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"An Optimal Auction When Resale Cannot be Prohibited"

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An Optimal Auction When Resale Cannot Be Prohibited*

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Abstract

A criticism of auction theory was that resales may wash away the impact of auction design, with bidders able to buy the good later. This paper thus considers a seller's auction-design problem when he cannot prohibit resales. The "optimal" auction in the existing theory—which does not consider resales—would subsidize a disadvantaged bidder to intensify the bidding competition. The paper proves that this "optimal" auction fails to be optimal, and then designs a new mechanism that is optimal. The "optimal" auction fails for two reasons. First, a bidder can distort his bid to manipulate prices at resales. Second, a bidder can buy the good from the subsidized bidder instead of competing with him. To fix these problems, the new mechanism designed here uses a novel payment scheme to cut off the unwanted information linkage between auctions and resales, and it subsidizes the disadvantaged bidder—even more than the "optimal" auction would do—so that he will choose to resell the good at prices desirable for the initial seller. Expecting such prices, the advantaged bidder cannot profit from distorting bids. Motivating the disadvantaged bidder by the subsidy and resale profit, the new mechanism intensifies the bidding competition and optimizes for the initial seller. Due to the subsidy, with a positive probability the good ends with a bidder who values it less than others. Consequently, an optimally designed auction can still manipulate where a good will eventually go, despite bidders' free access to resales.

Key Words: auction, optimal auction, resale, secondary market, collusion
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1 Introduction

An *optimal* auction is a mechanism designed to maximize a seller's expected profit. Although the literature on optimal auctions has become a basic ingredient of mechanism design, it assumes that bidders cannot trade among themselves after an auction. While this assumption may be plausible when a government auctions off non-transferable licenses, it is usually difficult, if not impossible, for a seller to prevent resales (Internet auctions, U.S. Treasury bill auctions, etc.).[1]

Intuitively, the no-resale assumption is crucial to the design of optimal auctions. As well-known in auction theory, an important feature of an optimal auction is that it subsidizes a disadvantaged bidder, thereby intensifying the bidding competition and raising revenues. Specifically, when bidders' valuations are distributed differently, with positive probability an optimal auction sells the good to a bidder who values it less than someone else.[2] With resales allowed, however, a subsidized low-value bidder can resell the good to others, who may want to wait for resales instead of bidding in the auction. If such a scenario turns out to be an equilibrium, then the "optimal" auction in the existing theory would hurt the seller instead of optimizing for him.

This paper thus considers optimal auction design when resale cannot be prohibited. The environment is a two-stage game with one good and two bidders. At stage one, the initial owner of the good chooses a mechanism to sell the good to at most one of the bidders. At stage two, the *winner*—the bidder who bought the good—posts a *resale price* to resell the good to the other bidder (the *loser*). The paper assumes that players' *types*—the utilities from consuming the good—are private and independently drawn from different distributions. Once a stage-one mechanism is chosen, the continuation game is analyzed according to perfect Bayesian equilibrium (PBE). A *dynamically optimal* auction will mean a stage-one mechanism whose continuation game has a PBE that gives the initial owner the highest expected payoff among all stage-one mechanisms whose continuation games have PBEs. In contrast, the "optimal" auction in the existing theory will be called *statically optimal auction*. This model departs from the optimal-auction literature by assuming that resale cannot be prohibited, and it inherits from this literature the assumption that any seller, initial or secondary, can commit to withholding the good.

The first question is whether resale really matters to optimal auction design. The paper finds that the answer is "yes." Specifically, if the initial owner sells the good via the statically optimal auction, there is no equilibrium that supports truthful bidding (Proposition 5.1); instead, there is a "collusive" equilibrium where the bidder who is not subsidized buys the good from the subsidized bidder instead of competing with him. (Proposition 5.2).

[1] Even if a government prohibits the transfer of licenses, resale can take the form of post-auction mergers among the bidders.

[2] It is not uncommon that actual auctions subsidize designated bidders. An example is the U.S. "regional narrowband" spectrum auction in 1994, where the government offered bidding credits to women- and minority-owned firms.

The reason why this mechanism fails is two-fold, and both are due to the payment scheme. First, from the amount of his payment a winner learns too much information about the loser, which the loser can manipulate to reduce resale prices. This leads to the impossibility of truthful bidding. Second, the mechanism charges a winner according to the loser's bid. The subsidized bidder can thus bid high, thereby crowding out the rival and profiting from resale. The other bidder shies away because he strictly prefers buying the good from resale to winning in the auction. This leads to the collusive equilibrium.

The next and more fundamental question is what an optimal auction looks like when resale cannot be prohibited. With the failure of the statically optimal auction, one might be tempted to conclude that it is no good for a seller to subsidize a bidder. The aforementioned reason for its failure, however, points to a different suspect, the payment scheme. A tentative idea is therefore to restructure the payment scheme to implement the subsidization rule. The trouble is how to strike a balance between subsidizing a disadvantaged bidder on one hand and providing sufficient incentive for the other bidder to participate on the other hand. With resale allowed, it seemed difficult to attract the other bidder away from the resale.

Yet the above idea worked. This paper designs a mechanism, called M^Z, and proves that it is optimal when resale cannot be prohibited (Proposition 6.1). The mechanism has two features. First, the subsidized bidder enjoys even more subsidies than he does in the statically optimal auction. Specifically, it is more probable for M^Z to sell the good to the subsidized lower-type bidder than it is for the statically optimal auction. Second, the payment scheme looks a little bit like a hybrid of the first-price and second-price auctions: On one hand, if the subsidized bidder wins, his payment depends only on his own bid; on the other hand, the payment from the other bidder, if he beats the subsidized rival, depends only on the bid from the subsidized bidder.

To understand this mechanism, let us analyze an example.

Example 1 *Bidder 1's type is uniformly distributed on* $[0, 10]$ *(in dollars), bidder 2's type is commonly known to be \$2, and the initial owner's type is commonly known to be zero.*

The statically optimal auction in this example, which can be calculated easily (Eqs. (2), (3), and (4)), is: offer the good to bidder 1 at price \$6; if bidder 1 rejects, sell the good to bidder 2 at price \$2. Notice that the mechanism subsidizes bidder 2 by selling him the good when bidder 1's bid is between \$2 and \$6.

This mechanism would have generated an expected revenue of \$3.6 had resale been prohibited. With resale allowed, it is obvious that there is no equilibrium that supports truthful bidding. Otherwise, upon winning in the auction, bidder 2 would learn that bidder 1's type is at most \$6 and would therefore offer a resale price \$4, but then bidder 1 with types in $(4, 10]$ would rather lie in the auction and buy the good at resale. To induce truthful bidding from bidder 1, we need bidder 2 to set a resale price at least as high as \$6. But how can we induce bidder 2 to do that? The solution is to subsidize him further—sell the good to him whenever bidder 1's bid is below \$10. Then, upon winning, bidder 2 learns only that bidder 1's type is bounded above by \$10, and his optimal resale price is \$6, which is exactly the price we want to charge bidder 1. Thus, the purpose of the first feature of the mechanism M^Z is to induce the subsidized bidder, who acts as a middleman, to charge the other bidder the price we desire.

Obviously, the above technique of further subsidization would not achieve its purpose if bidder 2 can infer bidder 1's type from the amount of his payment. This brings us to one aspect of the second feature: the subsidized bidder's payment should be independent of the bid from the other bidder. In Example 1, bidder 2's payment, if he wins, is \$3.6, regardless of bidder 1's bid. Here the number 3.6 comes from bidder 2's expected profit from resale. To induce truthful bidding from bidder 1, the mechanism sells the good to him at the price \$6 if he does bid \$10. Offered the same price in the auction and in the anticipated resale, bidder 1 cannot profit from lying in the auction. This brings us to the other aspect of the second feature of the mechanism: the payment from the bidder who is not subsidized should be independent of his own bid. (When the type of the subsidized bidder is nondegenerate, the construction of the payment scheme becomes nontrivial. That is because bidder 2's expected profit from resale is unknown to others, and bidder 1, who is not supposed to act as a middleman, may want to win and profit from resale.)

Example 1, while trivial, illustrates how the new mechanism M^Z works: It restructures the payment scheme to cut off the unwanted information linkage between auction and resale, and it subsidizes a bidder so that he, through optimally choosing a resale price, implements the allocation rule intended by the statically optimal auction.

Why is the mechanism M^Z optimal? The reason is that the initial owner cannot do better when resale cannot be prohibited than he could have done when resale can be prohibited (Lemmas 3.1 and 3.2). Thus, the expected revenue (\$3.6 in Example 1) the statically optimal auction generates when resale can be prohibited is an upper bound for the incentive-feasible expected revenues in our environment. The mechanism M^Z is therefore optimal because the expected revenue it generates is equal to that upper bound. In Example 1, it is obvious that M^Z yields an expected revenue \$3.6. In general, M^Z attains to the upper bound because the mechanism, coupled with the equilibrium path in resale, selects the final owner of the good and allocates surplus across bidders in exactly the same way as the statically optimal auction does in the setting where resale can be prohibited. Then the revenue equivalence theorem applies. (See Lemmas 3.4 and 6.3 for a proof.)

The literature on auctions with resales is scarce. The only other paper about optimal auctions with resales that I know is Ausubel and Cramton [1]. They assume that resale always achieves Pareto efficiency. This paper studies an environment where this assumption need not hold, and it proves that resale does not achieve efficiency whether the initial owner uses the statically optimal or the dynamically optimal auctions.[3] The other analytical works

[3] Interestingly, when the statically optimal auction is used, resale not only reinforces the misallocating rule intended by the statically optimal auction, but also misallocates with a greater probability than the intended level. In Figure 3, for instance, the statically optimal auction intends to misallocate the good to the lower-type bidder 2 when the type profile belongs to the area $ABLH$, while the collusive equilibrium misallocates

on auctions with resales have provided only positive analyses of several auction forms. These works include Haile [5, 6, 7], Bikhchandani and Huang [2], Bose and Deltas [3], Gupta and Lebrun [4], Milgrom [9], and Kamien, Li, and Samet [8].

Section 2 first formulates an auction environment where resale cannot be prohibited—a busy ready is advised to skip to Section 7 after the first four paragraphs of Section 2—and then formulates the equilibrium concept and the auction-design problem. Section 3 examines the relationship between the auction-design problem in our environment and that in the "textbook" environment where resale can be prohibited. Section 4 calculates the optimal resale price for a winner of an auction. Section 5 answers why the "optimal" auction in the literature fails in our environment. Section 6 constructs a new mechanism and proves that it is optimal in our environment. Section 7 analyzes a nontrivial example.

2 The Model

There are three players and one indivisible good. The good is initially owned by player 0, called *initial owner*. The other players, denoted by 1 and 2, are the *bidders* for the good. For each bidder i, the other bidder is denoted by $-i$.

For each $i = 0, 1, 2$, player i's utility from consuming the good is t_i. If he makes a money payment p (a negative p means receiving the payment), then his payoff is

$$t_i \mathbf{1}_{\text{consume}} - p,$$

where $\mathbf{1}_{\text{consume}}$ is one if he consumes the good and zero if otherwise. We will call t_i player i's *type*. The initial owner's type t_0 is common knowledge. For each $i = 1, 2$, bidder i's type t_i is his private information; players other than i have a common prior that t_i is independently drawn from a distribution F_i, with density function f_i and support $[\underline{t}_i, \bar{t}_i]$.

We model the environment as the following two-stage game. At stage 1, the initial owner chooses a mechanism, say M, to sell the good to at most one bidder. Such a bidder will be called *winner*, and the other bidder will be called *loser*. Each bidder chooses an action, called *bid*, allowed by the mechanism. In addition to being resource-feasible, the mechanism must allow non-participation and cannot have allocations contingent on the events after the initial owner exits the game. We will call such a mechanism *stage-one mechanism.* The identity of the winner is the only new common knowledge after M is carried out.

a. If the mechanism M does not pick any winner, then the initial owner consumes the good and the game ends.

throughout the larger area $AEDH$. The reason is that the collusive equilibrium eliminates the competition between bidders. If they were competing, the initial owner's incentive to subsidize bidder 2 would be curbed by his uncertainty about bidder 2's type, because raising bidder 2's winning probabilities requires raising the bidder's informational surplus. This brake against misallocation is gone, now that bidder 2 becomes the monopolistic middleman.

b. If the mechanism M picks a winner, say w, then the initial owner trades with the winner according to M and exits the game. Each bidder i's net payment p_1^i to others at stage one, called *stage-one payment*, adds to i's private information. This ends stage one. At stage two, the winner w posts a *resale price* p_2 for the good. The loser, denoted by l, can either accept or reject the resale price. The two players trade according to the price p_2 if l accepts it, and otherwise no trade occurs. The final owner then consumes the good and the entire game ends.

The game modeled above reflects two important features. One is that the initial owner cannot prohibit resales. This is captured by the assumption that the allocation of a stage-one mechanism cannot be contingent on the events after stage one. The other feature is that each owner, whether initial or secondary, can commit to withholding the good.[4] (A busy reader is advised to skip to Section 7 from here.)

2.1 Strategies and Beliefs

Once the initial owner has chosen any stage-one mechanism M, a *strategy profile* of the continuation game in our two-stage game is a list $\langle s_1, s_2, \wp_2, r\rangle$, and a *belief system* is a pair $\mu := \langle \mu_w, \mu_l\rangle$, with the following meaning:

- Bidder i $(i = 1, 2)$ bids $s_i(t_i)$ in the stage-one mechanism M given his type t_i.
- A winner w chooses a resale price $\wp_2(w, b_w, p_1^w, t_w)$ given his type t_w and the history that he won at the stage-one mechanism M by action b_w and paid p_1^w.
- The loser l chooses a response $r(b_l, p_1^l, p_2, t_l) \in \{\text{accept}, \text{reject}\}$, given his type t_l and the history that he had bid b_l and lost at stage one, paid p_1^l, and the winner has offered a price p_2.
- The cumulative distribution function (cdf) $\mu_w(b_w, p_1^w)$ denotes a winner w's updated belief about the loser l's type, given the history that he has bid b_w, won at stage one, and paid p_1^w.
- The cdf $\mu_l(b_l, p_1^l)$ denotes a loser l's updated belief about winner w's type, given the history that he has bid b_l, lost at M, and paid p_1^l.[5]

[4] One would wish to allow a secondary owner to choose *any* resale mechanism in addition to take-it-or-leave-it offers. Such a generalization would complicate the equilibrium concept. The reason is that a secondary owner may choose a mechanism whose continuation game has no equilibrium, while an equilibrium that respects the multistage structure of our game should specify players' best responses at all possible scenarios. Equilibrium-lacking mechanisms cannot be ruled out a priori, because the sequential rationality of a strategy profile depends on the belief system, which in turn should follow Bayes's rule according to the strategy profile, while the strategy profile must specify players' actions at each event. The set of resale mechanisms is thus restricted to take-it-or-leave-it offers.

[5] The loser's updated belief after seeing a resale price plays no role, with types independent across bidders.

We say that a belief system μ is *Bayesian* with respect to a strategy profile $\langle s_1, s_2, \wp_2, r\rangle$ if Bayes's rule is followed whenever possible, for any possible winner w, any type t_i of any bidder i, any bid b_i and any stage-one payment p_1^i possible in the mechanism M, and any resale price p_2:

a. After mechanism M is carried out, winner w forms belief $\mu_w(b_w, p_1^w)$ from the prior belief F_l about loser l's type, the fact that w's bid b_w beats l's bid $s_l(t_l)$ in mechanism M, and the fact that M mandates w to pay p_1^w according to the bids $(b_w, s_l(t_l))$.

b. After mechanism M is carried out, loser l forms his belief $\mu_l(b_l, p_1^l)$ from the prior belief F_w about w's type, the fact that w's bid $s_w(t_w)$ beats l's bid b_l in mechanism M, and the fact that M mandates l to pay p_1^l according to the bids $(s_w(t_w), b_l)$.

Notice that a player's belief updated in the above manner does not depend on his type. The reason is that a player learns about the other's type from the observed allocation outcomes mandated by the mechanism, which receives only the bids (b_1, b_2) from the bidders. This paper therefore denotes a player i's updated belief by $\mu_i(b_i, p_1^i)$.

