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Lindahl's Solution and Values for a Public-Goods Example

by

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Abstract: An example of an economy with a public good is presented with a non-atomic measure space of agents and money (transferable utility medium). Three solutions are computed: the Lindahl solution, the Shapley value, and the Harsanyi-Selten value. The three are found to differ significantly in their assignment of the societal benefits attributable to the presence of the public good.

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While much is known about the coincidence of the core, the set of competitive allocations, and the set of Shapley-value allocations for private-goods-only economies with a non-atomic measure space of agents (see Hildenbrand (1974), Aumann and Shapley (1974), and Aumann (1974)), comparatively little is known about such economies with public goods. It is known that for economies with public goods and a non-atomic measure space of agents the set of Lindahl allocations (a natural extension of the notion of competitive allocation) is contained in the core (also extended in a natural way) under mild assumptions. Muench (1972) showed, however, that the containment may be strict. Nothing has evidently been shown concerning the relationship between the set of Lindahl allocations and the set of Shapley-value allocations for such economies. A modification of Shapley value has been suggested by Harsanyi (1959) and Selten (1964) for games with side payments in which the characteristic function inadequately represents threats (as is evidently the case with public-goods-economies; see Rosenthal (1972)), but the relationships between the set of Harsanyi-Selten-value allocations, the set of Shapley-value allocations, and the set of Lindahl allocations have not been explored for economies with public goods and a non-atomic measure space of agents, even when side payments in transferable utility are possible.

In this note we present a simple example of such an economy in which there are two types of traders. In this example both types of traders are necessary for any benefits to be made available from the production of the one public good. The Shapley value results in an even split of the benefits between the types; the Lindahl solution gives all the benefits to the agents of type 2; and the Harsanyi-Selten value gives 3/4 of the benefits to the agents of type 1 and the remainder to the agents of type 2.

While the economy may not meaningfully be called symmetric between types, there is a certain similarity in the powers of the two types: only type 2 agents have endowments which are useful for the production of the public good, and only type 1 agents gain utility from the public good. For this reason, the Shapley value seems to me to represent what is most fair, at least for this example. Thus, the general equity of the other two concepts is called into question.

Example

$(A,\mathcal{A},\mu)$ is a measure space of agents; where $(A,\mathcal{A})$ is isomorphic to the closed unit interval with its Borel subsets, and μ is nonnegative and non-atomic. (A_1,A_2) forms a partition of A with $\mu(A_1) = \mu(A_2) = 1$. All agents within A_1 (resp. A_2) are of the same type. There are two private goods and one public good. Each infinitesimal agent (ds) in A_1 has utility function $u(x,\xi,y,s) = \xi + y$, where y is the aggregate level of the public good produced, and $(x,\xi)\mu(ds)$ is the private-good bundle consumed by ds. Similarly each agent in A_2 has utility function $u(x,\xi,y,s) = \xi$. Each agent of A_1 is endowed with private bundle $(0,1)\mu(ds)$, while each agent in A_2 is endowed with $(1,1)\mu(ds)$. The aggregate production set is

$$Z = \{(z,\eta,y): \quad z + y \leq 0, y \geq 0, \eta \leq 0\}$$

where $(-z)$ represents net inputs of the first private good to production, y is as above, and η represents net outputs of the second private good (which can be thought of as money).

Shapley Value

Let S be any coalition, $S_1 = S \cap A_1$, $S_2 = S \cap A_2$. The side-payment characteristic function for S is given by

$$v(S) = \max \left(\int_{S_1} (y + \xi)\, d\mu + \int_{S_2} \xi\, d\mu \right)$$

$$\text{subject to } \int_S x\, d\mu - \int_{S_2} 1\, d\mu + y \leq 0$$

$$\int_S \xi\, d\mu \leq \mu(S)$$

$$y \geq 0 \text{ and } x(s) \geq 0 \text{ a.e. in } S,$$

which is easily seen to be $\mu(S) + \mu(S_1)\mu(S_2)$. The symmetry axiom for Shapley value guarantees that the total payoffs to A_1 and A_2 at any value allocation be equal for such a game. In this case, both types receive total payoffs of 3/2.

Strictly speaking, the side-payment formulation for this economy makes sense only if negative values for ξ are possible (even if, as here, they don't actually arise). Otherwise, the values being computed here and below are properly λ-transfer values. (See Shapley (1969).)

