The Dynamics of the Nash Equilibrium Correspondence and n-Player Stochastic Games

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Abstract

A *quitting game* is a sequential game where each player has two actions: to *continue* or to *quit*. The game terminates once at least one player quits. The payoff depends on the subset of players who quit at the termination stage, and is 0 if no one ever quits.

For every continuation payoff x we assign a one-shot game, where the payoff if everyone continues is x. We study the dynamics of the correspondence that assigns to every continuation payoff the set of equilibrium payoffs in the corresponding one-shot game.

The study presented here has an implication on the approach one should take in trying to prove, or disprove, the existence of an equilibrium payoff in n-player stochastic games. It also shows that the minimal length of the period of a periodic δ-equilibrium in 3-player quitting games needs not be uniformly bounded for $\delta > 0$.

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1 Introduction

Whether or not any n-player undiscounted stochastic game admits a uniform equilibrium payoff is still an open problem, even though a lot of progress was achieved in recent years. Mertens and Neyman (1981) proved the existence of the value in zero-sum games. Existence of an equilibrium payoff was proved by Vrieze and Thuijsman (1989) for two-player non zero-sum absorbing games,[1] by Vieille (2000) for two-player stochastic games, and by Solan (1999) for three-player absorbing games.

In all of these existence proofs one approximates the game by a sequence of auxiliary games that admit a stationary equilibrium. By studying the asymptotic behavior of a sequence of stationary equilibria in the auxiliary game, as the approximation becomes better, one constructs an equilibrium payoff in the original undiscounted game.

Flesch et al (1997) studied an example of a three-player quitting game,[2] that admits no stationary ϵ-equilibrium, and the only equilibria in this game have a periodic flavor. As it turns out, periodic equilibria are a very useful concept, that helped solving a couple of classes of stochastic games. It was used in Solan's (1999) study of three-player absorbing games, and in Solan and Vieille's (1998) study of quitting games.

Solan and Vieille (1998) studied an example of a four-player quitting game where the approximation technique fails. Nevertheless, they have succeeded to prove that a class of n-player quitting games admits an equilibrium payoff using the following technique. With every vector $x \in \mathbf{R}^n$ associate the one-shot game $G(x)$ with continuation payoff x; that is, if everyone continues, the payoff is given by x. For every $\epsilon > 0$ let $E_\epsilon(x)$ be the set of all ϵ-equilibrium payoffs in $G(x)$, where the corresponding ϵ-equilibrium strategy profile is terminating with probability at least ϵ; that is, the probability that everyone continues is smaller than $1 - \epsilon$. Solan and Vieille proved that, given any periodic point of E_ϵ, one can construct an $\epsilon^{1/6}$-equilibrium in the original quitting game. They also found conditions on

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[1] *Absorbing games* are stochastic games where all states but one are absorbing.

[2] *Quitting games* are absorbing games where each player has two possible actions, to *continue* or to *quit*, and the game is absorbed with probability 1 once at least one player quits.

the payoff function that ensure that such a periodic point exists.

Since the method of approximating games is seemingly not powerful enough to deal with general n-player stochastic games, one needs to develop new techniques, and the approach used in Solan and Vieille (1998) may serve as a starting point.

In the present note we restrict ourselves to quitting games, and we study the dynamics of the correspondence E_0; that is, the correspondence that assigns to every vector $x \in \mathbf{R}^n$ the set of equilibrium payoffs in the one shot game with continuation payoff x.

If x is large enough (that is, x^i is large for every i), then the profile "everyone continues" is an equilibrium in $G(x)$, and therefore x is a fixed point of E_0. Nevertheless, the profile "everyone continues" needs not be an equilibrium in the original quitting game.

A sequence $x_1, x_2, \ldots, x_K = x_1$ of vectors in $\mathbf{R}^n$ is a *period* of E_0 if $x_k \in E_0(x_{k+1})$ for every $k = 1, 2, \ldots, K$ (addition modulo K). It is a *non trivial* period if for at least one index k, in the equilibrium strategy profile in $G(x_{k+1})$ that yields payoff x_k at least one player quits with positive probability. It is easy to verify that every non trivial period of E_0 corresponds to an equilibrium in the quitting game. Thus, if one's goal is equilibrium payoffs in the original quitting game, one should look for non trivial periods of E_0.

