Discussion Paper No. 133

ON THE STRUCTURE OF SEMIMARKOV PROCESSES AND THEIR CONVERSION TO CHUNG PROCESSES*

by

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*A preliminary version of this was presented at the Fourth Conference on Stochastic Processes and Their Applications in Toronto, Canada, August, 1974.

**Research supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR-74-2733. The United States Government is authorized to reproduce and distribute reprints for governmental purposes.

February 1975

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Abstract

Semimarkov processes with discrete state spaces are considered without restrictions on their probability laws. They admit states where every visit lasts a positive time even though there may be infinitely many such visits in a finite interval. These are called unstable holding states as opposed to the stable holding states which are encountered in Markov processes. Further, it is possible to have instantaneous states at which the behavior is that at an ordinary instantaneous state of a Chung process, or that at a sticky boundary point, or that at a non-sticky point. To convert such processes to Chung processes, each unstable state is split into infinitely many stable ones, and then a random time change is effected whereby some sets of constancy are dilated, and sojourn intervals are altered to have exponentially distributed lengths.

ON THE STRUCTURE OF SEMIMARKOV PROCESSES

AND THEIR CONVERSION TO CHUNG PROCESSES[1]

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1. INTRODUCTION

While introducing semimarkov[3] processes, LÉVY (1954) asserted that any such process can be transformed into a Markov process through a random time change using a suitable strictly increasing time scale. The assertion implies that the qualitative structure of the sample paths of a semimarkov process is the same as that of a Markov process. The only differences, therefore, would be restricted to "the probability laws governing the sojourns" at a holding state: they are exponential for Markov processes and arbitrary for semimarkov processes.

This, however, is not true. The source of error is the implicit assumption that "sojourn times at a fixed state" are well-defined random variables. It turns out that there are states such that every visit lasts a positive time but that between any two sojourn intervals there is a third. Of course, then, the left end points of those intervals cannot be well-ordered, which implies that we need to talk of random variables such as the length of the first (or second or third...) sojourn interval whose length exceeds ϵ. And the law of such intervals, of course, depends on ϵ. Here is an illustration.

(1.1) EXAMPLE. Consider a Brownian motion on $(-\infty,\infty)$ starting from 0. Let $\overline{M}$ be the set of all t at which the process is at the point 0. This is a closed set. Let K_0 be the smallest right-closed set whose closure is $\overline{M}$, i.e. K_0 is the set of all t in $\overline{M}$ which are not isolated on the right. Let K_1 be the complement of K_0 in $[0,\infty)$. Define a stochastic process (X_t) by

$$X_t = \begin{cases} 0 & \text{if } t \in K_0, \\ \\ 1 & \text{if } t \in K_1. \end{cases}$$

The process (X_t) is semimarkov in all the senses which we are aware of: LÉVY (1954), YACKEL[4] (1968), PYKE and SCHAUFELE (1964) p. 1740. State 0 is instantaneous, and state 1 is holding. The set K_1 is a countable union of intervals each of which has positive length, but any neighborhood of $t = 0$ contains infinitely many of those intervals. No two of these sojourn intervals have an end point in common, and between any two intervals there lies a third.

We note that the phrase "length of a sojourn interval at state 1" has no definite meaning, and it is faulty to speak of its probability law. However, there is a measure λ on $(0,\infty)$, whose total mass is infinite, and which governs the sojourns at 1 in the following sense: Consider those sojourn intervals whose lengths exceed ϵ; the lengths of the first, the second,... such intervals are independent and identically distributed as

$$\lambda(B)/\lambda((\epsilon,\infty)), \quad B \subset (\epsilon,\infty),\ B \text{ Borel}.$$

It is well known that the set K_0 is the range of a right continuous increasing additive process with zero drift. The measure λ above is the Lévy measure of this additive process.

Our object is to clarify the qualitative structure of semimarkov processes on discrete spaces, and to give a rectification of Lévy's assertion.

We will classify the states of a semimarkov process X first as holding versus instantaneous just as with Markov processes. A holding state is either stable or unstable depending on whether it is entered only finitely often or not during a finite interval. An instantaneous state i is attractive or repellent depending on whether its set of constancy $\{t : X_t = i\}$ is perfect or discrete (perfect means right closed and has no isolated points, discrete means every point is isolated). An attractive state is light or heavy according as the Lebesgue measure of its set of constancy is 0 or positive.

Every Chung process (see the definition below) is semimarkov. In a Chung process, every holding state is stable, and every instantaneous state is heavy. Supposing that a Chung process has a purely atomic boundary, in the terminology of CHUNG (1968), a sticky boundary point is like a light attractive state, and a non-sticky boundary point is like a repellent state.

The most important qualitative difference between semimarkov and Chung processes is due to unstable states. While converting a semimarkov process to a Chung process, each unstable state i is "split" into infinitely many stable ones (in addition to dilating light attractive states and replacing sojourn intervals by exponentially distributed ones). Section 5 sketches the steps necessary to reduce this conversion problem to that treated by YACKEL (1968). In fact, we have nothing to add if every state is either stable or heavy attractive--a state of affairs insured by YACKEL by requiring that the basic semigroup be "strong."

Finally, we would like to introduce the following much needed definition.

(1.2) CHUNG PROCESSES. Let (X_t) be a (continuous time-parameter) Markov process defined on a complete probability space, and taking values in a discrete state space E (a countable set with the discrete topology which is further compactified by adding an extra point if not already compact). Then X is said to be a <u>Chung process</u> provided that

a) the transition function of X be standard, and

b) the sample paths be "right lower semicontinuous," that is, for any t and ω, the sample path $X_s(\omega)$ has at most one limiting value in E as s decreases to t; $X_t(\omega)$ is equal to that limiting value in E if it exists, and is equal to the "point at infinity" if there is no such limit in E.

Chung processes are important because Chung, Doob, Feller, Kolmogorov, and Lévy have worked on them; they are interesting because their sample paths have intricate discontinuity properties not possessed by "standard" Markov processes on more general state spaces; and they do not fit neatly into the general potential-theoretic framework. Finally, on sheer volume of publications concerning them alone, Chung processes deserve a name of their own amongst other processes such as Brownian motion, Feller processes, Hunt process. We hope that the wisdom, and the justice implicit in, this definition will be recognized.

Our terminology for elements of stochastic processes in general follows DELLACHERIE (1972). In addition, we write $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_t = (t,\infty)$ for any $t \in \mathbb{R}_+$; and $\underline{\underline{\mathbb{R}}}_+$ and $\underline{\underline{\mathbb{R}}}_t$ for the associated σ-algebras of all Borel subsets.

2. SEMIMARKOV PROCESSES

In this section we describe the processes we are interested in. A more fundamental definition (which seems to have been what Lévy had in mind) will be given elsewhere along with comparisons with other definitions.

