ASYMPTOTICALLY OPTIMAL MARKET MECHANISMS

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February 2001
revised May 2001

Abstract. Because rational agents use their private information strategically in many trading environments any budget balanced, incentive compatible, and individually rational market mechanism will be inefficient.

This paper is concerned with the arising inefficiencies as the number of traders becomes large. We prove that the absolute inefficiency of a sequence of budget balanced market mechanisms can not converge to zero faster then $\frac{c_o}{m}$, where m is the size of the market and c_o is a number explicitly determined in the paper. We propose a simple modification of the Vickrey-Clarke-Groves mechanism which is budget balanced, individually rational, implementable in dominant strategies, and asymptotically optimal in the sense that it achieves the above rate of convergence.

As a side product of our analysis we get other asymptotic results describing the trade off between revenue and efficiency. For example, we prove that, as the market size m goes to infinity, the minimal deficit needed to implement the efficient allocation rule converges to a number d_o, which is also explicitly determined in the paper.

Keywords: Asymmetric information, large markets, asymptotic efficiency, Myerson-Satterthwaite Theorem, Vickrey-Clarke-Groves Mechanism.

1. Introduction

1.1. **Problem.** It is difficult to imagine a trading environment in which buyers and sellers do not have some private information about their own valuations of the traded goods. Because traders will use this private information strategically usually there will be no incentive compatible and individually rational trading mechanism which is both budget balanced and efficient. This result was first proved by Myerson and Satterthwaite[1] for the case of one buyer and one seller but remains qualitatively

I thank Levon Barseghyan, Jeff Ely, Burcay Erus, Eddie Dekel, Peter Klibanoff, Francesca Molinari, Lyndon Moore, Marcin Peski, Maxim Sinitsyn, and Michael Whinston for useful comments. I am most grateful to Asher Wolinsky for his valuable advice. I would like to thank Mark Satterthwaite for stimulating my interest in the topic, helpful comments, and making available computational resources of the KGSM to calculate the numerical example in Section 5.1.

[1]See Myerson and Satterthwaite (1983).

true if many buyers and sellers interact.[2] In this paper we will be interested in the effects of strategic use of private information as the number of traders becomes large.

The fact that it will usually be impossible to implement an incentive compatible and individually rational mechanism which is both budget balanced and efficient means that a mechanism designer will have to choose between a more efficient mechanism which runs a higher deficit and a less efficient one which creates a higher revenue. This poses several questions. What is the exact trade off between revenue and efficiency ? How significant is the whole problem quantitatively ? In particular the following two dual questions seem to be of primary interest:

1. How costly is it to implement the efficient allocation ?
2. What is the least size of inefficiency obtainable with a budget balanced mechanism ?

From the point of view of mechanism design two other questions are relevant:

3. What is a simple mechanism implementing the efficient allocation at minimal cost ?
4. What is a simple, budget balanced mechanism which achieves maximal efficiency ?

As for the third question, the work of Krishna and Perry (1998) implies that for any market size the Vickrey-Clarke-Groves mechanism, in which the expected gains of the buyer with the lowest valuation and of the seller with the highest valuation are equal to zero, minimizes the expected deficit.[3] The mechanism is not only simple, but also implementable in dominant strategies.

We will try to give answers to the remaining three questions in terms of asymptotic results for large numbers of buyers and sellers. There are several reasons why we think such asymptotic results are relevant in this context.

First, many interesting trading situations involve large numbers of buyers and sellers. Examples are stock markets and the rapidly increasing trade conducted via the internet. Secondly, by looking at asymptotic results we get simpler and more explicit statements than those obtainable for a fixed market size even under very strong assumptions. Finally, it turns out that our results give reasonable predictions even for very small markets as illustrated by a numerical example. This is, of course, crucial as the value of any asymptotic result depends on how good it approximates the environments of interest.

1.2. **Model.** We will consider a simple framework where the demand and supply sides are given by $k_1 \cdot m$ sellers and $k_2 \cdot m$ buyers. Here the numbers k_1 and k_2 will be assumed fixed, while the number m will be referred to as market size and allowed to vary.

We assume that each seller owns a single unit of some homogeneous good and each buyer is interested in purchasing exactly one unit. Both buyers and sellers

[2]Proposition 4 and Remark 4 (both in Section 3) give two versions of the Myerson-Satterthwaite Impossibility Theorem for the case of many buyers and sellers.

[3]The exact conditions under which this statement holds are given in Section 3.

are risk neutral and have privately known valuations of owning a unit of the good, which are drawn independently using a distribution F for sellers and G for buyers.

We will assume that the distributions F and G have the property that there is a unique point p^* such that expected supply at price p^* is equal to expected demand at this price.[4] Moreover we will assume that F and G have bounded supports, are twice continuously differentiable in some neighborhood of p^*, and that the densities $F'(p^*)$ and $G'(p^*)$ are greater then zero.

1.3. **Results.** Our analysis starts with a look at the Vickrey - Clarke - Groves mechanism where the expected gains of a buyer with lowest valuation and of the seller with highest valuation are equal to zero. Our first result is that, as the market size m goes to infinity, the deficit of this mechanism converges to

$$\frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}.$$

This is interesting because, as mentioned above, any mechanism implementing the efficient outcome must have a higher or equal deficit than that mechanism.[5] Therefore, the above expression also describes the cost of implementing the efficient allocation.

To answer the second question about the minimal inefficiency obtainable with a budget balanced mechanism we need a measure of inefficiency. We define the absolute inefficiency of a mechanism to be equal to the difference between the expected gains of trade created under the efficient allocation rule, and the expected gains of trade realized by the mechanism.[6] We prove that, as the market size m goes to infinity, the minimal absolute inefficiency obtainable with a budget balanced mechanism converges to zero at a rate of

$$\frac{1}{2 \cdot m} \cdot \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{(k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))^3}. \tag{1}$$

To provide a satisfactory answer to the last question is harder. For the case of regular[7] distributions F and G the work of Gresik and Satterthwaite (1989) proves the existence of a constrained efficient mechanism[8] and describes it.[9] For general distributions F and G the existence and form of a constrained efficient mechanism is a complicated issue. Even in the case of regular distributions F and

[4]This is equivalent to the requirement that the equation $k_1 \cdot F(p^*) = k_2 \cdot (1 - G(p^*))$ has a unique solution.

[5]The exact conditions under which this statement holds are given in Section 3.

[6]There is another measure of inefficiency commonly used in the literature. The relative inefficiency of a mechanism is defined to be the absolute inefficiency of the mechanism divided by the expected gains of trade obtainable under the efficient allocation rule . All our results could be easily restated in terms of relative inefficiencies as explained in Remark 2, Section 2.

[7]Gresik and Satterthwaite require that: (i) the supports of F and G are identical and equal to some bounded interval $[a, b]$ (ii) F and G have continuous and bounded first and second order derivatives on (a, b); (iii) the densities F' and G' are bounded away from 0 on (a, b) (iv) the functions $x + F(x)/F'(x)$ and $x - (1 - G(x))/G'(x)$ are increasing on (a, b).

[8]By constrained optimal mechanism we mean the budget balanced, incentive compatible and individually rational mechanism with minimal inefficiency.

[9]See also Satterthwaite and Wiliams (1999).

G the constrained efficient mechanism seems too complex to be implementable in practice.

This suggests the quest for a mechanism which does not run a deficit and which is strongly asymptotically efficient in the sense that its absolute inefficiency converges to zero at the optimal rate given by Formula (1). We prove that an intuitive modification of the Vickrey - Clarke - Groves mechanism does the job. The modification lies in the introduction of a fixed trading fee charged from those players who actually conduct a trade. Using an appropriate fee we are able to eliminates the deficit. The new mechanism is simple, implementable in dominant strategies and requires the mechanism designer to choose just one parameter - the size of the fee.

We can use our results to conclude that several mechanisms studied previously in the literature are weakly asymptotically efficient in the sense that as the market size m gets large their absolute inefficiency converges to zero at a rate of the same order as the optimal one.

Rustichini, Satterthwaite and Williams (1994) showed that under appropriate regularity conditions the absolute inefficiency of k-double auctions converges to zero at a rate of order $O(\frac{1}{m})$.[10] Another mechanism achieving inefficiencies converging at a rate of this order was proposed by McAfee (1992).[11] Our results (see Formula (1)) imply that the minimal inefficiency obtainable by a budget balanced mechanism for market size m can not converge to zero faster then at a rate of order $O(\frac{1}{m})$. Therefore, we can conclude that both k-double auctions and McAfee's mechanism are weakly asymptotically efficient.

Yoon (2000a) considers a modification of the Vickrey-Clarke-Groves mechanism which is different from ours. In our mechanism a fixed trading fee is charged, ex post, from those traders who actually conduct a trade. Yoon, on the other hand, considers a modification of the Vickrey-Clarke-Groves mechanism, where the players are invited to pay a fixed participation fee at the interim stage. Then a Vickrey-Clarke-Groves mechanism is implemented which includes only those players who paid the fee.[12] Yoon proves that if the participation fee is chosen to achieve budget balance the absolute inefficiency of this mechanism is at most of order $O(1)$. Although this result does not establish whether the mechanism is weakly asymptotically efficient or not Yoon (2000a) uses simulations to argue that asymptotically the mechanism has a higher inefficiency then the mechanism proposed by McAfee.

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[10]The results of Gresik and Satterthwaite (1989) as well as Rustichini, Satterthwaite and Wiliams (1994) are formulated using relative inefficiencies. The relative inefficiency of a direct mechanism is defined as the fraction of expected gains from trade, which could be achieved by the efficient allocation rule, but are not achieved by the mechanism. Here their results are stated using absolute inefficiencies. Remark 2 in Section 2 explains the connection between the two measures of inefficiency.

[11]The environment studied by McAfee differs from ours, it has a nonstrategic "expert" whose only role is to receive a steady flow of income generated by the mechanism. It is, however, easy to implement the allocation rule of McAfee's mechanism in our environment, simply by redistributing the expected income of the expert equally among all players.

[12]Yoon (2000b) considers a generalization of this mechanism to the case multi unit demand.

An example of a mechanism which is clearly not weakly asymptotically efficient is the mechanism charging a fixed price of p^* considered by Hagerty and Rogerson (1985). As the market size m increases the absolute inefficiency of this mechanism does not converges to zero but instead converges to infinity at a rate of order $O(\sqrt{m})$.

1.4. **Other Related Literature.** We are not aware of any paper studying asymptotic properties of the minimal deficit needed to implement the efficient allocation rule.

As for the rate at which the minimal inefficiency of a budget balanced mechanisms converges to zero as the market size becomes bigger, our work is closely related to a paper by Satterthwaite and Williams (1999). Satterthwaite and Williams build on the work of Gresik and Satterthwaite (1989) on constraint efficient mechanisms and on the work of Rustichini, Satterthwaite and Williams (1994) on k-double auctions. They show that in an environment where the number of buyers and sellers are equal and the distributions F and G are both uniform on $[0, 1]$ the minimal inefficiency of a budget balanced mechanisms converges to zero at a rate of order $O(\frac{1}{m})$. Then they use this result to show that k-double auctions satisfy a notion of worst-case asymptotic optimality over sets of environments including the environment mentioned above.

Our results about the minimal inefficiency of a sequence of budget balanced mechanisms extend those of Satterthwaite and Williams (1999) in two directions. First, our results are more general, since we drop the assumptions that the number of buyers and sellers are equal and that the distributions F and G are both uniform on $[0, 1]$. Instead, we allow different numbers of buyers and sellers and quite general distributions.[13] Secondly, we prove stronger results, which describe not only the order of the rate of convergence, but also the exact asymptotic behavior as for example in Formula (1). This is important, because it means that our asymptotic results can be used to predict the constrained optimal inefficiency of a budget balanced mechanism for any fixed market size m.[14]

Less related to our work are several papers (for example Gul and Postlewaite (1992)) considering a more general framework and dealing with the qualitative question whether convergence to efficiency does occur as the market size becomes large.

1.5. **Organization of the paper.** The next section formally introduces the framework and some notation.

Section 3 concerns the Vickrey-Clarke-Groves mechanism and the related issue of the minimal deficit required to implement the efficient allocation rule.

[13]The requirements on F and G were stated in Section 1.2 of this introduction.

[14]In terms of techniques our proofs differ from those in Satterthwite,Wiliams in two major aspects. First, as we rely less on combinatorial arguments we are able to successfully 'localize' our discussion around the point p^*. Secondly, the introduction of the Vickrey-Clarke-Groves mechanism with trading fees is helpful. After developing a good understanding of this mechanism we use it as a comparison device to obtain our results about the rate of convergence of the minimal inefficiency obtainable with a budget balanced mechanisms.

In Section 4 we introduce a new class of mechanisms - the Vickrey-Clarke-Groves mechanisms with a fixed trading fee - which generalizes the Vickrey-Clarke-Groves mechanisms discussed in Section 3. Next, in Theorems 1 and 2, we describe how the deficit and absolute inefficiency of the mechanism depend on the choice of the trading fee. This allows us to see how the Vickrey-Clarke-Groves mechanisms with an appropriately chosen trading fee resolves the conflict between revenue and efficiency. Then, in Theorem 3, we prove that no mechanism can do asymptotically better.

Section 5 discusses several smaller topics related to our main results from Section 3 and Section 4. First, we present a numerical example illustrating our results. Then we consider the question whether it could be worthwhile for a social planner to implement the efficient allocation and finance the deficit via taxes. Finally we briefly discuss the difference between requiring ex ante and ex post budget balance. A short conclusion follows in Section 6.

2. Framework and Notation

2.1. **Buyers and Sellers.** The supply and demand side of the model are given by a set S of $k_1 \cdot m$ sellers and a set B of $k_2 \cdot m$ buyers. Here the numbers k_1 and k_2 will be considered fixed while m will be referred to as the market size and will be allowed to vary. Sellers and buyers have valuations drawn independently from some probability distributions F and G respectively. Each player maximizes a von Neumann-Morgenstern utility function of the form

$$Q_i \cdot v_i + T_i,$$

where v_i is the privately known valuation of a unit of the good, Q_i is the expected probability of ending up with a unit of the good, and T_i the expected net transfer payment to agent i. Note that although we are mainly interested in the case where the valuations are non-negative our framework allows negative valuations.

Consider a fixed price p. For any market size m the expected supply at price p is equal to

$$E(|\{i \in S : v_i < p\}|) = m \cdot k_1 \cdot F(p).$$

while expected demand is given by

$$E(|\{i \in B : v_i > p\}|) = m \cdot k_2 \cdot (1 - G(p)).$$

Notice therefore that the expected demand at price p is equal to expected supply if and only if

$$m \cdot k_1 \cdot F(p) = m \cdot k_2 \cdot (1 - G(p)).$$

Dividing both sides of the equation by m we see that this condition does not depend on the market size m.

We will assume that the distributions F and G have bounded supports and that there is a unique number p^* such that the expected demand at price p^* is equal to expected supply, i.e.

$$k_1 \cdot F(p^*) = k_2 \cdot (1 - G(p^*)). \tag{2}$$

Remark 1. *The above assumption does not seem too restrictive. Indeed, consider, for example, the case where the number of buyers and sellers are equal, i.e.* $k_1 = k_2$*; F and G are continuous; and the supports of F and G are proper intervals* $[a, b]$ *and* $[c, d]$*. Then, one of the following three cases can occur:*

1. $c \geqslant b$*, the valuation of a buyer is always smaller then the valuation of a seller. In this case the efficient allocation requires each seller to keep his good. It is trivial to implement it.*

2. $a \geqslant d$*, the valuation of a seller is always smaller then the valuation of a buyer. In this case the efficient allocation requires that each buyer gets a good. We can implement the efficient allocation with a mechanism which gives the goods to all buyers for a fixed price of* p^**, where* p^* *is any number in* $[d, a]$*.*

3. The support of F and G have an intersection with a non-empty interior. This is the case where the Myerson-Satterthwaite Impossibility Theorem holds and in which we are interested. Note that in this case our assumption will always be satisfied, there is a unique price p^* *satisfying Equation* (2)*.*

2.2. **Market Mechanisms.** Appealing to the Revelation Principle we can limit ourselves to mechanisms that induce buyers and sellers to truthfully reveal their valuations. More precisely, for any fixed market size $m \in \mathbb{N}$ a (direct) market mechanism M^m consists of a pair of functions (q, t), where q and t are called respectively allocation and transfer payment rule. For any profile of valuations $v = (v_i)_{i \in B \cup S}$ and player $i \in B \cup S$ the number $q_i(v)$ is the probability that player i ends up with one unit of the good and $t_i(v)$ is the net payment received by player i.

As $q_i(v)$ are probabilities, they have to lie in the interval $[0, 1]$. In addition, the usual feasibility assumption

$$\sum_{i \in B \cup S} q_i(v) = k_1 \cdot m$$

has to be satisfied. The individual rationality conditions translate in our framework into

$$E(q_i(v) \cdot v_i + t_i(v) \mid v_i) \geqslant 0 \text{ for } i \in B$$

and

$$E(q_i(v) \cdot v_i + t_i(v) \mid v_i) \geqslant v_i \text{ for } i \in S.$$

The incentive compatibility constraint for a player $i \in B \cup S$ is equivalent to

$$E(q_i(v) \cdot v_i + t_i(v) \mid v_i) \geqslant E(q_i(v_{-i}, v_i') \cdot v_i + t_i(v_{-i}, v_i') \mid v_i)$$

for all types v_i', v_i that are possible for player i.

For any mechanism $M = (q, t)$ we define the deficit of the mechanism M to be equal to

$$\text{Expected Deficit}(M) = E\big(\sum_{i \in B \cup S} t_i(v)\big)$$

We will say that a mechanism is (ex ante) budget balanced if its expected deficit is equal to zero. In Section 5.3 we discuss the connection to ex post budget balance, which requires that the deficit is always equal to zero.

2.3. **Inefficiency.** For any fixed market size m and mechanism $M = (q, t)$ the gains of trade from using the allocation rule q if the profile of valuations of all players is $v = (v_i)_{i \in B \cup S} \in \mathbb{R}^{B \cup S}$ are defined as

$$\text{Exp. Gains of Trade}(q) = E\big(\sum_{i \in B \cup S} v_i \cdot q_i(v) - \sum_{i \in S} v_i\big).$$

It will be useful to define q^{eff} to be the efficient allocation rule which maximizes gains of trade. This means that q^{eff} almost surely assigns the $k_1 \cdot m$ goods to the $k_1 \cdot m$ players with the highest valuations.

To measure the inefficiency of a particular mechanism M^m we compare the expected gains of trade created by the mechanism with the expected gains from trade implementable by a social planner in the case of full information. Define $I(M)$, the absolute inefficiency of a direct, incentive compatible mechanism $M = (q, t)$ as the difference between the expected gains from trade created by the efficient allocation rule and the expected gains from trade achieved by the mechanism M^m i. e.,

$$I(M^m) = \text{Exp. Gains of Trade}(q^{eff}) - \text{Exp. Gains of Trade}(q)$$

Now the asymptotic properties of a sequence of mechanisms $(M^m)_{m \in \mathbb{N}}$ specifying one mechanism for each possible market size $m \in \mathbb{N}$ can be understood in terms of the asymptotic properties of the sequence $(I(M^m))_{m \in \mathbb{N}}$.