More generally, one should look for non trivial *inverse iterates*; that is, sequences $(x_k)_{k \in \mathbf{N}}$ such that (i) $x_k \in E_0(x_{k+1})$ for every $k \in \mathbf{N}$, and (ii) if for every $k \in \mathbf{N}$ α_k is an equilibrium strategy profile in $G(x_{k+1})$ that yields payoff x_k, then the strategy profile $(\alpha_k)_{k \in \mathbf{N}}$ in the original quitting game is terminating with probability 1.

We show, by studying an example of a three-player quitting game, that the correspondence E_0 needs not have non-trivial inverse iterates, even when the game admits an equilibrium payoff. In particular, it follows that three-player quitting games do not necessarily admit 0-equilibria.

This result shows that a simpler technique than the one used in Solan and Vieille (1998), namely, the search for a non trivial periodic point of E_0, is bound to fail. Thus, the result is useful both for those who try to prove that every stochastic game admits an equilibrium payoff, as well as for those who look for a counter example. For the first group it says

that looking for non-trivial periods of the correspondence E_0 is probably not the right path, whereas for the second group it says that even if E_0 does not have a non trivial period, the game may still admit a uniform equilibrium payoff.

Our result also complements that of Solan (1999, 2000). Solan (1999) proves that every three-player absorbing game admits a δ-equilibrium where the equilibrium path is periodic, and Solan (2000) proves that every absorbing team game[3] admits a δ-equilibrium where the equilibrium path is periodic, and the length of the period is 1 or 2. Our example shows that in three-player absorbing games the length of the period cannot be uniformly bounded (even if payoffs are bounded).

2 The Example

For every $\epsilon \geq 0$, let G_ϵ be a three-player quitting game with the following payoff matrix:

	C			C		Q		C		Q		

	C	Q			C	Q
C	$0,0,0$	$0,1,3$ *		$3,0,1$ *	$1,1+\epsilon,0$ *
Q	$1,3,0$ *	$1+\epsilon,0,1$ *		$0,1,1+\epsilon$ *	$0,0,0$ *

Figure 1

An asterisked entry is absorbing with probability 1, and the non-asterisked entry is absorbing with probability 0.

The game G_ϵ is a perturbation of the game G_0, which was studied by Flesch et al (1997).

A strategy for player i in G_ϵ is a sequence $(\alpha_k^i)_{k \in \mathbf{N}}$, where α_k^i is the probability that player i quits at stage k, provided the game has not terminated before. A strategy profile is a vector of strategies, one for each player.

For every continuation payoff $y \in \mathbf{R}^3$, let $G^\epsilon(y)$ be the one-shot game derived from G_ϵ with a continuation payoff y; that is, a one-shot game where each player has two possible

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[3]A *team game* is a game where the set of players is divided into two subsets, and the payoffs of players in the same subset coincide.

actions, to continue or to quit, the payoff if everyone continues is y, and all other payoffs are as appears in Figure 1. Let $E^\epsilon(y)$ be the set of all Nash equilibria of the game $G^\epsilon(y)$.[4]

A mixed strategy for player i in $G^\epsilon(y)$ is represented by a number $\alpha^i \in [0, 1]$, which is the probability that player i quits. A mixed strategy profile is a vector $\alpha = (\alpha^i)_{i=1}^3 \in [0, 1]^3$.

A sequence $(y(k), \alpha(k))_{k \in \mathbf{N}}$ where for every $k \in \mathbf{N}$, $y(k) \in \mathbf{R}^3$ and $\alpha(k) \in [0, 1]^3$ is *admissible* in G_ϵ if for every $k \in \mathbf{N}$, $\alpha(k)$ is an equilibrium in the game $G^\epsilon(y(k+1))$ that yields payoff $y(k)$. It is *admissible for y in G_ϵ* if it is admissible in G_ϵ and $y = y(1)$. It is *completely absorbing* if $\prod_{k \in \mathbf{N}} \prod_{i=1}^3 (1 - \alpha^i(k)) = 0$.