Let E be a discrete set (countable with discrete topology). If E is finite, let $\overline{E} = E$. If E is not finite, let $\overline{E} = E \cup \{\varphi\}$ be its one point compactification. Let $(\Omega, \underline{\underline{M}}, P)$ be a complete probability space, and let $X = (X_t)_{t>0}$ be a stochastic process defined on $(\Omega, \underline{\underline{M}})$ and taking values in $\overline{E}$. Let $\underline{\underline{H}}_t^0 = \sigma(X_s;\ 0 < s \leq t)$, $\underline{\underline{H}}_\infty^0 = \sigma(X_s;\ s > 0)$; let $\underline{\underline{H}}_\infty$ be the completion of $\underline{\underline{H}}_\infty^0$, and let $\underline{\underline{H}}_t$ be the σ-algebra generated by $\underline{\underline{H}}_{t+}^0 = \bigcap_{s>t} \underline{\underline{H}}_s^0$ and all the negligible sets in $\underline{\underline{H}}_\infty$. We will refer to the family $\underline{\underline{H}} = (\underline{\underline{H}}_t)$ as the history generated by X. Stopping times and progressive sets etc. are all relative to this history unless specified otherwise.

For each $\omega \in \Omega$ let $\overline{M}(\omega)$ be the set of all $t > 0$ at which the path $X(\omega) : s \to X_s(\omega)$ is not continuous plus those $t > 0$ at which $X(\omega)$ is continuous and has the value φ. The random set $\overline{M}$ will be called the discontinuity set of X.

Fix $\omega \in \Omega$. The set $\overline{M}(\omega)$ is closed in $(0,\infty)$. Its complement (if not empty) is a countable union of open intervals. Each such open interval is said to be contiguous to $\overline{M}(\omega)$. We define $L(\omega)$ to be the set of all left end points of the intervals contiguous to $\overline{M}(\omega)$.

For every $t > 0$ and $\omega \in \Omega$ we define

$$(2.1) \qquad S_t(\omega) = t - \sup\{s \leq t : s \in \overline{M}(\omega)\}.$$

We call $S_t(\omega)$ the <u>sojourn value</u> at t for the path $X(\omega)$. If t is a time of discontinuity for $X(\omega)$, then $S_t(\omega) = 0$. Otherwise, if t belongs to the contiguous interval $(a(\omega), b(\omega))$ (on which, by the way $\overline{M}(\omega)$ is defined, the path $X(\omega)$ remains constant), the sojourn value at t is $S_t(\omega) = t - a(\omega)$.

Throughout this paper we assume that the following axioms hold.

(2.2) <u>Regularity</u>: For almost every ω and every $t > 0$,

a) if $t \in L(\omega)$, then the path $X(\omega)$ is right continuous at t;

b) if $t \in \overline{M}(\omega)\setminus L(\omega)$, and if there exists a sequence $(t_n) \subset \overline{M}(\omega)\setminus L(\omega)$ decreasing to t and such that $X_{t_n}(\omega) = i$ for every n for some $i \in E$, then $X_t(\omega) = i$.

(2.3) <u>Point φ</u>: For every $t > 0$, $P\{X_t = \varphi\} = 0$.

(2.4) <u>Progressive measurability</u>: The process X is progressively measurable with respect to the history $\underline{\underline{H}}$.

(2.5) Regeneration. The process $(X,S) = (X_t, S_t)_{t>0}$ is a time-homogeneous Markov process which further enjoys the strong Markov property at every stopping time T of $\underline{\underline{H}}$ such that $X_T \in E$ a.s. on $\{T < \infty\}$.

Then X is a semimarkov process in the strict sense.

Regularity axiom (2.2) is close to a separability axiom. By (2.2a) every interval of constancy has the form [ ). Axiom (2.2b) is in fact an axiom on random sets on which X is in a fixed instantaneous state. These axioms might require the state space to be closed, and thus the need for compactifying the state space. Hence (2.3) delimits φ to the role of "an extra point added for reasons of smoothness." Axiom (2.4) is technical. The crucial assumption is (2.5): First, note that each S_t is $\underline{\underline{H}}_t$ measurable, and that (X,S) is progressively measurable relative to $\underline{\underline{H}}$. So, it is meaningful to talk of strong Markov property for (X,S). This property is of particular interest for stopping times T whose graphs [T] are contained in $\overline{M}$, i.e., $T(\omega) \in \overline{M}(\omega)$ for almost every $\omega \in \{T < \infty\}$. Then, $S_T = 0$, and the strong Markov property at T may be re-phrased as follows. For every state $i \in E$ there exists a probability measure P^i on $(\Omega, \underline{\underline{H}}^0_\infty)$ such that

$$(2.6)\qquad E[W\circ\theta_T \mid \underline{\underline{H}}_T] = E^i[W] \text{ a.s. on } \{X_T = i\}$$

for any bounded $\underline{\underline{H}}^o_\infty$ measurable random variable W and any stopping time T such that $[T] \subset \overline{M}$. Here and below (θ_t) are the usual shift operators defined on Ω such that

$$X_t\circ\theta_s = X_{t+s}, \qquad s \geq 0,\ t > 0.$$

(2.7) EXAMPLE. Consider the process X of Example (1.1). Then, it is well known that S is a strong Markov process, (see MEYER (1970) for instance). For any stopping time T, on $\{S_T > 0\}$ we have $X_T = 1$. There is no stopping time T such that $[T] \subset L$ other than $T = \infty$. Therefore, for any stopping time T, on $\{T < \infty,\ S_T = 0\}$ we have $X_T = 0$. It follows that the strong Markov property of S implies that of (X,S); that is, axiom (2.5) is satisfied. It is easy to check the other three. So, X is semimarkov.

(2.8) CHUNG PROCESSES. Suppose X is a Chung process. Then right lower semicontinuity implies (2.2). Point φ being "fictitious," (2.3) is true. Axiom (2.4) follows from right lower semicontinuity again. And it is known that (2.5) holds. So, X is semimarkov.

(2.9) REGENERATIVE SETS. Let K_0 be a regenerative set in the sense of MAISONNEUVE (1971); (Markov random set is another equivalent term; and the set on which a regenerative phenomenon in the sense of KINGMAN (1972) is equal to 1 is a particular regenerative set). Define X to be 0 on K_0 and 1 on $\mathbb{R}_+ \backslash K_0$. The resulting process X is semimarkov. Example (1.1) is a special case.

We end this section by giving a regularization of sample paths, due to LÉVY (1954), which simplifies certain questions.

For each $i \in \overline{E}$ and $\omega \in \Omega$ define

$$(2.10) \qquad K_i(\omega) = \{t > 0 : X_t(\omega) = i\}.$$

The random set K_i is called the <u>constancy set</u> for i.

(2.11) DEFINITION. A point $i \in \overline{E}$ is said to be instantaneous if for a.e. ω the set $K_i(\omega)$ has an empty interior. It is said to be a holding point if for a.e. ω the set $K_i(\omega)$ is a countable union of intervals each of which has a finite positive length. It is said to be absorbing if, for a.e. ω, $K_i(\omega)$ is either empty or consists of one infinite interval.

By FUBINI's theorem, axiom (2.3) implies that the point φ is instantaneous. If X is a Chung process then every point i is either instantaneous or holding or absorbing. However, in general, a semimarkov process may have states which are neither instantaneous nor holding nor absorbing. The following "splitting of states" is to eliminate this unpleasantness.

We define,

$$(2.12) \qquad R_t(\omega) = \inf\{s > t : s \in \overline{M}(\omega)\} - t .$$

Then, R_t is the remaining time to be spent in the state being occupied at t.

