step 1: Relating atoms.

Let t be an atom of F^i, for some $i \in I$. Set $S^* = \{i \in I, \Delta F^i_t > 0\}$ be the set of i's such that t is an atom of F^i.

We show that for every n there is $\tau(t;n) \in \mathbf{R}^+$ such that

A.i) $\lim_{n\to\infty} \tau(t;n) = t$,

A.ii) $\lim_{n\to\infty} \Delta F^i_{\tau(t;n)}(n) = \Delta F^i_t$ for each $i \in I$, and

A.iii) $\lim_{n\to\infty} F^i_{\tau(t;n)}(n) = F^i_t$ for each $i \in I$.

Let $\epsilon \in (0,t)$ satisfy $\Delta F^i_t > (2I+5)\epsilon$ for every $i \in S^*$.[14] In addition, we assume that both $t+\epsilon$ and $t-\epsilon$ are continuity points of F^i.

For n large enough, $F^i_{t+\epsilon}(n) - F^i_{t-\epsilon}(n) \geq F^i_{t+\epsilon} - F^i_{t-\epsilon} - \epsilon \geq \Delta F^i_t - \epsilon$. Let $\tau^i(t;n)$ be the infimum over all discontinuities of $F^i(n)$ in the interval $[t-\epsilon, t+\epsilon]$, and set $\tau(t;n) = \min_{i\in S^*} \tau^i(t;n)$. Since $F(n) \in \mathcal{G}$, one has

$$\sum_{s\in[t-\epsilon,t+\epsilon]} \Delta F^i_s(n) \geq F^i_{t+\epsilon}(n) - F^i_{t-\epsilon}(n) - 2I\epsilon, \text{ and } \sum_{s\in(\tau(t;n),t+\epsilon]} \Delta F^i_s(n) \leq 2\epsilon. \tag{7}$$

Eq. (7) implies that $\Delta F^i_{\tau(t;n)} \geq F^i_{t+\epsilon}(n) - F^i_{t-\epsilon}(n) - 2(I+1)\epsilon \geq \Delta F^i_t - (2I+3)\epsilon$. Therefore, for $i \in S^*$, $\Delta F^i_{\tau(t;n)} > 0$, so that $\tau^i(t;n) = \tau(t;n)$, and moreover $\Delta F^i_{\tau(t;n)}(n) \geq \Delta F^i_t - 5\epsilon$.[15] Therefore,

$$\liminf_n \Delta F^i_{\tau(t;n)}(n) \geq \Delta F^i_t - 5\epsilon. \tag{8}$$

[14] If $t = 0$, the condition $\epsilon < t$ is omitted, and in the sequel $t - \epsilon$ is replaced by t.

[15] For further use, we note the following additional consequence. Strictly speaking, the sequence $(\tau(t;n))_n$ depends on ϵ, and should rather be denoted by $(\tau^\epsilon(t;n))_n$. For $\epsilon' < \epsilon$, one has $\tau^\epsilon(t;n) \leq \tau^{\epsilon'}(t;n)$ whenever the two sides are well-defined. The last inequality in the text implies that $\tau^\epsilon(t;n) = \tau^{\epsilon'}(t;n)$ for n large enough. In that sense, the sequence $(\tau^\epsilon(t;n))_n$ is (asymptotically) independent of ϵ.

This implies that $\lim_{n\to\infty} \tau(t;n) = t$, so that (A.i) holds. Indeed, otherwise there would be a subsequence of $(\tau(t;n))_n$ – still denoted $(\tau(t;n))_n$ – such that $\lim_{n\to+\infty} \tau(t;n) = t' \neq t$. By repeating the above argument with $\epsilon' \in (0,\epsilon)$ small enough so that $t' \notin [t-\epsilon', t+\epsilon']$, we would construct another sequence $(\tau'(t;n))_n$ such that $\lim_{n\to+\infty} \Delta F^i_{\tau'(t;n)}(n) = \Delta F^i_t$, for each $i \in I$ – a contradiction to the second inequality in (7). By weak convergence, (A.i) implies that (A.ii) holds whenever $\Delta F^i_t = 0$, or, equivalently, whenever $i \notin S^*$.

We now prove that (A.ii) holds for $i \in S^*$ as well. Since $F^i_{t+\epsilon} - F^i_{t-\epsilon} \leq \Delta F^i_t + I\epsilon$, one has $\Delta F^i_{\tau(t;n)} \leq F^i_{t+\epsilon}(n) - F^i_{t-\epsilon}(n) \leq \Delta F^i_t + (I+1)\epsilon$, provided n is large enough. Therefore, $\limsup_n \Delta F^i_{\tau(t;n)}(n) \leq \Delta F^i_t + 2\eta$, which, together with (8), and since ϵ is arbitrary, yields

$$\lim_{n\to+\infty} \Delta F^i_{\tau(t;n)-}(n) = \Delta F^i_t, \text{ for each } i \in I, \tag{9}$$

so that (A.ii) holds.