Let
$$F_\epsilon = \{y \in \mathbf{R}^3 \mid \text{There is an admissible sequence for } y \text{ in } G_\epsilon\}.$$

In words, F_ϵ is the set of all vectors in $\mathbf{R}^3$ that are the first element in some inverse iterate of E^ϵ. By definition, if $(y(k), \alpha(k))_{k \in \mathbf{N}}$ is an admissible sequence in G_ϵ then $y(k) \in F_\epsilon$ for every $k \in \mathbf{N}$. Note that there may be several admissible sequences in G_ϵ for the same vector $y \in \mathbf{R}^3$.

A vector $y \in F_\epsilon$ is *trivial* (in G_ϵ) if every corresponding admissible sequence is not completely absorbing; that is, for every admissible sequence $(y(k), \alpha(k))_{k \in \mathbf{N}}$ for y in G_ϵ, $\prod_{k \in \mathbf{N}} \prod_{i=1}^3 (1 - \alpha^i(k)) > 0$.

One can verify that for every $\epsilon > 0$, any vector $y \in (1 + \epsilon, \infty)^3$ is trivial in G_ϵ. Indeed, such a y is in $E^\epsilon(x)$ if and only if $x = y$, and the corresponding equilibrium is $\alpha = (0, 0, 0)$.

It is clear that every non-trivial vector $y \in F_\epsilon$ corresponds to (at least one) equilibrium in G_ϵ; if $(y(k), \alpha(k))_{k \in \mathbf{N}}$ is a completely absorbing admissible sequence for y in G_ϵ, then the strategy profile $(\alpha(k))_{k \in \mathbf{N}}$ is an equilibrium in G_ϵ, and $y = y(1)$ is the corresponding equilibrium payoff.

Recalling the notion of equilibrium payoff (see, e.g., Mertens, Sorin and Zamir (1994, Section VII.4)), one can provide a stronger definition for trivial vectors: a vector $y \in F_\epsilon$ is trivial if there exists $\delta > 0$ such that every admissible sequence $(y(k), \alpha(k))_{k \in \mathbf{N}}$ for y in G_ϵ

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[4]The correspondence E^ϵ is the correspondence E_0 that was mentioned in the introduction for the game G_ϵ.

satisfies $\prod_{k\in\mathbf{N}}\prod_{i=1}^{3}(1-\alpha^i(k)) > \delta$. The results remain valid with this stronger definition. However, since it is not clear whether an analogue of Lemma 3.2 below is still valid, the proofs of the theorems is more involved.

Our first result is:

THEOREM 2.1 *For every $\epsilon > 0$ sufficiently small, F_ϵ contains only trivial vectors.*

Recall that by Solan (1999), the game G_ϵ admits a uniform equilibrium payoff. Thus, even if the game admits an equilibrium payoff, E^ϵ needs not have a non trivial inverse iterate.

In all the classes of non zero-sum stochastic games where the existence of an equilibrium payoff was proven, one can find δ-equilibrium strategy profiles where the equilibrium path is periodic. For absorbing team games one can even find δ-equilibria where the length of the period is bounded by 2 (see Solan (2000)). It is therefore natural to ask whether the minimal length of the period can be uniformly bounded in other classes of stochastic games as well. As our second theorem claims, this is not the case in three-player quitting games.

For every $\epsilon > 0$ and every $\delta > 0$, let $d(\epsilon,\delta)$ be the minimal period of a periodic δ-equilibrium of G_ϵ.

THEOREM 2.2 *For every $\epsilon > 0$ sufficiently small, $\liminf_{\delta\to 0} d(\epsilon,\delta) = +\infty$.*

Since the proof of Theorem 2.2 is similar in spirit to that of Theorem 2.1, we only provide a rough sketch for it.