(2.13) PROPOSITION. For every $\omega \in \Omega$ and $t > 0$ define

$$X_t'(\omega) = \begin{cases} ai & \text{if } R_t(\omega) = +\infty \\ hi & \text{if } R_t(\omega) \in (0,\infty) \\ ii & \text{if } R_t(\omega) = 0 \end{cases}$$

when $X_t(\omega) = i$. Then, the process X' is again a semimarkov process, and every state i' is either instantaneous, or holding and not absorbing, or absorbing.

This proposition can be found in LÉVY (1954). Throughout the remainder of this paper we assume that the regularization implicit in this proposition is already carried out, so that, we have

(2.14) Regularity. Each point $i \in \overline{E}$ of the process X is either absorbing, or holding, or instantaneous.

3. BEHAVIOR AT A HOLDING POINT

Throughout this section i is a fixed holding point. For a.e. ω, the set $K_i(\omega) = \{t > 0 : X_t(\omega) = i\}$ is a countable union of intervals. By axiom (2.2b), each component interval has the form [ ) except possibly the first which may have the form $(0,t)$. Since it is possible that $K_i(\omega) \cap B$ have infinitely many components for even bounded intervals B, first we consider those component intervals of $K_i(\omega)$ whose lengths (strictly) exceed ϵ. Here $\epsilon > 0$ is fixed. Let $L_{\epsilon n}(\omega)$ and $R_{\epsilon n}(\omega)$ be the left end point and the right end point of the n^{th} such interval; if there is no such interval we set $L_{\epsilon n}(\omega) = R_{\epsilon n}(\omega) = +\infty$.

Since X is progressively measurable, the random set K_i is progressive, and each $R_{\epsilon n}$ is a stopping time. Note that, in general, $L_{\epsilon n}$ is not a stopping time, but that $L_{\epsilon n} + \epsilon$ is. It is clear that

$$(3.1) \qquad X_{L_{\epsilon n}+\epsilon} = i, \; S_{L_{\epsilon n}+\epsilon} = \epsilon \quad \text{on } \{L_{\epsilon n} < \infty\}.$$

Let $A \subset \overline{E}$ and $B \in \underline{\underline{\mathbb{R}}}_\epsilon$; (recall that $\mathbb{R}_t = (t,\infty)$ for any $t \in \mathbb{R}_+ = [0,\infty)$, and that $\underline{\underline{\mathbb{R}}}_t$ is the set of all Borel subsets of $\mathbb{R}_t$). By the strong Markov property at $L_{\epsilon n} + \epsilon$,

$$(3.2) \qquad P\{R_{\epsilon n}-L_{\epsilon n} \in B,\; X_{R_{\epsilon n}} \in A \,|\, \underline{\underline{H}}_{L_{\epsilon n}+\epsilon}\} = L_\epsilon(i,A,B)$$

a.s. on $\{L_{\varepsilon n} < \infty\}$ for some measure $L_\varepsilon(i,\cdot)$ on the Borel subsets of $\overline{E} \times \mathbb{R}_\varepsilon$, where we write $L_\varepsilon(i,A,B)$ for $L_\varepsilon(i,A\times B)$ and below we will write $L_\varepsilon(i,j,B)$ for $L_\varepsilon(i,\{j\}\times B)$. If ε is such that $L_{\varepsilon,1} = +\infty$ a.s., we set $L_\varepsilon(i,\cdot) = 0$.

We now explore the shape of $L_\varepsilon(i,\cdot)$ as ε varies. Define

$$n_\varepsilon(i,t) = L_\varepsilon(i,\overline{E},\mathbb{R}_t), \quad t \geq \varepsilon. \tag{3.3}$$

Let $0 < \delta < \varepsilon$ and $B \in \mathbb{R}_\varepsilon$, and suppose that $L_\varepsilon(i,j,B) > 0$. Note that, $\{R_{\delta 1}-L_{\delta 1} \in B\}$ implies $\{R_{\delta 1} = R_{\varepsilon 1},\ L_{\delta 1} = L_{\varepsilon 1}\}$. Therefore, applying the strong Markov property at $L_{\delta 1} + \delta$ to the formula (3.2) for $L_\delta(i,j,B)$, we obtain

$$n_\delta(i,\varepsilon)L_\varepsilon(i,j,B) = L_\delta(i,j,B), \tag{3.4}$$

which is also true when $L_\delta(i,j,B) = 0$. In particular,

$$n_\delta(i,\varepsilon)n_\varepsilon(i,t) = n_\delta(i,t), \quad 0 < \delta \leq \varepsilon \leq t. \tag{3.5}$$

This implies that

$$n_\varepsilon(i,t) = n(i,t)/n(i,\varepsilon), \quad 0 < \varepsilon \leq t \tag{3.6}$$

for some right continuous decreasing function $t \to n(i,t)$ defined on $\mathbb{R}_0$. Using this fact in (3.4) shows that

$$(3.7)\quad L_\varepsilon(i,j,B) = L(i,j,B)/n(i,\varepsilon),\ 0 < \varepsilon,\ j \in \overline{E},\ B \in \underline{\underline{R}}_\varepsilon,$$

for some measure $L(i,\cdot,\cdot)$ on $\overline{E} \times \mathbb{R}_0$ satisfying

$$(3.8)\qquad L(i,\overline{E},\mathbb{R}_t) = n(i,t), \quad t > 0.$$

The measure $L(i,\cdot)$ is the i-entry of a kernel L which plays the same role in semimarkov processes as <u>Lévy kernels</u> do in the theories of Hunt processes and Markov additive processes, (see ÇINLAR (1975)). Moreover, the functions $n(i,\cdot)$ figure as densities for some measure N which is <u>invariant</u> for (X,S); (this is to explain the particular choice of letters L, n etc.)

(3.9) EXAMPLE. Consider the semimarkov process X of Example (1.1). For the holding state 1 we have

$$L(1,0,B) = \lambda(B), \quad L(1,1,B) = 0.$$

(3.10) EXAMPLE. Suppose X is a Chung process and let i be a holding point. Then $n(i,t) = c\cdot e^{-\lambda(i)t}$ for some finite constant $\lambda(i)$ and some constant $c > 0$. Therefore,

$$(3.11)\qquad L(i,j,dt) = cK(i,j)\lambda(i)e^{-\lambda(i)t}dt$$

for some numbers $K(i,j) \geq 0$, $K(i,i) = 0$. In terms of the generator G of X, (that is, $G(i,j)$ is the derivative at $t = 0$ of the transition function $P_t(i,j)$,) we have

$$\lambda(i) = -G(i,i); \quad \lambda(i)K(i,j) = G(i,j), \quad j \neq i.$$

Returning to the general case we note that $L(i,\cdot)$ and $n(i,\cdot)$ are defined only up to multiplication by a constant. If the limit $n(i,0+)$ of $n(i,t)$ as $t \downarrow 0$ is finite, then a convenient normalization is effected by choosing the multiplicative constant so that

$$n(i) = \lim_{t \downarrow 0} n(i,t) = 1. \tag{3.12}$$

Otherwise, if $n(i,0+) = \infty$, we set

$$n(i) = \lim_{t \downarrow 0} n(i,t) = +\infty. \tag{3.13}$$

(3.14) DEFINITION. A holding state is said to be stable if $n(i) = 1$ and unstable if $n(i) = \infty$.

It follows from (3.11) that for a Chung process every holding state is stable. The following proposition shows that the behavior of a semimarkov process at a stable holding state is qualitatively the same as that of a Chung process at such a state. We will later see that the converse of this proposition is also true, (see Proposition (3.29)).