Remark 2. *The absolute inefficiency measures non-realized gains of trades in terms of units of the transfer payments. Thus, if transfer payments are calculated in dollars the absolute inefficiency will tell us how many dollars could have been gained under the efficient allocation rule.*

Another notion of inefficiency often used in the literature measures the inefficiency as a percentage of expected gains of trade achievable under the efficient allocation rule. More precisely, $\zeta(M^m)$, the relative inefficiency of a market mechanism $M = (q, t)$ is defined to be the fraction of the expected gains from trade which could be achieved in the full information case, but are not achieved by the mechanism M. Thus,

$$\begin{aligned}\zeta(M) &= \frac{\textit{Exp. Gains of Trade}(q^{eff}) - \textit{Exp. Gains of Trade}(q)}{\textit{Exp. Gains of Trade}(q^{eff})} \\ &= \frac{I(M)}{\textit{Exp. Gains of Trade}(q^{eff})}.\end{aligned}$$

The law of large numbers implies that $\frac{\textit{Exp. Gains of Trade}(q^{eff})}{m}$ converges as m goes to infinity to

$$k_2 \cdot (1 - G(p^*)) \cdot E(v_b \,|\, v_b > p^*) + k_1 \cdot (1 - F(p^*)) \cdot E(v_s \,|\, v_s > p^*) - k_1 \cdot E(v_s)$$

where $b \in B$ is a buyer, $s \in S$ is a seller. Using this it is easy to restate our results in terms of relative inefficiencies instead of absolute inefficiencies.

2.4. **Comparing mechanisms.** We will be interested in asymptotic properties of market mechanisms as the number of traders becomes large. In particular, we would like to compare the efficiency of two sequences $(M^m)_{m\in\mathbb{N}}$ and $(N^m)_{m\in\mathbb{N}}$, where for each $m \in \mathbb{N}$, M^m and N^m are mechanisms for market size m. For this we will use the sequences $(I(M^m))_{m\in\mathbb{N}}$ and $(I(N^m))_{m\in\mathbb{N}}$. We will say that the sequence $(M^m)_{m\in\mathbb{N}}$ is asymptotically less efficient then $(N^m)_{m\in\mathbb{N}}$ if and only if

$$\liminf_{m\to\infty}\frac{I(M^m)}{I(N^m)} > 1.$$

Similarly, we will say that $(M^m)_{m\in\mathbb{N}}$ is asymptotically at least as efficient as $(N^m)_{m\in\mathbb{N}}$ if and only if

$$\limsup_{m\to\infty}\frac{I(M^m)}{I(N^m)} \leqslant 1.$$

Finally, we will say that a sequence of budget balanced mechanisms M^m is strongly asymptotically optimal (in the class of all budget balanced mechanisms) if it is at least as efficient as any other sequence of budget balanced mechanisms N^m.

Remark 3. *The definitions of absolute and relative inefficiencies of a mechanism imply that for any two mechanisms*

$$\frac{I(M)}{I(N)} = \frac{\zeta(M)}{\zeta(N)}.$$

Thus, the above definitions could have been as well stated in terms of relative inefficiencies.

A weaker notion of asymptotic efficiency can be defined analogously. A sequence $(M^m)_{m\in\mathbb{N}}$ is weakly at least as efficient as $(N^m)_{m\in\mathbb{N}}$ if and only if

$$\limsup_{m\to\infty}\frac{I(M^m)}{I(N^m)} < \infty.$$

A sequence of budget balanced mechanisms M^m is weakly asymptotically optimal if it is weakly at least as efficient as any other sequence of budget balanced mechanisms N^m.[15]

3. The cost of efficiency

For a market of size m we will define VCG^m to be the Vickrey-Clarke-Groves mechanism where the expected gains of a buyer with lowest valuation and of a seller with highest valuation are equal to zero. The significance of the mechanism VCG^m lies in the fact (Krishna, Perry (1998)) that, under weak assumptions, any efficient mechanism will run a deficit at least as big as the deficit of the VCG^m. Thus, by calculating the expected deficit of the VCG^m we will de facto calculate the minimal cost of implementing the efficient outcome.

[15]The notion of weak asymptotic optimally is only used in the introduction of this paper to state implications of our results for mechanisms studied previously in the literature.

3.1. **Definition of the $\mathbf{VCG^m}$.** Fix a market size $m \in \mathbb{N}$. We will denote by VCG^m the Vickrey-Clarke-Groves mechanism in which the expected gains of a buyer with lowest valuation and of a seller with highest valuation are equal to zero.

This means that the allocation rule of the VCG^m is equal to the efficient allocation rule q^{eff}, which assigns the $k_1 \cdot m$ objects to the $k_1 \cdot m$ players with highest valuations v_i. For the sake of concreteness we will assume that the allocation rule assigns an object with equal probability to two players in case of ties.

The payment rule t^{VCG^m} is defined by

$$t_i^{VCG^m}(v) = \begin{cases} SW_{-i}(v) - SW(\underline{v}_b, v_{-i}) & \text{if } i \in B \\ SW_{-i}(v) - SW(\overline{v}_s, v_{-i}) + \overline{v}_s & \text{if } i \in S \end{cases} \tag{3}$$

where $\underline{v}_b$ is the smallest possible valuation of a buyer in the support of G, $\overline{v}_s$ is the highest possible valuation of a seller in the support of F, $SW(v)$ is the social welfare under the efficient allocation:

$$SW(v) = \sum_{i \in B \cup S} v_i \cdot q_i^{eff}(v),$$

and $SW_{-i}(v)$ is defined as

$$SW_{-i}(v) = \sum_{j \in B \cup S - \{i\}} v_j \cdot q_j^{eff}(v).$$

In other words, the payment of a player corresponds to the impact she has on social welfare compared with the situation if she had type $\underline{v}_b$ (in case of buyers) or $\overline{v}_s$ (in case of sellers).

3.2. **Further description of the $\mathbf{VCG^m}$.** To get a more explicit description of the payment rule t^{VCG^m} define $\underline{p}(v)$ to be the highest valuation among the players who end up without a good under the efficient allocation rule

$$\underline{p}(v) = \max\{v_i : i \in B \cup S \text{ and } q^{eff}(v) < 1\}.$$

Similarly define $\overline{p}(v)$ to be the lowest valuation among the players who end up with a good under the efficient allocation rule

$$\overline{p}(v) = \min\{v_i : i \in B \cup S \text{ and } q^{eff}(v) > 0\}.$$

Note that if the valuations of all players are different then $\underline{p}$ and $\overline{p}$ are respectively equal to the $(k_1 \cdot m)$-th highest and $(k_1 \cdot m + 1)$-th highest valuation.

Now Equation (3) implies that

$$t_i(v) = \begin{cases} -q_i^{eff}(v) \cdot \max(\underline{p}(v), \underline{v}_b) & \text{if } i \in B \\ (1 - q_i^{eff}(v)) \cdot \min(\overline{p}(v), \overline{v}_s) & \text{if } i \in S \end{cases} \tag{4}$$

where $\overline{v}_s$ is the highest valuation in the support of F and $\underline{v}_b$ is the lowest valuation in the support of G.

In other words, any buyer pays $\max(\underline{p}(v), \underline{v}_b)$ if he does get a good and zero if he does not get a good. Similarly, sellers get $\min(\overline{p}(v), \overline{v}_s)$ if they sell their good and zero if they do not.

3.3. **The Deficit of the VCG$^{\mathbf{m}}$.** The above characterization of the payment rule allows a simple formula for the expected deficit. For any profile v define $\tau^{eff}(v)$ to be equal to the number of trades under the efficient rule

$$\tau^{eff}(v) = \sum_{i \in B} q_i^{eff}(v) = \sum_{i \in S} (1 - q_i^{eff}(v)).$$

Now we can write the expected deficit of the VCG^m as

$$\text{(5)} \qquad \text{Expected Deficit}(VCG^m) = E(\tau^{eff}(v) \cdot (\min(\overline{p}(v), \overline{v}_s) - \max(\underline{p}(v), \underline{v}_b)))$$

The next proposition describes the behavior of the expected deficit as the market size m goes to infinity.

Proposition 1. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$. Then*

$$\textit{Expected Deficit}(VCG^m) \to \frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \quad \textit{as} \quad m \to \infty.$$

Proof. See appendix. □

In the appendix the above proposition is proven as a special case of Theorem 1 bellow, concerning the deficit of the more general Vickrey-Clarke-Groves mechanisms with trading fees. Here we would like to take a more informal path - we will sketch an informal derivation of Proposition 1 and in the next section use the result to provide intuition for Theorem 1.

To understand Proposition 1 recall that p^* was defined by the equation

$$m \cdot k_1 \cdot F(p^*) = m \cdot k_2 \cdot (1 - G(p^*)).$$

This means that the expected number of players with valuations higher or equal then p^* is equal to

$$k_1 \cdot m \cdot (1 - F(p^*)) + k_2 \cdot m \cdot (1 - G(p^*)) = m \cdot k_1.$$

As $\underline{p}$ and $\overline{p}$ where defined to be the $(k_1 \cdot m)$-th highest and $(k_1 \cdot m + 1)$-th highest valuation the law of large numbers suggests that that as the market size m becomes larger both $\underline{p}$ and $\overline{p}$ will converge in probability to p^*.

Consider Equation (5) describing the expected deficit. Our assumptions guarantee that $\underline{v}_b < p^* < \overline{v}_s$. The above observation that $\underline{p}$ and $\overline{p}$ will converge in probability to p^* suggests that

$$\text{(6)} \qquad \text{Exp. Deficit}(VCG^m) = \lim_{m \to \infty} E(\tau^{eff}(v) \cdot (\overline{p}(v) - \underline{p}(v)))$$

What can we say about the number of trades $\tau^{eff}(v)$? Note that, the number of trades τ is equal to the number of sellers with valuations higher then $\underline{p}$. As for large markets $\underline{p}$ will be usually close to p^* the expected number of trades should be approximately equal to the expected number of sellers with valuations higher then p^*, i.e.

$$\text{(7)} \qquad \tau^{eff}(v) \approx k_1 \cdot m \cdot F(p^*) \text{ for large } m.$$

Now consider the term $(\overline{p}(v) - \underline{p}(v))$. Note that $\overline{p}(v) - \underline{p}(v)$ is the distance between the $(k_1 \cdot m)$-th and $(k_1 \cdot m + 1)$-th valuation among the players. What is the expected distance of two neighboring valuations lying close to p^* ? To answer this question consider the average number of players with valuations in a small interval $[p - \frac{1}{2}h, p + \frac{1}{2}h]$. If p is close to p^* there will be approximately $k_1 \cdot m \cdot F'(p^*) \cdot h$ sellers and $k_2 \cdot m \cdot G'(p^*) \cdot h$ buyers with valuations in the interval. This means that the distance between two neighboring valuations will be approximately

$$\overline{p}(v) - \underline{p}(v) \approx \frac{1}{k_1 \cdot m \cdot F'(p^*) + k_2 \cdot m \cdot G'(p^*)} \text{ for large } m. \tag{8}$$

Substituting $\frac{\tau^{eff}(v)}{m}$ and $\overline{p}(v) - \underline{p}(v)$ from Equations (7) and (8) in Equation (6) we get that the expected deficit converges to $\frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$.

3.4. **The cost of efficiency.** As noted before, the significance of the mechanism VCG^m lies in the following proposition, which is a corollary of the results of Krishna, Perry (1998):

Proposition 2 (Krishna, Perry)**.** *Assume F and G have convex supports and are twice continuously differentiable on their supports. Then the deficit of any efficient, incentive compatible, and individually rational mechanism is at least as high as the deficit of the VCG^m.*

Proof. The proposition is a straightforward application of Theorem 1 from Krishna, Perry (1998). □

Remark 4. *The above proposition together with Formula* (5) *gives a multilateral version of the Myerson-Satterthwaite Impossibility Theorem. Indeed, assume the assumptions of Proposition 2 hold and the point p^* lies in the interior of the supports of F and G. Then Formula* (5) *implies that the expected deficit of the VCG^m is positive. Therefore Proposition 2 implies that there is no incentive compatible, individually rational and budget balanced mechanism implementing the efficient allocation.*

Propositions 1 and 2 together imply that if F and G have convex supports and are twice continuously differentiable on the whole support the minimal cost of implementing the efficient allocation will converge to $\frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$.

It turns out that this is true even if we relax the global assumptions on the distributions F and G.

Proposition 3. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*
Let (M^m) be a sequence such that M^m is an efficient, incentive compatible, and individually rational mechanisms for market size m. Then

$$\liminf_{m \to \infty} \textit{Expected Deficit}(M^m) \geqslant \frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}.$$

Proof. The proposition is a special case of Theorem 3 from the next section. See the appendix for details. □

Note that the above proposition implies that an approximation of a large market with a "limiting model" which has a continuum of buyers and sellers with deterministic valuations (distributed accordingly to F and G) is problematic. Indeed, in this "limiting model" it is possible to implement the efficient allocation using a budget balanced mechanism in which everybody trades at the price p^*. Proposition 3, however, shows that, in our framework, the deficit will remain bounded away from zero as the market becomes large .

As a direct corollary of Proposition 3 we get the following version of the Myerson-Satterthwaite Impossibility Theorem for the case where the valuations of buyers and sellers have distributions with possibly non-convex supports.

Proposition 4. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*
Then there exists a number m_o such that for all $m \geqslant m_o$ there is no incentive compatible, individually rational, and budget balanced mechanism implementing the efficient allocation rule for market size m.

Note that Proposition 4 shows that, qualitatively, inefficiencies will arise as the market size becomes large even if trade was efficient for small market sizes. This is interesting because it contradicts the common intuition that strategic use of private information becomes less relevant as the number of traders increases.

This ends our section on the implementation of the efficient allocation rule. We showed that under very weak assumptions the minimal deficit needed to implement the efficient allocation asymptotically has to be at least equal $\frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$. Moreover, the bound is sharp, since, as shown in Proposition 1, the deficit of the VCG^m converges to this value.

4. The VCG mechanism with a trading fee

In this section we introduce a modification of the Vickrey-Clarke-Groves mechanism, which is still individually rational and implementable in dominant strategies but no longer runs a deficit. The idea of the modification is quite simple - we will use a trading fee to reduce the deficit of the Vickrey-Clarke-Groves mechanism. The significance of the new mechanism lies in the fact that we can explicitly determine the asymptotic behavior of its inefficiency and then show that the inefficiency created by the introduction of the fee is small in the sense that the mechanism is strongly asymptotically optimal.

4.1. Idea.

We want to modify the VCG^m in a way which eliminates the deficit. We will achieve this by charging a fixed fee R for each trade which occurs. For the sake of concreteness imagine that each buyer has to pay R after he receives the good.[16] Thus a buyer who anticipates the fee will assign a new valuation $v_b^{(R)}$ to the good, which is given by

$$v_b^{(R)} = v_b - R \quad \text{for all } b \in B.$$

[16]Who pays the fee is not very important. If we would imagine a situation where the seller has to pay the fee after he sold his good the valuations $v_i^{(R)}$ would change, but the following construction would lead to the same allocation rule. This is also true if buyers and sellers actively engaging in trade pay respectively fees R_1 and R_2, where $R_1 + R_2 = R$.

The reason for this is that the allocation rule is fully determined by the relative valuations of buyers and sellers.

On the other hand, the valuations of the seller from keeping the good clearly are unaffected by some fees which buyers have to pay, thus

$$v_s^{(R)} = v_s \text{ for all } s \in S.$$

The $VCG^m(R)$ defined in this section can be seen as a two stage mechanism. The first stage consists of a Vickrey-Clarke-Groves mechanism, which is identical to the one discussed in Section 3 except that it uses the valuations $v_i^{(R)}$ instead of the real valuations. In the second stage all the buyers who got the good pay R. As players participating in the first stage use the valuations given by $v_i^{(R)}$ the construction makes sense.

4.2. **Definition.** The allocation rule of the $VCG^m(R)$ is defined to be the rule which assigns the $k_1 \cdot m$ goods to the $k_1 \cdot m$ players for which the valuations $v_i^{(R)}$ are the highest. Again, we will assume that the allocation rule is symmetric in the sense that it allocates the good with equal probabilities in case of ties. Note that for each profile of valuations $v = (v_i)_{i \in B \cup C}$ the allocation rule $q^{VCG^m(R)}$ solves the program

$$\max_q \sum_{i \in B \cup S} v_i^{(R)} \cdot q_i(v) \tag{9}$$

where the maximum is taken over all allocation rules q.

The formula for the transfer payments is given by:

$$t_i^{VCG^m(R)}(v) = \begin{cases} SW_{-i}^{(R)}(v) - SW^{(R)}(\underline{v}_b, v_{-i}) - q_i^{VCG(R)}(v) \cdot R & \text{if } i \in B \\ SW_{-i}^{(R)}(v) - SW^{(R)}(\overline{v}_s, v_{-i}) + \overline{v} & \text{if } i \in S \end{cases}$$

where

$$SW^{(R)}(v) = \sum_{i \in B \cup S} v_i^{(R)} \cdot q_i^{VCG(R)}(v)$$

and

$$SW_{-i_o}^{(R)}(v) = \sum_{i \in B \cup S - \{i_o\}} v_i^{(R)} \cdot q_i^{VCG(R)}(v)$$

for any player $i_o \in B \cup S$.

Proposition 5. *The mechanism $VCG^m(R^m)$ defined above is individually rational and implementable in dominant strategies for any $m \in \mathbb{N}$ and $R \in \mathbb{R}$.*

Proof. The derivation is routine. To verify the individual rationality condition of a buyer $i \in B$ notice that his utility after reporting his type truthfully is given by

$$v_i \cdot q_i^{VCG^m(R)}(v) + t_i(v) = SW^{(R)}(v) - SW^{(R)}(\underline{v}_b, v_{-i}).$$

As $SW^{(R)}(v)$ is equal to the sum of the $(k_1 \cdot m)$-highest valuations $v_i^{(R)}$ it is clearly increasing in v_i. Therefore $SW^{(R)}(v) - SW^{(R)}(\underline{v}_b, v_{-i}) \geqslant 0$ and the individual rationality constraint holds. The argument for a seller $i \in S$ is similar. [17]

[17] For a seller $i \in S$ the utility after reporting his type truthfully is given by:

$$v_i \cdot q_i^{VCG^m(R)} + t_i = SW^{(R)}(v) - SW^{(R)}(\overline{v}_s, v_{-i}) + \overline{v}_s.$$

To check that the mechanism is implementable in dominant strategies notice that the utility of a buyer $i \in B$ after reporting type v_i' is equal to $v_i \cdot q_i^{VCG^m(R)}(v_i', v_{-i}) + t_i(v_i', v_{-i})$ or

$$\sum_{i \in B \cup S} v_i^{(R)} \cdot q_i^{VCG^m(R)}(v_i', v_{-i}) - SW^{(R)}(\underline{v}_b, v_{-i}) \tag{10}$$

As the allocation rule $q_i^{VCG^m(R)}$ was a solution to the program (9) it is clear that $v_i' = v_i$ maximizes expression (10) for any profile v_{-i}. The argument for a seller $i \in S$ is identical. □

4.3. **Size of the fee.** The next theorem is a generalization of Proposition 1 and describes the deficit of the $VCG^m(R^m)$.