Finally, we show that $\lim_{n\to\infty} F^i_{\tau(t;n)-}(n) = F^i_{t-}$, for each $i \in I$, which, together with (A.ii), implies that (A.iii) holds. W.l.o.g., we may assume that the sequence $(\tau(t;n))_n$ is monotonic. Assume first that it is non-decreasing, and let $\epsilon > 0$ be given. Choose $t' < t$ such that $F^i_{t'-} \geq F^i_{t-} - \epsilon$. Then, for n large enough, one has by (6)

$$F^i_{t'-} - \epsilon \leq F^i_{t'-}(n) \leq F^i_{\tau(t;n)}(n) \leq F^i_{t-}(n) \leq F^i_{t-} + \epsilon.$$

If the sequence $(\tau(t;n))_n$ is non-increasing, then $F^i_{\tau(t;n)-}(n) = F^i_{t-}(n)$ for n large, hence by (6) the claim still holds.

Step 2: $\lim_{n\to\infty} \pi_S(n) = \pi_S$ **whenever** $|S| \geq 2$**.**

Suppose $S \subseteq I$ with $|S| \geq 2$. For the sake of clarity, we set $g^S_t := \prod_{j\notin S}(1 - F^j_t))$, and $h^S_t := \prod_{i\in S} \Delta F^i_t$ for $S \subset I$ and $t \in \mathbf{R}^+$.

Then

$$\pi_S = \sum_{t\in\mathbf{R}^+} \prod_{j\notin S}(1 - F^j_t) \prod_{i\in S} \Delta F^i_t = \sum_{t\in\mathbf{R}^+} g^S_t h^S_t,$$

and a similar expression holds for $\pi_S(n)$.

Fix $i \in S$, and let $\epsilon > 0$ be arbitrary. Let $A \subset \mathbf{R}^+$ be a finite set of atoms that almost exhausts the atoms of F^i: $\sum_{t\in A} \Delta F^i_t \geq \sum_{t\in\mathbf{R}^+} \Delta F^i_t - \epsilon$.

By (A.ii) and (A.iii), $\lim_{n\to+\infty} g^S_{\tau(t;n)}(n) h^S_{\tau(t;n)}(n) = g^S_t h^S_t$ for every $t \in \mathbf{R}^+$. In particular, since A is a finite set,

$$\lim_{n\to\infty} \sum_{t\in A} g^S_{\tau(t;n)}(n) h^S_{\tau(t;n)}(n) = \sum_{t\in A} g^S_t h^S_t. \tag{10}$$

Moreover,

$$\sum_{t\notin A} g^S_t h^S_t \leq \sum_{t\notin S} \Delta F^i_t < \epsilon. \tag{11}$$

For $n \in \mathbf{N}$ set $A_n := \{\tau(t;n) : t \in A\}$. Our goal is to prove that

$$\lim_{n\to\infty} \sum_{t \notin A_n} g_t^S h_t^S = 0, \tag{12}$$

which, together with (10) and (11) implies that $\lim_{n\to\infty} \pi_S(n) = \pi_S$, provided $|S| \geq 2$.

Let $\delta_n := \sup\{\Delta F_s^i(n) : s \notin A_n, i \in I\}$ (with $\sup \emptyset = 0$) be the maximal size of the remaining discontinuities, and let t_n achieve the supremum, up to $1/n$. We claim that $\lim_{n\to\infty} \delta_n = 0$. Indeed, since the support of F^i is included in $[0, I]$, the sequence (t_n) converges, up to a subsequence, to some $t \in \mathbf{R}^+$. If $\Delta F_t^i > 0$ for some $i \in I$, then $\lim_{n\to\infty} \Delta F_{t_n}^j(n) = 0$ since $t_n \neq \tau(t;n)$ for each n. If $\Delta F_t^i = 0$ then by weak convergence $\lim_{n\to\infty} \Delta F_t^i(n) = 0$. Therefore, $\lim_{n\to\infty} \delta_n = 0$.

For every two sequences $(x_k, y_k)_{k=1}^\infty$ such that $0 \leq x_k, y_k \leq \delta < 1$ and $\sum_k x_k, \sum_k y_k \leq 1$ one has $\sum_k x_k y_k \leq \delta$. Since $|S| \geq 2$, and since $g_t^S(n) h_t^S(n)$ is non-zero on at most a countable set of t's, (12) holds.