(3.15) PROPOSITION. Let i be a stable holding state. Then, for almost every ω, the set $K_i(\omega) \cap B$ has only finitely many connected components for any bounded interval B.

PROOF. It is enough to show this for $B = (s, s+t)$ where s is arbitrary and $t > 0$ fixed. Choose t such that $n(i,t) > 0$. Define

$$(3.16) \qquad T_{\epsilon n} = s + L_{\epsilon n} \circ \theta_s, \; U_{\epsilon n} = s + R_{\epsilon n} \circ \theta_s .$$

Note that the number of component intervals K_i has in $B = (s, s+t)$ is equal to, or one greater than,

$$(3.17) \qquad N = \lim_{\epsilon \downarrow 0} \sum_n 1_B(T_{\epsilon n}),$$

and that

$$(3.18) \qquad P\{N > k\} = \lim_{\epsilon \downarrow 0} P\{\sum_n 1_B(T_{\epsilon n}) > k\}.$$

So, we need to show that this goes to 0 as $k \to \infty$.

The event figuring on the right side of (3.18) implies the event that the lengths of the intervals $(T_{\epsilon 1}+\epsilon, U_{\epsilon 1}), \ldots, (T_{\epsilon k}+\epsilon, U_{\epsilon k})$ sum to something less than t, and this event in turn implies the event that each one of these k intervals is less than t in length. On the other hand, $T_{\varepsilon 1} + \varepsilon$, $T_{\varepsilon 2} + \varepsilon$, ... are all stopping times. By the strong Markov property applied thereat, the lengths of the k intervals in question are i.i.d. with the common distribution φ given by

$$\varphi(u) = 1 - n(i,\varepsilon+u)/n(i,\varepsilon), \quad u \geq 0.$$

Hence,

$$(3.19) \qquad P\{N > k\} \leq \lim_{\varepsilon \downarrow 0} \left[1 - \frac{n(i,\varepsilon+t)}{n(i,\varepsilon)}\right]^k .$$

By hypothesis of stability for i, $n(i,\varepsilon) \to 1$ as $\varepsilon \to 0$. So, the limit on the right is $[1-n(i,t)]^k$, which goes to 0 as $k \to \infty$ since $n(i,t) > 0$ by choice.

Let $R = R_0$, where R_t is defined by (2.12), that is,

$$(3.20) \qquad R = \inf \overline{M}, \ R_t = R \circ \theta_t = \inf \overline{M} \cap \mathbb{R}_t - t\,;$$

and for the fixed state i under examination, define

$$(3.21) \qquad V = \inf \overline{M} \cap K_i, \ V_t = t + V \circ \theta_t, \ t > 0.$$

We call V the time of first <u>visit</u> to i, V_t is the time of first visit to i after t. Note that $V_t(\omega)$ differs from the first hitting time $H_t(\omega)$ of i after t whenever $X_t(\omega) = i$: If $X_t(\omega) = i$ then $(t,t+\varepsilon) \subset K_i(\omega)$ for some $\varepsilon > 0$, which implies that $V_t(\omega) \geq t + \varepsilon$ whereas $H_t(\omega) = t$.

In interpreting the following recall that $X_V \circ \theta_t = X_{V_t}$ is the state at the time of first visit to i after t.

(3.22) THEOREM. For almost every ω and for every $t > 0$,

$$X_V \circ \theta_t(\omega) = i \qquad \text{if } i \text{ is stable,}$$

$$X_V \circ \theta_t(\omega) \in \overline{E}\setminus\{i\} \quad \text{if } i \text{ is unstable.}$$

PROOF. <u>Case of stable i</u>. By the preceding theorem, for every ω in a set Ω' of full measure, the left end points of the component intervals of $K_i(\omega)$ form a discrete set (i.e. every point is isolated). Therefore, $V_t(\omega)$ must be one of those left end points for $\omega \in \Omega'$, and by axiom (2.2a), $X(\omega)$ is equal to i at $V_t(\omega)$.

<u>Case of unstable i</u>. Let

(3.23) $$p(s) = P\{X_V \circ \theta_t = i,\ R \circ \theta_V \circ \theta_t > s\}.$$

By the strong Markov property applied at $V_t + s$, we have

$$(3.24) \qquad p(u) = p(s)n(i,u)/n(i,s), \qquad 0 < s < u.$$

Since i is unstable, $n(i,s) \to \infty$ as $s \to 0$. Since $p(s)$ is bounded, (3.24) can hold only if

$$(3.25) \qquad p(s) = 0 \quad \text{for all } s > 0.$$

Since i is a holding state,

$$R\circ\theta_V\circ\theta_t = R_{V_t} > 0 \text{ on } \{X_V\circ\theta_t = i\}.$$

Therefore, (3.25) can hold only if (cf. (3.23) also)

$$(3.26) \qquad P\{X_V\circ\theta_t = i\} = 0.$$

This shows that for fixed t there is a negligible set Ω_t such that $X_V\circ\theta_t \neq i$ outside Ω_t. Let

$$(3.27) \qquad \Omega_0 = \bigcup_{t \text{ rational}} \Omega_t.$$

Then Ω_0 is negligible. Pick $\omega \notin \Omega_0$ and $t > 0$. If $V_t(\omega) = t$ then $X_V\circ\theta_t(\omega) \neq i$ (since $X_t(\omega) = i$ implies $V_t(\omega) > t$). If $V_t(\omega) > t$, then there are rationals r in $(t, V_t(\omega))$, and for any such r we have $V_r(\omega) = V_t(\omega)$. So, $X_V\circ\theta_t(\omega) = X_V\circ\theta_r(\omega) \neq i$ since $\omega \notin \Omega_r$. This completes the proof.

The following shows that the converse of Proposition (3.15) is also true; a holding state i is stable if and only if, for almost every ω, $K_i(\omega)$ has only finitely many component intervals in a bounded time interval.

(3.28) PROPOSITION. Let i be an unstable holding state. Then, for a.e. ω, between any two component intervals of $K_i(\omega)$ there is a third component interval of $K_i(\omega)$.

PROOF. Let Ω_0 be the negligible set defined by (3.27), and suppose $\omega \notin \Omega_0$. Let $[p,q)$ and $[u,v)$ be two component intervals of $K_i(\omega)$, suppose $q < u$. Pick $t \in (q,u)$. Clearly, $V_t(\omega) \leq u$. But the equality cannot hold, since then $X(V_t(\omega),\omega) = X_u(\omega) = i$ which contradicts the choice of ω. So, $V_t(\omega) < u$ and therefore there must exist $r,s \in (V_t(\omega),u)$ such that $[r,s)$ is a component interval of $K_i(\omega)$.

The following is a rewording of the preceding proposition. Recall that L is the set of all left end points of the intervals contiguous to $\overline{M}$. Therefore $L \cap K_i$ is the set of all left end points of the component intervals of K_i.

(3.29) COROLLARY. Let i be an unstable holding point. Then, for a.e. ω and every $t > 0$, if $(t,t+\varepsilon)$ contains one point of $L(\omega) \cap K_i(\omega)$, it contains infinitely many points of $L(\omega) \cap K_i(\omega)$.

We have seen in Theorem (3.22) that, at the time of a visit to an unstable state i, X is always at some other state. From the definition of visiting time and the preceding corollary, it follows that at such a time X must be in an instantaneous state.