2.2 The Definition of Equilibrium

Some notations are needed before we define the equilibrium concept. Denote a stage-one mechanism by a list

$$M := (B_i, q_i, \wp_{1i})_{i=1}^2;$$

bidder i is allowed to submit bids from the set B_i; given any profile (b_i, b_{-i}) of bids, bidder i wins at stage one with probability $q_i(b_i, b_{-i})$; at stage one, his net payment to others is $\wp_{1i}(b_i, b_{-i}, \text{win})$ if he wins at stage one, and is $\wp_{1i}(b_i, b_{-i}, \text{lose})$ if otherwise. (Hence the initial owner's revenue is $\sum_i[\wp_{1i}(b_i, b_{-i}, \text{win}) + \wp_{1i}(b_i, b_{-i}, \text{lose})]$.) For the continuation game following the choice of a stage-one mechanism, denote a strategy-belief profile by a list

$$e := (\langle s_1, s_2, \wp_2, r\rangle, \mu).$$

Given a pair $\langle M, e\rangle$ of mechanism and strategy-belief profile, one easily calculates:

- Bidder i's expected stage-one payment $P_i^e(b_i)$, calculated at the beginning of stage one and given the prior belief F_{-i}, if he bids b_i and bidder $-i$ plays strategy s_{-i}.
- A type-t_w winner w's expected profit $W_w^e(p_2; b_w, p_1^w, t_w)$ from resale, calculated at the end of stage one, if at stage one he bid b_w, won, and paid p_1^w, and at stage two he posts a resale price p_2 and has updated belief $\mu_w(b_w, p_1^w)$ about the loser's type.
- A type-t_l loser l's expected profit $L_l^e(\chi, p_2, t_l)$ from resale, given his response χ to the resale price p_2 at stage two.

- A type-t_i bidder i's expected payoff $\pi_i^e(b_i, \wp_2', r'; t_i)$, calculated at the beginning of stage one, if he bids b_i at stage one and plays strategies $\wp_2'$ (if he wins) and r' (if he loses), and if bidder $-i$ abides to the strategy profile in e.

To calculate $\pi_i^e(b_i, \wp_2', r'; t_i)$, we subtract i's expected stage-one payment $P_i^e(b_i)$ and add:

- The expected value (based on the prior F_{-i}) of the winning probability $q_i(b_i, s_{-i}(t_{-i}))$ times bidder i's expected gain upon winning, which is

$$t_i + W_i^e(\wp_2'[i, b_i, p_1^i(e,+), t_i]; b_i, p_1^i(e,+), t_i),$$

where $p_1^i(e,+) := \wp_{1i}(b_i, s_{-i}(t_{-i}), \text{win})$ denotes i's stage-one payment if he wins.

- The expected value (based on the prior F_{-i}) of $(1-q_i(b_i, s_{-i}(t_{-i})))q_{-i}(b_i, s_{-i}(t_{-i}))$ times bidder i's expected gain when he loses the good to bidder $-i$ at stage one, which is the expected value (based on the updated belief $\mu_i(b_i, p_1^i(e,-))$) of

$$L_i(r'[b_i, p_1^i(e,-), p_2(e,-), t_i]; p_2(e,-), t_i),$$

where $p_1^i(e,-) := \wp_{1i}(b_i, s_{-i}(t_{-i}), \text{lose})$ is i's stage-one payment if he does not win, and $p_2(e,-) := \wp_2(-i, s_{-i}(t_{-i}), p_1^{-i}(e,+), t_{-i})$ is the resale price posted by winner $-i$.

We are now ready to define the equilibrium concept. Once a stage-one mechanism M is chosen, a *perfect Bayesian equilibrium* (PBE) of the continuation game is a strategy-belief profile $e := (\langle s_1, s_2, \wp_2, r\rangle, \mu)$ that satisfies the following conditions:

1. The belief system μ is Bayesian with respect to the strategy profile $\langle s_1, s_2, \wp_2, r\rangle$.
2. The strategy profile $\langle s_1, s_2, \wp_2, r\rangle$ is sequentially rational with respect to μ:
   a. Given any resale price p_2 at stage two, a loser l with any type t_l accepts the offer if $p_2 \le t_l$ and rejects it if otherwise.
   b. Given any stage-one winner's bid b_w, stage-one payment p_1^w, and type t_w, the resale price $\wp_2(w, b_w, p_1^w, t_w)$ maximizes the winner's expected profit $W_w^e(\cdot; b_w, p_1^w, t_w)$ from resale, given his updated belief $\mu(b_w, p_1^w)$.
   c. For each $i \in \{1,2\}$ and given any type t_i, bidder i's strategy $\langle s_i(t_i), \wp_2, r\rangle$ maximizes his expected payoff $\pi_i^e(b_i, \wp_2', r'; t_i)$ over all possible stage-one bids b_i, resale price strategies $\wp_2'$, and response strategies r'.

This definition remains unchanged if Condition 2c is replaced with the following condition:

2c′ For each $i \in \{1,2\}$ and given any type t_i, bidder i's stage-one bid $s_i(t_i)$ maximizes his expected payoff $\pi_i^e(b_i, \wp_2, r; t_i)$ over all possible stage-one bids b_i.

One readily sees that "Conditions 2a, 2b, and 2c" (*sequential rationality*) is equivalent to "Conditions 2a, 2b, and 2c′" (*myopic rationality*).[6] This paper will therefore use Conditions 2c and 2c′ interchangeably.

2.3 The Initial Owner's Problem

When the continuation game following the choice of a stage-one mechanism M has a perfect Bayesian equilibrium (PBE) e, we call $\langle M, e\rangle$ *mechanism-PBE pair.* The initial owner's problem is to maximize his expected payoff over all possible mechanism-PBE pairs. The stage-one mechanism in such a seller-optimal pair is said to be *dynamically optimal.*

Removing the resale stage from our auction-resale game, which we call *dynamic environment,* one obtains an auction environment where the initial owner can prohibit resales. We call it the *static environment.* Given a mechanism in a static environment, its *Bayesian Nash equilibrium (BNE)* is defined as usual. The initial owner's problem in a static environment is to maximize his expected payoff over all possible mechanism-BNE pairs. This is the problem considered in the existing optimal-auction theory. The mechanism in such a seller-optimal mechanism-BNE pair is said to be *statically optimal.*

3 Preliminary Analysis

Let us start by examining the relationship between the initial owner's problem in our dynamic setting and that in the static setting. In the spirit of the revelation principle in multistage games (Myerson [11]), the first observation is that any incentive-feasible mechanism in the dynamic environment is also incentive-feasible in the static environment.

Lemma 3.1 *For any mechanism-PBE pair $\langle M, e\rangle$ there exists a mechanism-BNE pair $\langle M', s'\rangle$ that induces the same final allocation (the final owner of the good, and the transfers among the players) in the static environment as $\langle M, e\rangle$ does in the dynamic environment.*

Proof: If $\langle M, e\rangle$ is a mechanism-PBE pair, then its allocation outcome is equivalent to the following two-phase revelation game M' coupled with truth-telling: At phase one, each bidder secretly reports his type to a neutral mediator, who then plays the stage-one mechanism M on the bidder's behalf according to the strategy in e, and then allocates according to M. At phase two, each bidder reports his type again to the neutral mediator; this report is allowed

[6] Sequential rationality implies myopic rationality because 2c implies 2c′. For the converse, "2a and 2b" implies that, given any history up to the beginning of stage two, a myopically rational strategy profile maximizes a player's expected payoff from then on. Thus, if a myopically rational strategy profile is not sequentially rational, then it must have prescribed a dominated stage-one bid for a bidder with some type. But that would violate Condition 2c′.

to differ from the previous one. The mediator, remembering each bidder's phase-one report and phase-one allocation, sets a resale price and a response according to the strategy in e, and then allocates accordingly. Obviously, truth-telling is supported by a PBE of this revelation game M'. Consequently, the sequential rationality of PBE implies that truth-telling is a BNE of M'. ■

The above lemma implies that the initial owner cannot do better in our dynamic environment than in the corresponding static environment.

Lemma 3.2 *An upper bound of the initial owner's expected payoff from any mechanism-PBE pair in the dynamic environment is his maximum expected payoff over all mechanism-BNE pairs in the static environment.*

Proof: This follows from Lemma 3.1 and the initial owner's problems defined in Subsection 2.3. ■

The above lemma indicates a sufficient condition for a mechanism to be dynamically optimal: delivering an expected revenue equal to the level achieved by the statically optimal auction in the static environment. We hence review the statically optimal auction here.

For each bidder $i \in \{1,2\}$, define the *virtual utility* by

$$V_i(t_i) := t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}, \ \forall t_i \in [\underline{t}_i, \bar{t}_i]. \tag{1}$$

We denote the following mechanism (Myerson [10]) by $M^* := (B_i^*, q_i^*, \wp_{1i}^*)_{i=1}^2$:

1. Each bidder i's message space B_i^* is the support of his type.
2. Given any message profile $(\hat{t}_1, \hat{t}_2)$, a bidder i ($\forall i = 1,2$) wins the good with probability

$$q_i^*(\hat{t}_1, \hat{t}_2) := \begin{cases} 1 & \text{if } V_i(\hat{t}_i) > V_{-i}(\hat{t}_{-i}) \text{ and } V_i(\hat{t}_i) \geq t_0 \\ 0 & \text{if } V_i(\hat{t}_i) < V_{-i}(\hat{t}_{-i}) \text{ or } V_i(\hat{t}_i) < t_0 \\ 1 - q_{-i}^*(\hat{t}_1, \hat{t}_2) \in (0,1) & \text{if } V_i(\hat{t}_i) = V_{-i}(\hat{t}_{-i}) \geq t_0 \end{cases} \tag{2}$$

   and pays the initial owner

$$\wp_{1i}^*(\hat{t}_1, \hat{t}_2, \text{win}) := \inf\left\{\hat{t}_i \in \text{support } (F_i) : \ V_i(\hat{t}_i) \geq \max\{V_{-i}(\hat{t}_{-i}), t_0\}\right\}; \tag{3}$$

$$\wp_{1i}^*(\hat{t}_1, \hat{t}_2, \text{lose}) := 0. \tag{4}$$

As well-known in auction theory, the mechanism M^* is statically optimal, given the monotone-hazard-rate assumption that each virtual utility function V_i is strictly increasing. We therefore call M^* the *statically optimal auction*.

The following lemma spells out the allocation rule of the statically optimal auction. Figure 1 illustrates the case of $\underline{t}_i = 0$ ($\forall i = 1,2$). The auction offers bidder-specific *reserve* *prices*: $V_i^{-1}(t_0)$ for bidder i. If bidder i's bid is below the reserve price offered to him, then he is not picked as the winner. If both bidders bid above their reserve prices, then the good is allocated according to the *dividing curve* HL: bidder 1 wins if the bid profile lies to the right of the curve, and bidder 2 wins if otherwise. Notice that, with a positive probability, the good goes to the bidder who values it less than the other. In Figure 1, this event corresponds to the area $KLHI$, where bidder 2 wins despite his lower types. For simplicity, we assume:

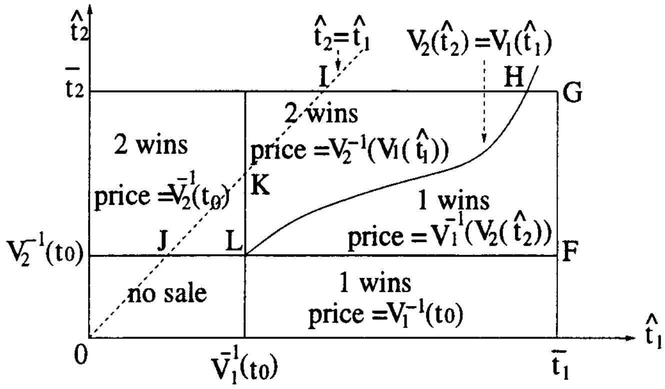

Figure 1: The Statically Optimal Auction M^*

Assumption 1 (Nontriviality) *For each bidder* $i \in \{1,2\}$, $\bar{t}_i > t_0 \geq V_i(\underline{t}_i)$.

The first inequality in this assumption says that there are gains from trade between the initial owner and the bidders. The second inequality implies that the reserve price for a bidder in the statically optimal auction M^* is not below the bidder's lowest possible type. Consequently, for each $i = 1, 2$, the point $V_i^{-1}(t_0)$ exists and is equal to the reserve price M^* offers to bidder i.

Lemma 3.3 *Assume for each* $i \in \{1,2\}$ *that the function* V_i *is strictly increasing. For any profile* $(\hat{t}_1, \hat{t}_2)$ *of* reported *types, the mechanism* M^* *allocates according to* exactly *one of the following six cases:*

$$\left[\hat{t}_1 < V_1^{-1}(t_0) \;\&\; \hat{t}_2 < V_2^{-1}(t_0)\right] \;\Rightarrow\; \textit{no trade}; \tag{5}$$
$$\left[\hat{t}_1 \geq V_1^{-1}(t_0) \;\&\; \hat{t}_2 < V_2^{-1}(t_0)\right] \;\Rightarrow\; \textit{bidder 1 wins \& pays } V_1^{-1}(t_0); \tag{6}$$
$$\left[\hat{t}_1 < V_1^{-1}(t_0) \;\&\; \hat{t}_2 \geq V_2^{-1}(t_0)\right] \;\Rightarrow\; \textit{bidder 2 wins \& pays } V_2^{-1}(t_0); \tag{7}$$
$$V_1^{-1}(t_0) \leq \hat{t}_1 < V_1^{-1}(V_2(\hat{t}_2)) \;\Rightarrow\; \textit{bidder 2 wins \& pays } V_2^{-1}(V_1(\hat{t}_1)); \tag{8}$$
$$V_2^{-1}(t_0) \leq \hat{t}_2 < V_2^{-1}(V_1(\hat{t}_1)) \;\Rightarrow\; \textit{bidder 1 wins \& pays } V_1^{-1}(V_2(\hat{t}_2)); \tag{9}$$
$$V_1(\hat{t}_1) = V_2(\hat{t}_2) \geq t_0 \;\Rightarrow\; \textit{a randomly picked winner } i \textit{ pays } V_i^{-1}(V_{-i}(\hat{t}_{-i})). \tag{10}$$

Proof: Since V_i are strictly increasing, the inverse functions V_i^{-1} exist and the function $V_2^{-1} \circ V_1$ is strictly increasing. This, coupled with the fact that the point $(V_1^{-1}(t_0), V_2^{-1}(t_0))$

lies on the curve $\hat{t}_2 = V_2^{-1} \circ V_1(\hat{t}_1)$, implies that the six cases listed above constitute a partition of the message space $[\underline{t}_1, \bar{t}_1] \times [\underline{t}_2, \bar{t}_2]$. Thus, the allocation outcome of the mechanism M^* falls into exactly one of the six cases. The payments in that list follow from Eq. (3). This proves the lemma. ■

The statically optimal auction M^* is important in our context, because a stage-one mechanism will be dynamically optimal if its continuation game has a PBE that induces the same allocation rule and the same surplus for lowest types in the dynamic setting as M^* does in the static setting.

Lemma 3.4 *Assume that the functions V_i are strictly increasing. Let $\langle M, e\rangle$ be a mechanism-PBE pair. Let the pair induce the probability $\tilde{q}_i(t_1, t_2)$ with which bidder i becomes the final owner of the good given the type profile (t_1, t_2) ($\forall t_1 \forall t_2$). Let the pair induce the surplus $\tilde{U}_i(\underline{t}_i)$ for bidder i with type $\underline{t}_i$ ($i = 1, 2$).*

a. *If $\tilde{q}_i(t_1, t_2) = q_i^*(t_1, t_2)$ for almost all type profiles (t_1, t_2) ($\forall i = 1, 2$) and if $\tilde{U}_i(\underline{t}_i) = 0$ for each $i \in \{1, 2\}$, then the stage-one mechanism M is dynamically optimal.*

b. *The initial owner's expected payoff generated by $\langle M, e\rangle$ is below the upper bound in Lemma 3.2 unless $\tilde{q}_i(t_1, t_2) = q_i^*(t_1, t_2)$ for almost all type profiles (t_1, t_2) ($\forall i = 1, 2$).*

Proof: Let Π^* denote the expected payoff generated by the statically optimal auction M^* in the static environment. Lemma 3.1 implies that a mechanism-PBE pair $\langle M, e\rangle$ in the dynamic setting is allocation-equivalent to a mechanism-BNE pair $\langle M', e'\rangle$ in the static environment. Thus, $\langle M', e'\rangle$ induces the winner-selection rule $(\tilde{q}_i)_{i=1}^2$ and yields surplus $\tilde{U}_i(\underline{t}_i)$ for bidder i with type $\underline{t}_i$ ($i = 1, 2$). Given the hypotheses in Part (a), the revenue-equivalence theorem in the static setting implies that $\langle M', e'\rangle$ and hence $\langle M, e\rangle$ yield an expected payoff Π^* to the initial owner. By Lemma 3.2, Π^* is the highest expected payoff the initial owner can possibly obtain in the dynamic environment. Thus, the mechanism-PBE pair $\langle M, e\rangle$ maximizes his welfare in that environment, so M is dynamically optimal. This proves (a). For Part (b), by the standard proof in the optimal-auction theory and the strict monotonicity of V_i, the upper bound in Lemma 3.2, which is Π^*, is equal to the maximum value of

$$\mathrm{E}_{t_1,t_2}^{F_1 \times F_2} \sum_{i=1}^{2} q_i(t_1, t_2) V_i(t_i) \tag{11}$$

among all possible mappings $(q_i)_{i=1}^2 : (t_1, t_2) \mapsto (q_i(t_1, t_2))_{i=1}^2$ of winning probabilities. Since (11) is maximized only if the weighted sum $\sum_{i=1}^2 q_i(t_1, t_2) V_i(t_i)$ is maximized for almost all type profiles (t_1, t_2), and the weighted sum is maximized only if the winning probabilities q_i are identical to q_i^*, Part (b) follows. ■

4 The Choice of Resale Prices

Let us start with the bottom of the auction-resale game and characterize a winner's choice of resale prices. We make the following assumption of monotone hazard rate.

