Optimal Auction in a Multidimensional World*

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Abstract

A long-standing unsolved problem, often arising from auctions with multidimensional bids, is how to design seller-optimal auctions when bidders' private characteristics ("types") differ in many dimensions. This paper solves the problem, assuming bidder-types stochastically independent across bidders. First, it proves that in any optimal auction, with positive probability, the object is not sold. Thus, a standard auction without a reserve price is not optimal. Second, and more importantly, the paper obtains an explicit formula for optimal auctions in a class of environments. The optimal mechanism is almost equivalent to a Vickrey auction with reserve price, except that the bids are ranked by an optimal scoring rule, which assigns scores to the multidimensional bids. When the hazard rate of a statistic of bidder-types is monotone, this auction is optimal among all mechanisms. When the hazard rate is not monotone, this auction is optimal among all "scoring mechanisms," where a winner chooses a multidimensional payment bundle subject to a type-specific rule. Our optimal auction implies that an optimizing seller would not evaluate bids by her own preferences; instead, she would induce downward distortion of nonmonetary provisions from the first-best configuration. Applied to multidimensional nonlinear pricing, our design of optimal auction yields an explicit optimal pricing function.

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1 Introduction

In real-world auctions, we often observe bidders submit bids containing several provisions. Economists have documented such examples in electric power industries (Laffont and Tirole [12, Ch. 14] and Chao and Wilson [6]), national defense procurement (Laffont and Tirole [12, Ch. 14]), environmental reservation of cropland (Osborn, Llacuna, and Linsenbigler [17]), school milk procurement (Tichy [23]), and the disposal of noxious wastes (Lescop [13]). An important question in these settings is how to design a mechanism optimal for the seller. Theoretically, this question corresponds to the following problem: what is a (seller-)optimal auction when bidders' private characteristics vary in several dimensions? This turned out to be a long-standing unsolved problem.

The above problem was unsolved because the available technique for designing optimal auctions is based on the assumption that bidders' private characteristics ("types") vary only in one dimension. This unidimensional assumption rules out the multidimensional nature of the above problem. To see that, imagine a hypothetical example: Several health care insurance companies ("bidders") compete to provide insurance coverage for the employees of a large firm ("seller"). Each insurance company bids a health care package $(x_j)_{j=1}^m$ containing m provisions, as well as a money transfer y to the firm. If a winning bid is $((x_j)_{j=1}^m, y)$, then the firm gets a payoff $y - c\sum_{j=1}^m x_j$ for some parameter $c \in R$, and the winning insurer gets

$$\sum_i^m \vartheta_j x_j^{1/2} - y,$$

where ϑ_j is his privately known valuation of provision j in the package. Thus, a bidder's private characteristic is a vector $(\vartheta_j)_{j=1}^m$. A mechanism designer may want to tailor each provision x_j according to some function $\tilde{x}_j$ of the multidimensional bidder-type $(\vartheta_j)_{j=1}^m$, and the functions $\tilde{x}_j$ may be different across j. Such a multidimensional design would be absent in the usual model of optimal auction, where the term $\sum_i^m \vartheta_j x_j^{1/2}$ is replaced by either a scalar t or a product tx, with t being a scalar private valuation and x being a scalar index of "quality."

Given the practical significance of our multidimensional problem, researchers in mechanism design have long been trying to solve it. The main barrier to progress is the incentive-compatibility (IC) constraint complicated by the multidimensional bidder-type.[1] If the bidder-type were one-dimensional, we could represent the constraint as a tractable monotonicity condition, which requires that higher types be more likely to win. This monotonicity representation would enable us to obtain an optimal auction by the available technique (Myerson [15]), whether the IC constraint is binding or not.[2] With multidimensional bidder-types, however, the task of representing incentive-compatibility as a monotonicity condition

[1]For example, due to multidimensional types, a bidder can lie about his type in two ways. One is to report a type that has a different probability of winning than the true type. The other is to fake a type whose corresponding transaction is different from the true type, with the probability of winning unchanged. While the first way of lying is allowed in the unidimensional settings, the second is absent.