Step 3: $\lim_{n\to\infty} \pi_S(n) = \pi_S$ whenever $S = \{i\}$ is a singleton.

Let $\epsilon > 0$ be arbitrary. We prove that $\pi_{\{i\}} - 3\epsilon \leq \liminf_{n\to\infty} \pi_{\{i\}}(n)$ and $\limsup_{n\to\infty} \pi_{\{i\}}(n) \leq \pi_{\{i\}} + 3\epsilon$.

As in step 2, let $A \subset \mathbf{R}^+$ be a finite set such that $\sum_{t\in A} \Delta F_t^i \geq \sum_{t\in\mathbf{R}^+} \Delta F_t^i - \epsilon$. We assume that A contains 0 if $\Delta F_0^i > 0$.

Since A is finite, we may assume w.l.o.g. that for every n, the finite set $\{\tau(t;n), t \in A\}$ contains $|A|$ different elements.

Denote $\widehat{F}_t^i = F_t^i - \sum_{s<t, s\in A} \Delta F_s^i$ and $\widehat{F}_t^i(n) = F_t^i(n) - \sum_{s<t, s\in A} \Delta F_{\tau(s;n)}^i$. This is the part of F^i (resp. $F^i(n)$) without the atoms in A. Then $(\widehat{F}^i(n))$ weakly converges to $\widehat{F}^i$.

Choose a finite sequence $0 < t_1 < \ldots < t_K = I + 1$ such that

B.i) $\widehat{F}_{t_{k+1}}^i - \widehat{F}_{t_k}^i < \epsilon$ for each $k = 0, \ldots, K-1$ (with $\widehat{F}_{t_0}^i = 0$).

B.ii) $t_1, \ldots, t_K$ are continuity points of F^j, for every $j \in I$.

We now modify the distributions F^i and $(F^i(n))_{n\in\mathbf{N}}$, and construct completely atomic distributions $\overline{F}^i$, $\underline{F}^i$, $(\overline{F}^i(n))_{n\in\mathbf{N}}$, and $(\underline{F}^i(n))_{n\in\mathbf{N}}$ as follows.

- $\overline{F}^i$: every $t \in A$ is an atom of $\overline{F}^i$ with size ΔF_t^i. In addition, each $(t_k)_{k=1}^{K-1}$ is an atom; the weight of this atom is equal to $\widehat{F}_{t_{k+1}}^i - \widehat{F}_{t_k}^i$.
- $\underline{F}^i$: every $t \in A$ is an atom of $\overline{F}^i$ with size ΔF_t^i. In addition, each $(t_k)_{k=2}^{K}$ is an atom; the weight of this atom is equal to $\widehat{F}_{t_k}^i - \widehat{F}_{t_{k-1}}^i$.
- $\overline{F}^i(n)$ and $\underline{F}^i(n)$ are defined analogously w.r.t. $F^i(n)$.[16]

[16]Note that, for n large enough, the two sets $\{\tau(t;n), t \in A\}$ and $\{t_k, k = 1, \ldots, K\}$ are disjoint.

Thus, under $\overline{F}^i$ player i acts earlier than under F^i, whereas under $\underline{F}^i$ he acts later.

Observe that in this definition, we ignored the part of $\widehat{F}^i$ prior to time t_1, but by (B.i) this part has small weight. Let $\overline{\pi}_{\{i\}}$, $\underline{\pi}_{\{i\}}$, $\overline{\pi}_{\{i\}}(n)$ and $\underline{\pi}_{\{i\}}(n)$ be analogous to $\pi_{\{i\}}$ under $(\overline{F}^i, F^{-i})$, $(\underline{F}^i, F^{-i})$, $(\overline{F}^i(n), F^{-i}(n))$ and $(\underline{F}^i(n), F^{-i}(n))$ respectively.

By (B.i) we have

$$\overline{\pi}_{\{i\}} + \epsilon \geq \pi_{\{i\}} \geq \underline{\pi}_{\{i\}}, \text{ and } \overline{\pi}_{\{i\}}(n) + \epsilon \geq \pi_{\{i\}}(n) \geq \underline{\pi}_{\{i\}}(n) \quad \forall n \in \mathbf{N}. \tag{13}$$

Moreover,

$$\overline{\pi}_{\{i\}} - \underline{\pi}_{\{i\}} < 2\epsilon. \tag{14}$$

Since $\overline{F}^i$ is completely atomic, we can derive an explicit formula for $\overline{\pi}_{\{i\}}$:

$$\overline{\pi}_{\{i\}} = \sum_{k=1}^{K-1} \prod_{j \neq i} (1 - F^j_{t_k}) \Delta \overline{F}^i_{t_k} + \sum_{t \in A} \prod_{j \neq i} (1 - F^j_t) \Delta \overline{F}^i_t. \tag{15}$$