Theorem 1. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*
Let $\rho \in \mathbb{R}$ be a fixed number and R^m any sequence of fees such that

$$\lim_{m \to \infty} R^m \cdot m = \rho.$$

Then the expected deficit of the $VCG^m(R^m)$ converges to

$$k_1 \cdot F(p^*) \cdot \left(\frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} - \rho\right).$$

Proof. See Appendix □

To understand Theorem 1 consider the interpretation of the $VCG^m(R^m)$ as a two stage mechanism given in Section 4.1. The first stage consisted of a Vickrey-Clarke-Groves mechanism which used the valuations $v_i^{(R)}$ instead of the real valuations. Notice that under the assumptions of the theorem R^m converges to zero as the market size grows, therefore, the valuations $v_i^{(R)}$ will converge to the real valuations v_i. In this situation Proposition 1 suggests that the first stage will run a deficit of $\frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$. In the second stage all the buyers who got the good paid R. Therefore, the second stage creates a surplus which is equal to R^m times the number of conducted trades. As the number of conducted trades is approximately equal to $k_1 \cdot F(p^*) \cdot m$ and the fee R^m is assumed to be approximately equal to $\frac{\rho}{m}$ it seems plausible that the surplus of the second stage should be equal to $k_1 \cdot F(p^*) \cdot \rho$. Subtracting the surplus of the second stage from the deficit of the first stage we get the formula in the theorem for the total deficit.

Using the theorem we are able to determine the size of the fee needed in order to eliminate the deficit, as seen in the next corollary.

Corollary 1. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*

To check the individual rationality condition we have to verify that $SW^{(R)}(v) - SW^{(R)}(\overline{v}_s, v_{-i}) + \overline{v}_s \geqslant v_i$. But this follows from the observation that by changing the type of player i from $\overline{v}_s$ to v_i we can not decrease $SW^{(R)}$ by more than $\overline{v}_s - v_i$.

Then there exists a sequence of numbers R^m such that
1. The fees R^m satisfy

$$\lim_{m\to\infty} R^m \cdot m = \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}.$$

2. For all $m \in \mathbb{N}$

$$\text{Expected Surplus}(VCG^m(R^m)) \geqslant 0$$

Moreover, if F and G are continuous everywhere we can choose the fees R^m so that the mechanisms $VCG^m(R^m)$ are ex ante budget balanced for all $m \in \mathbb{N}$.

Proof. See Appendix □

4.4. **Calculation of Asymptotic inefficiency.** The following theorem characterizes the size of inefficiency caused by the introduction of fees.

Theorem 2. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*
Let $\rho \in \mathbb{R}$ be a fixed number and R^m any sequence of fees such that

$$R^m \cdot m \to \rho.$$

Then

$$I(VCG^m(R^m)) \cdot m \to \frac{1}{2}\frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot \rho^2,$$

as m converges to infinity.

Proof. See Appendix. □

The formal derivation of the above result is in the appendix. Here we will present an informal argument.[18] To calculate the absolute inefficiency of the $VCG^m(R^m)$ we need to compare its gains of trade with the gains of trade if the efficient allocation rule was implemented, for example, by the $VCG^m(0)$. What is the inefficiency connected with the introduction of a fee R^m charged from all buyers who bought a unit of the good? Consider the supply-demand diagram in Figure 1, which should be familiar to any undergraduate economics student.

FIGURE 1

The fee charged from consumers moves the demand curve downwards by an amount corresponding to the size of the fee. The effect of this is that some trades will not be realized. The gains of trade lost due to the fee correspond to the area of the small triangle shaded in the picture. To calculate the area of the triangle we need to know the slopes of the demand and supply curves around the equilibrium price p. To determine these slopes think how a small change of the price from p to $p+h$ would affect expected supply. As the valuations of buyers are distributed according to distribution F the probability that a seller has a valuation between p and $p+h$ is equal to $F(p+h) - F(p)$. As there are $k_1 \cdot m$ sellers the expected change in supply is equal to $k_1 \cdot m \cdot (F(p+h) - F(p))$. Letting h go to zero we obtain that the slope

[18] The argument presented here is due to Asher Wolinsky.

of the supply curve is $\frac{1}{k_1 \cdot m \cdot F'(p^*)}$ at an equilibrium price p^*. Similar analysis gives a slope of the demand curve equal to $-\frac{1}{k_2 \cdot m \cdot G'(p^*)}$. Simple geometry now implies that the area of the triangle is equal to

$$\frac{1}{2} \cdot R^m \cdot \frac{R^m}{\frac{1}{k_1 \cdot m \cdot F'(p^*)} + \frac{1}{k_2 \cdot m \cdot G'(p^*)}}.$$

Using the fact that $\lim_{m\to\infty} R^m \cdot m = \rho$ we get the formula from the theorem.

For the important special case where R^m are chosen to achieve budget balance as described in Corollary 1 we get the following result.

Corollary 2. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*
Then there exists a sequence of numbers R^m such that
1. The absolute inefficiency of the $VCG^m(R^m)$ converges to

$$\lim_{m\to\infty} I(VCG^m(R^m)) \cdot m = \frac{1}{2} \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{(k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))^3}.$$

2. For all $m \in \mathbb{N}$

$$\textit{Expected Surplus}(VCG^m(R^m)) \geqslant 0$$

Moreover, if F and G are continuous everywhere we can choose the fees R^m so that the mechanisms $VCG^m(R^m)$ are ex ante budget balanced for all $m \in \mathbb{N}$.

Proof. The result follows immediately from Corollary 1 and Theorem 2. □

Remark 5. *In the case where F and G are not continuous it is still easy to construct a sequence of mechanisms M^m which are individually rational, implementable in dominant strategies, ex ante budget balanced and whose relative inefficiency converges to zero at the same rate as in Corollary 2. Indeed, to construct such a sequence it is enough to take a sequence of fees R^m as described in Corollary 2 and then define the mechanism M^m to be identical to the $VCG^m(R^m)$ except that the payment to each player is increased by $\frac{\textit{Expected Surplus}(VCG^m(R^m))}{k_1 \cdot m + k_2 \cdot m}$.*
To get a sequence of ex post budget balanced mechanism with the above properties another modification is necessary. This modification is described in Section 5.3.

4.5. **Asymptotic optimality.** Theorems 1 and 2 describe how deficit and inefficiency of the $VCG^m(R^m)$ depend on the choice of the fees R^m. Putting the two results together we see the trade off between revenue and efficiency. Let $\rho \in \mathbb{R}$ be an arbitrary number. If we want the deficit of a sequence of mechanisms to converge to some number smaller than

$$k_1 \cdot F(p^*) \cdot (\rho - \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)})$$

we have to choose fees R^m such that $\rho \geqslant \liminf_{m\to\infty} R^m \cdot m$ and, therefore, incur an inefficiency of at least

$$\frac{1}{2} \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot \rho^2.$$

Can we find a sequence of mechanisms which resolves the conflict between revenue and efficiency in a better way? The following theorem says that asymptotically this is not the case.

Theorem 3. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*
Assume M^m is any sequence of incentive compatible, individually rational market mechanisms such that for some $\rho \in \mathbb{R}$

$$\limsup_{m\to\infty} \textit{Expected Deficit}(M^m) = k_1 \cdot F(p^*) \cdot \left(\frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} - \rho\right)$$

as m goes to infinity. Then

$$\liminf_{m\to\infty} I(M^m) \cdot m \geqslant \frac{1}{2} \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot \rho^2.$$

Proof. See Appendix. □

As a direct implication we get the following important corollary

Corollary 3. *Assume F and G are twice continuously differentiable in some neighborhood of p^* and $G'(p^*), F'(p^*) > 0$.*
Assume M^m is any sequence of incentive compatible, individually rational, market mechanisms such that

$$\textit{Expected Surplus}(M^m) \geqslant 0$$

for all $m \in \mathbb{N}$. Then

$$\liminf_{m\to\infty} I(M^m) \cdot m \geqslant \frac{1}{2} \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{(k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))^3}.$$

Moreover, the bound is sharp.

Proof. The first statement of the corollary is a trivial consequence of Theorem 3 for the case where ρ is set to be equal to $\frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$. The sharpness of the bounds follows from Corollary 2. □

Corollaries 2 and 3 imply that the Vickrey-Clarke-Groves mechanism with an appropriately chosen trading fee is strongly asymptotically optimal in the class of mechanisms which do not run an expected deficit.

Why can't we do better than with the $VCG^m(R^m)$? To get a good understanding for this result the reader might want to consult the proof of Theorem 3 in the appendix. In short, the intuition is as follows. Not realizing a trade involving a buyer with a very high valuation or a seller with a very low valuation will lead to substantial losses in potential gains of trade. Therefore, somebody trying to design a mechanisms outperforming the $VCG^m(R^m)$ has very little freedom when choosing how to allocate the goods for players with valuations far away from p^*.[19] As a consequence, it is not possible to outperform the $VCG^m(R^m)$ by changing the allocation rule for players with valuations far away from p^*.

On the other hand, choosing how to allocate the goods among buyers and sellers with valuations very close to p^* in some sense boils down to the question to which

[19]See Section A.5 of the appendix for details.

extent the allocation rule should discriminate between buyers and sellers. In other words if a seller has a valuation p close to p^* and a buyer has a valuation $p+h$, the question is about the size of h necessary for the mechanism to assign the good to the buyer rather then to the seller. Note that a rule which does not discriminate at all against buyers will run a deficit and that a rule which discriminates against buyers will not be efficient. More generally, the mechanism designer faces a trade off between efficiency and deficit – a rule which discriminates more against buyers will create a higher revenue but at the same time be less efficient. As a result the mechanism designer trying to maximize efficiency will choose the fee which achieves budget balance. The fact that the same fee is sometimes used to compare valuations of buyers and sellers which are 'far away from p^*' is irrelevant.

5. Comments

5.1. **Numerical Example.**

It turns out that our asymptotic results can have surprisingly good predictive power even for relatively small market sizes.

Consider the case where there are twice as many buyers as sellers (i.e. $k_1 = 1$, $k_2 = 2$) and the valuations of buyers and sellers are drawn using quadratic distributions[20] with supports $[\$0, \$100]$ for sellers and $[\$0, \$120]$ for buyers (see Figure 2).

FIGURE 2

This means that the price p^* at which expected supply is equal to expected demand is approximately equal to \$69. At this price the probability of trading would be approximately 77% for sellers and 39% for buyers.[21]

Using Corollary 3 we can predict that if there are 2 buyers and 4 sellers the absolute inefficiency of the constrained optimal mechanism[22] is equal to[23]

$$\frac{1}{2 \cdot m} \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{(k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))^3} \simeq \$1.5$$

what corresponds to approximately 2.3% of the total gains from trade obtainable under the efficient allocation rule.

The absolute inefficiency of the constrained optimal mechanism calculated in our simulations is approximately equal to \$1.3 what corresponds to approximately 2.0% of the total gains from trade.

Using our asymptotic results we were able to predict the inefficiency of the constraint optimal mechanism with an accuracy of approximately 20 cents which corresponded to 0.3% of the total expected gains of trade. It seems as if our results

[20]We consider bell-shaped density functions because of the common intuition that they are relevant in economics. The choice of quadratic distributions is connected with computational issues.

[21]The actual probability of engaging in trade under the efficient allocation rule will depend slightly on the market size m.

[22]By constrained optimal mechanism we mean the budget balanced, incentive compatible and individually rational mechanism with minimal inefficiency.

[23]Notice that the prediction is the same whether we use $k_1 = 1$, $k_2 = 2$, $m = 2$ or $k_1 = 2$, $k_2 = 4$, $m = 1$.

have good predictive power even for markets involving as few as 6 players. In this context it is noteworthy that, as the market size m increases, our predictions should become more precise, while the numerical calculations will become more and more difficult due to the, so called, "curse of dimensionality".

Table 1 illustrates the convergence of some values of interest as the market size m goes to infinity.

TABLE 1

The first two columns of the table correspond respectively to the market size m and to the deficit of the VCG^m. The numbers in the third column are equal to m times the absolute inefficiency of the $VCG^m(R^m)$, where the fees R^m are chosen to achieve budget balance. Finally the numbers in the last column are equal to m times the absolute inefficiency of the constraint optimal mechanism.[24]

5.2. **Application.** Let us use our understanding of the trade off between revenue and efficiency to answer a 'real world' question:

> Should a benevolent government use tax money to finance the deficit of an efficient trading mechanism in the case of a large number of traders ?

First, consider the case where the government can costlessly collect lump sum taxes to finance the deficit of an efficient market mechanism. Then doing so clearly increases total social welfare independently of the market size m.

Usually, however, the collection of taxes is connected with costs on society arising from behavioral disturbances and organizational issues. Assume, therefore, that the cost of collecting an additional dollar of tax money is equal to a fixed value $\lambda > 1$. In this case financing the deficit of an efficient mechanism via taxes causes a cost on society which by Proposition 3 is asymptotically at least

$$(\lambda - 1) \cdot \frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} > 0$$

On the other hand, the benefits of implementing the efficient outcome instead of an asymptotically efficient budget balanced mechanism are equal to the absolute inefficiency of that mechanism and, thus, by Corollary 3, approximately equal to

$$\frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{2m \cdot (k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))^3} \to 0$$

Therefore, it is clear that financing the deficit using tax money will decrease total social welfare as long as the market size m is large enough.

[24]The values in Table 1 were calculated using numerical methods and, therefore, might carry errors. The main source of such errors should be the Monte Carlo methods used to calculate various integrals. Various and repeated calculations suggest that these errors should not be bigger then $\pm\$0.1$ for all the values reported in Table 1.

5.3. **Budget balance.** Until now we have only considered mechanisms which are ex ante budget balanced in the sense that the expected deficit is equal to zero:

$$E(\sum_{i \in B \cup S} t_i) = 0. \tag{11}$$

We might consider a stronger notion of budget balance, where we require that

$$\sum_{i \in B \cup S} t_i = 0 \text{ a. s.} \tag{12}$$

It is well known that in environments with a finite number of players and quasi-linear utility functions the two requirements are equivalent in the sense that for any incentive compatible and individually rational mechanism satisfying Equation (11) there is an outcome equivalent, incentive compatible, and individually rational mechanism satisfying Equation (12). The intuition for this is quite simple. As each of the players cares only about $E(t_i \mid v_i)$ the expected payment given his own type, it is possible to change the payments to achieve Equation (12) without changing the values $E(t_i \mid v_i)$.

Proposition 6. *Assume that (q, t) is a market mechanism which is ex ante budget balanced, i. e. $E(\sum_{i \in I} t_i) = 0$.*
Define a new payment rule s by

$$s_i(v) = E(t_i(v) \mid v_i) - \frac{\sum_{j \in B \cup S - \{i\}} E(t_j(v) \mid v_j)}{k_1 \cdot m + k_2 \cdot m - 1} + E(t_i(v)).$$

Then the mechanism (q, s) is ex post budget balanced (i.e. $\sum_{i \in I} s_i = 0$ a.s.) and

$$E(s_i(v) \mid v_i) = E(t_i(v) \mid v_i) \text{ a.s. for all } i \in B \cup S.$$

In particular, the mechanism (q, s) is individually rational and incentive compatible if and only if the mechanism (q, t) has the respective properties.

Proof. Straightforward. □

In the case of our trading mechanisms we can apply the proposition to the $VCG^m(R^m)$, where the fees where chosen to achieve budget balance. As a result, we obtain an incentive compatible, individually rational mechanism which is ex post budget balanced in the sense of Equation (12). The mechanism obtained in this way will no longer be implementable in dominant strategies.[25]

6. Conclusion

In most economic situations agents have some private information about their own preferences. The strategic use of this private information means that in many trading situations any incentive compatible, individually rational mechanisms will either run a deficit or be inefficient. Indeed, we saw in Proposition 4 that in our environment this will always be the case as long as the numbers of buyers and sellers is large enough.

[25] As a matter of fact an ex post budget balanced mechanism can not be implementable in dominant strategies. See Green and Laffont, 1977.

This paper tried to study the conflict between efficiency and budget balance using asymptotic results for markets involving many buyers and sellers. We introduced a new class of mechanisms, the Vickrey-Clarke-Groves mechanism with a fixed trading fee. Theorems 1 and 2 described how the expected deficit and inefficiency depend on the size of the fee. This allowed us to understand the trade off between revenue and efficiency for the Vickrey-Clarke-Groves mechanisms with a fixed trading fee. Then Theorem 3 proved that no other mechanisms can resolve this trade off in a way which is asymptotically superior. Using Theorems 1, 2 and 3 we were able to deduce results for two important special cases. Section 3 dealt with the problem of implementing the efficient allocation using a mechanism minimizing expected deficit. The corollaries of Section 4 dealt with the problem of finding a budget balanced mechanisms minimizing expected inefficiency.

What this paper did not attempt to do is to study the strategic use of private information from the point of view of a positive theorist. Our results are able to answer questions about inefficiencies connected with asymmetric information if the market mechanism happens to be constrained optimal. The question what market mechanisms we might expect in an unregulated competitive environment with buyers, sellers and intermediaries was not addressed here.

Appendix

A.1. **Notation and Conventions.** All results of this paper required that F and G are twice continuously differentiable in some neighborhood of p^* and that the densities $G'(p^*)$ and $F'(p^*)$ are positive. We will make this assumption throughout this appendix, without explicitly stating it each time in the following.

The appendix will make extensive use of conditional expectations. As those conditional expectations are defined up to a null event we will understand that any statements involving one or more conditional expectations are ment to hold almost surely. Assume for example that we say that

$$E(q^{VCG^m(R^m)} \mid v_i = p) \cdot m^2 < C_1 \text{ for all } p \leqslant p^* - \delta.$$

This means that there exists a function $Q : \mathbb{R} \to \mathbb{R}$ such that

$$Q(p) \cdot m^2 < C_1 \text{ for all } p \leqslant p^* - \delta$$

and $Q(v_i)$ is equal to $E(q^{VCG^m(R^m)}(v) \mid v_i)$ with probability one. The above convention will allow us to omit most 'a.s' in the following.

It will be useful to denote by α the value

$$\alpha = k_1 \cdot F(p^*) = k_2 \cdot (1 - G(p^*)).$$

Recall that for a fixed market size m the number $\alpha \cdot m$ is equal to expected demand and expected supply at price p^*.

In the following we will continue to use the convention that m always denotes the market size. In particular M^m will always be a mechanism for the environment of market size m, q^m always an allocation rule for the environment of market size m, etc.

A.2. **Localization.** In this subsection we will prove several results which will be used in the proofs of Theorems 1 and 2 to 'localize' our discussion around the point p^*.