(3.30) COROLLARY. Let i be an unstable holding state. Then, for any $t > 0$, $X_V \circ \theta_t$ is in the set of instantaneous states almost surely.

It follows that, if the semimarkov process X has an unstable state, then it must also have some instantaneous states. This, however, does not require that the state space be infinite. In Example (1.1), for instance, state 1 is an unstable holding state.

(3.31) REMARK. Consider the transition function (P_t) of the Markov process (X,S), and let P_{is} be the probability law corresponding to the entrance law $P_t((i,s),\cdot)$, $t > 0$. If i is a holding state and $s > 0$, then it is clear that

$$(3.32) \qquad \lim_{t \to 0} (X_t, S_t) = (i,s) \quad \text{a.s. } P_{is}.$$

If i is a stable holding point, the same is true for $s = 0$ also. However, if i is an unstable point, this limit no longer exists for $s = 0$. In fact, when i is unstable, the point $(i,0)$ is a branching point for the semi-group $(P_t)_{t>0}$ and the branching distribution is concentrated on the set $\{(j,0) : j \in \overline{E} \text{ instantaneous}\}$.

4. INSTANTANEOUS STATES

Throughout this section i will be a fixed instantaneous state in E. We will deal with the special point φ separately at the end.

Consider the set $K_i = \{t : X_t = i\}$. For each ω, we define $\overleftarrow{K_i}(\omega)$ to be the set of all limit points of $K_i(\omega)$ from the right side, that is, $t \in \overleftarrow{K_i}(\omega)$ if and only if there is a sequence $(t_n) \subset K_i(\omega)$ which is strictly decreasing to t. Define $\overrightarrow{K_i}(\omega)$ similarly as the set of all limit points from the left. Finally, let $K_i^m(\omega)$ be the minimal right closed set whose closure is the same as the closure of $K_i(\omega)$.

(4.1) LEMMA. a) The random set K_i is progressively measurable (relative to $\underline{\underline{H}}$). b) For almost every ω, $K_i(\omega)$ is right closed, that is, $\overleftarrow{K_i}(\omega) \subset K_i(\omega)$. c) The random set K_i^m is progressive.

PROOF. a) The first statement follows from the progressive measurability of X itself.

b) By the definition of an instantaneous state $K_i \subset \overline{M}$ almost surely. By axiom (2.2a), $L \cap K_i = \emptyset$ almost surely; hence, $K_i \subset \overline{M}\setminus L$. Now axiom (2.2b) applies to show that $K_i(\omega)$ is right closed for a.e. ω.

c) We have, for any ω,

$$(4.2) \qquad K_i^m(\omega) = K_i(\omega)\setminus(\overrightarrow{K_i}(\omega)\setminus\overleftarrow{K_i}(\omega));$$

that is, K_i^m is obtained from K_i by removing from K_i those non-isolated points of K_i which are isolated on the right. Since K_i is progressive, so are $\overrightarrow{K_i}$ and $\overleftarrow{K_i}$ (see DELLACHERIE (1972), p. 126). Now (4.2) implies the same for K_i^m.

(4.3) THEOREM. There is a right continuous increasing additive process $(A_s)_{s\in\mathbb{R}_+}$ (where A_0 is not necessarily zero) such that, for a.e. ω,

$$(4.4) \qquad K_i^m(\omega) = \{t : A_s(\omega) = t \text{ for some } s\}.$$

PROOF is immediate from the characterization theorem of MAISONNEUVE (1971) for regenerative sets whose conditions on K_i^m are satisfied by the following facts: K_i^m is progressive; K_i^m is right closed and minimal. For every $t \in K_i^m$, we have $X_t = i$ and $S_t = 0$, (the latter is because $K_i^m \subset K_i \subset \overline{M}$). Therefore, by the strong Markov property, for any stopping time T with $[T] \subset K_i^m$, the past $\underline{\underline{H}}_T$ is independent of the future $\sigma(X_{T+t};\ t \geq 0)$ which includes the "future of K_i^m after T."

It follows from the well-known facts about increasing additive processes that

$$(4.5)\qquad E[\exp\{-\lambda(A_s-A_0)\}] = \exp[-s\lambda N^{\lambda}(i)],$$

where

$$(4.6)\qquad \lambda N^{\lambda}(i) = \lambda a(i) + \int_{(0,\infty]} (1-e^{-\lambda y})L(i,dy)$$

for some constant $a(i) \in \mathbb{R}_+$ and some Lévy measure $L(i,\cdot)$ on $\overline{\mathbb{R}}_0 = (0,\infty]$. We define (recall that $\overline{\mathbb{R}}_t = (t,\infty]$)

$$(4.7)\qquad n(i,t) = L(i,\overline{\mathbb{R}}_t), \quad t \geq 0;$$

then $N^{\lambda}(i)$ is the Laplace transform of the measure $N(i,\cdot)$ which has an atom of weight $a(i)$ at $t = 0$ and is absolutely continuous on $(0,\infty)$ with density $n(i,\cdot)$.

(4.8) DEFINITION. The instantaneous state i is said to be <u>attractive</u> if $n(i,0) = +\infty$ and <u>repellent</u> if $n(i,0) < \infty$. If i is attractive, it is further said to be <u>light</u> if $a(i) = 0$ and <u>heavy</u> if $a(i) > 0$. State i is said to be <u>recurrent</u> if $n(i,+\infty) = 0$ and transient if $n(i,\infty) > 0$.

The next proposition provides the meanings behind these definitions. We omit the proof, which follows from Theorem (4.3), the well known facts about additive processes, and the observations that $K_i \backslash K_i^m$ is at most countable and its every point is a limit point of K_i.

(4.9) PROPOSITION. The following statements are true for a.e. ω in $\{K_i \neq \emptyset\}$.

a) If i is attractive $K_i(\omega)$ is everywhere dense in itself, and therefore, is perfect (in the right-topology). If i is repellent, then every point of $K_i(\omega)$ is isolated, and therefore, $K_i(\omega)$ is at most countable.

b) If i is repellent, or attractive and light, then the Lebesgue measure of $K_i(\omega)$ is zero. If i is heavy, then the Lebesgue measure of $K_i(\omega)$ is strictly positive.

c) If i is recurrent then $K_i(\omega)$ is unbounded; if i is transient then $K_i(\omega)$ is bounded.

In the case of a repellent state, if $T_0, T_1, \ldots$ are the successive points of K_i, we see that each T_n is a stopping time and the process (T_n) is a delayed (and possibly transient) renewal process. If X is a Chung process, then every instantaneous state is attractive and heavy. However, repellent and light attractive states do come up within the boundary theory for Chung processes. The behavior at a repellent state is very close to that of a Chung process at a non-sticky boundary point (though not exactly the same, as we shall see below), and the behavior at a light attractive state is the same as that of a Chung process at a sticky boundary point; (for the terms see CHUNG (1968)).

The following clarifies the picture at a repellent state.

(4.10) THEOREM. Let i be a repellent state. Then, for almost every ω and for every $t \in K_i(\omega)$, the path $X(\omega)$ admits φ as a limit point from the right at t; moreover, if $X(\omega)$ admits $j \in E$ as a limit point from the right side at t, then j must be an unstable holding point.