[2]Here we use the phrase "binding IC constraint" in the context where the agents' types are continuously distributed. When types are unidimensionally and discretely distributed, the phrase would mean that the

has been difficult. For example, we do not know a priori how to determine one type is "higher" than the other.[3]

In the mean time, the techniques recently developed in a related field, nonlinear pricing under multidimensional private information, suggests the possibility of a breakthrough in our optimal auction problem. In the health care example, the setup of the nonlinear pricing problem corresponds to the special case where there is only one insurance company bidding to provide insurance coverage. Thus, the firm seeking insurance coverage does not need to select a winner. Consequently, all that the firm needs to design is a pricing function that maps each health care package $(x_j)_{j=1}^m$ to a money payment $\tilde{y}((x_j)_{j=1}^m)$ (Rochet [19]).

In the setup of nonlinear pricing, McAfee and McMillan [14] characterized the incentive-compatibility (IC) constraint as a system of partial differential equations. These equations come from the first- and second-order *necessary* conditions of IC. To make the conditions also sufficient for IC, McAfee and McMillan assumed a "generalized single crossing property," which guarantees an agent's local optimum to be his global optimum. The authors, however, noted that their characterization assumes that the allocation of a mechanism is a differentiable function of agents' types. Since this differentiability assumption usually does not hold in auction settings (Figure 1), their result has not been applied to auctions.[4]

A breakthrough in multidimensional nonlinear pricing problems is done by Armstrong [1]. He established an exclusion result, which says that a profit-maximizing multiproduct monopolist would exclude a positive measure of consumer-types. This result shows that the dimensionality of private information does matter, because the exclusion result can be ruled out when consumer-types are one-dimension. Armstrong further proved that, in some settings, the monopolist's optimal pricing function depends only on her cost ("cost-based tariff"). Armstrong achieved that by adding two assumptions. One is "multiplicative separability" ([1, Eqs. (18) and (23)]). Due to this assumption, the incentive-compatibility (IC) constraint under any cost-based tariff becomes a monotonicity condition with respect to a one-dimension statistic of the consumer-type. The other assumption is the monotonicity of the hazard rate of that statistic ([1, Eq. (22)]), which guarantees that the IC constraint of a profit-maximizing cost-based tariff is non-binding. That paper did not characterize an optimal pricing scheme when the IC constraint is binding.

Rochet and Choné [20] took on the multidimensional nonlinear pricing problem through the dual approach. That is, they described a mechanism as its associated *surplus function*, which maps an agent's type to his payoff when everyone reports the true information. The IC constraint became a convexity condition of the surplus function. Rochet and Choné proved

"downward" constraints are binding, where a downward constraint means a high-type agent is not tempted to act as a low-type.

[3] A knee-jerk response to the multidimensional problem may be simply to rank a bid according to the seller's payoff from it. But such a ranking criterion may be suboptimal, as the theory of optimal auctions has long recognized in the case of unidimensional types. That is also true for multidimensional types, as this paper proves (Proposition 4.2).

[4] McAfee and McMillan [14] did work out an example that can be interpreted as an auction, but their solution there did not use their characterization result.

the existence and uniqueness of a monopolist's optimal mechanism for both binding and non-binding IC constraints. They further proved that optimal mechanisms with non-binding IC constraints are exceptional rather than generic. Rochet and Choné, however, noted that their dual approach does not provide a procedure to construct an optimal mechanism.

To pass from multidimensional nonlinear pricing to our optimal auction problem, one must confront an additional question: how to select a winning bidder. Any auction mechanism, by definition, must answer this question one way or another. In multidimensional settings, although the received auction theory has not answered this question, economists have observed from actual auctions the usage of "scoring rules" to select winners. Here a seller announces a minimum score and a rule that assigns scores to bids; after bids are submitted, she sells the good to a bidder whose bid is scored highest and above the minimum level. (See the sources cited at the beginning paragraph for examples.) With bids varying in several dimensions, the design of a scoring rule has been a central and difficult issue among policy makers. In a cropland reservation bidding program from 1986 to 1998, the U.S. government had been revising its scoring rule each year, and researchers in that program are still debating an appropriate rule.[5] In the California electricity wholesale market, the choice of a scoring rule that appeared to be wrong from hindsight had led to severe consequences (Chao and Wilson [6]).