Assume R^m is an arbitrary sequence of positive fees. Consider the mechanisms $VCG^m(R^m)$. We will introduce some notation similar to the one introduced in Section 3 to describe the VCG^m mechanism. First define $\tau^m(v)$ to be equal to the number of trades in mechanism $VCG^m(R^m)$, i.e.

$$\tau^m(v) = \sum_{i \in B} q_i^{VCG^m(R^m)}(v).$$

Next define $\overline{p}^m(v)$ to be the lowest valuation among the players who end up with a good under the mechanism $VCG^m(R^m)$, i.e.

$$\overline{p}^m(v) = \inf\{v_i^{(R^m)} : i \in B \cup S \text{ and } q^{VCG^m(R^m)}(v) > 0\}$$

and $\underline{p}^m(v)$ to be the highest valuation among the players who end up without a good under the mechanism $VCG^m(R^m)$, i. e.

$$\underline{p}^m(v) = \sup\{v_i^{(R^m)} : i \in B \cup S \text{ and } q^{VCG^m(R^m)}(v) < 1\}.$$

Note that, for a given $m \in \mathbb{N}$, if the $v_i^{(R^m)}$ valuations of all players are different then $\underline{p}^m$ and $\overline{p}^m$ are respectively equal to the $(k_1 \cdot m)$-th highest and $(k_1 \cdot m+1)$-th highest valuation $v_i^{(R^m)}$.

For each $m \in \mathbb{N}$ the definitions of τ^m, $\overline{p}^m$, and $\underline{p}^m$ depend on the fee R^m. In some situations it will be useful to explicitly state this dependence. In this cases we will write $\tau^{(R^m)}$, $\overline{p}^{(R^m)}$, and $\underline{p}^{(R^m)}$ instead of τ^m, $\overline{p}^m$, and $\underline{p}^m$.

Consider a fixed sequence of fees R^m such that $R^m \to 0$ as $m \to \infty$. The weak law of large numbers suggests that $\frac{\tau^m}{m}$ should converge in probability to α and both $\underline{p}^m$ and $\overline{p}^m$ should converge in probability to p^*. As a matter of fact we will show that the convergence is very fast. To do this we will use the following lemma.

Lemma 1 (Chernoff-Hoeffding additive bounds). *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with values in $\{0,1\}$. Then for any $\gamma > 0$,*

$$Pr(|\sum_{i=1}^{n} \frac{X_i}{n} - E(\sum_{i=1}^{n} \frac{X_i}{n})| > \gamma) < 2e^{-2\gamma^2 n}$$

Proof. See Theorem A.4 in Alon, Spencer, and Erdos (1992) page 235. Apply it once to $X_i - E(X_i)$ and once to $-(X_i - E(X_i))$. □

The following lemma is the main result of this subsection.

Lemma 2. *For any $\varepsilon > 0$ and $m \in \mathbb{N}$ define E_ε^m to be the event that $\underline{p}^m, \overline{p}^m \in [p^* - \varepsilon, p^* + \varepsilon]$ and $\frac{\tau^m}{m} \in [\alpha - \varepsilon, \alpha + \varepsilon]$.*
Then for any $r \in \mathbb{N}$, and $\varepsilon > 0$

$$m^r \cdot (1 - \Pr(E_\varepsilon^m)) \to 0$$

as $m \to \infty$.

Proof. Fix an $\varepsilon > 0$ and a $m \in \mathbb{N}$.

Choose a $\delta > 0$ small enough so that: (i) $\delta < \varepsilon$; (ii) $k_1 \cdot F(p^* - \delta) > \alpha - \varepsilon$ and $k_2 \cdot (1 - G(p^* + \delta)) > \alpha - \varepsilon$; (iii) $k_1 \cdot F(p^* + \delta) < \alpha + \varepsilon$ and $k_2 \cdot (1 - G(p^* - \delta)) < \alpha + \varepsilon$.

The remainder of the proof proceeds in five steps.

Step 1: For any $m \in \mathbb{N}$ consider the event

$$A^m = \{\underline{p}^m < p^* - \delta\}.$$

Define a^m and b^m, to be respectively the number of sellers and buyers with valuations $v_i^{(R^m)}$ smaller than $p^* - \delta$, i.e.

$$a^m = \sum_{i \in S} 1_{v_i \leqslant p^* - \delta} \quad \text{and} \quad b^m = \sum_{i \in B} 1_{v_i \leqslant p^* - \delta}.$$

Clearly a^m and b^m have binomial distributions with means $m \cdot k_1 \cdot F(p^* - \delta)$ and $m \cdot k_2 \cdot G(p^* - \delta + R^m)$ respectively. On the other hand the definition of A^m implies that

$$A^m = \{\frac{a^m + b^m}{m} \geqslant k_2\}. \tag{13}$$

Recall that $k_1 \cdot F(p^*) + k_2 \cdot G(p^*) = k_2$. As R^m converges to zero there exist a $\gamma > 0$ and a $m_o \in \mathbb{N}$ such that for all $m \geqslant m_o$

$$k_1 \cdot F(p^* - \delta) + k_2 \cdot G(p^* - \delta + R^m) < k_2 - 2\gamma.$$

Lemma 1 implies that $Pr(|\frac{a^m}{m} - k_1 \cdot F(p^* - \delta)| > \gamma) \leqslant 2e^{-2\gamma^2 m}$ and $Pr(|\frac{b^m}{m} - k_2 \cdot G(p^* - \delta + R^m)| > \gamma) \leqslant 2e^{-2\gamma^2 m}$. Thus $Pr(\frac{a^m + b^m}{m} \geqslant k_2) \leqslant 4e^{-2\gamma^2 m}$ for all $m \geq m_o$. But then Equation (13) implies that $Pr(A^m) \leqslant 4e^{-2\gamma^2 m}$ for all $m \geqslant m_o$. In particular,

$$\lim_{m\to\infty} \Pr(A^m) \cdot m^r = 0.$$

Step 2: An analogous reasoning as in Step 1 shows that if B^m is defined to be

$$B^m = \{\overline{p}^m > p^* + \delta\}$$

than $\lim_{m\to\infty} \Pr(B^m) \cdot m^r = 0$.

Step 3: Now define

$$C^m = \{\underline{p}^m > p^* - \delta \text{ and } \frac{\tau^m}{m} < \alpha - \varepsilon\}.$$

Notice that $\frac{\tau^m}{m} < \alpha - \varepsilon$ means that the numbers of sellers with valuations smaller than or equal to $\underline{p}^m$ is smaller than $(\alpha - \varepsilon) \cdot m$. But then $\underline{p}^m > p^* - \delta$ means that the number of sellers with valuations smaller then $p^* - \delta$ must also be smaller than $(\alpha - \varepsilon) \cdot m$. Therefore

$$C^m \subseteq \{\frac{a^m}{m} < \alpha - \varepsilon\}.$$

As $E(\frac{a_m}{m}) = k_1 \cdot F(p^* - \delta) > \alpha - \varepsilon$ we can again use Lemma 1 to show that

$$\lim_{m\to\infty} \Pr(A^m) \cdot m^r = 0.$$

Step 4: An analogous reasoning as in Step 3 shows that for

$$D^m = \{\overline{p}^m < p^* + \delta \text{ and } \frac{\tau^m}{m} > \alpha + \varepsilon\}$$

we have $\lim_{m\to\infty} \Pr(D^m) \cdot m^r = 0$.

Step 5: Note that the complement of the event E^m_ε is contained in $A^m \cup B^m \cup C^m \cup D^m$. In particular

$$\begin{aligned} (1 - \Pr(E^m_\varepsilon)) \cdot m^r &\leqslant \Pr(A^m \cup B^m \cup C^m \cup D^m) \cdot m^r \leqslant \\ &\leqslant (\Pr(A^m) + \Pr(B^m) + \Pr(C^m) + \Pr(D^m)) \cdot m^r. \end{aligned}$$

Steps 1 to 4 therefore imply that $\lim_{m\to\infty}(1 - \Pr(E^m_\varepsilon)) \cdot m^r = 0$. □

The following corollary describes an application of Lemma 2 which will be used extensively in the following.

Corollary 4. *Let χ_m be a sequence of functions such that for each $m \in \mathbb{N}$ the function χ_m assigns a real number $\chi_m(v)$ for each profile of valuations v in the environment of market size m. Assume moreover that there is a number r such that the functions $\frac{\chi_m}{m^r}$ are uniformly bounded for all $m \in \mathbb{N}$.*[26] *Then*

$$\limsup_{m\to\infty} E(\chi_m(v)) = \limsup_{m\to\infty} E(\chi_m(v) \,|\, E^m_\varepsilon)$$

[26]This means that $\exists_{K\in\mathbb{R}} \forall_{m\in\mathbb{N}} \forall_v \ |\frac{\chi_m(v)}{m^r}| < K$.

and

$$\liminf_{m\to\infty} E(\chi_m(v)) = \liminf_{m\to\infty} E(\chi_m(v) \mid E_\varepsilon^m).$$

Proof. Notice that $E(\chi_m(v))$ is equal to

$$\Pr(E_\varepsilon^m) \cdot E(\chi_m(v)|E_\varepsilon^m) + ((1 - \Pr(E_\varepsilon^m)) \cdot m^r) \cdot E(\frac{\chi_m(v)}{m^r}|\neg E_\varepsilon^m).$$

The statement of the corollary follows from Lemma 2 as the sequence of expected values $E(\frac{\chi_m(v)}{m^r}|\neg E_\varepsilon^m)$ is bounded. □

As a first application of Corollary 4 we prove the following result which will be used in the proof of Theorem 2.

Corollary 5. *Assume $\tilde{F}$ and $\tilde{G}$ is a pair of distributions with bounded supports such that $\tilde{F}$ and $\tilde{G}$ are equal to F and G in some neighborhood of p^*.*

Consider the environment where the valuations of buyers and sellers are distributed respectively according to $\tilde{F}$ and $\tilde{G}$. Compare this new environment with the original environment where the valuations of buyers and sellers are distributed respectively according to F and G.

Then the values

$$\limsup_{m\to\infty} \quad I(VCG^m(R^m)) \cdot m \in [-\infty, +\infty],$$
$$\liminf_{m\to\infty} \quad I(VCG^m(R^m)) \cdot m \in [-\infty, +\infty]$$

are the same in both environments.

Proof. We will organize the proof in several steps.

Step 1: Until now we were never concerned with the way the random valuations of the players are generated. The reason for this is, of course, that it did not matter as long as the valuations had a given joint distribution.

To compare the two environments in the proposition it will be, however, useful to choose a particular probabilistic structure. Let x_i for $i \in B \cup S$ be independent random variables uniformly distributed on $(0, 1)$. Define

$$v_i(x_i) = \begin{cases} \inf\{y : F(y) \geqslant x_i\} & \text{for } i \in S \\ \inf\{y : G(y) \geqslant x_i\} & \text{for } i \in B \end{cases}$$

and

$$\tilde{v}_i(x_i) = \begin{cases} \inf\{y : \tilde{F}(y) \geqslant x_i\} & \text{for } i \in S \\ \inf\{y : \tilde{G}(y) \geqslant x_i\} & \text{for } i \in B. \end{cases}$$

Note that $v_i(x_i)$ are independent and distributed according to F (for $i \in S$) and G (for $i \in B$). Similarly $\tilde{v}_i(x_i)$ are independent and distributed according to $\tilde{F}$ (for $i \in S$) and $\tilde{G}$ (for $i \in B$). For the rest of the proof we will assume that the random valuations v_i and $\tilde{v}_i$ are given by $v_i(x_i)$ and $\tilde{v}_i(x_i)$ respectively.

Step 2: Choose an $\varepsilon > 0$ small enough so that (i) $\tilde{F}$ and $\tilde{G}$ are respectively equal to F and G on the interval $[p^* - 3\varepsilon, p^* + 3\varepsilon]$, (ii) F and G are differentiable on the interval $[p^* - 3\varepsilon, p^* + 3\varepsilon]$ (iii) F' and G' are positive in the interval $[p^* - 3\varepsilon, p^* + 3\varepsilon]$. Next, choose a number m_o such that for all $m \geqslant m_o$ the fees R_m are smaller then ε.

Consider a player $i \in B \cup S$ such that $v_i^{(R^m)}(x) \in [p^* - 2\cdot\varepsilon, p^* + 2\cdot\varepsilon]$. Note that $v_i^{(R^m)}(x) \in [p^* - 2\cdot\varepsilon, p^* + 2\cdot\varepsilon]$ implies that $v_i(x) \in [p^* - 3\cdot\varepsilon, p^* + 3\cdot\varepsilon]$. As (i)-(iii) imply that on $[p^* - 3\varepsilon, p^* + 3\varepsilon]$ the functions F and G are increasing and respectively equal to $\tilde{F}$ and $\tilde{G}$ we can conclude that $v_i(x) = \tilde{v}_i(x)$.

An analogous reasoning shows that $\tilde{v}_i^{(R^m)}(x) \in [p^* - 2\cdot\varepsilon, p^* + 2\cdot\varepsilon]$ implies that $v_i(x) = \tilde{v}_i(x)$.

Step 3: Fix a market size $m \geqslant m_o$. Consider a fixed profile of valuations $v(x)$. Assume $i \in B \cup S$ is a player such that $v_i^{(R^m)}(x) \in [p^* - 2\cdot\varepsilon, p^* + 2\cdot\varepsilon]$. We claim that

$$\forall_{j \in B\cup S} \quad v_i^{(R^m)}(x_i) < v_j^{(R^m)}(x_i) \quad \Leftrightarrow \quad \tilde{v}_i^{(R^m)}(x_i) < \tilde{v}_j^{(R^m)}(x_i) \tag{14}$$

$$\forall_{j \in B\cup S} \quad v_i^{(R^m)}(x_i) > v_j^{(R^m)}(x_i) \quad \Leftrightarrow \quad \tilde{v}_i^{(R^m)}(x_i) > \tilde{v}_j^{(R^m)}(x_i). \tag{15}$$

To see that this is true consider the case where i is a buyer.[27] We will prove (14), the proof of (15) is analogous. Note that the fact that F is strictly increasing and equal to $\tilde{F}$ on $[p^* - 3\cdot\varepsilon, p^* + 3\cdot\varepsilon]$ means that for $p \in [p^* - 3\cdot\varepsilon, p^* + 3\cdot\varepsilon]$ there is a well defined inverse $F^{-1}(p) = \tilde{F}^{-1}(p)$. Thus, we can use Step 2 to conclude that $F^{-1}(v_i^{(R^m)}(x)) = \tilde{F}^{-1}(\tilde{v}_i^{(R^m)}(x))$.

To prove (14) consider first the case where $j \in B \cup S$ is another buyer. Then $v_i^{(R^m)}(x_i) < v_j^{(R^m)}(x_j)$ is equivalent to $v_i(x_i) < v_j(x_i)$. As the distribution G is strictly increasing in $[p^* - 3\cdot\varepsilon, p^* + 3\cdot\varepsilon]$ the definitions of Step 1 imply that $v_i(x_i) < v_j(x_j)$ is equivalent to $x_i < x_j$. An analogous reasoning proves that $\tilde{v}_i(x_i) < \tilde{v}_j(x_j)$ is equivalent to $x_i < x_j$. Thus (14) holds for this case.

Consider now the case where $j \in B \cup S$ is a seller. This means that $v_j^{(R^m)}(x_j) = v_j(x_j)$. As F is strictly increasing on $[p^* - 3\cdot\varepsilon, p^* + 3\cdot\varepsilon]$ the definitions of Step 1 imply that $v_i(x_i) < v_j(x_j)$ if and only if $F^{-1}(v_i(x_i)) < x_j$. Similarly $\tilde{v}_i(x_i) < \tilde{v}_j(x_j)$ if and only if $\tilde{F}^{-1}(\tilde{v}_i(x_i)) < x_j$. As we have already argued that $F^{-1}(v_i^{(R^m)}(x)) = \tilde{F}^{-1}(\tilde{v}_i^{(R^m)}(x))$ the statement follows also for this case.

Step 4: Recall that $q^{VCG^m(R^m)}$ assigns the goods to the players with the highest valuations $v_i^{(R^m)}$ (in the original environment) or $\tilde{v}_i^{(R^m)}$ (in the new environment). Assume $\underline{p}^m(v(x)), \overline{p}^m(v(x)) \in [p^* - 2\cdot\varepsilon, p^* + 2\cdot\varepsilon]$. Then Step 2 and 3 imply that $\underline{p}^m(v(x)) = \underline{p}^m(\tilde{v}(x))$, $\overline{p}^m(v(x)) = \overline{p}^m(\tilde{v}(x))$, and $\tau^m(v(x)) = \tau^m(\tilde{v}(x))$.

Step 5: Define the event $\tilde{E}_\varepsilon^m$ in the new environment analogous to E_ε^m in the original environment. Step 4 implies that $E_\varepsilon^m \subseteq \widetilde{E_\varepsilon^m}$. As the role of the two environments is totally symmetric an analogous reasoning proves that $\widetilde{E_\varepsilon^m} \subseteq E_\varepsilon^m$. Thus $\widetilde{E_\varepsilon^m} = E_\varepsilon^m$.

Step 6: Recall that

$$I(M) = E\Big(\sum_{i \in B\cup S} (q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i\Big).$$

Note that for $m \geqslant m_o$ the expected value

$$E\Big(\sum_{i \in B\cup S} (q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i \,|\, E_\varepsilon^m\Big)$$

[27] The argument for the case where i is a seller is similar.

is equal to

$$= E(\sum_{i\in B\cup S}(q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i \cdot 1_{v_i\in[p^*-2\varepsilon,p^*+2\varepsilon]} \,|\, E_\varepsilon^m).$$

But by Step 4 and Step 5 the last expression does not depend on the choice of the environment. Thus

$$\limsup_{m\to\infty} m \cdot E(\sum_{i\in B\cup S}(q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i \,|\, E_\varepsilon^m)$$

and

$$\liminf_{m\to\infty} m \cdot E(\sum_{i\in B\cup S}(q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i \,|\, E_\varepsilon^m)$$

also do not depend on the choice of the environment. Corollary 4 however implies that in both environments

$$\limsup_{m\to\infty} m \cdot E(\sum_{i\in B\cup S}(q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i) =$$

$$= \limsup_{m\to\infty} m \cdot E(\sum_{i\in B\cup S}(q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i \,|\, E_\varepsilon^m)$$

and

$$\liminf_{m\to\infty} m \cdot E(\sum_{i\in B\cup S}(q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i) =$$

$$= \liminf_{m\to\infty} m \cdot E(\sum_{i\in B\cup S}(q_i^{VCG^m(0)} - q_i^{VCG^m(R^m)}) \cdot v_i \,|\, E_\varepsilon^m).$$

Thus

$$\limsup_{m\to\infty} I(VCG^m(R^m)) \cdot m \quad \text{and} \quad \liminf_{m\to\infty} I(VCG^m(R^m)) \cdot m$$

do not depend on the environment. □

A.3. **Proof of Theorem 1.** Consider an $\varepsilon > 0$ small enough so that F and G are twice continuously differentiable in the interval $[p^* - \varepsilon, p^* + \varepsilon]$ and F' and G' are positive in $[p^* - \varepsilon, p^* + \varepsilon]$. Define τ^m, $\overline{p}^m$, $\underline{p}^m$ and E_ε^m for all $m \in \mathbb{N}$ as in the last subsection.