3 Analysis

Flesch et al (1997) studied the game G_0. The following Lemma summarizes several of their results that are used below.

LEMMA 3.1 *Let $(y(k), \alpha(k))_{k\in\mathbf{N}}$ be an admissible sequence (not necessarily completely absorbing) in G_0. Then*

   1. If $\alpha^i(k) \in (0,1)$ for each $i = 1,2,3$ then $\min_i\{y^i(k+1)\} < \min_i\{y^i(k)\}$.

2. *There is $k \in \mathbf{N}$ such that either $\alpha^1(k) = 0$, or $\alpha^2(k) = 0$, or $\alpha^3(k) = 0$.*

3. *If $\alpha^i(k) = 0$ for some $i = 1, 2, 3$ then there exists $j \neq i$ such that $\alpha^j(k) = 0$ as well.*

Moreover,

4. *For every $\delta > 0$ sufficiently small, the game G_0 does not admit any stationary δ-equilibrium.*

LEMMA 3.2 *If $(y(k), \alpha(k))_{k \in \mathbf{N}}$ is a completely absorbing admissible sequence in G_ϵ, then for every $n \in \mathbf{N}$, the sequence $(y(k), \alpha(k))_{k=n}^\infty$ is a completely absorbing admissible sequence for $y(k)$, provided ϵ is sufficiently small.*

Proof: By definition, $(y(k), \alpha(k))_{k=n}^\infty$ is an admissible sequence for $y(k)$. Since $(y(k), \alpha(k))_{k \in \mathbf{N}}$ is completely absorbing, it is sufficient to prove that $\alpha^i(k) < 1$ for every $i = 1, 2, 3$ and every $k \in \mathbf{N}$. Assume to the contrary that $\alpha^i(k) = 1$ for some i, k. Then $\alpha(k)$ is a 3ϵ-equilibrium in G_ϵ, hence a 4ϵ-equilibrium in G_0, which contradicts Lemma 3.1(4) if ϵ is sufficiently small. ∎

The following two lemmas are easy. The first is a simple matter of continuity, while the second follows from the payoff matrix in Figure 1.

LEMMA 3.3 *Let $(\epsilon_n, x_n, y_n, \alpha_n)_{n \in \mathbf{N}}$ be a sequence such that (i) $\epsilon_n \in (0, 1)$, $x_n, y_n \in \mathbf{R}^3$, $\alpha_n \in [0, 1]^3$ for every $n \in \mathbf{N}$, (ii) the limits $x = \lim_{n \to \infty} x_n$, $y = \lim_{n \to \infty} y_n$ and $\alpha = \lim_{n \to \infty} \alpha_n$ exist, while $\lim_{n \to \infty} \epsilon_n = 0$, and (iii) for every $n \in \mathbf{N}$, α_n is an equilibrium in $G^{\epsilon_n}(x_n)$ that yields payoff y_n. Then α is an equilibrium in $G^0(x)$ that yields payoff y.*

LEMMA 3.4 *Let $x, y \in \mathbf{R}^3$, $\epsilon \in [0, 1)$ and α be an equilibrium in $G^\epsilon(x)$ that yields payoff y. If $\alpha^i > 0$ for every $i = 1, 2, 3$ then $y^i < 1 + \epsilon$ for every $i = 1, 2, 3$.*

LEMMA 3.5 *If $x < (1, 1, 1)$ then for every $\epsilon \in (0, 1/3)$ and every $y \in E^\epsilon(x)$, $y < (1, 1, 1)$.*

Proof: Assume to the contrary that the lemma is not true, and let $x, y \in \mathbf{R}^3$ and $\epsilon \in (0, 1/3)$ satisfy (i) $x < (1, 1, 1)$, (ii) $y \in E^{\epsilon}(x)$, and (iii) $y^1 \geq 1$. Let $\alpha \in [0, 1]^3$ be an equilibrium in $G^{\epsilon}(x)$ that yields payoff y.