Assumption 2 (Hazard Rate) *For each player $i \in \{1, 2\}$ such that $\bar{t}_i > \underline{t}_i$, the density function f_i is positive on $[\underline{t}_i, \bar{t}_i]$ and*

$$\frac{1 - F_i(t_i)}{f_i(t_i)^2} f_i'(t_i) \geq -1, \ \forall t_i \in (\underline{t}_i, \bar{t}_i). \tag{12}$$

Define

$$V_{i,a}(t_i) := t_i - \frac{F_i(a) - F_i(t_i)}{f_i(t_i)}, \ \forall a \leq \bar{t}_i, \forall t_i \leq a. \tag{13}$$

Obviously, $V_{i,\bar{t}_i} = V_i$ for each $i = 1, 2$. Assumption 2 immediately implies:

$$V_{i,a}(a) = a \text{ and } V_{i,a}(t_i) < t_i, \ \forall i \in \{1, 2\}, \forall a \leq \bar{t}_i, \forall t_i < a. \tag{14}$$

Assumption 2 allows an easy characterization of a winner's optimal resale price. Denote $F_i(\cdot|E)$ for the distribution of type t_i conditional on the event E.

Lemma 4.1 *Suppose Assumption 2. Let $i \in \{1, 2\}$ and take any interval $[z, a] \subseteq [\underline{t}_i, \bar{t}_i]$.*

a. The function $V_{i,a}$ is strictly increasing for each $i = 1, 2$.

b. If a winner w's posterior belief about the stage-one loser l's type is $F_l[\cdot|z \leq$ (or $<$) $t_l \leq$ (or $<$) $a]$, then w's profit-maximizing resale price, given his type t_w, is

$$\begin{cases} V_{l,a}^{-1}(t_w) & \text{if } V_{l,a}(z) \leq t_w \leq a \\ z & \text{if } t_w \leq V_{l,a}(z) \\ \text{any number in } [t_w, \infty) & \text{if } t_w \geq a. \end{cases} \tag{15}$$

Proof: To prove Part (a), calculate that

$$\frac{d}{dt} V_{i,a}(t) = 2 + \frac{F_i(a) - F_i(t)}{f_i(t)^2} f_i'(t) \geq 2 - \frac{F_i(a) - F_i(t)}{f_i(t)^2} |f_i'(t)| \geq 2 - \frac{1 - F_i(t)}{f_i(t)^2} |f_i'(t)|.$$

If $f_i'(t) \geq 0$, the equality of the above calculation implies the strict monotonicity of function $V_{i,a}$. If $f_i'(t) < 0$, the above calculation, coupled with Eq. (12), implies the strict monotonicity of $V_{i,a}$. Thus, Part (a) follows and the inverse $V_{i,a}^{-1}$ exists.

We next prove Part (b), which says that the resale price (15) maximizes the winner's expected profit

$$(p_2 - t_w)(1 - [\mu_w(b_w, p_1^w)](p_2)) \tag{16}$$

from resale, over all resale prices $p_2 \geq t_w$. If $t_w \geq a$, then the winner knows that there is no gain from trading with the loser, hence any resale price above or equal to t_w is optimal. This gives the third branch of (15). Consider the other case $t_w < a$. By the hypothesis in Part (b) about the posterior belief, the derivative of (16) with respect to resale price p_2 is

$$\frac{f_l(p_2)}{F_l(a) - F_l(z)}(t_w - V_{l,a}(p_2)).$$

Obviously, this derivative is positive when $V_{l,a}(p_2) < t_w$, equal to zero when $V_{l,a}(p_2) = t_w$, and negative when $V_{l,a}(p_2) > t_w$. If $V_{l,a}(z) \leq t_w < a$, then the point $V_l^{-1}(t_w)$ exists and is optimal, hence the first branch of (15). If instead $V_{l,a}(z) \geq t_w$, then the optimal resale price is the lowest level in the range of the loser's type, which is z, hence the second branch of (15). ■

5 Why the Statically Optimal Auction Fails

Does resale possibility matter to optimal auction design? To answer this question, we investigate the performance of the statically optimal auction M^* in our dynamic setting. This section shows that M^* fails to be dynamically optimal. The reason is two-fold. First, bidders do not bid truthfully in the auction M^*, because it gives a winner some information that bidders want to manipulate (Subsection 5.1). Second, the option of resales motivates bidders to collude in M^*, resulting in low revenues (Subsection 5.2). Thus, resale possibility requires a new design of optimal auctions.

5.1 Manipulating the Information Linkage

To understand why there is no perfect Bayesian equilibrium (PBE) where bidders bid truthfully in the statically optimal auction M^*, recall the intended allocation rule of M^* illustrated by Figure 1. A crucial feature of the allocation rule is that the winner's payment depends on the loser's bid. Consequently, if truthful bidding were an equilibrium, the winner would infer the loser's type from the stage-one payment and then set the resale price accordingly. Knowing that, the loser would have an incentive to distort his bid, thereby manipulating the resale price. In Figure 2, for example, bidder 1 gains a payoff represented by the area $EFLN$ from bidding t_1' instead of his true type t_1.

An exception of this negative result is when the bidders have an identical distribution of types. In that case, the statically optimal auction M^* does not misallocate the good, i.e., the curve GFL coincides with the 45-degree line in Figure 2. One can then show that a winner who thinks that the loser had bid truthfully would set the resale price to be the winner's type (Eq. (15)), so the loser is indifferent between truth and lies. To rule out this special case, we make the following assumption:

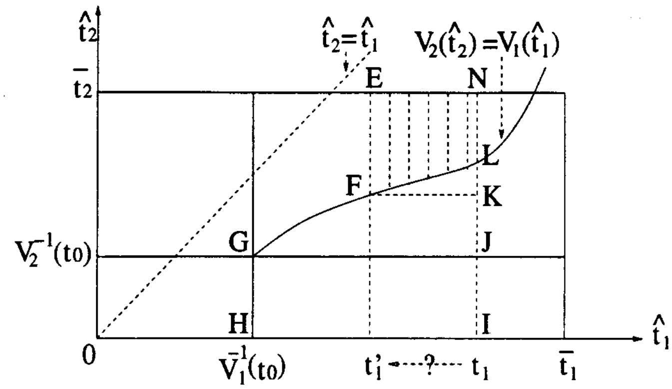

Figure 2: Bidder 2 lies in the statically optimal auction.

Assumption 3 (Partiality) *For all* $t_2 \in [\underline{t}_2, \bar{t}_2]$, $\underline{t}_1 \leq t_2 \leq \bar{t}_1$ *and* $V_2(t_2) > V_1(t_2)$.

This assumption captures the most interesting feature of the statically optimal auction: subsidizing one bidder (bidder 2) in order to intensify the bidding competition. In Figure 2, Assumption 3 implies that the curve GFL lies to the right of the 45-degree line. Consequently, the mechanism M^* may misallocate the good to bidder 2 when his reported type is below bidder 1's, while it never does so for bidder 1.

Proposition 5.1 *Suppose Assumptions 1, 2, and 3. If the statically optimal auction* M^* *is the stage-one mechanism, then the continuation game has no perfect Bayesian equilibrium (PBE) that induces truthful bidding at stage one.*

Proof: Suppose that there is a PBE inducing truthful bidding at stage one. We will show that bidder 1 would rather lie at stage one, for a positive measure of his types. Pick any $t_1 \in (\max\{\bar{t}_2, V_1^{-1}(t_0)\}, V_1^{-1}(V_2(\bar{t}_2)))$. (This interval is nondegenerate by Assumptions 1 and 3.) Consider a bid $\hat{t}_1 \in (V_1^{-1}(t_0), t_1)$ for bidder 1 with type t_1:

1. Suppose bidder 2's type $t_2 < V_2^{-1}(V_1(\hat{t}_1))$. Since bidder 2 bids truthfully and $\hat{t}_1 < t_1$, bidder 1 wins and pays the same amount whether he bids $\hat{t}_1$ or t_1 (Eqs. (6) and (9)). Furthermore, no resale occurs in this case, because $t_1 > \hat{t}_1 > V_1^{-1}(V_2(t_2))$ and $V_1^{-1}(V_2(t_2)) \geq t_2$ (Assumption 3). Thus, the bids t_1 and $\hat{t}_1$ are indifferent here.

2. Suppose bidder 2's type $t_2 > V_2^{-1}(V_1(t_1))$. With $t_1 > V_1^{-1}(t_0)$, we know $V_2(t_2) > t_0$, so bidder 2 wins at stage one whether bidder 1 bids $\hat{t}_1$ or t_1 (Eq. (8)). If bidder 1 bids $\hat{t}_1$, then bidder 2's stage-one payment is $V_2^{-1}(V_1(\hat{t}_1))$ (Eq. (8)), from which bidder 2 infers that bidder 1's type is $\hat{t}_1$ ($V_2^{-1} \circ V_1$ is strictly monotone by Assumption 2); the resale price will then be $\hat{t}_1$ and bidder 1's payoff from the bid $\hat{t}_1$ will be $t_1 - \hat{t}_1$. If instead bidder 1 bids truthfully t_1, by the same token, the resale price will be t_1 and so bidder 1's payoff will be zero. Since $t_1 > \hat{t}_1$, the truthful bid t_1 is dominated here.

3. Suppose bidder 2's type $t_2 \in (V_2^{-1}(V_1(\hat{t}_1)), V_2^{-1}(V_1(t_1)))$. If bidder 1 bids $\hat{t}_1$, by the reasoning in the previous case, he will lose at stage one and buy the good from bidder 2 at price $\hat{t}_1$. If instead bidder 1 bids truthfully t_1, then he will win at stage one and buy the good at price $V_1^{-1}(V_2(t_2))$ (by Eq. (9), where $t_2 > V_2^{-1}(t_0)$ because $t_2 > V_2^{-1}(V_1(\hat{t}_1))$ and $\hat{t}_1 > V_1^{-1}(t_0)$). Since $\hat{t}_1 < V_1^{-1}(V_2(t_2))$ here, the truthful bid t_1 is again dominated.

As the other cases (e.g., $t_2 = V_2^{-1}(V_1(t_1))$) are probability-zero events to bidder 1, it is dominated for him to bid truthfully. This contradicts the supposition that truthful bidding is an equilibrium. ■

5.2 Collusion in the Statically Optimal Auction

Now that the equilibrium for which the statically optimal auction M^* is intended cannot survive when resale is allowed, we need to evaluate the performance of this auction by finding an equilibrium. This section shows that M^* has a an equilibrium where bidders lie at stage one and trade subsequently, resulting in low revenues for the initial owner.

The driving force behind this "collusive" equilibrium is the resale option, which allows the bidders to trade after the auction M^* if they could collude in M^*. They can collude in M^* because the mechanism M^* determines a winner's payment by the rivals' bids. Thus, one bidder (say bidder 2) can bid his highest possible type, unless his actual type is so low that the loss at stage one exceeds the profit from resales. Knowing that, bidder 1 would shade his bid down to the stage-one reserve price offered to him—just to assure a sale at stage one—and further shade his bid if his type is below that reserve price. He could have competed with bidder 2 at stage one, by bidding sufficiently high. But doing so is dominated, because if he wins at stage one, bidder 1 must pay a price determined by bidder 2's bid, which is 2's highest possible type. If bidder 1 waits until stage two, then he can win at a lower price. Knowing bidder 1's reaction, bidder 2 has no incentive to shade his bid from the highest possible type, since 2 is going to pay only his stage-one reserve price if he wins. Inducing an allocation rule different from the one intended by the mechanism M^* (Figure 3), this equilibrium gives the initial owner a lower expected payoff in the dynamic setting than it does in the static setting (Lemma 3.4 (b)).

We emphasize here that it is the resale option that drives this collusive equilibrium. As well-known, a Vickrey auction in the static environment also has a "static" collusive equilibrium, where one bidder bids high and the others bid low, regardless of their types. That static collusive equilibrium, however, does not survive in our environment when the reserve price offered to the high bidder is nontrivial (the case "$V_i^{-1}(t_0) > \underline{t}_i$" in Assumption 1), for then a low-type high bidder will get a negative payoff. In contrast, the collusive equilibrium here survives nontrivial reserve prices, because the high bidder expects a profit from resale. The static collusive equilibrium also relies on the assumption that the high bidder can submit a bid above all possible types of others. This assumption does not hold in our environment, because bidder 2's highest bid $\bar{t}_2$ can fall below bidder 1's highest bid $\bar{t}_1$ (the case "$\bar{t}_2 < \bar{t}_1$"

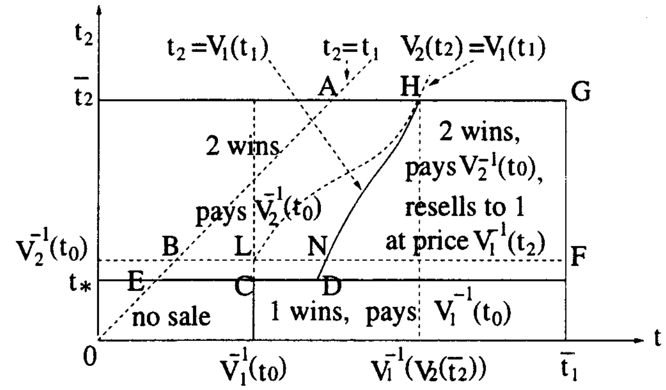

Figure 3: Collusion in the Statically Optimal Auction M^*

in Assumption 3). In contrast, the collusive equilibrium here does not need that assumption, because the low bidder, even with extremely high types, gets a lower price from the high bidder at resale than from the initial owner. Here, the incentive of collusion comes from the option of resales.

Our collusive equilibrium requires a weaker assumption than Assumption 3:

Assumption 4 (Weak Partiality) *For all* $t_2 \in [\underline{t}_2, \bar{t}_2]$, $\underline{t}_1 \leq t_2 \leq \bar{t}_1$ *and* $V_2(t_2) \geq V_1(t_2)$.

Weaker than Assumption 3, this assumption allows the curve HL to coincide with the 45-degree line AB in Figure 3. An immediate consequence of Assumption 4 is that the statically optimal auction M^* offers a lower reserve price to bidder 2 than to bidder 1.

Lemma 5.1 *By Assumptions 2 and 4,* $V_2^{-1}(t_0) \leq V_1^{-1}(t_0)$.

Proof: Suppose not, then

$$t_0 = V_2(V_2^{-1}(t_0)) \geq V_1(V_2^{-1}(t_0)) > V_1(V_1^{-1}(t_0)) = t_0,$$

where the first inequality follows from Assumption 4 and the second inequality follows from the monotonicity of V_1 (Assumption 2). This contradiction proves the lemma. ∎

The rest of this section constructs the equilibrium via four steps. Step 1 constructs a strategy profile. Step 2 derives the belief system from the strategy profile according to Bayes's rule. Step 3 proves that bidder 2's best response is to abide to the strategy profile. Step 4 proves the analogous claim for bidder 1. Proposition 5.2 wraps up the result.

5.2.1 Step 1: The Strategy Profile

We first define bidder 1's bidding strategy:

$$s_1^\diamond(t_1) := \begin{cases} t_1 & \text{if } t_1 < V_1^{-1}(t_0) \\ V_1^{-1}(t_0) & \text{if } t_1 \geq V_1^{-1}(t_0). \end{cases} \tag{17}$$

A winner's resale price strategy $\wp_2^\diamond$ and a loser's response strategy $r^\diamond$ are easily defined by Conditions 2a and 2b of Subsection 2.2, once the belief system is defined.