Notice that the definition of the transfer payment rule in Section 4.2 implies that the expected deficit is equal to

$$\text{Exp. Deficit}(VCG^m(R^m)) = E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)}) = \tag{16}$$

$$= E(\tau^m \cdot (\min(\overline{p}^m, \overline{v}^s) - \max(\underline{p}^m, \underline{v}_b - R^m) - R^m),$$

where $\overline{v}^s$ is the highest valuation in the support of F and $\underline{v}_b$ is the lowest valuation in the support of G. The definition of E_ε^m together with the above formula implies that for large enough m

$$(\alpha - \varepsilon) \cdot E(m \cdot (\overline{p}^m - \underline{p}^m) \,|\, E_\varepsilon^m) - (\alpha + \varepsilon) \cdot m \cdot R^m \leqslant$$

$$\leqslant E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)} \,|\, E_\varepsilon^m) \leqslant$$

$$\leqslant (\alpha + \varepsilon) \cdot E(m \cdot (\overline{p}^m - \underline{p}^m) \,|\, E_\varepsilon^m) - (\alpha - \varepsilon) \cdot m \cdot R^m.$$

The above together with $\lim_{m\to\infty} m \cdot R^m = \rho$ implies that

$$(\alpha - \varepsilon) \cdot \liminf_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m) \,|\, E_\varepsilon^m) - (\alpha + \varepsilon) \cdot \rho \leqslant$$
$$\leqslant \liminf_{m\to\infty} E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)} \,|\, E_\varepsilon^m) \leqslant \limsup_{m\to\infty} E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)} \,|\, E_\varepsilon^m) \leqslant$$
$$\leqslant (\alpha - \varepsilon) \cdot \limsup_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m) \,|\, E_\varepsilon^m) - (\alpha + \varepsilon) \cdot \rho.$$

Using Corollary 4 we get that

$$(\alpha - \varepsilon) \cdot \liminf_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m)) - (\alpha + \varepsilon) \cdot \rho \leqslant$$
$$\leqslant \liminf_{m\to\infty} E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)}) \leqslant \limsup_{m\to\infty} E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)}) \leqslant$$
$$\leqslant (\alpha - \varepsilon) \cdot \limsup_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m)) - (\alpha + \varepsilon) \cdot \rho.$$

As the above holds for arbitrary small $\varepsilon > 0$ we can conclude that

$$\alpha \cdot \liminf_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m)) - \alpha \cdot \rho \leqslant$$
$$\leqslant \liminf_{m\to\infty} E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)}) \leqslant \limsup_{m\to\infty} E(\sum_{i\in B\cup S} t_i^{VCG^m(R^m)}) \leqslant \tag{17}$$
$$\leqslant \alpha \cdot \limsup_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m)) - \alpha \cdot \rho.$$

We will prove the following lemma.

Lemma 3. *Under the assumptions of the theorem* $\lim_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m))$ *exists and*

$$\lim_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m)) = \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$$

Before we prove Lemma 3 notice that Lemma 3 together with Formula (17) implies that the expected deficit of the $VCG^m(R^m)$ converges to

$$\alpha \cdot \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} - \alpha \cdot \rho.$$

as m goes to infinity.

Q.E.D.

Proof of Lemma 3. An informal argument for this result was given in section 3.3. Unfortunately the formalized version of this argument is rather tedious and will therefore be organized in several steps.

Step 1: For any $\varepsilon \geqslant 0$ small enough so that F and G are differentiable at all points in the interval $[p^* - \varepsilon, p^* + \varepsilon]$ define

$$\Delta_\varepsilon = \sup_{p\in[p^*-\varepsilon, p^*+\varepsilon]} \max(|F'(p) - F'(p^*)|, |G'(p) - G'(p^*)|).$$

As F and G are twice continuously differentiable in some neighborhood of p^* it is clear that the function $\Delta_{(\cdot)}$ is continuous at zero and $\lim_{\varepsilon\to 0} \Delta_\varepsilon = 0$.

Choose an $\varepsilon > 0$ small enough so that (i) F and G are differentiable in the interval $[p^* - 5 \cdot \varepsilon, p^* + 5 \cdot \varepsilon]$; (ii) $\Delta_{5\cdot\varepsilon} < \min(F'(p^*), G'(p^*))$.

Step 2: Recall that $\overline{p}^m$ is the lowest valuation out of the τ^m buyers and the $k_1 \cdot m - \tau^m$ sellers with valuations higher than $\underline{p}^m$. This means that $m \cdot (\overline{p}^m - \underline{p}^m)$ conditional on $\underline{p}^m$ and $\frac{\tau^m}{m}$ has the distribution given by:

$$\Pr(m \cdot (\overline{p}^m - \underline{p}^m) \leqslant x \,|\, \underline{p}^m = p, \frac{\tau^m}{m} = a) =$$

$$= 1 - \Pr(m \cdot (\overline{p}^m - \underline{p}^m) \geqslant x \,|\, \underline{p}^m = p, \frac{\tau^m}{m} = a) =$$

$$= 1 - \frac{(1 - F(p + \frac{x}{m}))^{m \cdot (k_1 - a)}}{(1 - F(p))^{m \cdot (k_1 - a)}} \frac{(1 - G(p + \frac{x}{m} + R^m))^{m \cdot a}}{(1 - G(p + R^m))^{m \cdot a}} = \tag{18}$$

$$= 1 - (1 - \frac{F(p + \frac{x}{m}) - F(p)}{1 - F(p)})^{m \cdot (k_1 - a)} \cdot (1 - \frac{G(p + \frac{x}{m} + R^m) - G(p + R^m)}{1 - G(p + R^m)})^{m \cdot a}$$

Step 3: Choose an m_o such that for all numbers $m \geqslant m_o$ the fees R^m are smaller then ε.

For any $m \geqslant m_o$ define the distribution functions $K_\varepsilon^m, L_\varepsilon^m : \mathbb{R} \to \mathbb{R}$ by

$$K_\varepsilon^m(x) = 1 - (1 - \frac{x \cdot (F'(p^*) + \Delta_{5 \cdot \varepsilon})}{m \cdot (1 - F(p^* + \varepsilon))})^{m \cdot (k_1 - \alpha + \varepsilon)} \cdot (1 - \frac{x \cdot (G'(p^*) + \Delta_{5 \cdot \varepsilon})}{m \cdot (1 - G(p^* + 2 \cdot \varepsilon)))})^{m \cdot (\alpha + \varepsilon)}$$

and

$$L_\varepsilon^m(x) = 1 - (1 - \frac{x \cdot (F'(p^*) - \Delta_{5 \cdot \varepsilon})}{m \cdot (1 - F(p^* - \varepsilon))})^{m \cdot (k_1 - \alpha - \varepsilon)} \cdot (1 - \frac{x \cdot (G'(p^*) - \Delta_{5 \cdot \varepsilon})}{m \cdot (1 - G(p^* - 2 \cdot \varepsilon)))})^{m \cdot (\alpha - \varepsilon)}$$

respectively.

Using Equation (18) it is trivial to verify that for $p \in [p^* - \varepsilon, p^* + \varepsilon]$, $a \in [\alpha - \varepsilon, \alpha + \varepsilon]$ and $x \leq 2 \cdot \varepsilon \cdot m$

$$K_\varepsilon^m(x) \leqslant \Pr(m \cdot (\overline{p}^m - \underline{p}^m) \geqslant x \,|\, \underline{p}^m = p, \frac{\tau^m}{m} = a) \leqslant L_\varepsilon^m(x).$$

This equality will be exploited in Step 5.

Step 4: Define K_ε^∞ and L_ε^∞ to be exponential distributions with means

$$\frac{1}{\frac{(F'(p^*) + \Delta_{5 \cdot \varepsilon}) \cdot (k_1 - \alpha + \varepsilon)}{1 - F(p^* + \varepsilon)} + \frac{(G'(p^*) + \Delta_{5 \cdot \varepsilon}) \cdot (\alpha + \varepsilon)}{1 - G(p^* + 2 \cdot \varepsilon)}}$$

and

$$\frac{1}{\frac{(F'(p^*) - \Delta_{5 \cdot \varepsilon}) \cdot (k_1 - \alpha - \varepsilon)}{1 - F(p^* - \varepsilon)} + \frac{(G'(p^*) + \Delta_{5 \cdot \varepsilon}) \cdot (\alpha - \varepsilon)}{1 - G(p^* - 2 \cdot \varepsilon)}}$$

respectively, i.e.

$$K_\varepsilon^\infty(x) = 1 - e^{-x \cdot (\frac{(F'(p^*) + \Delta_{5 \cdot \varepsilon}) \cdot (k_1 - \alpha + \varepsilon)}{1 - F(p^* + \varepsilon)} + \frac{(G'(p^*) + \Delta_{5 \cdot \varepsilon}) \cdot (\alpha + \varepsilon)}{1 - G(p^* + 2 \cdot \varepsilon)})}$$

and

$$L_\varepsilon^\infty(x) = 1 - e^{-x \cdot (\frac{(F'(p^*) - \Delta_{5 \cdot \varepsilon}) \cdot (k_1 - \alpha - \varepsilon)}{1 - F(p^* - \varepsilon)} + \frac{(G'(p^*) + \Delta_{5 \cdot \varepsilon}) \cdot (\alpha - \varepsilon)}{1 - G(p^* - 2 \cdot \varepsilon)})}.$$

Using the definitions of K_ε^m and L_ε^m it is routine[28] to verify that for any $x \geqslant 0$

$$K_\varepsilon^m(x) \underset{m \to \infty}{\longrightarrow} K_\varepsilon^\infty(x) \text{ and } L_\varepsilon^m(x) \underset{m \to \infty}{\longrightarrow} L_\varepsilon^\infty(x).$$

[28]Use the fact that for any $x \in \mathbb{R}$ the sequence $(1 - \frac{x}{m})^m$ converges (monotonically) to e^{-x} as m goes to infinity.

In other words K_ε^m and L_ε^m converge in distribution to K_ε^∞ and L_ε^∞, as m goes to infinity.

Step 5: Consider the distribution of $m \cdot (\overline{p}^m - \underline{p}^m)$ conditional on $\underline{p}^m$, $\frac{\tau^m}{m}$ and E_ε^m. Note that

$$\Pr(m \cdot (\overline{p}^m - \underline{p}^m) \leqslant x \,|\, E_\varepsilon^m,\ \underline{p}^m = p, \frac{\tau^m}{m} = a) =$$

$$= 1 - \Pr(m \cdot (\overline{p}^m - \underline{p}^m) \geqslant x \,|\, p^* + \varepsilon > \overline{p}^m,\ \underline{p}^m = p, \frac{\tau^m}{m} = a) =$$

$$= 1 - \frac{\Pr(m \cdot (p^* + \varepsilon - p) \geqslant m \cdot (\overline{p}^m - \underline{p}^m) \geqslant x \,|\, \underline{p}^m = p, \frac{\tau^m}{m} = a)}{\Pr(m \cdot (p^* + \varepsilon - p) \geqslant m \cdot (\overline{p}^m - \underline{p}^m) \,|\, \underline{p}^m = p, \frac{\tau^m}{m} = a)} =$$

$$= 1 - \max(1 - \frac{\Pr(m \cdot (\overline{p}^m - \underline{p}^m) \leqslant x \,|\, \underline{p}^m = p, \frac{\tau^m}{m} = a)}{\Pr(m \cdot (\overline{p}^m - \underline{p}^m) \leqslant m \cdot (p^* + \varepsilon - p) \,|\, \underline{p}^m = p, \frac{\tau^m}{m} = a)}, 0).$$

The above equality together with the inequalities from Step 4 imply that for $p \in [p^* - \varepsilon, p^* + \varepsilon]$, $a \in [\alpha - \varepsilon, \alpha + \varepsilon]$ and $\frac{x}{m} \leqslant 2 \cdot \varepsilon$

$$1 - \max(1 - \frac{K_\varepsilon^m(x)}{L_\varepsilon^m(m \cdot (p^* + \varepsilon - p))}, 0) \overset{(*)}{\geqslant}$$

$$\overset{(*)}{\geqslant} \Pr(m \cdot (\overline{p}^m - \underline{p}^m) \leqslant x \,|\, E_\varepsilon^m,\ \underline{p}^m = p, \frac{\tau^m}{m} = a) \overset{(**)}{\geqslant} \tag{19}$$

$$\overset{(**)}{\geqslant} 1 - \max(1 - \frac{L_\varepsilon^m(x)}{K_\varepsilon^m(m \cdot (p^* + \varepsilon - p))}, 0).$$

We will show that (**) holds also if $p \in [p^* - \varepsilon, p^* + \varepsilon]$, $a \in [\alpha - \varepsilon, \alpha + \varepsilon]$ but $\frac{x}{m} \geq 2 \cdot \varepsilon$? Notice that in this case $x \geq 2 \cdot \varepsilon \cdot m \geqslant p^* + \varepsilon - p$. This has several consequences. First

$$\Pr(m \cdot (\overline{p}^m - \underline{p}^m) \leqslant x \,|\, E_\varepsilon^m,\ \underline{p}^m = p, \frac{\tau^m}{m} = a) = 1$$

Secondly

$$K_\varepsilon^m(m \cdot (p^* + \varepsilon - p)) \geqslant K_\varepsilon^m(x) \geqslant L_\varepsilon^m(x).$$

and therefore

$$1 - \frac{L_\varepsilon^m(x)}{K_\varepsilon^m(m \cdot (p^* + \varepsilon - p))} \geq 0.$$

Thus for $p \in [p^* - \varepsilon, p^* + \varepsilon]$, $a \in [\alpha - \varepsilon, \alpha + \varepsilon]$ Equation (**) holds even if $\frac{x}{m} \geq 2 \cdot \varepsilon$.

Define for $m \geqslant m_o$ the distribution functions $S_\varepsilon^m, T_\varepsilon^m : \mathbb{R} \to \mathbb{R}$ by

$$S_\varepsilon^m(x) = \begin{cases} 1 & \text{if } \frac{x}{m} \geq 2 \cdot \varepsilon \\ 1 - \max(1 - \frac{1 - K_\varepsilon^m(x)}{1 - L_\varepsilon^m(m \cdot (p^* + \varepsilon - p))}, 0) & \text{otherwise} \end{cases}$$

and

$$T_\varepsilon^m(x) = 1 - \max(1 - \frac{1 - L_\varepsilon^m(x)}{1 - K_\varepsilon^m(m \cdot (p^* + \varepsilon - p))}, 0).$$

The results of this step can now be summarized in the observation that for $p \in [p^* - \varepsilon, p^* + \varepsilon]$, $a \in [\alpha - \varepsilon, \alpha + \varepsilon]$ and $x \geq 0$

$$S_\varepsilon^m(x) \geqslant \Pr(m \cdot (\overline{p}^m - \underline{p}^m) \leqslant x \,|\, E_\varepsilon^m,\ \underline{p}^m = p, \frac{\tau^m}{m} = a) \geqslant T_\varepsilon^m(x)$$

In other words, for any $p \in [p^* - \varepsilon, p^* + \varepsilon]$ and $a \in [\alpha - \varepsilon, \alpha + \varepsilon]$ the distribution $m \cdot (\overline{p}^m - \underline{p}^m)$ conditional on E_ε^m, $\underline{p}^m = p$ and $\frac{\tau^m}{m} = a$ stochastically dominates S_ε^m, but is itself stochastically dominated by T_ε^m.

Before we can use the above first order stochastic dominance relationship we need to establish a uniform bound on the distributions T_ε^m.

Step 6: Notice that the definition of T_ε^m implies that

$$T_\varepsilon^m(x) \geqslant 1 - (1 - L_\varepsilon^m(x)) = L_\varepsilon^m(x).$$

The definition of L_ε^m implies that for any fixed x the sequence $L_\varepsilon^m(x)$ is decreasing. In particular, $L_\varepsilon^m(x) \geqslant \lim_{m\to\infty} L_\varepsilon^m(x) = L_\varepsilon^\infty(x)$.

Therefore we can conclude that

$$T_\varepsilon^m(x) \geqslant L_\varepsilon^\infty(x).$$

for all $m \geqslant m_o$.

Step 7: Let s_ε^m and t_ε^m be respectively the mean of the distributions T_ε^m and S_ε^m.

Step 5 implies that

$$s_\varepsilon^m(x) \leqslant E(m \cdot (\overline{p}^m - \underline{p}^m) \,|\, E_\varepsilon^m,\ \underline{p}^m = p, \frac{\tau^m}{m} = a) \leqslant t_\varepsilon^m(x).$$

As, by the law of iterated expectations, the expected value $E(m \cdot (\overline{p}^m - \underline{p}^m))$ is equal to

$$E(E(m \cdot (\overline{p}^m - \underline{p}^m) \geqslant x \,|\, E_\varepsilon^m,\ \underline{p}^m = p, \frac{\tau^m}{m} = a))$$

we can conclude that

$$s_\varepsilon^m(x) \leqslant E(m \cdot (\overline{p}^m - \underline{p}^m)) \leqslant t_\varepsilon^m(x).$$

In particular,

$$\begin{aligned}
(20) \qquad & \liminf_{m\to\infty} s_\varepsilon^m(x) \leqslant \liminf_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m)) \leqslant \\
& \leqslant \limsup_{m\to\infty} E(m \cdot (\overline{p}^m - \underline{p}^m)) \leqslant \limsup_{m\to\infty} t_\varepsilon^m(x).
\end{aligned}$$

The definitions of S_ε^m and T_ε^m in Step 5 imply immediately that for any x

$$\lim_{m\to\infty} S_\varepsilon^m(x) = \lim_{m\to\infty} K_\varepsilon^m(x) = K_\varepsilon^\infty(x)$$

and

$$\lim_{m\to\infty} T_\varepsilon^m(x) = \lim_{m\to\infty} L_\varepsilon^m(x) = L_\varepsilon^\infty(x)$$

In other words S_ε^m and T_ε^m converge in distribution to $K_\varepsilon^\infty(x)$ and $L_\varepsilon^\infty(x)$.