Since $x^i < 1$ for every $i = 1, 2, 3$, it cannot be the case that at most two players quit with positive probability under α. Indeed, if a single player i quits with positive probability, this player expects to receive $x^i < 1$ by continuing and 1 by quitting. If two players quit with positive probability, say players i and $i + 1 \mod 3$, then player i expects to receive at least 1 by quitting, and less than 1 by continuing.

Thus, $\alpha^i \in (0, 1)$ for every i. Since $y^1 \geq 1$ and $\alpha^1 \in (0, 1)$, we get

$$x^1(1 - \alpha^2)(1 - \alpha^3) + 3(1 - \alpha^2)\alpha^3 + \alpha^2\alpha^3 = (1 - \alpha^3) + \epsilon\alpha^2(1 - \alpha^3) \geq 1.$$

Since $x^1 < 1$ and $\epsilon > 0$ the left hand-side equality implies that $1 - \alpha^2 + 2\alpha^3 - \alpha^2\alpha^3 > 1 - \alpha^3$, while the right hand-side inequality implies that $\epsilon\alpha^2(1 - \alpha^3) \geq \alpha^3$. These two inequalities imply that $3\alpha^3/(1 + \alpha^3) > \alpha^2 \geq \alpha^3/\epsilon(1 - \alpha^3)$, and therefore $1/3 > \epsilon > \epsilon(1 - \alpha^3) > (1 + \alpha^3)/3 > 1/3$, a contradiction. ∎

Note that if $y \in F_{\epsilon}$ is not trivial, then y is in the convex hull of the payoffs in the entries of the matrix in the game G_{ϵ}. In particular,[5]

$$\sum_{i=1}^{3} y^i \leq 4 \text{ and } 0 \leq y^i \leq 3 \quad \forall i. \tag{1}$$

Let F be the limit set of all non trivial vectors in F_{ϵ}. That is,

$$F = \{y \in \mathbf{R}^3 \mid y = \lim_{n \to \infty} y_n, y_n \in F_{\epsilon_n} \text{ is non trivial}, \epsilon_n \to 0\}.$$

To prove Theorem 2.1, it is sufficient to prove that $F = \emptyset$. Note that by (1), for every $y \in F$, $\sum_{i=1}^{3} y^i \leq 4$, and $0 \leq y^i \leq 3$ for each $i = 1, 2, 3$.

Define $\Delta = \left\{y \in \mathbf{R}^3 \mid \sum_{i=1}^{3} y^i = 4\right\}$. Our next goal is to prove:

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[5]Actually, for every $\epsilon \geq 0$, the stationary strategy $\alpha^i = 1/2$ guarantees player i an expected payoff $1/2$. It follows that the max-min value of each player is at least $1/2$. In particular $y^i \geq 1/2$ for every player i and every non-trivial vector.

LEMMA 3.6 $F \subseteq \Delta$.

Proof: Assume to the contrary that there is $y \in F$ such that $\sum_{i=1}^{3} y^i < 4 - 4\rho$, for some $\rho > 0$.

Let $y_n \to y$ be a sequence such that $y_n \in F_{\epsilon_n}$ is non trivial and $\epsilon_n \to 0$. For every $n \in \mathbf{N}$, let $(y_n(k), \alpha_n(k))_{k \in \mathbf{N}}$ be a completely absorbing admissible sequence for y_n in G_{ϵ_n}. By taking a subsequence, assume w.l.o.g. that $\sum_{i=1}^{3} y_n^i < 4 - 4\rho$ for every $n \in \mathbf{N}$.

We first claim that if y is chosen appropriately, we can assume w.l.o.g. that $\alpha_n^1(1), \alpha_n^2(1) \geq \rho/4$ for every $n \in \mathbf{N}$.

To prove this claim, we will find (i) $y' \in F \setminus \Delta$ such that $d(y', \Delta) \geq \rho^2/16$ (y' may be different from y), (ii) some sequence $(y_n')_{n \in \mathbf{N}}$ such that $y_n' \in F_{\epsilon_n}$ and $y_n' \to y'$, and (iii) for every $n \in \mathbf{N}$ a completely absorbing admissible sequence $(y_n'(k), \alpha_n'(k))_{k \in \mathbf{N}}$ for y_n' in G_{ϵ_n}, such that $\alpha_n'^1(1), \alpha_n'^2(1) \geq \rho/4$.