To define bidder 2's bidding strategy, look at the function

$$g(t) := t - V_2^{-1}(t_0) + [1 - F_1(V_1^{-1}(t))][V_1^{-1}(t) - t], \ \forall t \in [\underline{t}_2, \bar{t}_2]. \tag{18}$$

Lemma 5.6 will show that $g(t)$ is a type-t bidder 2's surplus if he bids above the reserved price $V_2^{-1}(t_0)$ offered to him by M^*. Until then, we need only to notice the following fact:

Lemma 5.2 *By Assumptions 2 and 4, the function g is well-defined, and there exists a unique point $t_* \in [\underline{t}_2, V_2^{-1}(t_0)]$, with*

$$t_* := \inf\{t \in [\underline{t}_2, V_2^{-1}(t_0)] : g(t) \geq 0\}, \tag{19}$$

such that $g(t) > 0$ if $t \in [t_*, \bar{t}_2]$ and $g(t) < 0$ if $t \in [\underline{t}_2, t_*)$.

Proof: The function g is well-defined if any $t_2 \in [\underline{t}_2, \bar{t}_2]$ belongs to the range of V_1^{-1}, i.e., $V_1(\underline{t}_1) \leq t_2 \leq V_1(\bar{t}_1)$. The first inequality follows from the fact that $V_1(\underline{t}_1) < \underline{t}_1 \leq t_2$ (Eq. (14) and Assumption 4); the second inequality follows from the fact that $V_1(\bar{t}_1) = \bar{t}_1 \geq t_2$ (Eq. (14) and Assumption 4). Thus, the function g is well-defined. To prove the rest of the lemma, one can show that the derivative $g'(t) = F_1(V_1^{-1}(t)) > 0$, so g is strictly increasing. Furthermore, $g(V_2^{-1}(t_0)) \geq 0$ by Eq. (18). Thus, the point t_* defined by Eq. (19) is either the unique root of the equation $g(t) = 0$ in $[\underline{t}_2, V_2^{-1}(t_0)]$ or the end point $\underline{t}_2$. ■

Lemma 5.2 implies that the following bidding strategy of bidder 2 is well-defined:

$$s_2^\diamond(t_2) := \begin{cases} t_2 & \text{if } t_2 < t_* \\ \bar{t}_2 & \text{if } t_2 \geq t_*. \end{cases} \tag{20}$$

The next lemma highlights some properties of the bidding strategies $(s_i^\diamond)_{i=1}^2$.

Lemma 5.3 *Denote $\hat{t}_i$ ($\forall i = 1, 2$) for bidder i's bid in M^*. Assumption 2 implies:*

a. If $\hat{t}_2 > V_2^{-1}(t_0)$ and bidder 1 plays $s_1^\diamond$, then bidder 2 wins and pays $V_2^{-1}(t_0)$ in M^*.

b. If bidder 2 wins in M^* and bidder 1 plays $s_1^\diamond$, then bidder 2 pays $V_2^{-1}(t_0)$ at stage one.

c. *If bidder 1 wins in M^*, his payment $p_1^1 \neq V_1^{-1}(t_0)$, and bidder 2 plays $s_2^\diamond$, then bidder 2's bid is $\bar{t}_2$.*

Proof: To prove Part (a), we calculate that

$$V_1(s_1^\diamond(t_1)) \leq V_1(V_1^{-1}(t_0)) = t_0 < V_2(\hat{t}_2),$$

where the first inequality follows from Eq. (17) and the monotonicity of V_1 (Assumption 2), and the last inequality follows from $\hat{t}_2 > V_2^{-1}(t_0)$ and the strict monotonicity of V_2 (Assumption 2). By Eq. (8), the above calculation implies Part (a). Part (b) follows from the fact that $s_1^\diamond(t_1) \leq V_1^{-1}(t_0)$ $(\forall t_1)$ and Eqs. (7), (8), and (10). For Part (c), notice from "Eqs. (6), (9), and (10)" that "bidder 1 wins and $p_1^1 \neq V_1^{-1}(t_0)$" implies bidder 2's bid $\hat{t}_2 > V_2^{-1}(t_0)$. If $\hat{t}_2 \neq \bar{t}_2$, then Eq. (20) would imply that $\hat{t}_2 = t_2 < t_* \leq V_2^{-1}(t_0)$, a contradiction. Thus, bidder 2's bid $\hat{t}_2$ must be $\bar{t}_2$ if he plays strategy $s_2^\diamond$. This proves Part (c). ■

5.2.2 Step 2: The Belief System

The following lemma calculates the posterior belief μ_w of a winner $w \in \{1,2\}$. A loser's posterior belief $\mu_l^\diamond$ plays a minor role (needed only when the loser calculates his expected payoff upon losing at stage one) and can be calculated likewise.

Lemma 5.4 *For any stage-one winner w and any of his reported type $\hat{t}_w \geq V_w^{-1}(t_0)$ and stage-one payment p_1^w, define*

$$\mu_w^\diamond(\hat{t}_w, p_1^w) := \begin{cases} F_1 & \text{if } w = 2 \\ F_2(\cdot|t_2 < t_*) & \text{if } w = 1 \text{ and } p_1^w = V_1^{-1}(t_0) \\ F_2(\cdot|t_2 \geq t_*) & \text{if } w = 1 \text{ and } p_1^w = V_1^{-1}(V_2(\bar{t}_2)) \\ F_2 & \text{if } w = 1 \text{ and } p_1^w \notin \{V_1^{-1}(t_0), V_1^{-1}(V_2(\bar{t}_2))\}. \end{cases} \tag{21}$$

By Assumptions 1, 2, and 4, the belief system $\mu^\diamond$ is Bayesian with respect to the strategy profile containing $(s_i^\diamond)_{i=1}^2$, provided that M^* is the stage-one mechanism.

Proof: When bidder 2 wins at stage one ($w = 2$), there are two possibilities:

1. Bidder 2's stage-one payment $p_1^2 \neq V_2^{-1}(t_0)$. From this payment, bidder 2 learns that bidder 1 must have deviated from $s_1^\diamond$ (Lemma 5.3 b). Thus, Bayes's rule has no restriction on bidder 2's posterior belief in this case.

2. Bidder 2's stage-one payment $p_1^2 = V_2^{-1}(t_0)$. Since bidder 1 is expected to play the strategy $s_1^\diamond$, "Eqs. (7), (8), and (10)" implies that bidder 1 cannot beat bidder 2 and that bidder 2's stage-one payment is $V_2^{-1}(t_0)$, regardless of bidder 1's type. (This statement depends on the assumption that bidders do not know whether the winner is chosen by a tie. Dropping this assumption would merely add one case to Eq. (21).) Consequently, bidder 2 learns nothing new from his winning status and payment in this case.

Thus, the first branch of Eq. (21) obeys Bayes's rule. Let us turn to the case when bidder 1 wins at stage one ($w = 1$). Again there are two possibilities:

1. Bidder 1's stage-one payment $p_1^1 = V_1^{-1}(t_0)$. The fact that he won implies that bidder 2's bid did not exceed $V_2^{-1}(t_0)$ (Lemma 5.3 a). By Assumption 1 and the strict monotonicity of V_2 (Assumption 2), $V_2^{-1}(t_0) < \bar{t}_2$. Thus, the definition (20) of $s_2^\diamond$ implies that $t_2 < t_*$, hence the second branch of Eq. (21).
2. The stage-one payment $p_1^1 \neq V_1^{-1}(t_0)$. From his winning status and payment, bidder 1 learns that bidder 2, if playing $s_2^\diamond$, must have bid $\bar{t}_2$ (Lemma 5.3 c), hence the third branch of Eq. (21). In this case, bidder 1's payment must be $V_1^{-1}(V_2(\bar{t}_2))$. If not, then bidder 1 knows that bidder 2 must have deviated from $s_2^\diamond$, so Bayes's rule has no restriction on bidder 2's posterior belief, hence the last branch of Eq. (21).

With all cases exhausted, we have shown that the posterior belief $\mu_w^\diamond$ obeys Bayes's rule. ■

Lemma 5.5 *Suppose Assumptions 1 and 2. Given M^* as the stage-one mechanism and given the belief system characterized by Eq. (21), if bidder 2 wins at stage one then he will set the resale price as $V_1^{-1}(t_2)$ for any type t_2 of his.*

Proof: Winning at stage one, bidder 2 updates his belief about bidder 1's type as F_1 (Eq. (21). Thus, by Eq. (15), bidder 2 will set the resale price to be $V_1^{-1}(t_2)$ if $V_1(\underline{t}_1) \leq t_2 \leq \bar{t}_1$, while the negation of "$V_1(\underline{t}_1) \leq t_2 \leq \bar{t}_1$" is impossible by Assumption 1. ■

5.2.3 Step 3: Bidder 2's Best Response

This subsection shows that bidder 2's strategy $s_2^\diamond$ is a best response to the strategy-belief profile $e^\diamond := (\langle s_1^\diamond, s_2^\diamond, \wp^\diamond, r^\diamond, \rangle, \mu^\diamond)$.

Lemma 5.6 *Suppose Assumptions 1, 2, and 4. If bidder 1 plays $s_1^\diamond$, then bidder 2's expected payoff from bidding above $V_2^{-1}(t_0)$ in mechanism M^* is equal to $g(t_2)$, given any of his type $t_2 \in [\underline{t}_2, \bar{t}_2]$ and the belief system $\mu^\diamond$.*

Proof: Bidding above $V_2^{-1}(t_0)$, bidder 2 buys the good at price $V_2^{-1}(t_0)$ at stage one (Lemma 5.3 a). By Lemma 5.5, bidder 2 will set the resale price to be $V_1^{-1}(t_2)$ if his type is t_2. Thus, bidder 2's expected profit at the resale stage is $[1 - F_1(V_1^{-1}(t_2))][V_1^{-1}(t_2) - t_2]$, and so his expected payoff from bidding above $V_2^{-1}(t_0)$ is equal to $g(t_2)$. ■

Lemma 5.7 *Suppose Assumptions 2 and 4. If bidder 1 plays the strategy $s_1^\diamond$ and wins in M^*, then he will set the resale price to be equal to his type, and no resale occurs at stage two if bidder 2's type $t_2 < V_2^{-1}(t_0)$.*

Proof: Since bidder 1 plays the strategy $s_1^\diamond$ and wins, both his bid and payment must be $V_1^{-1}(t_0)$ (Lemma 3.3 and Eq. (17)). His posterior belief is therefore $F_2(\cdot|t_2 < t_*)$ (Eq. (21)). As he bids $V_1^{-1}(t_0)$ according to $s_1^\diamond$, his actual type must be $t_1 \geq V_1^{-1}(t_0)$. Since $V_1^{-1}(t_0) \geq V_2^{-1}(t_0)$ (Lemma 5.1) and $V_2^{-1}(t_0) \geq t_*$ (Eq. (19)), we have $t_1 \geq t_*$. It thus follows from Eq. (15) that bidder 1 sets his resale price to be equal to his type t_1. Consequently, if bidder 2's type is below $V_2^{-1}(t_0)$, and hence below t_1, no resale will occur. ■

Lemma 5.8 *Suppose Assumptions 1, 2, and 4. Given the stage-one mechanism M^*, the strategy $s_2^\diamond$ (Eq. (20)) is bidder 2's best response to the strategy-belief profile $e^\diamond$.*

Proof: First, suppose that bidder 2's type is $t_2 < t_*$. If he bids $s_2^\diamond(t_2)$, then he loses at stage one, since $s_2^\diamond(t_2) < t_* \leq V_2^{-1}(t_0)$ (Eqs. (20) and (19)); furthermore, bidder 2's expected payoff is zero, since there will not be any resale in this case (Lemma 5.7). If instead he deviates from $s_2^\diamond(t_2)$, then the only deviation that can make a difference is to bid above $V_2^{-1}(t_0)$, thereby obtaining an expected payoff $g(t_2)$ (Lemma 5.6), which is negative since $t_2 < t_*$. Thus, deviation from the strategy $s_2^\diamond$ is dominated for any type $t_2 < t_*$.

Second, suppose that bidder 2's type is $t_2 \geq t_*$. We claim that he prefers winning the good at stage one to buying it at stage two. The reason is that his payment is equal to $V_2^{-1}(t_0)$ if he wins at stage one (Lemma 5.3 b) and equal to t_1 (bidder 1's type) if he buys the good at stage two (Lemma 5.7). Since $V_2^{-1}(t_0) \leq V_1^{-1}(t_0) \leq t_1$ (Lemma 5.1 and Eq. (17)), bidder 2 will pay less at stage one than at stage two (except for the probability-zero event "$t_1 = V_1^{-1}(t_0)$"). Thus, it is dominated for bidder 2 to bid less than or equal to $V_2^{-1}(t_0)$. (The bid $V_2^{-1}(t_0)$ leads to a tie when bidder 1 bids $V_1^{-1}(t_0)$, by Eq. (10).) For any bid above $V_2^{-1}(t_0)$, bidder 2's expected payoff is constantly $g(t_2)$ (Lemma 5.6). Thus, bidding $\bar{t}_2$ is optimal for him. ■

5.2.4 Step 4: Bidder 1's Best Response

This subsection shows that bidder 1's strategy $s_1^\diamond$ is a best response to the strategy-belief profile $e^\diamond$.

Lemma 5.9 *Suppose Assumptions 1, 2, and 4. If bidder 2 plays strategy $s_2^\diamond$, then bidding $\hat{t}_1 \geq V_1^{-1}(V_2(\bar{t}_2))$ is dominated for bidder 1, given the belief system $\mu^\diamond$ in Eq. (21).*

Proof: Let t_1 denote bidder 1's type and suppose that he bids $\hat{t}_1 \geq V_1^{-1}(V_2(\bar{t}_2))$. If he loses, then the bid $\hat{t}_1$ makes no difference from the bid prescribed by $s_1^\diamond$, since bidder 2 will update to F_1 in either case (Eq. (21)). If bidder 1 wins and bidder 2's type is below t_*, again the bid $\hat{t}_1$ makes no difference, for his payment will be $V_1^{-1}(t_0)$ whether he deviates or not.

The only case left is that bidder 1 wins and bidder 2's type is at least t_*. By the definition of $s_2^\diamond$, bidder 1's payment is $V_1^{-1}(V_2(\bar{t}_2))$. From this amount, bidder 1 updates his belief into $F_2(\cdot|t_2 \geq t_*)$ (Eq. (21)). By Eq. (15), he will set the resale price as follows:

1. If $V_2(t_*) \leq t_1 < \bar{t}_2$, then the resale price is $V_2^{-1}(t_1)$. By Eq. (14) and Assumption 4, $V_2^{-1}(t_1) < \bar{t}_2 \leq V_1^{-1}(V_2(\bar{t}_2))$, so bidder 1's stage-one payment exceeds his stage-two revenue. Thus, he gets negative expected payoff from bidding $\hat{t}_1$, while bidding according to $s_1^\diamond$ guarantees a nonnegative payoff.

2. If $V_2(t_*) > t_1$, then the resale price is t_*. By Eq. (19), $t_* \leq V_2^{-1}(t_0)$; since $V_2^{-1}(t_0) \leq V_1^{-1}(t_0)$ (Lemma 5.1) and $V_1^{-1}(t_0) < V_1^{-1}(\bar{t}_2)$ (Assumption 1), the resale price t_* is less than the stage-one payment $V_1^{-1}(V_2(\bar{t}_2))$. Thus, the bid $\hat{t}_1$ yields a negative expected payoff for bidder 1 and hence is dominated.

3. If $t_1 \geq \bar{t}_2$, then the resale price is t_1, so bidder 1's profit from resale is zero. Thus, his overall expected payoff is $t_1 - V_1^{-1}(V_2(\bar{t}_2))$. In contrast, if he bids according to $s_1^\diamond$, which is $V_1^{-1}(t_0)$ in this case, bidder 2 of type t_2 would have won at stage one and would resell the good at the resale price $V_1^{-1}(t_2)$ (Lemma 5.5), thereby giving bidder 1 a payoff $t_1 - V_1^{-1}(t_2)$. Since $t_2 \leq \bar{t}_2$, $V_1^{-1}(t_2) < V_1^{-1}(V_2(\bar{t}_2))$ (except for the zero-probability event "$t_2 = \bar{t}_2$"), so bidder 1 could have paid less for the good than the amount $V_1^{-1}(V_2(\bar{t}_2))$. Thus, the bid $\hat{t}_1$ is again dominated.