As we established in Step 6 that the distributions S_ε^m and T_ε^m are uniformly dominated by L_ε^∞, which is an exponential distribution and therefore integrable we can use the Dominated Convergence Theorem to conclude that

$$\lim_{m\to\infty} s_\varepsilon^m = \frac{1}{\frac{(F'(p^*)+\Delta_{5\cdot\varepsilon})\cdot(k_1-\alpha+\varepsilon)}{1-F(p^*+\varepsilon)} + \frac{(G'(p^*)+\Delta_{5\cdot\varepsilon})\cdot(\alpha+\varepsilon)}{1-G(p^*+2\cdot\varepsilon)}}$$

and

$$\lim_{m\to\infty} t_\varepsilon^m = \frac{1}{\frac{(F'(p^*)-\Delta_{5\cdot\varepsilon})\cdot(k_1-\alpha-\varepsilon)}{1-F(p^*-\varepsilon)} + \frac{(G'(p^*)+\Delta_{5\cdot\varepsilon})\cdot(\alpha-\varepsilon)}{1-G(p^*-2\cdot\varepsilon)}}$$

Plugging in the above expressions in Equation (20) we get that

$$\frac{1}{\frac{(F'(p^*)+\Delta_{5\cdot\varepsilon})\cdot(k_1-\alpha+\varepsilon)}{1-F(p^*+\varepsilon)}+\frac{(G'(p^*)+\Delta_{5\cdot\varepsilon})\cdot(\alpha+\varepsilon)}{1-G(p^*+2\cdot\varepsilon)}} \leqslant$$
$$\leqslant \liminf_{m\to\infty} E(m\cdot(\overline{p}^m-\underline{p}^m)) \leqslant \limsup_{m\to\infty} E(m\cdot(\overline{p}^m-\underline{p}^m)) \leqslant$$
$$\leqslant \frac{1}{\frac{(F'(p^*)-\Delta_{5\cdot\varepsilon})\cdot(k_1-\alpha-\varepsilon)}{1-F(p^*-\varepsilon)}+\frac{(G'(p^*)+\Delta_{5\cdot\varepsilon})\cdot(\alpha-\varepsilon)}{1-G(p^*-2\cdot\varepsilon)}}$$

As the above analysis holds for arbitrary small $\varepsilon>0$ we can conclude that the limit $\lim_{m\to\infty} E(m\cdot(\overline{p}^m-\underline{p}^m))$ exists and is equal to

$$\lim_{m\to\infty} E(m\cdot(\overline{p}^m-\underline{p}^m)) = \frac{1}{\frac{(F'(p^*))\cdot(k_1-\alpha)}{1-F(p^*)}+\frac{(G'(p^*))\cdot\alpha}{1-G(p^*)}}$$

□

A.4. **Proof of Theorem 2.** By Corollary 5 we can assume without loss of generality that: (i) the supports of the distributions F and G are identical and equal to some interval $[\underline{v},\bar{v}]$; (ii) F and G are continuous on $[\underline{v},\bar{v}]$ and twice continuously differentiable on $(\underline{v},\bar{v})$; (iii) $\sup_{p\in(\underline{v},\bar{v})} F'(p)<\infty$ and $\sup_{p\in(\underline{v},\bar{v})} G'(p)<\infty$; (iv) $\sup_{p\in(\underline{v},\bar{v})} \frac{G'(p)}{1-G(p)}<\infty$.

Consider a fixed market size $m\in\mathbb{N}$. Recall that the definition of $I(VCG^m(R))$ implies that

$$I(VCG^m(R)) = E(\sum_{i\in B\cup S} q^{VCG^m(0)}(v)\cdot v_i - \sum_{i\in B\cup S} q^{VCG^m(R^m)}(v)\cdot v_i).$$

Notice that for any $R\in\mathbb{R}$:

$$\sum_{i\in B\cup S} q^{VCG^m(R)}(v)\cdot v_i = \sum_{i\in B\cup S} q^{VCG^m(R)}(v)\cdot v_i^{(R)} + \tau^{(R)}\cdot R,$$

where $\tau^{(R)}$ stands for the number of trades in the $VCG^m(R)$.

Recall that $q^{VCG^m(R)}$ is the allocation rule maximizing $\sum_{i\in B\cup S} q(v)\cdot v_i^{(R)}$. As the distributions F and G are continuously differentiable $E(\sum_{i\in B\cup S} q^{VCG^m(R)}(v)\cdot v_i)$ is differentiable. Thus the envelope theorem implies that

$$\frac{d}{dR}E(\sum_{i\in B\cup S} q^{VCG^m(R)}(v)\cdot v_i) = R\cdot\frac{d}{dR}E(\tau^{(R)}).$$

Therefore the fundamental theorem of calculus implies that

$$\begin{aligned} I(VCG^m(R^m)) &= m\cdot E(\sum_{i\in B\cup S} q^{VCG^m(0)}\cdot v_i) - E(\sum_{i\in B\cup S} q^{VCG^m(R^m)}\cdot v_i) \\ &= -\int_{r\in[0,R^m]} r\cdot m\cdot\frac{d}{dR}E(\tau^{(r)})\cdot dr \qquad (21) \\ &= -\int_{s\in[0,m\cdot R^m]} \frac{s}{m}\cdot\frac{d}{dR}E(\tau^{(\frac{s}{m})})\cdot ds. \end{aligned}$$

Fix a value $s\in\mathbb{R}$ and consider the mechanisms $VCG^m(\frac{s}{m})$. Note that, as the distributions F and G are continuous, the event that two players have the same valuation has probability zero. In particular, with probability 1 there will be a unique player which valuation is equal to $\underline{p}^{VCG^m(R^m)}$. For any market size m let $\underline{i}^m \in B \cup S$ be a random variable such that $v_{\underline{i}^m} = \underline{p}^{VCG^m(\frac{s}{m})}$ with probability 1. By the above remark $\underline{i}^m$ is determined up to a set of probability zero.

We will need the following two lemmas.

Lemma 4. *For any* $m \in \mathbb{N}$ *and* $\frac{s}{m} \in \mathbb{R}$

$$\frac{1}{m}\frac{d}{dR}E(\tau^{(R)} \mid \frac{\tau^{(\frac{s}{m})}}{m}, \underline{p}^m)_{R=\frac{s}{m}} = \begin{cases} -\frac{\Pr(\underline{i}^m \in S \mid \frac{\tau^{(\frac{s}{m})}}{m}, \underline{p}^m)\cdot a \cdot G'(\underline{p}+\frac{s}{m})}{(1-G(\underline{p}+\frac{s}{m}))} & \text{if } \underline{p}^m + \frac{s}{m} < \overline{v} \\ 0 & \text{if } \underline{p}^m + \frac{s}{m} \geqslant \overline{v} \end{cases}$$

Proof. Note that

$$\frac{d}{dR}E(\tau^{(R)} \mid \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p) =$$
$$= -\lim_{h \to 0}\frac{1}{h}E(\tau^{(R)} - \tau^{(R+h)} \mid \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p)$$

As $\tau^{(R)} - \tau^{(R+h)}$ can take only values in $\{0, 1, 2, \ldots, \min(k_1 \cdot m, k_2 \cdot m)\}$ we can rewrite the above expression as

$$= -\lim_{h \to 0}\frac{1}{h}\sum_{n=0}^{k_1 \cdot m} n \cdot \Pr(\tau^{(R)} - \tau^{(R+h)} = n \mid \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p) =$$

and exchange the (finite) sum and the limit

$$= -\sum_{n=0}^{k_1 \cdot m} n \cdot \lim_{h \to 0}\frac{\Pr(\tau^{(R)} - \tau^{(R+h)} = n \mid \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p)}{h}. \tag{22}$$

Consider $\Pr(\tau^{(R+h)} - \tau^{(R)} = n \mid \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p)$ for some fixed $n \in \mathbb{N}$. For the number of trades to decrease by n it must be that there are n buyers who get the good under the fee R, but do not get the good under the fee $R^m + h$. Of course there must be instead n sellers who get the good under the fee $R^m + h$ and did not get the good under the fee R. As the distributions F and G are continuous we can assume that there is just one player which valuation is equal to $\underline{p}$. Then a player $i \in B \cup S$ gets a good under the fee R^m if and only if his valuation $v_i^{R^m}$ is strictly bigger then $\underline{p}^m$. Recall that an in increase in the fee R meant that the valuations v_i^R of all buyers decrease by h while the valuations of sellers remain constant. Thus for the number of trades to decrease by n it must be that there are at least n buyers with valuations $v_i^{(\frac{s}{m})}$ in $(p, p+h]$ and n sellers with valuations $v_i^{(\frac{s}{m})}$ in $[p-h, p]$.

Consider first the case where the player with valuation $\underline{p}^m$ which we denoted $\underline{i}^m$ is a buyer. As there must be at least n buyers with valuations $v_i^{(\frac{s}{m})}$ in $(p, p+h]$ and n sellers with valuations $v_i^{(\frac{s}{m})}$ in $[p-h, p]$ assumption (iii) (see beginning of the proof of the theorem) implies that

$$\Pr(\tau^{(R)} - \tau^{(R+h)} = n \mid \underline{i}^m \in B, \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p) \leqslant O(h^{2 \cdot n}). \tag{23}$$

Now consider the case where $\underline{i}^m \in S$. Again there must be at least n buyers with valuations $v_i^{(\frac{s}{m})}$ in $(p, p+h]$ and n sellers with valuations $v_i^{(\frac{s}{m})}$ in $[p-h, p]$.

As $\underline{i}^m$ is a seller with a valuation in $[p-h,p]$ we get

$$\Pr(\tau^{(R)}-\tau^{(R+h)}=n\,|\,\underline{i}^m\in S,\frac{\tau^{(\frac{s}{m})}}{m}=a,\underline{p}^m=p)\leqslant O(h^{2\cdot n-1}) \tag{24}$$

Clearly $\Pr(\tau^{(R+h)}-\tau^{(R)}=n\,|\,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)$ can be rewritten as a the sum

$$\Pr(\underline{i}^m\in B\,|\,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)\cdot\Pr(\tau^{(R)}-\tau^{(R+h)}=n\,|\,\underline{i}^m\in B,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)+$$
$$+\Pr(\underline{i}^m\in S\,|\,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)\cdot\Pr(\tau^{(R)}-\tau^{(R+h)}=n\,|\,\underline{i}^m\in S,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m).$$

Thus we can use Equations (23) and (24) in Expression (22) to get that

$$\frac{d}{dR}E(\tau^{(R)}\,|\,\frac{\tau^{(\frac{s}{m})}}{m}=a,\underline{p}^m=p)=$$
$$=-\Pr(\underline{i}^m\in S\,|\,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)\cdot\lim_{h\to0}\frac{\Pr(\tau^{(R)}-\tau^{(R+h)}=1\,|\,\underline{i}^m\in S,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)}{h}$$
$$=-\Pr(\underline{i}^m\in S\,|\,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)\cdot\lim_{h\to0}\frac{\Pr(\tau^{(R)}-\tau^{(R+h)}\geqslant1\,|\,\underline{i}^m\in S,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)}{h}$$

Let us look at $\Pr(\tau^{(\frac{s}{m})}-\tau^{(\frac{s}{m}+h)}\geqslant1\,|\,\underline{i}^m\in S,\frac{\tau^{(\frac{s}{m})}}{m}=a,\underline{p}^m=p)$. As $\underline{i}$ is a seller the number of trades will decrease by at least 1 if and only if there is a buyer with a valuation in $[p,p+h]$. Therefore

$$\Pr(\tau^{(\frac{s}{m})}-\tau^{(\frac{s}{m}+h)}\geqslant1\,|\,\underline{i}^m\in S,\frac{\tau^{(\frac{s}{m})}}{m},\underline{p}^m)=\begin{cases}1-\frac{(1-G(p+\frac{s}{m}+h))^{am}}{(1-G(p+\frac{s}{m}))^{am}} & \text{if } \underline{p}^m+\frac{s}{m}<\overline{v}\\ 0 & \text{if } \underline{p}^m+\frac{s}{m}\geqslant\overline{v}\end{cases}$$

The statement of the lemma follows. $\square$

Lemma 5. *The probabilities* $\Pr(\underline{i}^m\in S)$ *converge to*

$$\frac{\alpha\cdot\frac{F'(p^*)}{F(p^*)}}{(\alpha\cdot\frac{F'(p^*)}{F(p^*)}+(k_2-\alpha)\cdot\frac{G'(p^*)}{G(p^*)})}.$$

as $m\to\infty$.

Proof. Define $\underline{p}_b^m$ to be the highest valuation $v_i^{(R^m)}$ among all of the buyers with valuations $v_i^{(R^m)}<\overline{p}^m$, i.e.

$$\underline{p}_b^m(v)=\sup\{v_i^{(R^m)}:i\in B\text{ and }v_i^{(R^m)}<\overline{p}^m\}$$

Similarly define $\underline{p}_s^m$ to be the highest valuation $v_i^{(R^m)}$ among all of the sellers with valuations $v_i^{(R^m)}<\overline{p}^m$, i.e.

$$\underline{p}_b^m(v)=\sup\{v_i^{(R^m)}:i\in S\text{ and }v_i^{(R^m)}<\overline{p}^m\}$$

As $\underline{p}^m$ is the highest valuation $v_i^{(R^m)}$ among all players with valuations $v_i^{(R^m)}<\overline{p}^m$, i.e.

$$\underline{p}^m(v)=\sup\{v_i^{(R^m)}:i\in B\cup S\text{ and }v_i^{(R^m)}<\overline{p}^m\}$$

the following equivalence holds almost surely

$$\underline{i}^m\in S\Leftrightarrow\underline{p}_s^m>\underline{p}_b^m\Leftrightarrow m\cdot(\overline{p}^m-\underline{p}_s^m)<m\cdot(\overline{p}^m-\underline{p}_b^m). \tag{25}$$

Note however that the definition of $\underline{p}_s^m$ and $\underline{p}_b^m$ immediately allows us to write down formulas for the conditional probabilities

$$\Pr(m \cdot (\overline{p}^m - \underline{p}_b^m) < x \,|\, \overline{p}, \frac{\tau^m}{m}) = 1 - \frac{G(\overline{p}^m - \frac{x}{m})^{k_2 \cdot m - \tau^m}}{G(\overline{p})^{k_2 \cdot m - \tau^m}}$$

and

$$\Pr(m \cdot (\overline{p}^m - \underline{p}_s^m) < x \,|\, \overline{p}, \frac{\tau^m}{m}) = 1 - \frac{F(\overline{p}^m - \frac{x}{m})^{\tau^m}}{F(\overline{p})^{\tau^m}}$$

Note that therefore $m \cdot (\overline{p}^m - \underline{p}_b^m)$ and $m \cdot (\overline{p}^m - \underline{p}_s^m)$ conditional on $\overline{p} = p$ and $\frac{\tau^m}{m} = a$ converge in distribution to exponential distributions with means $\frac{G(p)}{(k_2 - a) \cdot G'(p)}$ and $\frac{F(p)}{a \cdot F'(p)}$. Using the same methods as in the proof of Lemma 3 we can conclude that

$$\lim_{m \to \infty} \Pr(\underline{p}_s^m > \underline{p}_b^m \,|\, \overline{p} = p, \frac{\tau^m}{m} = a) = \frac{\frac{a \cdot F'(p)}{F(p)}}{\frac{(k_2 - a) \cdot G'(p)}{G(p)} + \frac{a \cdot F'(p)}{F(p)}}$$

Applying Corollary 4 for arbitrary small $\varepsilon > 0$ we get the hypothesis. □

Consider a fixed $m \in \mathbb{N}$. Note that Lemma 4 together with assumption (v) on the distribution G (see the beginning of the proof) guarantee that

$$\frac{1}{m} \frac{d}{dR} E(\tau^{(R)}) = \frac{1}{m} E(\frac{d}{dR} E(\tau^{(R)} \,|\, \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p)).$$

Lemma 4 therefore implies that $\frac{1}{m} \cdot \frac{d}{dR} E(\tau^{(R)})$ is uniformly bounded for all $m \in \mathbb{N}$. The Dominated Convergence Theorem together with Equation (21) and $\lim_{m \to \infty} R^m \cdot m = \rho$ therefore implies that

$$\liminf_{n \to \infty} I(VCG^m(R^m)) = -\int_{s \in [0,\rho]} s \cdot \liminf_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})}) \cdot ds$$

and

$$\limsup_{n \to \infty} I(VCG^m(R^m)) = -\int_{s \in [0,\rho]} s \cdot \limsup_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})}) \cdot ds.$$

To complete the proof it is enough to show that for all $s \in [0, \rho]$

$$\frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})}) \to \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{(k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))} \tag{26}$$

as $m \to \infty$.

Consider a fixed $s \in [0, \rho]$. As $\frac{d}{dR} E(\tau^{(R)} \,|\, \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p)$ is uniformly bounded Corollary 4 implies that for any $\varepsilon > 0$.

$$\begin{aligned}
\liminf_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})}) &= \liminf_{m \to \infty} \frac{1}{m} E(\frac{d}{dR} E(\tau^{(R)} \,|\, \frac{\tau^{(\frac{s}{m})}}{m} = a, \underline{p}^m = p)) \\
&= \liminf_{m \to \infty} \frac{1}{m} E(\frac{d}{dR} E(\tau^{(R)} \,|\, \frac{\tau^{(\liminf_{m \to \infty} \frac{s}{m})}}{m} = a, \underline{p}^m = p) \,|\, E_\varepsilon^m) \\
&= \liminf_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})} \,|\, E_\varepsilon^m)
\end{aligned}$$

and similarly

$$\limsup_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})}) = \limsup_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})} \,|\, E_\varepsilon^m)$$

Using Lemma 4 together with Lemma 5 and Corollary 4 we therefore get that

$$\frac{\alpha \cdot \frac{F'(p^*)}{F(p^*)}}{(\alpha \cdot \frac{F'(p^*)}{F(p^*)} + (k_2 - \alpha) \cdot \frac{G'(p^*)}{G(p^*)})} \cdot \frac{(\alpha - \varepsilon) \cdot \min_{p \in [p^* - 2\varepsilon, p^* + 2\varepsilon]} G'(p)}{\max_{p \in [p^* - 2\varepsilon, p^* + 2\varepsilon]]}(1 - G(p))} \leqslant$$

$$\leqslant \liminf_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})}) \leqslant \limsup_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})}) \leqslant$$

$$\leqslant \frac{\alpha \cdot \frac{F'(p^*)}{F(p^*)}}{(\alpha \cdot \frac{F'(p^*)}{F(p^*)} + (k_2 - \alpha) \cdot \frac{G'(p^*)}{G(p^*)})} \cdot \frac{(\alpha + \varepsilon) \cdot \max_{p \in [p^* - 2\varepsilon, p^* + 2\varepsilon]} G'(p)}{\min_{p \in [p^* - 2\varepsilon, p^* + 2\varepsilon]]}(1 - G(p))}$$

As the above holds for arbitrary small $\varepsilon > 0$ we get that $\lim\limits_{m \to \infty} \frac{1}{m} \frac{d}{dR} E(\tau^{(\frac{s}{m})})$ exists and is equal to

$$\frac{\alpha \cdot \frac{F'(p^*)}{F(p^*)}}{(\alpha \cdot \frac{F'(p^*)}{F(p^*)} + (k_2 - \alpha) \cdot \frac{G'(p^*)}{G(p^*)})} \cdot \frac{\alpha \cdot G'(p^*)}{(1 - G(p^*))} = \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{(k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))}$$

We proved Equation (26) from which statement of the theorem followed.

Q.E.D.

A.5. **Localization revisited.** In Section 7.2 we proved some results which allowed us to 'localize' our discussion in the proofs of Theorems 1 and 2 around the point p^*. The following two lemmas will play a similar role in the proof of Theorem 3. Before we can state the lemmas we need to introduce another definition.