For every $n \in \mathbf{N}$ let π_n be the probability that under $(\alpha_n(k))_{k \in \mathbf{N}}$, in the stage of absorption at least two players play Q.

If for every $k \in \mathbf{N}$, $\alpha_n^i(k) \geq \rho/4$ for at most one player i, then $\pi_n < 4 \times \rho/4 = \rho$. In particular, it follows that $\sum_{i=1}^{3} y_n^i > 4 - 4\rho$ — a contradiction.

Therefore, for every $n \in \mathbf{N}$ there is $k_n \in \mathbf{N}$ such that $\alpha_n^i(k_n) \geq \rho/4$ for at least two players.

Define for every $n \in \mathbf{N}$ an admissible sequence $(y_n'(k), \alpha_n'(k))_{k \in \mathbf{N}}$ by $y_n'(k) = y_n(k_n + k - 1)$ and $\alpha_n'(k) = \alpha_n(k_n + k - 1)$. Since $(y_n(k), \alpha_n(k))$ is completely absorbing, and by Lemma 3.2, $(y_n'(k), \alpha_n'(k))$ is completely absorbing as well.

By taking a subsequence, we can assume w.l.o.g. that $y' = \lim_{n \to \infty} y_n'(1) = \lim_{n \to \infty} y_n(k_n)$ exists. Since $\alpha_n'^i(1) \geq \rho/4$ for at least two players, and since $\sum_{i=1}^{3} y_n'^i(2) \leq 4$, it follows that $\sum_{i=1}^{3} y_n'^i(1) \leq 4 - (\rho/4)^2$, and therefore $d(y', \Delta) \geq \rho^2/16$.

The claim now follows since the number of players is finite, and the games G_ϵ are symmetric.

By taking a subsequence, we can assume w.l.o.g. that for every $k \in \mathbf{N}$, $y(k) = \lim_{n \to \infty} y_n(k)$ and $\alpha(k) = \lim_{n \to \infty} \alpha_n(k)$ exist. By Lemma 3.3, $y(k) \in E^0(y(k+1))$, and $\alpha(k)$ is the corresponding equilibrium.

Since $\alpha_n^1(1), \alpha_n^2(1) \geq \rho/4$ for every $n \in \mathbf{N}$, $\alpha^1(1), \alpha^2(1) \geq \rho/4$ as well. It follows from Lemma 3.1(3) that $\alpha^3(1) > 0$. By Lemma 3.4, $y(1) < (1,1,1)$.

Let $k > 1$ be the first stage such that $\alpha^i(k) = 0$ for at least one player i. By Lemma 3.1(2), such a finite k exists. By Lemma 3.1(3), $\alpha^i(k) = 0$ for at least two players, say 2 and 3. Moreover, by Lemma 3.1(1), $y^i(k) < 1$ for some i.

In particular, it follows that $\alpha^1(k) \neq 0$ (otherwise, $\alpha(k)$ is not an equilibrium in $G^0(y(k+1))$: player i can profit by quitting). The payoff matrix in Figure 1 implies that $y^1(k) = 1$ and $1 < y^3(k)$. Since there is i such that $y^i(k) < 1$, it follows that $y^2(k) < 1$.

Since $\alpha(k)$ is an equilibrium in $G^0(y(k+1))$ in which only player 1 quits with positive probability, it follows that

$$
\begin{aligned}
y^2(k) &= (1 - \alpha^1(k))y^2(k+1) + 3\alpha^1(k), \text{ and} \\
y^3(k) &= (1 - \alpha^1(k))y^3(k+1).
\end{aligned}
$$

In particular, $1 > y^2(k) > y^2(k+1)$, $1 < y^3(k) < y^3(k+1)$, and $y^1(k) = 1 = y^1(k+1)$. It follows that at stage $k+1$, player 1 is the unique player who quits with positive probability. Indeed, one can verify that if any other subset of players quit with positive probability, $y(k)$ cannot be an equilibrium payoff.