With all cases considered, we have proved the lemma. ■

Lemma 5.10 *Suppose Assumptions 1, 2, and 4. Given the stage-one mechanism M^*, the strategy $s_1^\diamond$ (Eq. (17)) is bidder 1's best response to the strategy-belief profile $e^\diamond$.*

Proof: First, suppose that bidder 1's type $t_1 < V_1^{-1}(t_0)$. According to the strategy $s_1^\diamond$, he would bid below $V_1^{-1}(t_0)$ and lose at stage one. Suppose that he deviates to a bid $\hat{t} \geq V_1^{-1}(t_0)$. By Lemma 5.9, we need only to consider $\hat{t} < V_1^{-1}(V_2(\bar{t}_2))$. If bidder 2's type $t_2 \geq t_*$, he will bid $\bar{t}_2$ (Eq. (20)) and beat bidder 1 at stage one ($\hat{t} < V_1^{-1}(V_2(\bar{t}_2))$), so bidder 1's deviation to $\hat{t}_1$ will make no difference. If instead bidder 2's type $t_2 < t_*$, he will bid below $V_2^{-1}(t_0)$, so bidder 1 will win and pay $V_1^{-1}(t_0)$ at stage one (Eq. (6)). Bidder 1 will then update to the belief $F_2(\cdot|t_2 < t_*)$ (Eq. (21)). Consequently, his revenue from resale is below t_*, while $t_* \leq V_2^{-1}(t_0) \leq V_1^{-1}(t_0)$ (Eq. (19) and Lemma 5.1). Thus, bidder 1 pays more than he will receive, so the deviation $\hat{t}_1$ is dominated.

Second, suppose that bidder 1's type $t_1 \geq V_1^{-1}(t_0)$. The strategy $s_1^\diamond$ prescribes bidding $V_1^{-1}(t_0)$. Suppose that he deviates to a bid $\hat{t} \neq V_1^{-1}(t_0)$. By Lemma 5.9, we need only to consider $\hat{t} < V_1^{-1}(V_2(\bar{t}_2))$. Furthermore, deviating to a bid $\hat{t} \in (V_1^{-1}(t_0), V_1^{-1}(V_2(\bar{t}_2)))$ makes no difference, since bidder 2 plays the strategy $s_2^\diamond$. Thus, the only deviation that can make a difference is $\hat{t} < V_1^{-1}(t_0)$. We hence consider such a deviating bid $\hat{t}$. If bidder 2's type $t_2 \geq t_*$, then bidder 2 will win whether bidder 1 deviates or not, so the deviation will make no difference. If instead $t_2 < t_*$, then the deviation $\hat{t}$ results in no sale at stage one (Eq. (5)), thereby yielding zero payoff for bidder 1. In contrast, bidding according to $s_1^\diamond$ would give him a positive payoff whenever $t_1 > V_1^{-1}(t_0)$. Thus, the deviation is dominated. ■

5.2.5 The Collusive Equilibrium

Having completed the above steps, we obtain a collusive equilibrium of the mechanism M^*. Here bidder 1 bids at most the reserve price offered to him by M^*, bidder 2 bids his highest possible type, and with a positive probability the bidders trade with each other after stage one. The following proposition summarizes this result and further asserts that the auction M^* gives the initial owner a lower expected payoff in our setting than it does in the static setting. The latter claim, coupled with a dynamically optimal auction constructed next (Proposition 6.1), implies that the statically optimal auction M^* is dynamically suboptimal.

Proposition 5.2 *Suppose Assumptions 1, 2, and 4. If the statically optimal auction M^* is the stage-one mechanism, then the continuation game has a perfect Bayesian equilibrium (PBE) $e^\diamond$, characterized by Eqs. (17), (19), (20), and (21), such that the pair $\langle M^*, e^\diamond \rangle$ induces a positive probability of resale and gives the initial owner a lower expected payoff than M^* would give in the static environment.*

Proof: By Lemmas 5.4, 5.8, and 5.10, the strategy-belief profile $e^\diamond$ constructed by Eqs. (17), (19), (20), and (21) is a PBE. We first prove that this equilibrium induces the allocation rule illustrated by Figure 3. If bidder 2's type $t_2 \geq t_*$, then by the strategies $s_1^\diamond$ and $s_2^\diamond$, bidder 2 buys the good at price $V_2^{-1}(t_0)$ at stage one. He will then set the resale price as $V_1^{-1}(t_2)$ (Lemma 5.5), so bidder 1 will be the final owner of the good if his type is above $V_1^{-1}(t_2)$. If $t_2 < t_*$ and bidder 1's type $t_1 < V_1^{-1}(t_0)$, then the bidding strategies imply that neither bidders get the good. If $t_2 < t_*$ and $t_1 \geq V_1^{-1}(t_0)$, then bidder 1 buys the good at price $V_1^{-1}(t_0)$ at stage one; no resale will occur in this case (Lemma 5.7 and $t_* \leq V_2^{-1}(t_0)$). The allocation rule illustrated by Figure 3 hence follows.

We next prove that this equilibrium induces resales with a positive probability. According to the allocation rule verified above, a resale occurs whenever the type profile (t_1, t_2) is in the area $HNFG$ of Figure 3. Furthermore, the interval $[V_2^{-1}(t_0), \bar{t}_2]$ is nondegenerate (Assumption 1) and hence has positive probability (F_2 has no gap by Assumption 2). This, coupled with the strict monotonicity of the function V_1 (Assumption 2), implies that the area $HNFG$ has a positive probability with respect to $F_1 \times F_2$. Thus, the equilibrium leads to a positive probability of resales.

We finally prove that the mechanism-PBE pair $\langle M^*, e^\diamond \rangle$ gives the initial owner an expected payoff lower than M^* would have done in the static setting. By Lemma 3.4 (b), it suffices to prove that the rule by which $\langle M^*, e^\diamond \rangle$ selects the final owner is different from $(q_i^*)_{i=1}^2$ (Eq. (2)) on a set of positive probability measure, which is obvious from an inspection of Figures 1 and 3. (The area HCD in Figure 3 suffices.) ■

6 A Dynamically Optimal Auction

The previous section has shown that the statically optimal auction M^* fails when resale cannot be prohibited. Then what is an optimal auction here? To answer this question, recall the two problems, besides resale option, that cause the failure of M^*. The first is that the payment scheme in M^* gives a winner too much information about the loser. The second problem is that M^* charges a winner according to the loser's bid, thereby allowing collusive bidding, as in a Vickrey auction. One idea to fix these problems is to change the payment scheme so that a bidder's payment depends only on his own bid. That would stop the information linkage and make a bidder pay for his own bid.

Even if an auction could fix the above problems, how do we know that it is dynamically optimal (i.e., optimal when resale cannot be prohibited)? The answer comes from the revenue-equivalence theorem, which corresponds to Lemma 3.4 (Section 3). By that lemma, it suffices to design a stage-one mechanism that has an equilibrium inducing the allocation rule $(q_i^*)_{i=1}^2$ intended by M^* and giving zero surplus to the lowest type of each bidder.

Motivated by the above ideas, this section constructs a dynamically optimal auction M^Z. Figure 4 illustrates its rules. The main idea of this design is that bidder 2, whom M^*

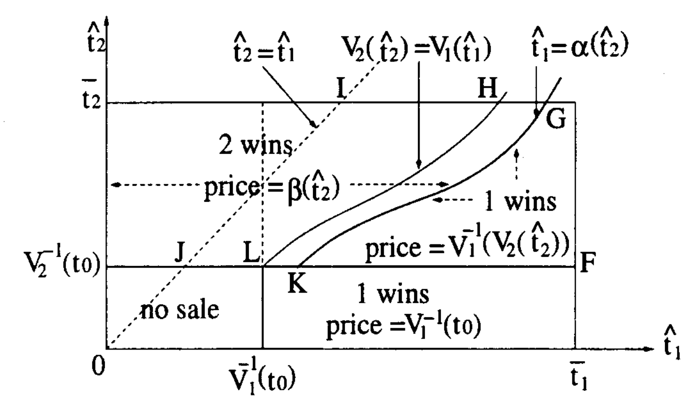

Figure 4: The Dynamically Optimal Auction M^Z

intends to subsidize, is expected to be a middleman sometimes. This motivates the following features of the mechanism M^Z. The first is that, when both bidders bid above their reserve prices, the good goes to bidder 2 if and only if the bid profile lies to the left of curve GK in Figure 4, while in M^* the dividing curve would have been HL. We choose the curve GK such that, when bidder 2 wins, he will sell the good to bidder 1 if and only if the type profile lies between curves HL and GK. This feature is to assure that the final owner of the good is selected in the way intended by M^*. The second feature is that bidder 2's payment $\beta(\hat{t}_2)$ depends only on his own bid $\hat{t}_2$. This is to cut off the unwanted information linkage across stages. We design the payment function β to induce truthful bidding from bidder 2. The third feature is that, when beating bidder 2 at stage one, bidder 1 pays as much as the resale price had bidder 2 won. This makes bidder 1 indifferent between winning the good at stage one and buying it from resale, thereby inducing truthful bidding from him.

The rest of this section presents the result in four steps. Step 1 constructs the mechanism M^Z. Step 2 defines a strategy-belief profile e^Z. Step 3 proves that truthful bidding in M^Z is bidder 1's best response to e^Z. Step 4 proves that truthful bidding in M^Z is bidder 2's best response to e^Z. Proposition 6.1 wraps up the result.

6.1 Step 1: Constructing the Mechanism M^Z

To construct the mechanism, we first specify the dividing curve GK in Figure 4, and then specify the payment function β. As introduced above, the dividing curve is represented by the equation $\hat{t}_1 = \alpha(\hat{t}_2)$, where we want to define the function $\alpha : [\underline{t}_2, \bar{t}_2] \to [\underline{t}_1, \bar{t}_1]$ by

$$V_{1,\alpha(x)}^{-1}(x) = V_1^{-1}(V_2(x)), \ \forall x \in [\underline{t}_2, \bar{t}_2]. \tag{22}$$

We will see from Lemma 6.5 that this equation says that a type-x bidder 2 who has bid truthfully and won at stage one will set $V_1^{-1}(V_2(x))$ as the resale price. Until then, we need only to know that $\alpha(x)$ is well-defined by Eq. (22):

Lemma 6.1 *By Assumptions 2 and 3, for any $x \in [\underline{t}_2, \bar{t}_2]$ there exists exactly one $\alpha(x) \in [V_1^{-1}(V_2(x)), \bar{t}_1]$ such that Eq. (22) holds.*

Proof: Pick any $x \in [\underline{t}_2, \bar{t}_2]$. For any $a \in [V_1^{-1}(V_2(x)), \bar{t}_1]$, we have $V_{1,a}(\underline{t}_1) \leq x$ (Eq. (14) and Assumption 3) and $x \leq V_1^{-1}(V_2(x)) \leq a$ (Assumption 3). The strict monotonicity of $V_{1,a}$ (Lemma 4.1 (a)) hence implies that $V_{1,a}^{-1}(x)$ exists and is unique. Thus, we can define a function γ_x by

$$\gamma_x(a) := V_{1,a}^{-1}(x), \ \forall a \in [V_1^{-1}(V_2(x)), \bar{t}_1]. \tag{23}$$

Eq. (22) then becomes $\gamma_x(\alpha(x)) = V_1^{-1}(V_2(x))$. To prove the existence of $\alpha(x)$, we apply the Intermediate-Value Theorem to function γ_x: First, the function γ_x is continuous by Assumption 2. Second, "$\gamma_x(V_1^{-1}(V_2(x))) < V_1^{-1}(V_2(x))$" is equivalent to "$x < V_1^{-1}(V_2(x))$" (by the definition of γ_x and Eq. (14)), which is equivalent to "$V_1(x) < V_2(x)$," which is true by Assumption 3. Third, "$\gamma_x(\bar{t}_1) \geq V_1^{-1}(V_2(x))$" is equivalent to "$V_1^{-1}(x) \geq V_1^{-1}(V_2(x))$" (by the definitions of γ_x and V_1), which is equivalent to "$x \geq V_2(x)$," which is true by Eq. (14). Thus, $\alpha(x)$ exists. To prove the uniqueness of $\alpha(x)$, we show that γ_x is strictly monotone: Pick any $a, a' \in [V_1^{-1}(V_2(x)), \bar{t}_1]$ such that $a > a'$. Denote $t := V_{1,a}^{-1}(x)$ and $t' := V_{1,a'}^{-1}(x)$. Suppose $t \leq t'$. Then

$$x = V_{1,a}(t) < V_{1,a'}(t) \leq V_{1,a'}(t') = x;$$

here the first inequality used Eq. (13), Assumption 2, and $a > a'$, and the second inequality used Lemma 4.1 (a) and "$t \leq t'$." This contradiction implies $t > t'$, as desired. ■

We next construct a function $\beta : [V_2^{-1}(t_0), \bar{t}_2] \to R$, which will determine bidder 2's payment if he wins at stage one. Define

$$\psi(x,t) := [F_1(\alpha(x)) - F_1(\gamma_t(\alpha(x)))][\gamma_t(\alpha(x)) - t], \ \forall x, t \in [\underline{t}_2, \bar{t}_2]. \tag{24}$$

We want to define β by the following differential equation system:

$$\forall t \in [V_2^{-1}(t_0), \bar{t}_2],\ [t - \beta(t)](F_1 \circ \alpha)'(t) - \beta'(t)F_1(\alpha(t)) + \frac{\partial}{\partial y}\psi(y,t)\Big|_{y=t} = 0; \tag{25}$$

$$[V_2^{-1}(t_0) - \beta(V_2^{-1}(t_0))]F_1(\alpha(V_2^{-1}(t_0))) + \psi(V_2^{-1}(t_0), V_2^{-1}(t_0)) = 0. \tag{26}$$

Subsection 6.4 will show that Eq. (25) is the first-order necessary condition for bidder 2 to bid truthfully, and Eq. (26) the condition that bidder 2 gets zero surplus if his type is equal to the minimum bid $V_2^{-1}(t_0)$. Until then, we need only to know that the function β is well-defined by Eqs. (25) and (26).

Lemma 6.2 *By Assumptions 2 and 3, there exists a unique solution* $\beta : [V_2^{-1}(t_0), \bar{t}_2] \to R$ *for Eqs. (25) and (26).*

Proof: By Assumptions 2 and 3, the function α is well-defined (Lemma 6.1). Furthermore, the function is differentiable by Eq. (23) and the differentiability of the virtual utility functions (Assumption 2). Thus, Eq. (25) is meaningful. As Eqs. (25) and (26) comprise a linear ordinary differential equation system, the lemma follows. ■

With functions α and β well-defined, we define the stage-one mechanism M^Z by the following rules for any profile $(\hat{t}_1, \hat{t}_2) \in \prod_{i=1}^{2}[\underline{t}_i, \bar{t}_i]$ of bids, which are independently submitted:

1. If $\hat{t}_1 < V_1^{-1}(t_0)$ and $\hat{t}_2 < V_2^{-1}(t_0)$, then the initial owner withholds the good.
2. If $\hat{t}_1 \geq V_1^{-1}(t_0)$ and $\hat{t}_2 < V_2^{-1}(t_0)$, then bidder 1 wins and pays $V_1^{-1}(t_0)$.
3. If $\hat{t}_1 > \alpha(\hat{t}_2)$ and $\hat{t}_2 \geq V_2^{-1}(t_0)$, then bidder 1 wins and pays $V_1^{-1}(V_2(\hat{t}_2))$.
4. If $\hat{t}_1 \leq \alpha(\hat{t}_2)$ and $\hat{t}_2 \geq V_2^{-1}(t_0)$, then bidder 2 wins and pays $\beta(\hat{t}_2)$.

Figure 4 illustrates the above rules.

6.2 Step 2: The Strategy-Belief Profile

Let us start by defining a strategy-belief profile $e^Z := (\langle s_1^Z, s_2^Z, \wp_2^Z, r^Z\rangle, \mu^Z)$ for the continuation game once the stage-one mechanism M^Z is chosen. Define the response strategy r^Z by Condition 2a for perfect Bayesian equilibrium (Subsection 2.2). For any bidder i, any winner w, any type $t_i \in [\underline{t}_i, \bar{t}_i]$, any possible bid $\hat{t}_w$ from the winner, and any possible stage-one payment p_1^w from him, define:

$$s_i^Z(t_i) := t_i; \tag{27}$$

$$\wp_2^Z(2, \hat{t}_2, p_1^2, t_2) := V_{1,\alpha(\hat{t}_2)}^{-1}(t_2); \tag{28}$$

$$\wp_2^Z(1, \hat{t}_1, p_1^1, t_1) := \begin{cases} \bar{t}_1 & \text{if } t_1 \geq p_1^1 \text{ or "} t_1 \geq V_2^{-1}(t_0) \ \& \ p_1^1 = V_1^{-1}(t_0) \text{"} \\ V^{-1}_{2,V_2^{-1}(t_0)}(t_1) & \text{if } V_{2,V_2^{-1}(t_0)}(\underline{t}_2) \leq t_1 < V_2^{-1}(t_0) \ \& \ p_1^1 = V_1^{-1}(t_0) \\ \underline{t}_2 & \text{if } V_{2,V_2^{-1}(t_0)}(\underline{t}_2) > t_1 \ \& \ p_1^1 = V_1^{-1}(t_0) \\ \max\{V_2^{-1}(V_1(p_1^1)), t_1\} & \text{if } t_1 < p_1^1 \ \& \ t_1 \geq V_2^{-1}(t_0) \ \& \ p_1^1 > V_1^{-1}(t_0); \end{cases} \tag{29}$$

$$\mu_w^Z(\hat{t}_w, p_1^w) := \begin{cases} F_1(\cdot | t_1 \leq \alpha(\hat{t}_w)) & \text{if } w = 2 \\ F_2(\cdot | t_2 \leq V_2^{-1}(t_0)) & \text{if } w = 1 \text{ and } p_1^w \leq V_1^{-1}(t_0) \\ \delta_{V_2^{-1}(V_1(p_1^w))} & \text{if } w = 1 \text{ and } p_1^w > V_1^{-1}(t_0). \end{cases} \tag{30}$$

Here δ_a denotes the Dirac measure at the point a.