In the following we will say that a market mechanism $M = (q, t)$ is symmetric if the allocation rule treats all buyers and all sellers symmetrically. Formally this means that if $\pi : B \cup S \to B \cup S$ is a permutation, such that $\pi(B) = B$ (and therefore $\pi(S) = S$) and $v = (v_i)_{i \in B \cup S}$ is a profile of valuations then

$$q_{\pi(i)}((v_{\pi(i)})_{i \in B \cup S}) = q_i((v_i)_{i \in B \cup S})$$

for all $i \in B \cup S$.

Before reading the remainder of the appendix the reader might want to have a look at Section A.1 to see the conventions we use to talk about conditional expectations.

Lemma 6. *Assume $(M^m)_{m \in \mathbb{N}}$ is a sequence of incentive compatible, symmetric, not necessarily individually rational mechanisms*[29] *such that*

$$\limsup_{m \to \infty} I(M^m) \cdot m < \infty.$$

Let $\varepsilon > 0$. Then there exists constants C_1, C_2 and C_3 such that

(i) *For all $p' \leqslant p^* - \varepsilon$ and $m \in \mathbb{N}$*

$$\max_{i \in B \cup S} E(q_i^m(v) \,|\, v_i = p') \cdot m^2 < C_1.$$

(ii) *For all $p' \geqslant p^* + \varepsilon$ and $m \in \mathbb{N}$*

$$\max_{i \in B \cup S} (1 - E(q_i^m(v) \,|\, v_i = p')) \cdot m^2 < C_2.$$

[29] Such that M^m is a mechanism for the environment of market size m.

(iii) *For all* $m \in \mathbb{N}$ *and for any* p' *and* p'' *such that either* $p' \leqslant p^* - \varepsilon$ *and* $p'' \leqslant p^* - \varepsilon$ *or* $p' \geqslant p^* + \varepsilon$ *and* $p'' \geqslant p^* + \varepsilon$

$$\max_{i \in B \cup S} |E(t(v_{-i}, v_i) \mid v_i = p') - E(t(v_{-i}, v_i) \mid v_i = p'')| \cdot m^2 < C_3.$$

Proof. We start with the proof of (i).

Notice that the incentive compatibility conditions imply that for any player $i \in B \cup S$ the conditional expectations $E(q_i^m(v) \mid v_i)$ are increasing. This means that to prove (i) it is sufficient to show that for any $p', p'' \leqslant p^* - \varepsilon$ the sequence $\max_{i \in B \cup S} E(q_i^m(v) \cdot 1_{[p',p'']}(v_i)) \cdot m^2$ is bounded. Suppose this is not true. Than for some $p', p'' \leqslant p^* - \varepsilon$ either

$$\text{the sequence } \max_{i \in B} E(q_i^m(v) \cdot 1_{[p',p'']}(v_i)) \cdot m^2 \text{ is unbounded} \tag{27}$$

or

$$\text{the sequence } \max_{i \in S} E(q_i^m(v) \cdot 1_{[p',p'']}(v_i)) \cdot m^2 \text{ is unbounded.}$$

For the sake of concreteness assume the first alternative.[30] Define the sequence Q^m to be equal to the expected number of goods distributed to buyers with valuations in $[p', p'']$, i.e.

$$Q^m = E(\sum_{i \in B} q_i^m(v) \cdot 1_{[p',p'']}(v_i)).$$

As the mechanism are assumed to be symmetric $E(q_i^m(v) \cdot 1_{[p',p'']}(v_i)) \cdot m^2$ takes the same value for all buyers $i \in B$. Therefore (27) implies that the sequence $Q^m \cdot m$ is also unbounded. Define $R^m = 0$ for all $m \in \mathbb{N}$. Next define E_ε^m for any $\varepsilon > 0$ and $m \in \mathbb{N}$ as in Section A.2. Using Corollary 4 we conclude that if we define the sequence $\tilde{Q}^m$ by

$$\tilde{Q}^m = E(\sum_{i \in B} q_i^m(v) \cdot 1_{[p',p'']}(v_i) \mid E^m_{\frac{\varepsilon}{3}}).$$

then the sequence $\tilde{Q}^m \cdot m$ is also unbounded.

Notice that conditional on $E^m_{\frac{\varepsilon}{3}}$ the efficient allocation rule assigns all goods to players with valuations bigger than $p^* - \frac{\varepsilon}{3}$. On the other hand, conditional on $E^m_{\frac{\varepsilon}{3}}$, the rule q^m assigns on average $\tilde{Q}^m$ goods to players with valuations in $[p', p'']$. This implies that, conditional on $E^m_{\frac{\varepsilon}{3}}$, the absolute inefficiency of the mechanism M^m must be at least

$$\tilde{Q}^m \cdot ((p^* - \frac{\varepsilon}{3}) - p'') \geqslant \tilde{Q}^m \cdot \frac{2}{3}\varepsilon.$$

But than the fact that $\tilde{Q}^m \cdot m$ is unbounded implies that the sequence

$$E(\sum_{i \in B \cup S} (q_i^{eff} - q_i^m) \cdot v_i \mid E^m_{\frac{\varepsilon}{3}}) \cdot m$$

is also unbounded. Using Corollary 4 we get that the sequence

$$E(\sum_{i \in B \cup S} (q_i^{eff} - q_i^m) \cdot v_i) \cdot m = I(M^m) \cdot m$$

is unbounded which contradicts assumption that $\limsup_{m \to \infty} I(M^m) \cdot m < \infty$.

The proof of part (ii) is analogous to the proof of part (i).

[30] The other case is identical.

To prove (iii) define the constant C_3 to be equal to

$$C_3 = 2 \cdot \max(C_1, C_2) \cdot \max(|\overline{v}_s|, |\underline{v}_s|, |\overline{v}_b|, |\underline{v}_b|),$$

where C_1 and C_2 are the constants from parts (i) and (ii) of the lemma, and $\overline{v}_s$, $\underline{v}_s$, $\overline{v}_b$ and $\underline{v}_b$ are respectively the highest and lowest valuations in the supports of F and G. Next, rewrite the incentive compatibility constrains as

$$\begin{aligned}(E(q_i^m(v) \,|\, v_i = p') - E(q_i^m(v) \,|\, v_i = p'')) \cdot p'' \leqslant \\ \leqslant E(t(v_{-i}, v_i) \,|\, v_i = p') - E(t(v_{-i}, v_i) \,|\, v_i = p'') \leqslant \\ (E(q_i^m(v) \,|\, v_i = p') - E(q_i^m(v) \,|\, v_i = p'')) \cdot p'\end{aligned}$$

and apply part (i) and (ii) of the lemma to conclude that

$$|E(t(v_{-i}, v_i) \,|\, v_i = p') - E(t(v_{-i}, v_i) \,|\, v_i = p'')| \leqslant \frac{C_3}{m^2}.$$

□

As an application we get the following lemma.

Lemma 7. *Assume $c_o, d_o \in \mathbb{R}$ are arbitrary numbers and $(M^m)_{m\in\mathbb{N}}$ is a sequence of incentive compatible, individually rational mechanisms*[31] *such that*

$$\limsup_{m\to\infty} I(M^m) \cdot m < \infty$$

Assume moreover, that $\tilde{F}$ and $\tilde{G}$ is a pair of distributions on $[0, \infty)$ with bounded and convex supports such that $\tilde{F}$ and $\tilde{G}$ are equal to F and G in some neighborhood of p^*,

$$F(x) \leqslant \tilde{F}(x) \text{ and } G(x) \leqslant \tilde{G}(x) \text{ for } x \leqslant p^*$$

and

$$F(x) \geqslant \tilde{F}(x) \text{ and } G(x) \geqslant \tilde{G}(x) \text{ for } x \geqslant p^*.$$

Consider sequences $(\tilde{M}^m)_{m\in\mathbb{N}}$ such that $\tilde{M}^m$ is a symmetric, incentive compatible and individually rational mechanisms in the environment of market size m where the distributions of buyers and sellers are distributed according to the distributions $\tilde{F}$ and $\tilde{G}$. Then there exists such a sequence $(\tilde{M}^m)_{m\in\mathbb{N}}$ satisfying

$$\limsup_{m\to\infty} I(\tilde{M}^m) \cdot m \leqslant \limsup_{m\to\infty} I(M^m) \cdot m$$

and

$$\liminf_{m\to\infty} \textit{Expected Deficit}(\tilde{M}^m) = \liminf_{m\to\infty} \textit{Expected Deficit}(M^m).$$

Proof. The proof proceeds in several steps.

Step 1: Notice that it is enough to prove the theorem for the case where the mechanisms M^m are symmetric. Indeed, assume a mechanism $M^m = (q^m, t^m)$ is not symmetric. Denote the set of all permutations $\pi : B \cup S \to B \cup S$ such that $\pi(B) = B$ by Π. Now, define a new mechanism where allocation rule and payment rule of a player $i \in B \cup S$ are given respectively by

$$\frac{1}{|\Pi|} \sum q_{\pi(i)}((v_{\pi(i)})_{i\in B\cup S}))$$

[31] Such that M^m is a mechanism for the environment of market size m.

and

$$\frac{1}{|\Pi|}\sum t_{\pi(i)}((v_{\pi(i)})_{i\in B\cup S}).$$

It is trivial to verify that the so defined mechanism is symmetric, incentive compatible, individually rational and has the same inefficiency and deficit as the original mechanism. As we can substitute all mechanisms in the sequence $(M^m)_{m\in\mathbb{N}}$ which are not symmetric with symmetric mechanism having the same inefficiency and deficit it is enough to prove the lemma for the case where the initial sequence $(M^m)_{m\in\mathbb{N}}$ consists only of symmetric mechanisms.

Step 2: Consider a fixed market size. Except in Corollary 5 we were never concerned with the way the random valuations of the players were generated as long as they have the desired joint distribution. Now we will again use the particular probabilistic structure introduced in Corollary 5, which will allow easy comparisons between the environment where the distributions of buyers and sellers are distributed according to F and G and the environment where they are distributed according to $\tilde{F}$ and $\tilde{G}$.

Let x_i for $i\in B\cup S$ be independent random variables uniformly distributed on $(0,1)$. Define

$$v_i(x_i)=\begin{cases}\inf\{y:F(y)\geqslant x_i\} & \text{for } i\in S\\ \inf\{y:G(y)\geqslant x_i\} & \text{for } i\in B\end{cases}$$

and

$$\tilde{v}_i(x_i)=\begin{cases}\inf\{y:\tilde{F}(y)\geqslant x_i\} & \text{for } i\in S\\ \inf\{y:\tilde{G}(y)\geqslant x_i\} & \text{for } i\in B.\end{cases}$$

Note that $v_i(x_i)$ are independent and distributed according to F (for $i\in S$) and G (for $i\in B$). Similarly $\tilde{v}_i(x_i)$ are independent and distributed according to $\tilde{F}$ (for $i\in S$) and $\tilde{G}$ (for $i\in B$). In the following we will assume that the valuations v_i and $\tilde{v}_i$ are given by $v_i(x_i)$ and $\tilde{v}_i(x_i)$ respectively.

For all $m\in\mathbb{N}$ define the mechanism $\tilde{N}^m=(\tilde{q}^m,\tilde{s}^m)$ as follows. For any player $i\in B\cup S$ the allocation rule $\tilde{q}^m$ is given by

$$\tilde{q}_i^m((\tilde{v}_i(x_i))_{i\in B\cup S})=q_i^m((v_i(x_i))_{i\in B\cup S})$$

while the transfer payment rule $\tilde{s}^m$ is given by

$$\tilde{s}_i^m(\tilde{v})=\tilde{q}_i(\tilde{v}_{-i},p^*)\cdot p^*+t_i^m(\tilde{v}_{-i},p^*)-\tilde{q}_i(\tilde{v})\cdot v_i+\int_{p^*}^{v_i}E(\tilde{q}_i(\tilde{v})\,|\,\tilde{v}_i=x)\cdot dx.$$

Step 3: Choose an $\varepsilon>0$ such that (i) in the interval $[p^*-\varepsilon,p^*+\varepsilon]$ both F and G are twice continuously differentiable, and have positive densities (ii) $F(p)=\tilde{F}(p)$, $G(p)=\tilde{G}(p)$ for all $p\in[p^*-\varepsilon,p^*+\varepsilon]$. Note that then our definition implies that for $x\in[p^*-\varepsilon,p^*+\varepsilon]$

$$E(\tilde{q}_i^m(\tilde{v})\,|\,\tilde{v}_i=x)=E(q_i^m(v)\,|\,v_i=x)$$

and

$$E(\tilde{s}_i^m(\tilde{v})\,|\,\tilde{v}_i=x)=E(t_i^m(v)\,|\,v_i=x).$$

Step 4: The fact that the mechanisms M^m are incentive compatible implies that $E(q_i(v)\,|\,v_i)$ is increasing. But then the definition of the allocation rule $\tilde{q}^m$

implies that $E(\tilde{q}_i(v)\,|\,v_i)$ is also increasing. In this situation it is straightforward to verify that the mechanism $\tilde{N}^m = (\tilde{q}^m, \tilde{s}^m)$ given by the above formulas are also incentive compatible.

Step 5: We will show that

$$\limsup_{m\to\infty} I(\tilde{N}^m)\cdot m \leqslant \limsup_{m\to\infty} I(M^m)\cdot m.$$

Recall that

$$I(M^m) = E(\sum_{i\in B\cup S} q_i^{eff}(v)\cdot v_i - \sum_{i\in B\cup S} q_i^m(v_i)\cdot v_i)$$

Consider a profile $(x_i)_{i\in B\cup S}$ such that $(v_i(x_i))_{i\in B\cup S}$ lies in the event E_ε^m, where E_ε^m is defined as in Section 7.2.[32] Consider a fixed player $i\in B\cup S$. Note that if $v_i(x_i)\in[p^*-\varepsilon, p^*+\varepsilon]$ then $v_i(x_i)=\tilde{v}_i(x_i)$ and therefore

$$q_i^{eff}((v_i(x_i))_{i\in B\cup S})\cdot v_i(x_i) - q_i^m((v_i(x_i))_{i\in B\cup S})\cdot v_i(x_i) =$$

$$= \tilde{q}_i^{eff}((\tilde{v}_i(x_i))_{i\in B\cup S})\cdot \tilde{v}_i(x_i) - \tilde{q}_i^m((\tilde{v}_i(x_i))_{i\in B\cup S})\cdot \tilde{v}_i(x_i).$$

On the other hand, if $v_i(x_i) < p^*-\varepsilon$ then our assumption that

$$F(p)\leqslant \tilde{F}(p) \text{ and } G(p)\leqslant \tilde{G}(p) \text{ for } x\leqslant p^*$$

implies that $v_i(x_i)\geqslant \tilde{v}_i(x_i)$. But in this case conditional on the event E_ε^m both $q_i^{eff}((v_i(x_i))_{i\in B\cup S})$ and $q_i^{eff}((\tilde{v}_i(x_i))_{i\in B\cup S})$ are equal to 1. Therefore the definition of $\tilde{q}^m$ implies

$$(q_i^{eff}((v_i(x_i))_{i\in B\cup S}) - q_i^m((v_i(x_i))_{i\in B\cup S}))\cdot v_i(x_i)\geqslant$$

$$\geqslant (\tilde{q}_i^{eff}((\tilde{v}_i(x_i))_{i\in B\cup S})\cdot \tilde{v}_i(x_i) - \tilde{q}_i^m((\tilde{v}_i(x_i))_{i\in B\cup S})\cdot \tilde{v}_i(x_i).$$

Finally, if $v_i(x_i) > p^*-\varepsilon$ a similar analysis gives

$$(q_i^{eff}((v_i(x_i))_{i\in B\cup S}) - q_i^m((v_i(x_i))_{i\in B\cup S}))\cdot v_i(x_i)\geqslant$$

$$\geqslant (\tilde{q}_i^{eff}((\tilde{v}_i(x_i))_{i\in B\cup S})\cdot \tilde{v}_i(x_i) - \tilde{q}_i^m((\tilde{v}_i(x_i))_{i\in B\cup S})\cdot \tilde{v}_i(x_i).$$

Combining the above we get that

$$E(\sum_{i\in B\cup S} q_i^{eff}(v)\cdot v_i - \sum_{i\in B\cup S} q_i^m(v_i)\cdot v_i\,|\,E_\varepsilon^m)\geqslant$$

$$\geqslant E(\sum_{i\in B\cup S} \tilde{q}_i^{eff}(\tilde{v}_i)\cdot \tilde{v}_i - \sum_{i\in B\cup S} \tilde{q}_i^m(\tilde{v}_i)\cdot \tilde{v}_i\,|\,E_\varepsilon^m).$$

Using Corollary 4 we conclude that

$$\limsup_{m\to\infty} I(\tilde{N}^m)\cdot m \leqslant \limsup_{m\to\infty} I(M^m)\cdot m.$$

Step 6: As the mechanisms M^m were assumed to be symmetric the definition of $\tilde{N}^m$ immediately implies that the mechanisms $\tilde{N}^m$ are also symmetric. Step 5 therefore implies that we can apply Lemma 6 not only to the sequence M^m, but also to the sequence $\tilde{N}^m$. Let C_1, C_2, C_3 and $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ be respectively the constants

[32] Notice that this condition does not depend which of the two environments is used to define E_ε^m.

from Lemma 6 for the mechanisms M^m and $\tilde{N}^m$. Now define the payment rule $\tilde{t}^m$ by

$$\tilde{t}_i^m(\tilde{v}) = \tilde{s}_i^m(\tilde{v}) + \frac{2 \cdot \max(C_3, \tilde{C}_3)}{m^2} + \frac{4 \cdot \max(C_1, C_2, \tilde{C}_1, \tilde{C}_2)}{m^2} \cdot \max(|\overline{v}_s|, |\underline{v}_s|, |\overline{v}_b|, |\underline{v}_b|)$$

and set $\tilde{M}^m = (\tilde{q}^m, \tilde{t}^m)$. Note that Step 4 implies that $\tilde{M}^m$ is incentive compatible and Step 5 implies that $\limsup_{m\to\infty} I(\tilde{M}^m) \cdot m < \infty$.

Step 7: The individual rationality constrains for the mechanisms $\tilde{M}^m$ follow directly from the individual rationality constrains for the mechanisms M^m for the types $p^* - \varepsilon$ and $p^* + \varepsilon$, Step 3 and Lemma 6. Similarly

$$\liminf_{m\to\infty} \text{Expected Deficit}(\tilde{M}^m) = \liminf_{m\to\infty} \text{Expected Deficit}(M^m)$$

follows from Lemma 6 and Step 3. □

A.6. **Proof of Theorem 3.** Notice that by Lemma 7 we can restrict our attention to the case where F and G have convex supports, are continuous, twice continuously differentiable on the interior of their supports, and have densities which are bound away from zero on the interior of the respective supports.

We will need the following lemma

Lemma 8. *For any market mechanism $M^m = (q,t)$ satisfying the above assumptions*

$$\textit{Exp. Defifcit}(M^m) \geqslant$$
$$\geqslant E\Big(\sum_{s\in S}\big(q_i(v)\cdot\big(v_i + \frac{F(v_i)}{F'(v_i)}\big) - \bar{v}_s\big) + \sum_{b\in B} q_i(v)\cdot\big(v_i - \frac{1-G(v_i)}{G'(v_i)}\big)\Big)$$

In case of $M^m = VCG^m(R^m)$ we can substitute the inequality with an equality.