REMARK. It is possible to have more than one unstable state as limit points from the right.

PROOF. Let Ω' be the set on which $K_i \subset \overline{M} \setminus L$ and $K_j \supset \overleftarrow{K_j}$ for every stable holding state j and for every instantaneous state j. By the definition of an instantaneous state i, by Proposition (3.15) on K_j for a stable holding state, and by Lemma (4.1) concerning K_j for j instantaneous, we have that $P(\Omega') = 1$.

Let $\omega \in \Omega'$ and let $t \in K_i(\omega)$. Since $t \in \overline{M}(\omega) \setminus L(\omega)$, there is a sequence $(t_n) \subset \overline{M}(\omega)$ strictly decreasing to t. If the sequence $(X_{t_n}(\omega))$ has no limit points in E, then there is a subsequence $(X_{s_n}(\omega))$ converging to φ since the state space $\overline{E}$ is compact.

Suppose that $(X_{t_n}(\omega))$ admits $j \in E$ as a limit point for some sequence (t_n) decreasing to t. Since $\omega \in \Omega'$, the facts that $t \in \overleftarrow{K_j}(\omega)$ and $t \in K_i(\omega)$ imply that j must be unstable. Let

V_s^j be defined by (3.21) for the present state j; and consider the sequence $(s_n) = (V_{t_n}^j(\omega))$. By Corollary (3.30), $X_{s_n}(\omega)$ is instantaneous for each n. Hence, the sequence $(X_{s_n}(\omega))$ cannot have any limit points in E, because (s_n) decreases to t and $\omega \in \Omega'$. So, $(X_{s_n}(\omega))$ must admit φ as a limit point.

The preceding proof goes through for $t \in K_i(\omega)$ where i is attractive and t is isolated on the right. We put this next along with some other supplementary facts about such points t. Note that, Theorem (4.3) specifies the structure of K_i^m completely, and hence, the next proposition completes the picture of K_i.

(4.11) PROPOSITION. Suppose i is attractive, and let $L_i = K_i \setminus K_i^m = \vec{K_i} \setminus \overleftarrow{K_i}$.

a) The random set L_i is progressive.

b) If T is a stopping time such that $[T] \subset L_i$, then $T = +\infty$ almost surely.

c) For a.e. ω, $L_i(\omega)$ is countable; if $t \in L_i(\omega)$, then there is $(t_n) \downarrow t$ such that $X_{t_n}(\omega) \to \varphi$; if $t \in L_i(\omega)$ and if there is $(t_n) \downarrow t$ such that $X_{t_n}(\omega) \to j$ for some $j \in E$, then j is unstable.

PROOF. a) follows from DELLACHERIE (1972) p. 126; the proof of c) is the same as that of Theorem (4.10). To show b) let T be such a stopping time. Then, $[T] \subset K_i$ and the strong Markov property at T implies (by the attractiveness of i) that $T(\omega) \in \overleftarrow{K}_i(\omega)$ for a.e. $\omega \in \{T < \infty\}$. Since $L_i(\omega) \cap \overleftarrow{K}_i(\omega) = \emptyset$, and since $T(\omega) \in L_i(\omega)$ for a.e. $\omega \in \{T < \infty\}$ by hypothesis, we must have $T = +\infty$ almost surely.

We have seen in the preceding propositions that if i is attractive and if a left end point t of an interval contiguous to $K_i(\omega)$ is such that $t \in K_i(\omega)$ and $t \in \overleftarrow{K}_j(\omega)$ for some j, then j must be unstable. The following strengthens this result by showing that, in that case, $t \in \overrightarrow{K}_j(\omega)$ also: in other words, there are infinitely many component intervals of $K_j(\omega)$ in any interval $(t-\varepsilon, t)$ with $\varepsilon > 0$.

(4.12) PROPOSITION. Suppose i is attractive, let $L_i = K_i \backslash K_i^m$ as before. Then for almost every ω,

$$L_i(\omega) \cap \overleftarrow{K}_j(\omega) \subset L_i(\omega) \cap \overrightarrow{K}_j(\omega).$$

PROOF. Let V_t^j be the time of first visit to j after t defined by (3.21), but for state j. For any fixed t, this is a stopping time, and therefore, $[V_t^j] \cap L_i = \emptyset$ almost surely by Proposition (4.11b). Let

$$\Omega' = \bigcup_r \{V_r^j \notin L_i\} \qquad (4.13)$$

where the union is over all rational numbers r. Then, $P(\Omega') = 1$.

Choose $\omega \in \Omega'$ and suppose $t \in L_i(\omega) \cap \overleftarrow{K_j}(\omega)$. Then, for any $s < t$, $s \leq V_s^j(\omega) \leq t$. On the other hand, if s is rational, $V_s^j(\omega)$ cannot be t since $\omega \in \Omega'$. Choose a sequence r_n of rationals strictly increasing to t. Then,

$$r_n \leq V_{r_n}^j(\omega) < t$$

for every n, which implies that for each n there is $s_n \in [r_n, t)$ such that $s_n \in K_j(\omega)$. Since $r_n \uparrow t$, this implies that $t \in \overrightarrow{K_j}(\omega)$.

In view of the ease with which Theorem (4.3) is obtained it is worth commenting upon its main ingredients. The obvious (and quite reasonable) ones are the progressive measurability and strong Markov property. Other than these, the most important is Axiom (2.2b), which is equivalent to saying that K_i is right closed for every instantaneous state i. For Chung processes this follows from right lower semicontinuity, which in turn is made possible by separability and stochastic continuity. However, for semimarkov processes, it is difficult to obtain the property (2.2b) by any reasonable hypothesis. The following is an example of a strong Markov process X which satisfies all the axioms except (2.2b), and for which

Theorem (4.3) fails, and the behavior at any of the three instantaneous states is radically different from that dexcribed above.

(4.14) EXAMPLE. Let (N_t^1), (N_t^2),... be independent Poisson processes each of which has parameter 1. Let Ω' be the set of all ω such that no two paths $t \to N_t^n(\omega)$ and $t \to N_t^m(\omega)$ have any jump time in common. Then $P(\Omega') = 1$ as is well known. We now define

$$X_t(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega' \text{ and } t \text{ is a jump time of } N^m(\omega) \\ & \quad \text{for some \underline{odd} integer } m; \\ 2 & \text{if } \omega \in \Omega' \text{ and } t \text{ is a jump time of } N^m(\omega) \\ & \quad \text{for some \underline{even} integer } m; \\ 0 & \text{otherwise.} \end{cases}$$

Then $P\{X_t = 0\} = 1$ for every t, and X is stochastically continuous. For every ω, $X(\omega)$ admits all three possible values as limiting values from the right (and left) at any time t. So, X is separable trivially. Let $\underline{\underline{H}}_t^n = \sigma(N_s^n;\ s \leq t)$, and set $\underline{\underline{H}}_t^0 = \underline{\underline{H}}_t^1 \otimes \underline{\underline{H}}_t^2 \otimes \ldots$. Then, X is strong Markov with respect to $(\underline{\underline{H}}_t^0)$. Every state is instantaneous, every t is a time of discontinuity, axiom (2.2a) holds trivially, axiom (2.2b) does not hold. Theorem (4.3) is not true for any state.