Similarly, in any stage $l > k$, $\alpha^2(l) = \alpha^3(l) = 0$ and $\alpha^1(l) > 0$.

For every $l \geq k$, $\alpha(l)$ is an equilibrium in $G^0(y(l+1))$ that yields expected payoff $y(l)$. It follows that $y^2(l) \geq 1 - \alpha^1(l)$. In particular, for every $l \geq k$ we have $\alpha^1(l) \geq 1 - y^2(l) \geq 1 - y^2(k) > 0$, hence $(y(k), \alpha(k))_{k \in \mathbf{N}}$ is completely absorbing. It follows that $y^3(k) = 0$, hence player 3 can quit at stage k and profit, contradicting the fact that $\alpha(k)$ is an equilibrium in $G^0(y(k+1))$. ∎

Proof of Theorem 2.1: Assume to the contrary that the theorem does not hold. Then there exists a sequence $\epsilon_n \to 0$ and a sequence $(y_n)_{n \in \mathbf{N}}$ such that $y_n \in F_{\epsilon_n}$. For every $n \in \mathbf{N}$

let $(y_n(k), \alpha_n(k))$ be a completely absorbing admissible sequence for y_n in G_{ϵ_n}. By Lemma 3.6, we can assume w.l.o.g. that $d(y_n, \Delta) < \rho$ for every $n \in \mathbf{N}$, where $\rho \in (0, 1)$ is arbitrary.

Lemmas 3.3 and 3.6 imply that it cannot be the case that for every $n \in \mathbf{N}$ there is $k_n \in \mathbf{N}$ such that $\alpha_n^i(k_n) > 0$ for all i. Indeed, otherwise, any accumulation point y of the sequence $y_n(k_n)$, as n goes to infinity, is in F. By Lemma 3.3 $\sum_{i=1}^{3} y^i \leq 3$, which contradicts Lemma 3.6.

Fix n sufficiently large such that $\epsilon_n \in (0, (1 - \rho)/3)$.

We will now show that there is a stage k such that $\alpha_n^i(k) > 0$ for exactly two players. As discussed above, if n is sufficiently large then for every $k \in \mathbf{N}$ there is at least one player i such that $\alpha_n^i(k) = 0$.

So assume that for every $k \in \mathbf{N}$, for at most one player i we have $\alpha_n^i(k) > 0$. Since y_n is not trivial, there is a stage k such that at that stage one player, say player 1, quits with positive probability. If player 1 is the only player who ever quits with positive probability under α_n, then, since y_n is not trivial, $y_n^3(1) = 0$, hence player 3 could have gained by quitting at stage 1. Hence there is a first stage l such that $\alpha_n^2(l) + \alpha_n^3(l) > 0$. Since player 1 is the only player who quits with positive probability until stage l, $y_n^1(l) = 1$ and $y_n^3(l) > 1$, which implies by the payoff matrix in Figure 1 that $\alpha_n^3(l) = 0$, hence $\alpha_n^2(l) > 0$. However, in this case player 1 can profit by quitting with probability 1 at stage l, and receiving more than 1.

By Lemma 3.2 we can assume w.l.o.g. that $\alpha_n^1(1), \alpha_n^2(1) > 0$, while $\alpha_n^3(1) = 0$. Indeed, since $y_n(1) \in F_{\epsilon_n}$, $y_n(k) \in F_{\epsilon_n}$ as well.