Proof. Standard. □

For each $m \in \mathbb{N}$ let us look at the program of finding an allocation rule $\tilde{q}$ maximizing:

$$\max_q E\Big(\sum_{i\in B\cup S} q_i(v)\cdot v_i\Big)$$

subject to the budget constraint

$$E\Big(\sum_{s\in S}\big(q_i(v)\cdot\big(v_i + \frac{F(v_i)}{F'(v_i)}\big) - \bar{v}_s\big) + \sum_{b\in B} q_i(v)\cdot\big(v_i - \frac{1-G(v_i)}{G'(v_i)}\big)\Big) \leqslant$$
$$\leqslant \text{Expected Deficit}(M^m)$$

This is a standard maximization program having a solution $\tilde{q}^m$ which is uniquely determined up to set of Lebesque measure zero. Note that Lemma 8 implies that the inefficiency of $\tilde{q}^m$ is at least as small as the inefficiency of M^m.

Let us describe a solution $\tilde{q}^m$ in more detail. Denote the multiplier of the budget constraint by λ^m. $\tilde{q}^m$ allocates the $k_1 \cdot m$ goods to the $k_1 \cdot m$ players with the highest virtual valuations

$$v_i + \frac{\lambda^m}{\lambda^m+1}\cdot\frac{F(v_i)}{F'(v_i)} \tag{28}$$

for sellers ($i \in S$) and

$$v_i - \frac{\lambda^m}{\lambda^m + 1} \cdot \frac{1 - G(v_i)}{G'(v_i)} \tag{29}$$

for buyers ($i \in B$).

As already said Lemma 8 implies that the inefficiency of $\tilde{q}^m$ is at least as small as the inefficiency of M^m. Theorem 1 and 2 therefore imply that

$$\liminf_{m\to\infty} I(\tilde{q}^m) \cdot m \leqslant \frac{1}{2} \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot \rho^2. \tag{30}$$

But this means that $\lambda^m \to 0$ as $m \to \infty$. We will need the following two results.

Lemma 9. *Let R^m be a sequence of fees such that $R^m \underset{m\to\infty}{\longrightarrow} 0$. Let $\varepsilon > 0$ and define the event E^m_ε as in Section 7.2. Then there exists an $m_0 \in \mathbb{N}$ such that for any $m \geqslant m_0$*

$$E(\tilde{q}^m(v) \,|\, E^m_\varepsilon, v_i = x) = 0 \textit{ for } x \leqslant p^* - 2\varepsilon$$

and

$$E(\tilde{q}^m(v) \,|\, E^m_\varepsilon, v_i = x) = 1 \textit{ for } x \geqslant p^* + 2\varepsilon.$$

Proof. Define $K = \sup_p \frac{F(p)}{F'(p)} + \sup_p \frac{1-G(p)}{G'(p)} < \infty$, where the first supremum is over the interior of the support of F and the second over the interior of the support of G. Note that the difference between the virtual valuations given by (28) and (29) and the valuations $v_i^{(R^m)}$ is at most $R^m + K \cdot \lambda^m$. Therefore the statement follows from the fact that $R^m, \lambda^m \to 0$. ☐

Lemma 10.

$$\limsup_{n\to\infty} \lambda^m \cdot m < \infty.$$

Proof. Assume the statement of the lemma is not true. Then there is a subsequence m_k such that $\lambda^{m_k} \cdot m_k \to \infty$. Choose an $\varepsilon > 0$ such that $\frac{F(v_i)}{F'(v_i)}$ and $\frac{1-G(v_i)}{G'(v_i)}$ are bounded away from zero on $[p^* - 2\varepsilon, p^* + 2\varepsilon]$ and define $K = \inf_{p\in[p^*-2\varepsilon,p^*+2\varepsilon]} \min(\frac{F(v_i)}{F'(v_i)}, \frac{1-G(v_i)}{G'(v_i)})$. Let $R^m = \frac{2\rho}{m}$ for all $m \in \mathbb{N}$ and define the event E^m_ε as in Section A.2. As $\lambda^{m_k} \cdot m_k \cdot K \to \infty$ and $R^{m_k} \cdot m_k \to 2\rho$ there exists a number k_o such that for $k \geqslant k_o$

$$\lambda^{m_k} \cdot K \geqslant R^{m_k}$$

But then, conditional on $E^{m_k}_\varepsilon$, the inefficiency of the allocation rule $\tilde{q}^{m_k}$ is higher then the inefficiency of the $VCG^{m_k}(R^{m_k})$.

Using Corollary 4 we conclude that

$$\liminf_{k\to\infty} I(\tilde{q}^{m_k})\cdot m_k \geqslant \liminf_{k\to\infty} I(VCG^{m_k}(R^{m_k}))\cdot m_k = \frac{1}{2} \frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot (2\rho)^2.$$

As this contradicts (30) the statement of the lemma must be true. ☐

Choose an $\varepsilon > 0$. Define

$$c^1_\varepsilon = \min_{p\in[p^*-2\varepsilon,p^*+2\varepsilon]} \frac{F(p)}{F'(p)} + \min_{p\in[p^*-2\varepsilon,p^*+2\varepsilon]} \frac{1-G(p)}{G'(p)}$$

and

$$c_\varepsilon^2 = \max_{p\in[p^*-2\varepsilon,p^*+2\varepsilon]} \frac{F(p)}{F'(p)} + \max_{p\in[p^*-2\varepsilon,p^*+2\varepsilon]} \frac{1-G(p)}{G'(p)}.$$

Note that for large enough market sizes $m \in \mathbb{N}$ the virtual utility functions given by (28) and (29) will be strictly increasing in v_i.[33] The fact that the virtual utility functions are strictly increasing (for $m \geqslant m_0$) means that the buyers which receive the good under the allocation rule $\tilde{q}^m$ correspond to the buyers with the highest valuations v_i. Similarly the sellers which receive the good under the allocation rule $\tilde{q}^m$ are the buyers with the highest valuations v_i. In this sense the allocation rule $\tilde{q}^m$ (for $m \geqslant m_0$) is similar to the efficient allocation rule or the allocation rule of the Vickrey-Clarke-Groves mechanism with a trading fee. Of course the allocation rule still differs in the extend it discriminates between buyers and sellers.

The above observation together with Corollary 4 and Lemma 9 implies that

$$\begin{aligned}
&\liminf_{m\to\infty} I(VCG^m(c_\varepsilon^2 \cdot \frac{\lambda^m}{\lambda^m+1}))\cdot m \leqslant \\
&\leqslant \liminf_{m\to\infty} I(\tilde{q}^m)\cdot m \leqslant \\
&\leqslant \liminf_{m\to\infty} I(VCG^m(c_\varepsilon^1 \cdot \frac{\lambda^m}{\lambda^m+1}))\cdot m.
\end{aligned} \tag{31}$$

Note that allocating a good to a seller $i_i \in S$ instead to a buyer $i_2 \in B$ will decrease the deficit if and only if $v_{i_1} + \frac{F(v_{i_1})}{F'(v_{i_1})}$ is bigger then $v_{i_2} - \frac{1-G(v_{i_2})}{G'(v_{i_2})}$. This together with Corollary 4 and Lemma 9 implies that

$$\begin{aligned}
&\limsup_{m\to\infty} \text{ Expected Deficit}(VCG^m(c_\varepsilon^1 \cdot \frac{\lambda^m}{\lambda^m+1})) \leqslant \\
&\leqslant \limsup_{m\to\infty} \text{ Expected Deficit}(M^m) \leqslant \\
&\leqslant \limsup_{m\to\infty} \text{ Expected Deficit}(VCG^m(c_\varepsilon^2 \cdot \frac{\lambda^m}{\lambda^m+1})).
\end{aligned}$$

The last inequality together with Lemma 10 and Theorem 2 implies that

$$\begin{aligned}
&\alpha(c_\varepsilon^1 \cdot \limsup_{m\to\infty} \frac{\lambda^m}{\lambda^m+1} - \frac{1}{k_1\cdot F'(p^*)+k_2\cdot G'(p^*)}) \leqslant \\
&\leqslant \alpha(\rho - \frac{1}{k_1\cdot F'(p^*)+k_2\cdot G'(p^*)}) \leqslant \\
&\leqslant \alpha(c_\varepsilon^2 \cdot \limsup_{m\to\infty} \frac{\lambda^m}{\lambda^m+1} - \frac{1}{k_1\cdot F'(p^*)+k_2\cdot G'(p^*)})
\end{aligned}$$

[33] The derivatives

$$(\frac{F(p)}{F'(p)})' \text{ and } (\frac{1-G(p)}{G'(p)})'$$

are bounded. Indeed, F' and G' are bounded away from zero and the first and second order derivatives of F and G are bounded.

Therefore the virtual utility functions

$$v_i + \frac{\lambda^m}{\lambda^m+1}\cdot\frac{F(v_i)}{F'(v_i)} \text{ and } v_i - \frac{\lambda^m}{\lambda^m+1}\cdot\frac{1-G(v_i)}{G'(v_i)}$$

will be strictly increasing as long as λ^m is chosen small enough.

As the above equation has to hold for all ε we get (by taking $\varepsilon \to 0$) that

$$\limsup_{m\to\infty} \frac{\lambda^m}{\lambda^m + 1} = \frac{\rho}{\lim\limits_{\varepsilon\to 0} c_\varepsilon^1} = \frac{\rho}{\frac{F(p^*)}{F'(p^*)} + \frac{1-G(p^*)}{G'(p^*)}}.$$

Now we can use equation (31) together with Theorem 3 to get that

$$\frac{1}{2}\frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot \left(\frac{c_\varepsilon^1}{\frac{F(p^*)}{F'(p^*)} + \frac{1-G(p^*)}{G'(p^*)}} \cdot \rho\right)^2 \leqslant \liminf_{m\to\infty} I(q^m) \cdot m$$

Noticing again, that the above inequality must hold for all ε and taking the limit as $\varepsilon \to 0$ we get that

$$\frac{1}{2}\frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot \rho^2 \leqslant \liminf_{m\to\infty} I(q^m) \cdot m$$

This together with

$$\liminf_{m\to\infty} I(\tilde{q}^m) \cdot m \leqslant \liminf_{m\to\infty} I(M^m) \cdot m$$

implies the statement of the theorem.

Q.E.D.

A.7. **Corollaries.** In this section we prove several corollaries and propositions which follow from Theorems 1, 2 and 3.

Proof of Proposition 1. The statement of the proposition follows immediately from Theorem 1 (see Section 4.3) for the case where $\rho = 0$ and $R_m = 0$ for all $m \in \mathbb{N}$. □

Proof of Proposition 3. We will show that proposition 3 follows from Theorem 3.

Assume the statement of the proposition is not true. Then there exists a sequence of mechanisms M^m satisfying the assumptions of the proposition such that

$$\liminf_{m\to\infty} \text{Expected Deficit}(M^m) < \frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}.$$

Choose a $c < \frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$ and a subsequence m_k such that the expected deficit of the subsequence M^{m_k} converges to c.

Now define $\tilde{M}^m$ to be the following sequence of market mechanisms. For all m appearing in the subsequence $(m_k)_{k\in\mathbb{N}}$ the mechanism $\tilde{M}^m$ is equal to M^m. For all $m \in \mathbb{N}$, which do not appear in the sequence $(m_k)_{k\in\mathbb{N}}$ the mechanism $\tilde{M}^m$ is defined to be the mechanism for market size m which 'does not do anything', i.e. allocates all the goods to the sellers and does not make any transfer payments. Clearly $\tilde{M}^m$ is a sequence of individually rational, incentive compatible mechanisms. Notice that for all $m \in \mathbb{N}$ which do not appear in the sequence $(m_k)_{k\in\mathbb{N}}$ the expected deficit of the mechanisms $\tilde{M}^m$ is equal to zero. Therefore

$$\limsup_{m\to\infty} \text{Expected Deficit}(\tilde{M}^m) = \max(0, c) < \frac{k_1 \cdot F(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}.$$

Define $\rho = \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} - \frac{\max(0,c)}{k_1 \cdot F(p^*)} > 0$. Apply Theorem 3 to the sequence $\tilde{M}^m$ to conclude that

$$\liminf_{m\to\infty} I(\tilde{M}^m) \cdot m \geqslant \frac{1}{2}\frac{k_1 \cdot F'(p^*) \cdot k_2 \cdot G'(p^*)}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} \cdot \rho^2 > 0.$$

This, however, contradicts the assumption that the mechanisms $\tilde{M}^{m_k} = M^{m_k}$ were efficient. $\square$

Proof of Corollary 1. Define

$$Q^m = \inf\{R \in [0, \infty) : \text{Expected Surplus}(VCG^m(R)) \geqslant 0\}.$$

Next choose sequences R^m and S^m, such

$$Q^m \cdot (1 + \frac{1}{m^2}) > R^m > Q^m > S^m > Q^m \cdot (1 - \frac{1}{m^2})$$

and Expected Surplus$^{VCG^m(R^m)} \geqslant 0$ for all $m \in \mathbb{N}$.

Consider the sequence $VCG^m(\frac{2}{m \cdot (k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))})$. Note that Theorem 1 implies that the expected surplus of the Vickrey-Clarke-Groves mechanism with a fee $\frac{2}{m \cdot (k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*))}$ converges to $\frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$ as m goes to infinity. As $\frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$ is strictly bigger then zero this implies that the sequence $m \cdot Q^m$ is bounded.

We will prove that

$$m \cdot Q^m \to \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}. \tag{32}$$

Suppose this is not the case. Then, the fact that $m \cdot Q^m$ is bounded implies the existence of a subsequence Q^{m_k} and a number $c \neq \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$ such that

$$m_k \cdot Q^{m_k} \to c \text{ as } k \to \infty.$$

The definition of R^m and S^m imply that $m_k \cdot R^{m_k}$ and $m_k \cdot S^{m_k}$ also converge to c. Using Theorem 1 we therefore get that

$$\begin{aligned}
&\lim_{m \to \infty} \text{Expected Surplus}(VCG^{m_k}(R^{m_k})) = \\
&= \frac{\alpha}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)} - \alpha \cdot c = \\
&= \lim_{m \to \infty} \text{Expected Surplus}(VCG^{m_k}(S^{m_k}))
\end{aligned}$$

As the definition of R^m and S^m imply that the mechanisms $VCG^m(R^m)$ do not run a deficit and the mechanisms $VCG^m(S^m)$ do not run a surplus we get that the above limits must be both equal zero. But this means that $c = \frac{1}{k_1 \cdot F'(p^*) + k_2 \cdot G'(p^*)}$. The contradiction proves that (32) indeed has to be true.

For the discontinuous case it is clear that the numbers R^m satisfy both conditions stated in the theorem. In the continuous case the definition of the numbers Q^m implies that the mechanisms $VCG^m(Q^m)$ are budget balanced. Therefore the fees Q^m fulfill the conditions of the theorem. $\square$

References

[1] Alon, N, J.H. Spencer, and P. Erdos (1992): *The Probabilistic Method*, John Wiley & Sons, New York.

[2] Green J., and J.-J. Laffont (1977): "Characterization of satisfactory Mechanisms for the Revelation of Preferences for Public Goods", *Econometrica*, 45, 427-438.

[3] Gresik, T., and M. Satterthwaite (1989): "The Rate at which a simple Market converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms", *Journal of Economic Theory* 48, 304-332.

[4] Gul, F., and A. Postlewaite (1992): "Asymptotic efficiency in Large Exchange Economies with asymmetric information.", *Econometrica* 60, 1273-1292.

[5] Hagerty, K., and W Rogerson (1985): "Robust Trading Mechanisms", *Journal of Economic Theory*, 42, 94-107.

[6] Krishna, V., and M. Perry (1998): "Efficient Mechanism Design", working paper.

[7] McAfee, R. (1992): "A Dominant Strategy Auction", *Journal of Economic Theory*, 56, 434-450.

[8] Myerson, R., and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory* 29, 265-81.

[9] Rustichini, A., M. Satterthwaite, and S. Wiliams (1994): "Convergence to Efficiency in a Simple Market with incomplete information", *Econometrica*, 62, 1041-1064.

[10] Satterthwaite, M., and S. Wiliams (1999): "The Optimality of a Simple Market Mechanism", *CMS-EMS* 1256, Northwestern University.

[11] Yoon, K. (2000a): "The Modified Vickrey Double Auction", *Journal of Economic Theory*, forthcoming.

[12] Yoon, K. (2000b): "Asymptotic Efficiency of Participatory Vickrey-Clarke-Groves Mechanisms", manuscript.

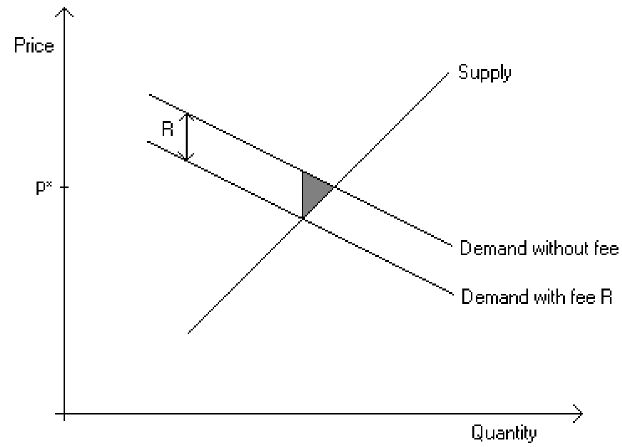

Figure 1. An informal derivation of Theorem 2.

Market Size	Deficit of VCG^m	$m \cdot I(VCG^m(R^m))$	$m \cdot I(Opt^m)$
m=1	\$18.1	\$2.3	\$2.2
m=2	\$19.7	\$2.7	\$2.6
m=3	\$20.1	\$2.8	\$2.7
m=5	\$20.4	\$2.9	\$2.9
m$\rightarrow \infty$	\$20.77	\$3.03	\$3.03

Table 1. Convergence of some values of interest

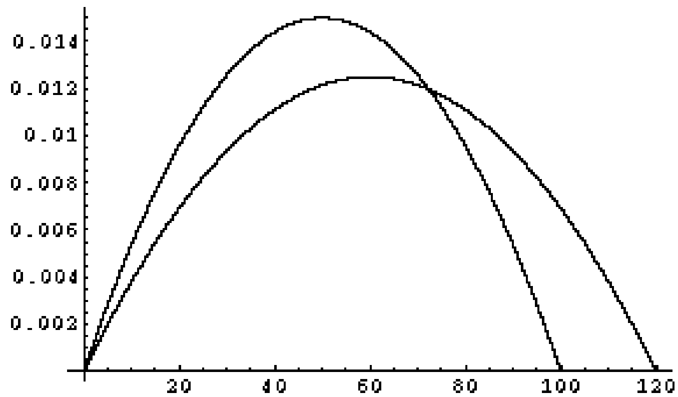

FIGURE 2. The densities F' and G' in the example.