It should be clear from the foregoing discussion that the behavior of $K_\varphi = \{t : X_t = \varphi\}$ is radically different from that at K_i for an instantaneous state $i \in E$ even though φ is an instantaneous point. This is because the set K_φ is not necessarily right closed; and because we do not have the strong Markov property at stopping times T at which X may be in φ.

5. CONVERSION TO CHUNG PROCESSES

We will now give the rectification of Lévy's assertion concerning the conversion of semimarkov processes to Markov processes. The version due to YACKEL (1968) contains the main idea in this regard and we have nothing to add if the semimarkov process X is such that every state is either stable or instantaneous and heavy. Yackel insures this state of affairs by requiring that the semi-group (P_t) corresponding to the process (X_t, S_t) be "strong". Our object in this section is to start from our semimarkov process X and to transform it to a Chung process.

Let C and D be the set of all stable and unstable states respectively, and put

$$(5.1) \quad \hat{D} = D \times \{1,2,\ldots\}, \; \hat{C} = C \cup \hat{D}, \; \hat{E} = (E \backslash D) \cup \hat{D}.$$

Define $f : \hat{E} \to E$ by setting

$$(5.2) \quad f(i) = i \text{ if } i \in E \backslash D, \; f(i,n) = i \text{ if } (i,n) \in \hat{D}.$$

In one of the steps below we need to use sequences W_i, $i \in \hat{C}$, independent of each other and of X, where each W_i is a sequence of i.i.d. exponential random variables. If $(\Omega, \underline{\underline{M}}, P)$ is not large enough to support them, it can be replaced by a suitably larger one in the usual ways known to all. We assume that it is large enough. Here, then, is the main result.

(5.3) THEOREM. There exist a strong Markov process $\hat{X}$ with state space $\hat{E}$ and a strictly increasing continuous process A such that

$$X_t = f(\hat{X}_{A_t}), \quad t > 0. \tag{5.4}$$

Moreover, considered as a semimarkov process on $\hat{E}$, the process $\hat{X}$ has no unstable states and no light attractive states.

(5.5) REMARK. The last statement of (5.3) means that, if X has no repellent states, then $\hat{X}$ is a Chung process. If X has repellent states, then putting

$$\tilde{X}_t(\omega) = \begin{cases} \hat{X}_t(\omega) & \text{if } X_t(\omega) \text{ is not repellent,} \\ \varphi & \text{if } X_t(\omega) \text{ is repellent,} \end{cases} \tag{5.6}$$

we obtain a Chung process $\tilde{X}$.

The remainder of this section is devoted to proving the above theorem and identifying $(\hat{X}_t)$ and (A_t).

(5.7) STEP 1. Dilation of K_i for light attractive i. Let A be the set of all light attractive states. For each i in A, let C^i be the local time constructed by MAISONNEUVE (1971) for the regenerative set K_i^m, and note that

$$(5.8) \qquad E[C_t^i] \leq e^t$$

(since the potential Φ generated by $e^{-s}dC_s^i$ is bounded by 1; see MAISONNEUVE (1971), p. 149). Let $(p_i)_{i\in A}$ be a sequence of strictly positive numbers p_i with $\Sigma\, p_i = 1$. Define[5]

$$(5.9) \qquad C_t^o = \sum_{i\in A} p_i C_t^i.$$

The following summarizes the relevant facts concerning the C^i.

(5.10) LEMMA. a) Each C^i $(i \in A)$ is a continuous (increasing) additive functional of (X,S). The support of the measure dC_s^i is $\overline{K}_i$ (--the closure of K_i; this is equal to $K_i \cup L_i$ in the notation of Proposition (4.11)).

b) C^o is an increasing continuous additive functional of (X,S). The support of dC_s^o is $(\cup_i \overline{K}_i) \cup K_\varphi$.

c) Almost surely, for each i in A and $t > 0$,

$$(5.11) \qquad \int_0^t 1_{K_i}(s)\, dC_s^o = p_i C_t^i.$$

PROOF. (a) is a restatement of the results shown by MAISONNEUVE. To show (b), first note that $E[C_t^o] \leq e^t$ by (5.8) and (5.9). So, (5.9) defines an increasing finite valued functional of (X,S). Additivity follows from that for the C^i. Left continuity for C^o follows from the fact that each C^i is increasing and continuous. If t is a time of right increase for $C^i(\omega)$, that is, if $C^i_{t+\varepsilon}(\omega) > C^i_t(\omega)$ for all $\varepsilon > 0$, then t belongs to $K_i^m(\omega) \subset K_i(\omega)$. Hence, a time t of right increase for $C^o(\omega)$ belongs to $K_i(\omega)$ for exactly one i. It follows that C^o is right continuous, and that its set of right increase is $\cup_i K_i^m$. The support of dC_s^o is just the closure of this; by axiom (2.2b), the closure is in $(\cup_i \overline{K}_i) \cup K_\varphi$.

Finally, (5.11) follows from (5.9), the fact that the set of right increase of C^o is $\cup_i K_i^m$, the fact that $K_i \backslash K_i^m$ is at most countable, the fact that the closure of $\cup_i K_i^m$ differs from $\cup_i K_i$ by at most countably many points, and the fact that C^o is continuous (and therefore dC_s^o puts no mass on a countable set).

Define, for $t > 0$,

(5.12) $\quad C_t = t + C_t^o, \; \tau_t = \inf\{s : C_s > t\};$

(5.13) $\quad X_t^1 = X_{\tau_t}, \; S_t^1 = S_{\tau_t}, \; \underline{\underline{H}}_t^1 = \underline{\underline{H}}_{\tau_t}.$

(5.14) LEMMA. X^1 is a semimarkov process (relative to $\underline{\underline{H}}^1$) with state space E. Each $i \in E\backslash A$ has the same classification for X^1 as it had for X. Each $i \in A$ is heavy attractive for X^1.

PROOF. C is a strictly increasing continuous additive functional of (X,S). Therefore, (τ_t) is continuous and strictly increasing; and each τ_t is a stopping time of $\underline{\underline{H}}$. It follows that the random time change (5.13) yields a progressive strong Markov process (X^1,S^1) relative to $\underline{\underline{H}}^1$. Regularity conditions (2.2) and (2.3) follows from the fact that (τ_t) is strictly increasing. This latter fact also implies that the qualitative properties of the sample paths of X^1 are exactly as those of the paths of X. Since C^o remains constant over K_i for $i \in E\backslash A$, the states in $E\backslash A$ have the same classification and in fact the Lebesgue measure of $K_i^1 \cap [0,C_t]$ is equal to the Lebesgue measure of $K_i \cap [0,t]$, where K_i^1 is the set of i-constancy of X^1. The statement concerning $i \in A$ follows from (5.11) and the fact that the Lebesgue measure of $K_i^1 \cap [0,C_t]$ is equal to the mass put on $K_i \cap [0,t]$ by dC_s, which is $p_i C_t^i$.

(5.15) STEP 2. <u>Splitting the unstable states</u>. For each unstable holding state i (of X^1 and therefore of X), let $K_{in}(\omega)$ be the union of those component intervals of $K_i(\omega)$ whose lengths belong to the interval $[1/n,1/(n-1))$, $n = 1,2,\ldots$. Then, each $K_{in}(\omega)$ has only finitely many components during a bounded interval, and we have

$K_i(\omega) = \cup_n K_{in}(\omega)$. Define

$$(5.16) \qquad X_t^2(\omega) = \begin{cases} (i,n) & \text{if } t \in K_{in}(\omega),\ (i,n) \in \hat{D}, \\ X_t^1(\omega) & \text{otherwise;} \end{cases}$$

and set $S_t^2(\omega) = S_t^1(\omega)$. Let $(\underline{\underline{H}}_t^2)$ be the right continuous complete history generated by (X_t^2).