Let m be the minimal integer for which either $\alpha_n^1(m) = 0$, or $\alpha_n^2(m) = 0$, or both. Let us first argue that such a m exists. Otherwise, for every $m \in \mathbf{N}$, $\alpha_n^1(m), \alpha_n^2(m) > 0$, hence $\alpha_n^3(m) = 0$. By the payoff matrix in Figure 1, the overall probability that under $(\alpha_n(m))_{m \in \mathbf{N}}$ player 1 will ever quit is at most $1/3$ (otherwise, player 2 can profit by never quitting). In particular, for some m, the overall probability that under α_n player 1 will ever quit after stage m is at most ϵ_n, while the overall probability that player 2 will ever quit after that stage is 1 (since the sequence is completely absorbing). In particular, player 1 is better of

by quitting at stage $m+1$ with probability 1.

Thus, for every l such that $1 \leq l < m$, we have $\alpha_n^1(l), \alpha_n^2(l) > 0$, while $\alpha_n^3(l) = 0$. Let $p = 1 - \prod_{l=1}^{m-1}(1 - \alpha_n^2(l))$ be the overall probability that player 2 quits during the first $m-1$ stages, provided player 1 does not quit. Since $y_n^1(1) > 1$, $y_n^1(m) < 1 + \epsilon_n$, and $y_n^1(1) = (1-p)y_n^1(m)$, it follows that $p < \epsilon_n/(1+\epsilon_n) < \epsilon_n$. Since $d(y_n(1), \Delta) < \rho$, $y_n^1(1) < 1 + \epsilon_n$ and $y_n^2(1) < 1$, it follows that $y_n^3(1) > 2 - \rho - \epsilon_n$. Since $p < \epsilon_n$, and whenever player 1 quits player 3 receives at most 1, it follows that $y_n^3(m) > 2 - \rho - 2\epsilon_n$.

So we have asserted that $y_n^1(m) > 1$, $y_n^2(m) < 1$, and $y_n^3(m) > 2 - \rho - 2\epsilon_n$. Let $S = \{i \in \{1,2,3\} \mid \alpha_n^i(m) > 0\}$. We will prove that $S = \emptyset$.

As already discussed, $S \neq \{1,2,3\}$. Since $y_n^i(m) \neq 1$ for every $i = 1,2,3$, it cannot be that $|S| = 1$. By the choice of m, $S \neq \{1,2\}$. Since $y_n^2(m) < 1$, $S \neq \{2,3\}$. Since $y_n^1(m) > 1$, $S \neq \{1,3\}$.

Thus, $S = \emptyset$, and therefore $y_n(m) = y_n(m+1)$. Since $y^2(m) < 1$ player 2 can quit at stage m and profit, contradicting the fact that $\alpha_n(m)$ is an equilibrium in $G^{\epsilon_n}(y_n(m+1))$.
∎

Sketch of the Proof of Theorem 2.2: Assume to the contrary that for some fixed $\epsilon > 0$ sufficiently small, $\liminf_{\delta \to 0} d(\epsilon, \delta) < +\infty$. Then there is $K \in \mathbf{N}$ and a sequence $\delta_n \to 0$ such that for every $n \in \mathbf{N}$ there is a periodic δ_n-equilibrium in G_ϵ with period K. Let $(\alpha_n(1), \ldots, \alpha_n(K))$ be the period of the δ_n-equilibrium, and let $(y_n(1), \ldots, y_n(K))$ be the corresponding sequence of payoffs. In particular, $\alpha_n(k)$ is an ϵ-equilibrium in the one-shot game $G^\epsilon(y_n(k+1))$ (addition modulo K), that yields expected payoff $y_n(k)$.

By taking a subsequence, we can assume w.l.o.g. that the limits $\alpha(k) = \lim_{n \to \infty} \alpha_n(k)$ and $y(k) = \lim_{n \to \infty} y_n(k)$ exist for every $k = 1, \ldots, K$. By an analogue of Lemma 3.3, $(y(k), \alpha(k))_{k \in \mathbf{N}}$ is an admissible sequence in G_0, where for $k > K$, $y(k) = y(k \bmod K)$ and $\alpha(k) = \alpha(k \bmod K)$.

As in the proof of Lemma 3.6 one can show that $y(k) \in \Delta$ for every $k \in \mathbf{N}$. A similar analysis to that done in the proof of Theorem 2.1 leads to a contradiction. ∎

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