The above strategy-belief profile means: Each bidder bids truthfully at stage one. If bidder 2 bids $\hat{t}_2$ and wins, then he updates that bidder 1's type is bounded from above by $\alpha(\hat{t}_2)$, and he sets the resale price as $V^{-1}_{1,\alpha(\hat{t}_2)}(t_2)$ if his own type is t_2. If bidder 1 wins, then he updates that bidder 2's type is equal to $V_2^{-1}(V_1(p_1^w))$ if the stage-one payment p_1^w is above $V_1^{-1}(t_0)$, and otherwise will infer that bidder 2's type is bounded from above by $V_1^{-1}(t_0)$; bidder 1 will set the resale price according to Eq. (29).[7] A loser accepts an offer from the winner if and only if the former's type is above the latter's resale price. (As in Section 5, a loser's posterior belief μ_l^Z plays a minor role and can be calculated likewise.)

If e^Z is a perfect Bayesian equilibrium (PBE), then the pair $\langle M^Z, e^Z \rangle$ will select the final owner in exactly the same way as the statically optimal auction M^* intends but fails to do:

Lemma 6.3 *Suppose Assumptions 1 and 3. If the strategy-belief profile e^Z characterized by Eqs. (27), (28), (29), and (30) is a PBE of the continuation game once M^Z is chosen, then $\langle M^Z, e^Z \rangle$ gives zero payoff to the lowest type of each bidder, and the way in which it selects the final owner is $(q_i^*)_{i=1}^2$ (defined in Eq. (2)), the same as the rule intended by the statically optimal auction M^*.*

Proof: By construction, the final-owner-selection rule of M^Z is the same as $(q_i^*)_{i=1}^2$ unless the profile of bids $(\hat{t}_1, \hat{t}_2)$ belongs to

$$S := \{(x, y) \in [\underline{t}_1, \bar{t}_1] \times [\underline{t}_2, \bar{t}_2] : y \geq V_2^{-1}(t_0); V_1^{-1}(V_2(y)) \leq x \leq \alpha(y)\}.$$

Since e^Z prescribes truthful bidding (Eq. (27)), at the end of stage one the winner is chosen according to $(q_i^*)_{i=1}^2$ unless the profile (t_1, t_2) of types belongs to S. If $(t_1, t_2) \in S$, then bidder 2 wins the good at stage one (Rule 4 of M^Z) and will set the resale price to be $V^{-1}_{1,\alpha(t_2)}(t_2)$ (Eq. (28)), which is equal to $V_1^{-1}(V_2(t_2))$ by Eq. (22). Consequently, bidder 1 ends up being the final owner, the same as $(q_i^*)_{i=1}^2$. Thus, the winner-selection rule $(q_i^*)_{i=1}^2$ is implemented if $\langle M^Z, e^Z \rangle$ comprise a mechanism-PBE pair.

[7] In cases of the first branch of Eq. (29), Lemma 6.5 will show that any resale price above or equal to bidder 1's type t_1, including the prescribed $\bar{t}_1$, is optimal. If instead Eq. (29) prescribes the resale price to be t_1, then bidder 2 with sufficiently high types might want to bid low and buy the good from bidder 1.

We next examine the payoff to the lowest type $\underline{t}_i$ of each bidder i. By Assumption 1, $\underline{t}_i \leq V_i^{-1}(t_0)$. If $\underline{t}_i < V_i^{-1}(t_0)$, then bidder i cannot win at stage one (Rules 1, 2, and 4). Neither can i buy the good from resale, because the resale price will be at least $V_i^{-1}(t_0)$, given truthful bidding at stage one (Eqs. (27), (28), and (29)). Thus, i has zero expected payoff. Consider the other case, $\underline{t}_i = V_i^{-1}(t_0)$. It suffices to prove that bidder i's expected payoff conditional on winning at stage one is zero. First, suppose that bidder 1 wins at stage one, then he breaks even at stage one (Rule 2), and he gains no profit from resale, since resale will not occur (the first branch of Eq. (29) and $\bar{t}_1 \geq t_2$ by Assumption 3). Second, suppose that bidder 2 wins at stage one, then he will set the resale price to be $V_1^{-1}(t_0)$ (Eq. (28)), hence his expected profit from resale is

$$[V_1^{-1}(t_0) - V_2^{-1}(t_0)][F_1(\alpha(V_2^{-1}(t_0))) - F_1(V_1^{-1}(t_0))]/F_1(\alpha(V_2^{-1}(t_0))),$$

which is exactly his stage-one payment $\beta(V_2^{-1}(t_0))$ (Eq. (26)). Thus, conditional on winning at stage one, each bidder i with type $V_i^{-1}(t_0)$ breaks even, as claimed. ■

The above lemma, coupled with Lemma 3.4 (a), implies that the mechanism M^Z will be dynamically optimal if e^Z is a PBE. The rest of this section will prove that. Let us begin by checking Bayes's rule and the resale prices in e^Z defined above.

Lemma 6.4 *By Assumption 2, the belief system μ^Z defined by Eq. (30) is Bayesian with respect to the strategy profile $\langle s_1^Z, s_2^Z, \wp_2^Z, r^Z \rangle$.*

Proof: If bidder 2 wins after bidding $\hat{t}_2$ at stage one, then from his winning status and Rule 4 of mechanism M^Z he learns that bidder 1's type does not exceed $\alpha(\hat{t}_2)$. That is all he can infer, since his stage-one payment $\beta(\hat{t}_2)$ contains no information about bidder 1. Hence the first branch of Eq. (30) follows Bayes's rule. When bidder 2 wins and needs to pay p_1^w at stage one, there are two possibilities. Either $p_1^w = V_1^{-1}(t_0)$ when bidder 2's bid is below $V_2^{-1}(t_0)$ (Rule 2 of M^Z), or $p_1^w = V_1^{-1}(V_2(\hat{t}_2))$ when bidder 2's bid $\hat{t}_2 \geq V_2^{-1}(t_0)$ (Rule 3). From his stage-one payment p_1^w, bidder 1 can tell these two cases apart, because "$\hat{t}_2 > V_2^{-1}(t_0)$" is equivalent to "$V_1^{-1}(V_2(\hat{t}_2)) > V_1^{-1}(t_0)$" (Lemma 4.1 (a)), which is equivalent to "$p_1^w > V_1^{-1}(t_0)$" (Rule 3). Furthermore, when $p_1^w > V_1^{-1}(t_0)$, bidder 1 not only knows that bidder 2's bid is above $V_2^{-1}(t_0)$, he also knows that bidder 2's bid is equal to $V_2^{-1}(V_1(p_1^w))$ by Rule 3 of M^Z and the strict monotonicity of $V_2^{-1} \circ V_1$ (Lemma 4.1 (a)). Since bidder 1 assumes that bidder 2 bids truthfully, the second and third branches of Eq. (30) follows Bayes's rule. ■

Lemma 6.5 *By Assumptions 2 and 3, the resale price strategy defined by Eqs. (28) and (29) is a best response to the strategy-belief profile e^Z defined by Eqs. (27), (28), (29), and (30).*

Proof: As proved in Lemma 4.1 (b), a winner's resale price is determined by Eq. (15). If bidder 2 bids $\hat{t}_2$ and wins, then his posterior belief is $F_1(\cdot|t_1 \leq \alpha(\hat{t}_2))$ (Eq. (30)), so Eq. (15) implies that the resale price is $V_{1,\alpha(\hat{t}_2)}^{-1}(t_2)$, unless his type t_2 is below $V_{1,\alpha(\hat{t}_2)}(\underline{t}_1)$ or above $\alpha(\hat{t}_2)$.

The first case is impossible by Assumption 3 and Eq. (14). The second case is impossible because $\alpha(\hat{t}_2) \geq V^{-1}_{1,\alpha(\hat{t}_2)}(t_2) > t_2$ (Eq. (14) and the definition of $V_{1,a}$). Thus, Eq. (28) gives bidder 2's optimal resale price. If bidder 1 bids $\hat{t}_1$ and wins, there are two cases.

a. The stage-one payment p^1_1 does not exceed bidder 1's type t_1. This contains two possibilities. First, $p^1_1 = V_1^{-1}(t_0)$. Then bidder 1 learns that $t_2 < V_2^{-1}(t_0)$ (Rule 2 of M^Z); since $V_2^{-1}(t_0) \leq V_1^{-1}(t_0)$ (Lemma 5.1) and $V_1^{-1}(t_0) = p^1_1 \leq t_1$, bidder 1 knows that $t_1 > t_2$, i.e., there is no gain from trade with bidder 2. Second, $p^1_1 > V_1^{-1}(t_0)$ (always $p^1_1 \geq V_1^{-1}(t_0)$ by the rules of M^Z). Then bidder 1 learns that $p^1_1 = V_1^{-1}(V_2(t_2))$ (Rule 3); since $t_1 \geq p^1_1$ and $V_1^{-1}(V_2(t_2)) \geq t_2$ (Assumption 3), again bidder 1 knows that there is no gain from trade with bidder 2. Thus, any resale price above t_1, hence $\bar{t}_1$, is optimal for bidder 1.

b. The stage-one payment $p^1_1 > t_1$. This contains three alternatives:

   i. Suppose $t_1 \geq V_2^{-1}(t_0)$ and $p^1_1 = V_1^{-1}(t_0)$. From "$p^1_1 = V_1^{-1}(t_0)$" bidder 1 learns that $V_2^{-1}(t_0) > t_2$, hence $t_1 > t_2$: zero gain from trade. Thus, the resale price $\bar{t}_1$ is optimal.

   ii. Suppose $t_1 < V_2^{-1}(t_0)$ and $p^1_1 = V_1^{-1}(t_0)$. From $p^1_1 = V_1^{-1}(t_0)$ bidder 1 learns that $V_2^{-1}(t_0) > t_2$. Consequently, "$t_1 < V_2^{-1}(t_0)$ and (15)" implies that the optimal resale price is $V^{-1}_{2,V_2^{-1}(t_0)}(t_1)$ if $t_1 \geq V_{2,V_2^{-1}(t_0)}(\underline{t}_2)$ and is $\underline{t}_2$ if otherwise.

   iii. Suppose $t_1 \geq V_2^{-1}(t_0)$ and $p^1_1 > V_1^{-1}(t_0)$. From $p^1_1 > V_1^{-1}(t_0)$ bidder 1 learns that $t_2 = V_2^{-1}(V_1(p^1_1))$, so an optimal resale price is either $V_2^{-1}(V_1(p^1_1)$ or t_1, whichever is higher.

Exhausting all cases, we have proved that Eq. (29) gives bidder 1's optimal resale price. ■

6.3 Step 3: Bidder 1's Best Response

Lemma 6.6 *By Assumptions 2 and 3, truthful bidding is bidder 1's best response to the strategy-belief profile e^Z defined by Eqs. (27), (28), (29), and (30).*

Proof: Let t_1 denote bidder 1's type and $\hat{t}_1$ his deviating bid. Notice that his bid affects only his winning status at stage one (Rules 2 and 3 of M^Z), and affects bidder 2's posterior belief only through the winning status (the first branch of Eq. (30)). Thus, bidding $\hat{t}_1$ instead of t_1 does not make a difference when both lead to the same winning status at stage one.

First, consider the case $t_1 \geq V_1^{-1}(t_0)$.

i. Suppose $\hat{t}_1 < t_1$. Then the deviation can possibly make a difference only when bidding t_1 leads to a win and $\hat{t}_1$ leads to a loss at stage one. Losing at stage one gives bidder 1 a payoff $\max\{0, t_1 - V_1^{-1}(V_2(t_2))\}$ via resale (the first branch of Eq. (28)), while winning at stage one gives bidder 1 a nonnegative resale profit plus the stage-one profit $t_1 - V_1^{-1}(V_2(t_2))$, which is positive because his winning status implies $t_1 > \alpha(t_2) \geq V_1^{-1}(V_2(t_2))$ (Rule 3 of M^Z and Lemma 6.1). Thus, bidding $\hat{t}_1 < t_1$ cannot improve upon truthful bidding.

ii. Suppose $\hat{t}_1 > t_1$. Then the deviation can make a difference only when bidding t_1 leads to a loss and $\hat{t}_1$ leads to a win at stage one. Again, losing at stage one gives bidder 1 a payoff $\max\{0, t_1 - V_1^{-1}(V_2(t_2))\}$ via resale. In contrast, winning at stage one gives him at most $t_1 - V_1^{-1}(V_2(t_2)) + \max\{0, t_2 - t_1\}$. If $t_2 - t_1 \leq 0$, then the deviation is not better than truthful bidding. If instead $t_2 - t_1 > 0$, then the deviation yields a negative payoff, since $V_1^{-1}(V_2(t_2)) > t_2$ by Assumption 3. Thus, the deviation is dominated.

Second, consider the case $t_1 < V_1^{-1}(t_0)$. Suppose that he deviates to bid $\hat{t}_1 \geq V_1^{-1}(t_0)$, which is the only deviation that can possibly make a difference. If bidder 2's type t_2 is below $V_2^{-1}(t_0)$, then at stage one bidder 1 buys the good at a price $V_1^{-1}(t_0)$, which is above his type t_1. This loss cannot be recovered at stage two, because the resale price cannot exceed t_2, while $t_2 < V_2^{-1}(t_0)$ (the second branch of Eq. (30)) and $V_2^{-1}(t_0) \leq V_1^{-1}(t_0)$ (Lemma 5.1, which holds since Assumption 3 implies Assumption 4). Thus, the deviation gives bidder 2 a negative expected payoff. If instead bidder 2's type $t_2 \geq V_2^{-1}(t_0)$, then the only case where bidder 1's deviation can make a difference is that he beats bidder 2 at stage one, i.e., $\hat{t}_1 > \alpha(t_2)$. But then bidder 1 has to pay $V_1^{-1}(V_2(t_2))$ at stage one, while the resale price is at most t_2, which is below $V_1^{-1}(V_2(t_2))$ (Assumption 3). The deviation again gives bidder 2 a negative expected payoff. Thus, the deviation is dominated. ■

6.4 Step 4: Bidder 2's Best Response

Finally, we turn to the bidding strategy of bidder 2. We will first calculate his expected payoff as a function of his bid, and then prove that truthful bidding maximizes that function. Recall the function ψ defined by Eq. (24).

Lemma 6.7 *Suppose Assumptions 2 and 3. Given the strategy-belief profile e^Z and given any type $t \in [\underline{t}_2, \bar{t}_2]$, bidder 2's expected payoff $u(x,t)$ from bidding any $x \in [\underline{t}_2, \bar{t}_2]$ at stage one is*

$$u(x,t) = \begin{cases} 0 & \text{if } x < V_2^{-1}(t_0) \\ (t - \beta(x))F_1(\alpha(x)) + \psi(x,t) & \text{if } x \geq V_2^{-1}(t_0). \end{cases} \tag{31}$$

Proof: If he bids below $V_2^{-1}(t_0)$, then bidder 2 does not win at stage one. His payoff is zero if bidder 1 does not win either. If bidder 1 wins, then bidder 2 knows that bidder 1's type t_1 is at least $V_1^{-1}(t_0)$ and stage-one payment $p_1^w = V_1^{-1}(t_0)$ (Rule 2 of M^Z), and hence bidder 2 knows that the resale price will be $\bar{t}_1$ (the first branch of Eq. (29)), which is above his type (Assumption 3). Consequently, bidder 2's expected payoff is also zero when bidder 1 wins. This gives the first branch of Eq. (31).