Therefore, an auction designer in our multidimensional setting has two tasks. One is to design a payment function that determines a winner's multidimensional payment package for each bidder-type. The other task is to design a winner-selection criterion. Although the first may benefit from the progress of the multidimensional nonlinear pricing literature, the second task is specific to the nature of auctions. To my knowledge, no one has offered a general design of optimal auctions when bids and bidder-types are both multidimensional.[6]

This paper therefore steps in and provides an explicit formula of optimal auctions in our multidimensional setting. Its main assumption is that bidder-types are stochastically

[5]This program is called Conservation Reserve Program, where the U.S. Department of Agriculture (USDA) retires erodible croplands from production by renting them from farmers. A participating farmer submits a bid that specifies the acreage and soil quality of the cropland, as well as the rent for the land and how the land will maintained during the retirement period. The USDA ranks the bids by a scoring rule that condenses a bid's provisions into a score (Osborn, Llacuna, and Linsenbigler [17, p5]). For the discussion about scoring rules in this program, see Reichelderfer and Boggess [18, p10], Barbarika, Osborn, and Heimlich [5, p122], and Babcock, Lakshminarayan, Wu, and Zilberman [3, 4].

[6]Che [7] considered auction settings where the bidder-type is one-dimension and the bid is two-dimension. In our health care example, his setting corresponds to an aforementioned special case, where a winner's payoff is $tx - y$. Che designed a scoring rule that achieves optimality among auctions where the IC constraint is non-binding and the trade always takes place.

Armstrong [2] solved the optimal auction problem in a two-object setting with binary bidder-type. That paper also demonstrated geometrically the complication of multidimensional auction design.

Jehiel, Moldovanu, and Stacchetti [9] considered multidimensional types from a different angle. Their focus was auctions where a bidder's payoff depends on the identity of the winner. In our health care example, their setting corresponds to the case where an insurance company's payoff is $t^i - y$, where t^i is a scalar value for the bidder if company i wins the competition. Due to such additively separable payoff functions, the authors characterized the IC constraint as a condition of monotonicity and integrability. They obtained a mechanism optimal among auctions where bids are one-dimensional and the good is always sold.

independent across bidders. A bidder's type is a vector $(\vartheta_j)_{j=1}^m$, continuously distributed; depending on the mechanism, his *transaction* with the seller contains a money transfer y and a nonmonetary bundle $(x_k)_{k=1}^l$. A winning bidder's payoff is $u((x_k)_{k=1}^l, (\vartheta_j)_{j=1}^m) - y$ for some function u. The model therefore contains the following frameworks as special cases: independent private value auctions (where $u((x_k)_{k=1}^l, (\vartheta_j)_{j=1}^m)$ is replaced by a scalar t), auctions of incentive contracts in Che [7] and Laffont and Tirole [11] (where the vectors $(x_k)_{k=1}^l$ and $(\vartheta_j)_{j=1}^m$ are respectively replaced by scalars x and t), and multidimensional nonlinear pricing (where the number of bidders is one).

The paper first proves that an optimal auction gives zero winning probability to a positive measure of bidder-types (Proposition 3.1). Consequently, an auction would not be optimal without an appropriate entrance fee or reserve price (Corollary 3.1). This exclusion result shows that the dimensionality of bidder-types does make a difference, since in the unidimensional framework, a standard auction can be optimal without any reserve price, as long as the scalar bidder-types are sufficiently large. The proof is an extension of Armstrong's proof in the framework of nonlinear pricing, except that our auction setting requires a more careful treatment of the differentiability condition of mechanisms.

The paper then takes on the unsolved optimal auction problem and obtains a general solution in a class of environments. The difficulties of this problem come from the coupling of two features of the model: (i) a bidder-type $(\vartheta_j)_{j=1}^m$ is a vector and (ii) the bidder-type is not additively separable from the transaction $((x_k)_{k=1}^l, y)$ in the preference $u((x_k)_{k=1}^l, (\vartheta_j)_{j=1}^m)$. Without the first feature, one can easily characterize the incentive-compatibility (IC) constraint tractably and then obtain optimal auctions. Without the second feature, one can simply apply the solution of Myerson [15] by substituting the valuation $u((\vartheta_j)_{j=1}^m)$ here for the one-dimension type there. When both features are present, there has not been a tractable representation for the IC constraint of an arbitrary mechanism.