Lindahl Solution

A Lindahl equilibrium for this economy is an allocation $(\bar{x}(\cdot), \bar{y}, \bar{\xi}(\cdot))$ and a nonnegative price system $(\bar{p}, \bar{q}(\cdot), 1)$ satisfying:

i) $(\bar{x}(s), \bar{y}, \bar{\xi}(s))$ maximizes $y + \xi$

$$\text{subject to } \bar{p}x + \bar{q}(s)y + \xi \leq 1$$

$$x, y \geq 0 \qquad \text{a.e. in } A_1.$$

ii) $(\bar{x}(s), \bar{y}, \bar{\xi}(s))$ maximizes ξ

$$\text{subject to } \bar{p}x + \bar{q}(s)y + \xi \leq \bar{p} + 1$$

$$x, y \geq 0 \qquad \text{a.e. in } A_2.$$

iii) $(\int_A \bar{x}\,d\mu, \bar{y}, \int_A \bar{\xi}\,d\mu)$ maximizes $y\int_A \bar{q}\,d\mu + \bar{p}(\int_A x\,d\mu - 1) + (\int_A \xi\,d\mu - 2)$

$$\text{subject to } (\int_A x\,d\mu - 1, y, \int_A \xi\,d\mu - 2) \in Z.$$

From (i), $\bar{q}(s) > 0$ a.e. in A_1, hence $\int_A \bar{q}\,d\mu > 0$. From (iii), $\int_A \bar{q}\,d\mu \leq \bar{p}$. Hence $\bar{p} > 0$. Hence $\bar{x}(s) = 0$ a.e. in A. Hence $\bar{y} = 1$. From (i), $\bar{q}(s) = 1$ and $\bar{\xi}(s) = 0$ a.e. in A_1. From (ii), $\bar{q}(s) = 0$ and $\bar{\xi}(s) = \bar{p} + 1$ a.e. in A_2. Hence $\bar{p} = 1$. Therefore, at all Lindahl equilibria, the utility payoff to A_1 is 1; and the utility payoff to A_2 is 2. (The Lindahl equilibria are all identical except for sets of agents of zero measure.)

Harsanyi-Selten Value

The Harsanyi-Selten value for the economy may be simply expressed as the Shapley value of a characteristic function g defined for each coalition S and its complement $\bar{S}$ by the equations

$$g(S) + g(\bar{S}) = v(N) \text{ and}$$

$$g(S) - g(\bar{S}) = H(S)$$

where v is the usual characteristic function and H(S) is the equilibrium value of the two-person zero-sum game played between S and $\bar{S}$ in which S maximizes (and $\bar{S}$ minimizes) the difference between the total utility payoffs to S and $\bar{S}$ in the economy. H(S) is therefore a measure of the relative advantage which S holds over $\bar{S}$ in terms of threats. Here v(N) = 3 and

$$H(S) = \mu(S) - \mu(\bar{S}) + \max_{0 \leq y \leq \mu(S_2)} \min_{0 \leq \bar{y} \leq \mu(\bar{S}_2)} \left(\int_{S_1} (y + \bar{y})\,d\mu - \int_{\bar{S}_1} (y + \bar{y})\,d\mu \right).$$

If $\mu(S_1) > 1/2$, $H(S) = 2\mu(S) - 2 + \mu(S_2)(2\mu(S_1) - 1)$. If $\mu(S_1) \leq 1/2$, $H(S) = 2\mu(S) - 2 + (1 - \mu(S_2))(2\mu(S_1) - 1)$. (If S has more type 1 agents than $\bar{S}$, then it produces all the public good it can; and $\bar{S}$ produces none. The reverse is true when $\bar{S}$ has more type 1 agents.) Thus,

$$g(S) = \begin{cases} 1/2 + \dfrac{\mu(S_2)}{2} + \mu(S_1) + \mu(S_1)\mu(S_2) & \text{if } \mu(S_1) > 1/2 \\ 2\mu(S_1) + \dfrac{3\mu(S_2)}{2} - \mu(S_1)\mu(S_2) & \text{if } \mu(S_1) \leq 1/2. \end{cases}$$

g satisfies the hypotheses of Theorem B in Aumann and Shapley (1974), which provides a formula for computing the Shapley value. Applying this formula we find that the Harsanyi-Selten value for the economy provides total payoffs of 7/4 to A_1 and 5/4 to A_2.

It is perhaps worth noting that the core of the side-payment game with characteristic function $v(S) = \mu(S) + \mu(S_1)\mu(S_2)$ is quite large and contains all three solutions. A question arises as to whether the special nature of the utility functions is the cause of the distance between the various solutions in this example. Of course, the separated ξ term is necessary for the side-payment analysis. Otherwise, however, it appears that the phenomenon remains after small perturbations. For example, if any single continuously differentiable concave, nondecreasing function of x and y is multiplied by a sufficiently small positive constant and added to each type's utility function, it is easy to see that all three solutions will be moved a distance which can be made arbitrarily small.

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