Now consider the case when bidder 2's bid x is at least $V_2^{-1}(t_0)$. This contains two alternatives. First, bidder 1 wins at stage one, i.e., $t_1 > \alpha(x)$ (Rule 3 of M^Z). Since $\alpha(x) \geq V_1^{-1}(V_2(x))$ (Lemma 6.1) and bidder 1's stage one payment is $p_1^1 = V_1^{-1}(V_2(x))$ (Rule 3), bidder 2 knows that $t_1 > p_1^1$ and hence the resale price will be $\bar{t}_1$ (the first branch of Eq. (29)), giving bidder 2 a zero payoff. The second alternative is that bidder 2 wins at stage one, i.e., $t_1 \leq \alpha(x)$. Then his stage-one profit is $t-\beta(x)$ (Rule 4 of M^Z), and his optimal resale price is $V_{1,\alpha(\hat{t}_2)}^{-1}(t_2)$ (Eq. (28)). This results in a resale if and only if $t_1 \geq V_{1,\alpha(\hat{t}_2)}^{-1}(t_2)$. Thus, bidder 2's expected stage-two profit, conditional on winning at stage one, is equal to $\psi(x,t)/F_1(\alpha(x))$ defined in Eq. (24), where function γ_t is defined by Eq. (23). Bidder 2's overall payoff is therefore equal to the second branch of Eq. (31). ■

Notice from Eq. (31) that Eq. (25) assures the first-order necessary condition for bidder 2 to bid truthfully if his type is above $V_2^{-1}(t_0)$. We next examine the second-order sufficient condition for truthful biding. Denote $\sigma(x,t)$ for bidder 2's optimal resale price if his stage-one bid is x and type is t, hence by Eq. (23)

$$\sigma(x,t) = \gamma_t(\alpha(x)) = V_{1,\alpha(x)}^{-1}(t), \ \forall x \in [V_2^{-1}(t_0), \bar{t}_2], \forall t \in [\underline{t}_2, \bar{t}_2]. \tag{32}$$

By Eqs. (32) and (13), we have

$$F_1(\alpha(x)) = F_1(\sigma(x,t)) + f_1(\sigma(x,t))[\sigma(x,t) - t], \ \forall x \in [V_2^{-1}(t_0), \bar{t}_2], \forall t \in [\underline{t}_2, \bar{t}_2]. \tag{33}$$

Since bidder 2 wants to maximize the payoff function u given his type, let us calculate its derivatives. (The derivative at an endpoint means the left- or right- derivative.)

Lemma 6.8 *Suppose Assumptions 2 and 3. For any $x \in [V_2^{-1}(t_0), \bar{t}_2]$ and any $t \in [\underline{t}_2, \bar{t}_2]$,*

$$\frac{\partial}{\partial x}u(x,t) = (F_1 \circ \alpha)'(x)[\sigma(x,t) - \sigma(x,x)]; \tag{34}$$

$$\frac{\partial}{\partial t}u(x,t) = F_1(\sigma(x,t)). \tag{35}$$

Proof: "Assumptions 2 and 3" implies that the function β is well-defined by the differential equation system (25) and (26), hence is differentiable. Thus, the function u is differentiable. By Eq. (25), we have

$$\frac{\partial}{\partial x}u(x,t) = (t-x)(F_1 \circ \alpha)'(x) + \frac{\partial}{\partial y}\psi(y,t)\Big|_{y=x} - \frac{\partial}{\partial y}\psi(y,x)\Big|_{y=x}.$$

By the definition (24) of ψ, the partial derivative $\frac{\partial}{\partial y}\psi(y,t)$ is equal to

$$(F_1 \circ \alpha)'(y)[\sigma(y,t) - t] + \{F_1(\alpha(y)) - f_1(\sigma(y,t))[\sigma(y,t) - t] - F_1(\sigma(y,t))\}\frac{\partial}{\partial y}\sigma(y,t),$$

which is equal to $(F_1 \circ \alpha)'(y)[\sigma(y,t) - t]$ by Eq. (33). Thus,

$$\frac{\partial}{\partial x}u(x,t) = (t-x)(F_1 \circ \alpha)'(x) + (F_1 \circ \alpha)'(x)[\sigma(x,t) - t] - (F_1 \circ \alpha)'(x)[\sigma(x,x) - x],$$

which implies Eq. (34). We next calculate

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \frac{\partial}{\partial t}\{tF_1(\alpha(x))+[F_1(\alpha(x))-F_1(\sigma(x,t))][\sigma(x,t)-t]\}\\ &= F_1(\sigma(x,t))+\{F_1(\alpha(x))-F_1(\sigma(x,t))-f_1(\sigma(x,t))[\sigma(x,t)-t]\}\frac{\partial}{\partial t}\sigma(x,t).\end{aligned}$$

Thus, Eq. (35) follows from Eq. (33). ■

To assure the second-order sufficient condition for truthful bidding, we want the derivative $\partial u/\partial x$ to satisfy a single-crossing property. Since $\sigma(x,t)-\sigma(x,x)$ has such a property, as will be proved below (Eqs (37) and (38)), Eq. (34) suggests that we need the function $F \circ \alpha$ to be increasing. For this purpose we add an assumption:

Assumption 5 (Monotone Resale Profit) *The derivative* $(V_1^{-1} \circ V_2)'(x) \geq 1$ *for all* $x \in (V_2^{-1}(t_0), \bar{t}_2)$.

This assumption says that the expression $V_1^{-1}(V_2(t_2)) - t_2$ is increasing in bidder 2's type t_2. By Eq. (28), this expression is equal to bidder 2's profit if he bids truthfully, acts as a middleman, and a resale occurs. Thus, Assumption 5 roughly corresponds to a monotonicity condition for the middleman's "type." In Figure 4, this assumption implies that the curve HL cannot be steeper than 45 degree. (To see that, differentiate the equation $\hat{t}_2 = V_2^{-1}(V_1(\hat{t}_1))$, which represents the curve HL.) Using Assumption 3, one can easily show that Assumption 5 is satisfied if bidder 2's virtual utility function V_2 is concave and increases as least as fast as bidder 1's virtual utility function V_1.

As desired, Assumption 5 implies the strict monotonicity of the function $F_1 \circ \alpha$:

Lemma 6.9 *By Assumptions 2, 3, and 5, the function* $F_1 \circ \alpha$ *is strictly increasing on* $[V_2^{-1}(t_0), \bar{t}_2]$.

Proof: By Eqs. (22) and (33), for any $x \in [V_2^{-1}(t_0), \bar{t}_2]$, we have

$$F_1(\alpha(x)) = F_1(V_1^{-1}(V_2(x))) + f_1(V_1^{-1}(V_2(x)))[V_1^{-1}(V_2(x)) - x].$$

Denote $t := V_1^{-1}(V_2(x))$ and $\eta(t) := t - V_2^{-1}(V_1(t))$. Since $V_1^{-1} \circ V_2 : [V_2^{-1}(t_0), \bar{t}_2] \to [V_1^{-1}(t_0), V_1^{-1}(\bar{t}_2)]$ is strictly increasing, it suffices to show that

$$G(t) := F_1(t) + f_1(t)\eta(t)$$

is strictly increasing for all $t \in [V_1^{-1}(t_0), V_1^{-1}(\bar{t}_2)]$. For this purpose, calculate the derivative

$$G'(t) = f_1(t)[1+\eta'(t)] + f_1'(t)\eta(t).$$

Notice that $1+\eta'(t) \geq 0$ by Assumption 5 and that $\eta(t) > 0$ by Assumption 3. Thus, we will be done if $f_1'(t) > 0$. Let us therefore consider the other case, $f_1'(t) \leq 0$. For the derivative $G'(t)$ to be positive, we need only

$$-f_1'(t) < f_1(t)[1+\eta'(t)]/\eta(t).$$

Since Eq. (12) already implies $-f_1'(t) \leq f_1(t)^2/[1-F_1(t)]$, we need only

$$f_1(t)/[1-F_1(t)] < [1+\eta'(t)]/\eta(t). \tag{36}$$

The definition of function η means $V_1(t) = V_2(t-\eta(t))$; Eq. (13) then implies

$$\frac{1-F_1(t)}{f_1(t)} = \eta(t) + \frac{1-F_1(t-\eta(t))}{f_1(t-\eta(t))}.$$

Thus, $[1-F_1(t)]/f_1(t) > \eta(t) \geq \eta(t)/[1+\eta'(t)]$ since $\eta'(t) \geq 0$ (Assumption 5). This implies (36), as desired. Thus, the function $F_1 \circ \alpha$ is strictly increasing. ■

We now prove that truthful bidding is bidder 2's best response at stage one.

Lemma 6.10 *By Assumptions 2, 3, and 5, truthful bidding is bidder 2's best response to the strategy-belief profile e^Z defined by Eqs. (27), (28), (29), and (30).*

Proof: For all $x \in [V_2^{-1}(t_0), \bar{t}_2]$ and all $t \in [\underline{t}_2, \bar{t}_2]$, we claim:

$$t > x \quad \Rightarrow \quad \sigma(x,t) > \sigma(x,x); \tag{37}$$

$$t < x \quad \Rightarrow \quad \sigma(x,t) < \sigma(x,x). \tag{38}$$

To prove them, suppose $\sigma(x,t) \leq \sigma(x,x)$. Then Eq. (32) and the monotonicity of $V_{1,\alpha(x)}$ (Lemma 4.1 (a)) implies

$$t = V_{1,\alpha(x)}(\sigma(x,t)) \leq V_{1,\alpha(x)}(\sigma(x,x)) = x.$$

This proves (37). One can prove (38) likewise. Thus, "Lemma 6.9 and Eqs. (34), (37), and (38)" implies: for all $x \in [V_2^{-1}(t_0), \bar{t}_2]$ and all $t \in [\underline{t}_2, \bar{t}_2]$,

$$\frac{\partial}{\partial x} u(x,t) \begin{cases} > 0 & \text{if } x < t \\ = 0 & \text{if } x = t \\ < 0 & \text{if } x > t. \end{cases} \tag{39}$$

Pick any type $t_2 \in [\underline{t}_2, \bar{t}_2]$ of bidder 2. There are two cases. First, $t_2 \geq V_2^{-1}(t_0)$. Then bidding his own type t_2 maximizes bidder 2's expected payoff $u(\cdot, t_2)$ among all bids above $V_2^{-1}(t_0)$. Bidding below $V_2^{-1}(t_0)$ is worse, because for bid $x < V_2^{-1}(t_0)$, his expected payoff

$$u(x,t_2) = 0 = u(V_2^{-1}(t_0), V_2^{-1}(t_0)) \leq u(V_2^{-1}(t_0), t_2) \leq u(t_2,t_2),$$

where the first equality uses Eq. (31), the second equality uses Eq. (26), the first inequality uses the fact that $u(V_2^{-1}(t_0), \cdot)$ is increasing (Eq. (35)), and the last inequality uses Eq. (39). The second case is that $t_2 < V_2^{-1}(t_0)$. Bidding below $V_2^{-1}(t_0)$ surely gives zero payoff to bidder 2 of such a type. Any bid x above $V_2^{-1}(t_0)$ is worse, because

$$u(x,t_2) < u(V_2^{-1}(t_0), t) < u(V_2^{-1}(t_0), V_2^{-1}(t_0)) = 0,$$

where the first inequality uses the third branch of Eq. (39), the second inequality uses Eq. (35), and the equality uses Eq. (26). Thus, truthful bidding maximizes bidder 2's expected payoff. ■

6.5 The Optimality of Mechanism M^Z

Having completed the above steps, we obtain a perfect Bayesian equilibrium (PBE) e^Z that supports truthful bidding at the mechanism M^Z. Furthermore, this mechanism-PBE pair yields the highest possible expected payoff for the initial owner in our dynamic environment.

Proposition 6.1 *By Assumptions 1, 2, 3, and 5, the stage-one mechanism M^Z constructed in Subsection 6.1 is dynamically optimal for the initial owner.*

Proof: "Lemmas 6.4, 6.5, 6.6, and 6.10" implies "Conditions 1, 2a, 2b, and 2c′" in the definition of PBE (Subsection 2.2). Thus, the strategy-belief profile e^Z characterized by Eqs. (27), (28), (29), and (30) is a PBE of the continuation game once M^Z is chosen. Consequently, "Assumption 1 and Lemma 6.3" implies that the conditions of Lemma 3.4 (a) are satisfied. Lemma 3.4 (a) then implies that the mechanism-PBE pair $\langle M^Z, e^Z\rangle$ gives the initial owner the highest possible expected payoff in our environment, i.e., the stage-one mechanism M^Z is dynamically optimal. ■

7 Examples

This section illustrates the results of this paper by the following example.

Example 2 *For each $i \in \{1,2\}$, bidder i's type t_i is uniformly distributed on $[0, \bar{t}_i]$, such that $\bar{t}_1 > \bar{t}_2$. The initial owner's type t_0 is commonly known to be zero.*

By Eq. (1), we calculate, for each $i \in \{1,2\}$ and each $t_i \in [0, \bar{t}_i]$:

$$\begin{aligned} V_i(t_i) &= 2t_i - \bar{t}_i; \\ \Delta &:= \bar{t}_1 - \bar{t}_2; \\ V_1^{-1}(V_2(t_2)) &= t_2 - \Delta/2. \end{aligned}$$

Notice that Assumptions 1, 2, 3, 4, and 5 are satisfied.

We can then calculate the statically optimal auction M^* (i.e., the "optimal" auction in the existing auction literature) by Eqs. (2), (3), and (4). The mechanism M^* sets $\bar{t}_i/2$ as the reserve price for bidder i. If both bidders bid above their reserve prices, then M^* allocates the good according to the line LH in Figure 5: If the bid profile $(\hat{t}_1, \hat{t}_2)$ lies in the area MLH, then bidder 2 wins and pays $\hat{t}_1 - (\bar{t}_1 - \bar{t}_2)/2$; if $(\hat{t}_1, \hat{t}_2)$ lies in the area $HLFG$, then bidder 1 wins and pays $\hat{t}_2 + (\bar{t}_1 - \bar{t}_2)/2$.

A crucial feature of this mechanism M^* is that it intends to subsidize a disadvantaged bidder in order to intensify the bidding competition. Here the disadvantaged bidder is

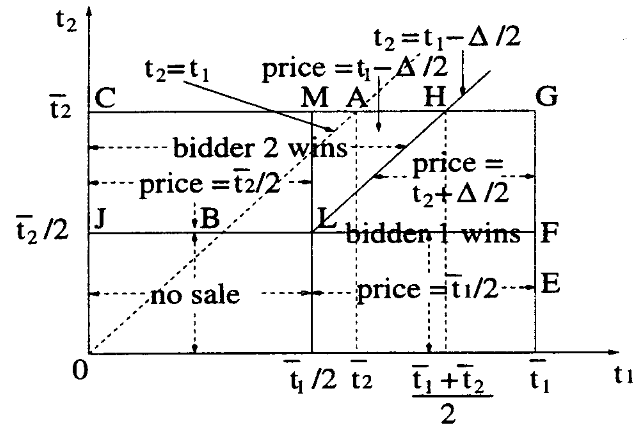

Figure 5: The Intended Rule of the Statically Optimal Auction (Example 2)

bidder 2, whose distribution of type is stochastically dominated by bidder 1's distribution. The intended subsidization corresponds to the event in which the mechanism M^* intends to sell the good to bidder 2 while bidder 1's type is higher than bidder 2's. This event is the area $ABLH$ in Figure 5. As proved in Myerson [10], this mechanism is optimal for the initial owner when resale can be prohibited. Hence we call it *statically optimal auction.*

A problem of M^* is that its subsidization scheme fails to intensify the bidding competition when resale cannot be prohibited. The intuition is that a bidder can buy the good from the subsidized bidder instead of competing with him. Specifically, if the initial owner sells the good via the mechanism M^*, there is no equilibrium that supports truthful bidding (Proposition 5.1); instead, there is a collusive equilibrium yielding low revenues for the initial owner (Proposition 5.2).

To see why truthful bidding is impossible for the statically optimal auction M^*, let us suppose that there were such an equilibrium. Consider the case where bidder 2 wins and pays p_1^2. If p_1^2 is above the reserve price $\bar{t}_2/2$, then bidder 2, trusting bidder 1 to be truthful, thinks that the type profile (t_1, t_2) belongs to the area MLH in Figure 5. Thus, he infers that bidder 1's type is $p_1^2 + \Delta/2$ and sets that as the resale price. But then bidder 1 with types in $(\bar{t}_2, (\bar{t}_1 + \bar{t}_2)/2)$ would strictly prefer bidding below his true type, thereby lowering p_1^2 and the resale price. This contradicts the supposition of truthful bidding.