One has a similar expression for $\underline{\pi}_{\{i\}}$. For $\overline{\pi}_{\{i\}}(n)$ one has

$$\overline{\pi}_{\{i\}}(n) = \sum_{k=1}^{K-1} \prod_{j \neq i} (1 - F^j_{t_k}(n)) \Delta \overline{F}^i_{t_k}(n) + \sum_{t \in A} \prod_{j \neq i} (1 - F^j_{\tau(t;n)}(n)) \Delta \overline{F}^i_{\tau(t;n)}(n). \tag{16}$$

By (A.ii) and (A.iii), since $(F^i(n))$ weakly converges to F^i, and since (t_k) are continuity points of F^i, $\lim_{n\to\infty} \Delta \overline{F}^i_{t_k}(n) = \Delta \overline{F}^i_{t_k}(n)$. Since the (t_k) are continuity points of $(F^j)_{j \neq i}$, $\lim_{n\to\infty} \overline{F}^j_{t_k}(n) = \overline{F}^j_{t_k}(n)$. Therefore, again using (A.ii) and (A.iii), we obtain $\lim_{n\to\infty} \overline{\pi}_{\{i\}}(n) = \overline{\pi}_{\{i\}}$. Similarly, one obtains $\lim_{n\to\infty} \underline{\pi}_{\{i\}}(n) = \underline{\pi}_{\{i\}}$. These two inequalities, together with (13) and (14), delivers the claim.

6 Comments and extensions

In this paper we analyzed continuous-time games of timing with complete information. In several classes of economic interest, we proved the existence of a subgame-perfect ϵ-equilibrium for each $\epsilon > 0$. We here conclude by discussing which insights can be gained for the analysis of discrete time games with short time periods, and some extensions of our results.

Let $\widehat{\sigma}$ be a subgame-perfect ϵ-equilibrium of a continuous-time game of timing. Consider a discrete-time version of the game, in which the players are allowed to stop only at times t_n, $n \in \mathbf{N}$, where $(t_n)_n$ is a strictly increasing sequence in $\mathbf{R}^+$. We denote by $\widehat{\tau}$ the discretized version of $\widehat{\sigma}$, defined as follows: at time t_n, assuming no player acted before, player i acts with probability $\sigma^i_{t_{n-1}}((t_{n-1}, t_n])$ (and acts with probability $\sigma^i_0(\{0\})$ at time zero, if $t_0 = 0$). In words, at t_n, player i

assigns to *act* the probability with which he would have acted between t_{n-1} and t_n, had he been allowed to act at any time. Assuming all functions u_S are continuous, it is easy to check that $\widehat{\tau}$ is, say, a subgame-perfect 2ϵ-equilibrium of the game in discrete time, provided $\sup_n |t_n - t_{n-1}|$ is small enough. Moreover, this result does not rely on the sequence (t_n) being known *ex ante*. Specifically, assume that the sequence (t_n) is a random sequence that increases a.s. to ∞, and assume that players get to know the value of t_n at time t_n only.[17] Since the probability to act at time t_n is computed *ex post*, as a function of the interval $(t_{n-1}, t_n]$, the profile $\widehat{\tau}$ is well-defined. Moreover, it is a subgame-perfect 2ϵ-equilibrium provided that, with high probability, $\sup_n |t_n - t_{n-1}|$ is small enough. Thus, our analysis of the continuous-time game gives an easy scheme for constructing approximate equilibria in a large class of discrete time scenarios.

Finally, we discuss weakenings of the complete information assumption. Our approach do not extend to games with asymmetric information. Nevertheless, it yields partial results in the case of games with symmetric incomplete information. In these games, u_S is a stochastic process, for each $S \subset I$, whose law is publicly known. At any time, all the players have the same information on the realization of the payoff processes.[18] These games were first introduced by Dynkin (1969) in a two-player zero-sum discrete-time setting. Since then, they have come to be known as Dynkin games in the theory of stochastic processes, and a very extensive literature has been devoted to the zero-sum case, see Solan and Vieille (2001) and the references therein. In a related work, we analyze two-player non-zero-sum games under the assumption that, for each S, the stochastic process u_S is right-continuous with left-limits and satisfies a weak integrability condition. We prove that techniques similar to the ones we developed in the present paper can be applied to prove the existence of an ϵ-equilibrium for each $\epsilon > 0$, see Laraki et al. (2002).

[17] We need not assume that the players have any prior information on the law of the sequence (t_n).

[18] For example, they may know past and present values of u_S, $S \subseteq I$, and therefore learn the paths $u_S(\cdot)$, for $\emptyset \neq S \subseteq I$, as time unfolds.

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