(5.17) LEMMA. X^2 is a semimarkov process (relative to $\underline{\underline{H}}^2$) with state space $\hat{E}$. Every $i \in \hat{C}$ is a stable holding state for X^2. Every $i \in E\backslash\hat{C}$ is either heavy attractive or repellent. Moreover, with f defined by (5.2), we have

$$(5.18) \qquad X_t^1(\omega) = f(X_t^2(\omega)), \quad t > 0,\ \omega \in \Omega.$$

PROOF will only be sketched. Note that the discontinuity set $\overline{M}^2$ for X^2 is the same as that for X^1, and therefore $S_t^2 = S_t^1$ and $R_t^2 = R_t^1 = \inf\{s > t : s \in \overline{M}^1\} - t$. Note again that $X_t^2(\omega) = (i,n)$ if and only if $X_t^1(\omega) = i \in D$ and $S_t^1(\omega) + R_t^1(\omega) \in [1/n, 1/(n-1))$. Now, progressive measurability for X^2 follows from that for X^1 and the fact that

$$t \to 1_{[1/n,1/(n-1))}(S_t^1+R_t^1)\,1_{\{i\}}(X_t^1).$$

is right continuous. The strong Markov property for (X^2,S^2) is seen as follows: The process $Y = (X^1,S^1,R^1)$ is strong Markov. For a stopping time T of $\underline{\underline{H}}^2$, first note that the future of (X^2,S^2) after T is conditionally independent of "the past $\underline{\underline{G}}_T$ of Y before T" given $Y_T = (X_T^1, S_T^1, R_T^1)$. And, conditional expectation of any function $g(Y_T)$ of Y_T given $\underline{\underline{H}}_T^2$ is some function of (X_T^2, S_T^2). Of course, $\underline{\underline{H}}_T^2 \subset \underline{\underline{G}}_T$. This completes the proof since the remaining assertions are trivial.

(5.19) STEP 3. Conversion to a Markov Process. This step is exactly the same as the method of YACKEL (1968) which works as long as there are no unstable or no light attractive states. For the sake of completeness we describe the random time change involved.

For $i \in C$ let $m(i)$ be a median value of the sojourn distribution $\mathbf{L}(i,\overline{E},\cdot)$; see Section 3 for the definition. For $i \in \hat{D}$, set $m(i,n) = 1/n$. For each $j \in \hat{C} = C \cup \hat{D}$, let $W_j = (W_{jk})$ be a sequence of i.i.d.r.v. having the exponential distribution with parameter $m(j)$. Further let W_j, $j \in \hat{C}$, be independent of each other and of X^2. (See the remarks preceding Theorem (5.3).) Let $[T_{jk}, U_{jk})$ be the k^{th} component interval of $\mathbf{K}_j^2 = \{t : X_t^2 = j\}$, and set

$$(5.20) \qquad B_t^j = \sum_{k=1}^{\infty} \int_0^t \frac{W_{jk}}{U_{jk}-T_{jk}} 1_{[T_{jk},U_{jk})}(s)\,ds.$$

Define,

$$(5.21) \qquad B_t^0 = \int_0^t 1_{E \setminus \hat{C}}(X_s^2)\,ds,$$

and set

$$(5.22) \qquad B_t = B_t^0 + \sum_{j \in \hat{C}} B_t^j;$$

$$(5.23) \qquad \tau_t = \inf\{s : B_s > t\}; \quad X_t^3 = X_{\tau_t}^2.$$

Define $(\underline{\underline{H}}_t^3)$ to be the history generated by X^3.

Since C is strictly increasing, τ is continuous; since C is continuous, τ is strictly increasing. The last fact implies that the qualitative structure of the paths of X^3 is the same as that of the paths of X^2. In particular, stable states of X^2 are stable for X^3, and similarly for the heavy attractive states and repellent states.

YACKEL's proof shows that X^3 is strong Markov. X^3 may differ from a Chung process only if there are repellent states for X (--then, they remain repellent for X^3). This process X^3 is the process $\hat{X}$ of Theorem (5.3). Finally, since C used in Step 1 and B used in this final step are both strictly increasing, (5.12) and (5.23) imply

$$(5.24) \qquad X_t = X_{C_t}^1; \quad X_t^2 = X_{B_t}^3.$$

This together with (5.18) shows that (5.4) holds with

(5.25) $$\hat{X}_t = X_t^3 \text{ and } A_t = B_{C_t}.$$

This completes the proof of Theorem (5.3).

Remark (5.5) makes use of Theorem (4.3): If $\hat{X}_t(\omega) = i$ where i is repellent, then φ is the only limiting value of $\hat{X}_s(\omega)$ as $s \downarrow t$. This is because $\hat{X}$ has no unstable states, and $\hat{E}$ is given the discrete topology. It is possible that for some sequence (t_k) decreasing to t, at which $\hat{X}(\omega)$ is i repellent, we have $\hat{X}_{t_k}(\omega) = (j,n_k)$ for some unstable state j of X. Giving $\hat{E}$ the discrete topology with φ as the only point at infinity, the limit of (j,n_k) is φ as $k \to \infty$. It is this fact which is the basis of our statement that the conversion of X to $\hat{X}$ alters the qualitative structure somewhat.

In order to keep the same qualitative behavior, then, we need to let $\hat{E}$ have the discrete topology but have as points at infinity the point φ plus every point $i \in D$. Now, $i \in D$ is the limit of any sequence of the form (i,n_k), $n_k \to \infty$ as $k \to \infty$. Indeed, starting with the Chung process $\tilde{X}$ described in Remark (5.5), while studying the behavior of $\tilde{X}$ at its entrance boundary, each point $i \in D$ will be a repellent boundary point.

6. CONCLUDING REMARKS

We mention the implications of several possible regularity conditions which are commonly introduced.

If every path $X(\omega)$ is a separable function, then there can be no repellent states, (but all other kinds are possible). If X is right separable, then there can be no repellent states, and K_i is minimal (i.e. $L_i = \emptyset$) for every instantaneous state i. If X is right continuous, then there are no instantaneous states and no unstable states.

The role of "right lower semicontinuity" is somewhat ambiguous. If we define it as usual, i.e. by the condition that

$$X_t = \liminf_{s \downarrow t} X_s \tag{6.1}$$

for some ordering of states in E so that φ always corresponds to $+\infty$, that ordering becomes important. For example, in Example (1.1), X satisfies (6.1). However, if we interchange the labels of the two states, (so that $0' = 1$ and $1' = 0$ and $0' < 1'$,) then X is not right lower semicontinuous; and moreover any attempt to make it so would set $X_t' = 0'$ for all t.

We may define a property RLSC by requiring that, for any t, X have at most one finite limiting value at t from the right, and that X be equal to that finite limit if it exists, and be otherwise φ. In the case of Chung processes this property "RLSC" is simply the right lower semicontinuity. For semimarkov processes, if X has the RLSC property, then there can be no unstable states and no repellent states.

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