Let us look at the collusive equilibrium when M^* is used to sell the good. Roughly speaking, the idea is that bidder 2, taking advantage of the subsidy, buys the good and resells it to the other bidder, who does not compete with bidder 2 in the auction. Specifically, bidder 1 expects high bids from bidder 2 and hence bids only $\bar{t}_1/2$ at stage one if his type is above that value, and bids below it if otherwise. Expecting low bids from bidder 1, bidder 2 bids his highest possible type $\bar{t}_2$ if his type is at least

$$t_* = \max\{0, \sqrt{2\bar{t}_1\bar{t}_2} - \bar{t}_1\}$$

and bids below $\bar{t}_2/2$ if otherwise. Here the cutoff t_* is obtained via finding the root of the function $g(t_2)$ (Eq. (18)), which is type-t bidder 2's surplus:

$$g(t_2) = [(t_2 + \bar{t}_1)^2 - 2\bar{t}_1\bar{t}_2]/(4\bar{t}_1), \ \forall t_2 \in [\underline{t}_2, \bar{t}_2].$$

(It is easy to verify that $0 \leq t_* \leq \bar{t}_2/2$.) Along the equilibrium path, bidder 2 wins at stage one if his type is at least the cutoff t_*, regardless of bidder 1's type. Consequently, bidder 2 learns nothing new about bidder 1 from his winning status or his payment to the initial owner. Bidder 2 hence sets the resale price p_2 to maximize his expected profit from resale, given his type t_2:

$$\max_{p_2 \geq t_2} (p_2 - t_2)(1 - p_2/\bar{t}_1).$$

The optimal resale price is therefore $(t_2 + \bar{t}_1)/2$. In other words, he sets the resale price according to line HN in Figure 6. From bidder 1's viewpoint, although this price line is

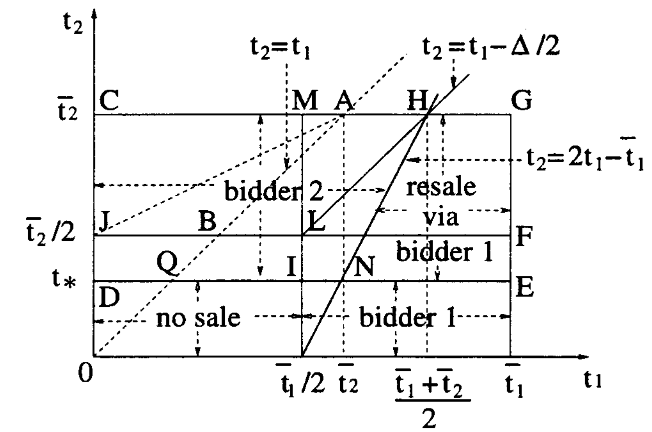

Figure 6: Collusion in the Statically Optimal Auction M^* (Example 2)

more stringent than the price line HL offered by the mechanism M^*, he would still prefer buying the good at resale, because competing with bidder 2 at stage one would result in an even higher payment $((\bar{t}_1 + \bar{t}_2)/2)$. At this equilibrium, the revenue for the initial owner is at most $\bar{t}_i/2$, the minimum bid for winner i. One can show that the initial owner's expected revenue is lower than the amount which the statically optimal auction M^* would have generated in the no-resale static environment.[8]

With the failure of the mechanism M^*, can a seller still benefit from subsidizing a disadvantaged bidder when resale cannot be prohibited? More fundamentally, what is an optimal auction when resale cannot be prohibited? To answer these questions, let us recall why M^* fails. The reason is two-fold. First, the payment scheme in M^* tells a winner too much information about the loser, which the loser wants to manipulate. This leads to the impossibility of truthful bidding in M^*. Second, M^* charges a winner according to the loser's bid; the subsidized bidder can therefore bid high, thereby crowding out the rival and

[8]Interestingly, the resale at this equilibrium does not correct the misallocation of the mechanism M^*; instead, the set (area $AQNH$) of type profiles where this equilibrium misallocates the good is larger than the one intended by M^*. See also Footnote 3.

profiting from resale. Thus, an idea to fix these problems is to change the payment scheme so that the subsidized bidder's payment depends only on his own bid.

Let us therefore try the following mechanism, denoted by M^Z. (Subsection 6.1 has the general definition.) Each bidder submits a bid independently; given the profile $(\hat{t}_1, \hat{t}_2)$ of bids, the rules are:

1. If $\hat{t}_1 < \bar{t}_1/2$ and $\hat{t}_2 < \bar{t}_2/2$, then the initial owner withholds the good.
2. If $\hat{t}_1 \geq \bar{t}_1/2$ and $\hat{t}_2 < \bar{t}_2/2$, then bidder 1 wins and pays $\bar{t}_1/2$.
3. If $\hat{t}_1 > \hat{t}_2 + \Delta$ and $\hat{t}_2 \geq \bar{t}_2/2$, then bidder 1 wins and pays $t_2 + \Delta/2$.
4. If $\hat{t}_1 \leq \hat{t}_2 + \Delta$ and $\hat{t}_2 \geq \bar{t}_2/2$, then bidder 2 wins and pays $\beta(\hat{t}_2)$, where the function β will be constructed below.

Figure 7 illustrates these rules. The mechanism has three features. First, when both bidders

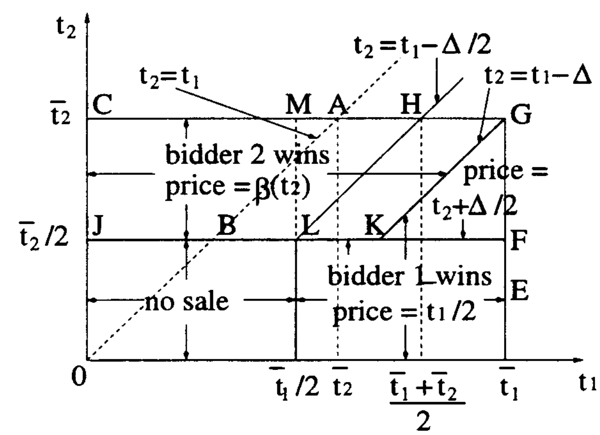

Figure 7: The Dynamically Optimal Auction M^Z (Example 2)

bid above their minimum bids $\bar{t}_i/2$, the good goes to bidder 2 if and only if the bid profile lies to the left of dividing line GK in Figure 7, instead of HL, which is the dividing line for M^*. Second, the payment $\beta(\hat{t}_2)$ from bidder 2, whom we hope to subsidize, depends only on his own bid $\hat{t}_2$. Third, upon beating bidder 2, bidder 1 pays $\hat{t}_2 + \Delta/2$ instead of $\hat{t}_2 + \Delta$ that the dividing line GK might seem to indicate.

Why is the mechanism M^Z designed like that? Due to the second feature, when bidder 2 wins in M^Z, the only news he obtains is that he has beat bidder 1, which, by the first feature, means that bidder 1's type is bounded above by $\hat{t}_2 + \Delta$. Consequently, given his bid $\hat{t}_2$ and type t_2, bidder 2 sets a resale price to solve

$$\max_{p_2 \geq t_2} (p_2 - t_2)[1 - p_2/(\hat{t}_2 + \Delta)],$$

so his resale price is $(t_2 + \hat{t}_2 + \Delta)/2$. This, if bidder 2 has bid truthfully, is exactly the price the statically optimal auction M^* intends to charge bidder 1. Thus, the first and second features induce bidder 2 to carry out the initial owner's desirable allocation rule at the resale stage. Due to the third feature, bidder 1 will pay the same price whether he beats bidder 2 in M^Z or he buys the good from bidder 2 at resale. Mainly because of that, bidder 1 cannot profit from misrepresenting his type. (Subsection 6.3 has the proof.)

We need to design the payment function β such that bidder 2 bids truthfully. We first examine bidder 2's payoff when he bids $\hat{t}_2$ below the minimum bid $\bar{t}_2/2$. Then he cannot win in M^Z (Rules 1 and 2). His expected payoff is obviously zero if bidder 2 does not win either. We claim that his expected payoff is zero even if bidder 2 wins. That is because bidder 2 knows that the winner bidder 1 will infer that bidder 2's bid—and hence bidder 2's type since bidder 1 assumes bidder 2 to be truthful—is below bidder 1's own type (Rule 3 and the fact that $\bar{t}_1 \geq \bar{t}_2$); consequently, bidder 2 expects bidder 1 to conclude that there is no gain from trades with bidder 2 and to set the resale price as $\bar{t}_1$, which bidder 2 cannot afford. (Footnote 7 discusses the case when bidder 1 chooses other resale prices.) Thus, bidder 2's expected payoff is zero, as claimed. We next examine bidder 2's payoff when $\hat{t}_2 \geq \bar{t}_2/2$. Then he wins in mechanism M^Z with probability $F_1(\hat{t}_2 + \Delta)$ (Rule 4 of M^Z). If he wins, then his expected resale profit (previous paragraph) is

$$\left(\frac{\hat{t}_2 - t_2 + \Delta}{2}\right)\left(\frac{(\hat{t}_2 + \Delta) - (t_2 + \hat{t}_2 + \Delta)/2}{\hat{t}_2 + \Delta}\right).$$

If bidder 2 loses, then his expected payoff is zero. The reason is similar to the case where he bids below his minimum bid $\bar{t}_2/2$. Therefore, before bidding, type-t_2 bidder 2's expected payoff $u(\hat{t}_2, t_2)$ from bidding $\hat{t}_2$ is zero if $\hat{t}_2 < \bar{t}_2$, and otherwise is

$$u(\hat{t}_2, t_2) = \frac{1}{\bar{t}_1}\left\{(t_2 - \beta(\hat{t}_2))(\hat{t}_2 + \Delta) + \frac{\hat{t}_2 - t_2 + \Delta}{2}\left[(\hat{t}_2 + \Delta) - \frac{t_2 + \hat{t}_2 + \Delta}{2}\right]\right\}.$$

To construct the function β, we calculate, for all $\hat{t}_2 \in [\bar{t}_2/2, \bar{t}_2]$ and all $t_2 \in [\underline{t}_2, \bar{t}_2]$:

$$\bar{t}_1 \frac{\partial}{\partial \hat{t}_2} u(\hat{t}_2, t_2) = -\beta'(\hat{t}_2)(\hat{t}_2 + \Delta) + t_2 - \beta(\hat{t}_2) + \frac{\hat{t}_2 - t_2 + \Delta}{2}; \tag{40}$$

$$\frac{\partial}{\partial t_2} u(\hat{t}_2, t_2) = \frac{\hat{t}_2 + t_2 + \Delta}{2}. \tag{41}$$

Thus, the first-order necessary condition for truthful bidding is $\frac{\partial}{\partial \hat{t}_2} u(\hat{t}_2, t_2)|_{\hat{t}_2 = t_2} = 0$. Since a truth-telling bidder 2 with type below $\bar{t}_2$ gets zero payoff, to assure incentive compatibility at the minimum bid $\bar{t}_2/2$ we also need $u(\bar{t}_2, \bar{t}_2) = 0$. Solving this differential equation system, we obtain

$$\beta(\hat{t}_2) = \frac{(2\hat{t}_2 + \Delta)^2 + \bar{t}_1}{8(\hat{t}_2 + \Delta)}, \; \forall \hat{t}_2 \in [\bar{t}_2/2, \bar{t}_2]. \tag{42}$$

One can check that β is strictly increasing and that $\beta(\bar{t}_2/2) > \bar{t}_2/2$. The latter fact makes sense because the payment scheme takes bidder 2's resale profit into account.

To verify that bidder 2 bids truthfully, we use Eqs. (40) and (42) to calculate:

$$\frac{\partial}{\partial \hat{t}_2} u(\hat{t}_2, t_2) = \frac{t_2 - \hat{t}_2}{2\bar{t}_1}, \ \forall \hat{t}_2 \in [\bar{t}_2/2, \bar{t}_2], \forall t_2 \in [\underline{t}_2, \bar{t}_2]. \tag{43}$$

Using Eqs. (41), (43), and the fact that $u(\bar{t}_2, \bar{t}_2) = 0$, one can prove that truthful bidding maximizes bidder 2's expected payoff for any type of his. (See the paragraph following Eq. (39) for a proof.)

So far we have demonstrated heuristically that, if the initial owner sells the good by the mechanism M^Z, then there is an equilibrium where bidders bid truthfully in M^Z and the final owner is selected in exactly the same way as the statically optimal auction M^* does in the resale-prohibiting static environment. How good is mechanism M^Z, then? It turns out that M^Z is optimal for the initial owner when resale cannot be prohibited.

To see why the mechanism M^Z is optimal, we first notice that the initial owner cannot do better in our dynamic environment (where resale cannot be prohibited) than in the static environment (where resale can be prohibited). That is because the initial owner in the static environment can replicate any mechanism in the dynamic environment by offering an option of resale (Lemma 3.1). Thus, the expected revenue Π^* generated by the statically optimal auction M^* in the static setting is the best the initial owner can possibly get in the dynamic setting. Therefore, M^Z is dynamically optimal if it gives the initial owner an expected revenue equal to Π^* in our dynamic setting. Comparing Figures 5 with 7, we see that the payment scheme in M^Z differs from that in M^* only when the bid profile belongs to the area $CJKG$. Thus, it suffices to show that, given truthful bidding, the expected revenues over this area are identical between the two mechanisms. For M^*, the expected revenue when the type profile belongs to the area $CJKG$ is

$$\int_{\bar{t}_2/2}^{\bar{t}_2} \frac{1}{\bar{t}_1} \left(\int_0^{\bar{t}_1/2} \frac{\bar{t}_2}{2} dt_1 + \int_{\bar{t}_1/2}^{t_2+\Delta/2} (t_1 - \Delta/2) dt_1 + \int_{t_2+\Delta/2}^{t_2+\Delta} (t_2 + \Delta/2) dt_1 \right) dt_2/\bar{t}_2.$$

For M^Z, the corresponding expected revenue is

$$\int_{\bar{t}_2/2}^{\bar{t}_2} \int_0^{t_2+\Delta} \beta(t_2) dt_1 dt_2/(\bar{t}_1 \bar{t}_2) = \frac{1}{8\bar{t}_1\bar{t}_2} \int_{\bar{t}_2/2}^{\bar{t}_2} ((2t_2 + \Delta)^2 + \bar{t}_1^2) dt_2,$$

where the equality uses Eq. (42). An elementary calculation shows that the two expected revenues are equal to each other, as desired. ("Lemmas 3.4 and 6.3" proves the general case by the revenue-equivalence theorem.) The mechanism M^Z is hence dynamically optimal.

The above example is a special case of Proposition 6.1, which designs a mechanism and proves that it is optimal when resale cannot be prohibited. Following are two more examples that fit the general result.

Example 3 *For each $i \in \{1, 2\}$, bidder i's type t_i is drawn from the distribution $F_i(t_i) := 1 - e^{-\lambda_i t_i}$ ($\forall t_i \in [0, \infty)$), such that $\lambda_2 > \lambda_1 > 0$. The initial owner's type is commonly known to be zero.*

By Eq. (1), we calculate, for each $i \in \{1,2\}$ and each $t_i \in [0,\infty)$:

$$\begin{aligned} V_i(t_i) &= t_i - 1/\lambda_i; \\ V_1^{-1}(V_2(t_2)) &= t_2 + 1/\lambda_1 - 1/\lambda_2. \end{aligned}$$

One easily calculates that Assumptions 1, 2, 3, 4, and 5 are satisfied.

Example 4 *Bidder 1's type is drawn from the distribution $F_i(t_1) := 1 - e^{-\lambda t_1}$ ($\forall t_1 \in [0,\infty)$); bidder 2's type is uniformly distributed on $[0, 1/\lambda]$, such that $\lambda > 0$. The initial owner's type is commonly known to be zero.*

By Eq. (1), we calculate, for each $t_1 \in [0,\infty]$ and each $t_2 \in [0, 1/\lambda]$:

$$\begin{aligned} V_1(t_1) &= t_1 - 1/\lambda; \\ V_2(t_2) &= 2t_2 - 1/\lambda; \\ V_1^{-1}(V_2(t_2)) &= 2t_2. \end{aligned}$$

It is easy to check that Assumptions 1, 2, 3, 4, and 5 are satisfied.

8 Concluding Remark

One thing held against auction theory was that a secondary market may wash away the significance of an auction. If a potential buyer can obtain the good from resale, why would he care about winning in the auction? If secondary markets will eventually deliver the good to the buyer who values it most, why should a social planner care about auction design? Auction theorists used to fend off these criticisms by assuming an exogenous friction to secondary markets. Needless to say, this assumption is neither theoretically elegant, nor realistically compelling, with the advancing information technology easing transaction costs. In this context, an insight contributed by this paper is: Even when resale has no friction at all, efficient allocation is not guaranteed, and a seller can still manipulate where the good will eventually go by designing an optimal auction. The paper therefore indicates that auction design is crucial to social planning even with frictionless secondary markets.

I hope that this paper could stimulate more works on auction design in a dynamic environment. Further extensions may include environments that have many bidders and multistage resales.

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