To bypass the above obstacle, this paper starts with a subset of mechanisms called *scoring mechanisms*: a winner is assigned a score and an additively separable scoring rule that maps bids to scores; the winner is to carry out a transaction whose score equals to the one assigned. In such a scoring mechanism, say ρ, a type-ϑ bidder behaves as if his payoff from winning is equal to a "private valuation" $\tau_\rho(\vartheta)$ minus a "payment" s, where s is the score assigned, and the *induced* valuation $\tau_\rho(\vartheta)$ is a scalar depending on his type and the mechanism. Such a separable structure enables us to characterize the IC constraint as a monotonicity condition with respect to this unidimensional induced valuation $\tau_\rho(\vartheta)$ (Lemma 4.3). We next add an assumption about the bidders' preferences and type distributions, which resembles the multiplicative separability assumption in Armstrong [1]. Due to this assumption, the induced valuation $\tau_\rho(\vartheta)$ in any scoring mechanism is monotone in a one-dimension statistic z of a bidder's type ϑ, independent of the mechanism. The IC constraint therefore becomes a monotonicity condition with respect to z.

Based on this tractable representation, we obtain a formula for optimal mechanisms by extending the technique of Myerson [15]. The optimal auction is almost equivalent to a Vickrey auction, except that the bids are ranked by an optimal scoring rule ρ^* (Equation (29)).

More precisely, the optimal mechanism is a "scoring-rule auction" (Theorem 4.1):

> The seller commits to the scoring rule ρ^*. Each bidder then independently pledges a score. The seller sells the good to a highest-score bidder if his score is positive, and withholds the good if otherwise. The winner carries out a transaction $((x_k)_k, y)$ such that its score $\rho^*((x_k)_k, y)$ is equal to either the second highest score pledged by the bidders or zero, whichever is larger.

This auction is optimal among all mechanisms when the hazard rate of the statistic z is monotone (non-binding IC constraint), and is optimal among all scoring mechanisms when otherwise (binding IC constraint).

A convenient feature of this mechanism is that the two tasks of auction design—to find a winner-selection criterion and choose a payment function that determines a winner's transaction—are fulfilled by one single construct: our scoring rule. Both tasks are delegated to the bidders via the bidding game.

The reason why our scoring-rule auction delivers optimality is roughly the following. Extending the usual steps of optimal auction design (Myerson [15, Section 4]), we know that the seller's equilibrium expected payoff cannot exceed a weighted sum

$$\sum_{\text{bidder } i} \text{prob}(i \text{ wins}) \, \text{MR}_i(\mathbf{x}^i, y^i)$$

at each possible state of the world, where $\text{MR}_i(\mathbf{x}^i, y^i)$ denotes the seller's marginal payoff from raising the probability with which bidder i wins, given the transaction $(\mathbf{x}^i, y^i)$. Thus, the best the seller could do is to (i) maximizes the marginal payoff MR_i and (ii) maximize the winning probabilities to those i whose $\max \text{MR}_i$ are positive and maximal among all bidders, subject to the IC constraint.

When the hazard rate of the statistic z is monotone, our scoring-rule auction implements both maximization operations without violating the IC constraint. The scoring rule induces a winner to choose the MR_i-maximizing transaction, thereby achieving operation (i). Furthermore, bidders with higher $\max \text{MR}_i$ bid higher scores in the auction, due to our scoring rule and the monotone hazard rate. Therefore, a winner's $\max \text{MR}_i$ is maximal among all bidders. Finally, the minimum score makes it unprofitable for a bidder to participate with a nonpositive $\max \text{MR}_i$. Thus, the scoring-rule auction achieves operation (ii) and reaches the upper bound of the above weighted sum, which is the highest the seller can get in any mechanism. Consequently, our auction game maximizes the seller's equilibrium expected payoff among all mechanisms.[7]

When the hazard rate of the statistic z is non-monotone, the IC constraint is binding when one attempts the above maximization operations. Since we manage to represent the

[7] More precisely, "all mechanisms" here means all the *regular* mechanisms satisfying the regularity condition in Section 2. This condition guarantees that the usual beginning step of optimal auction design is valid. The condition is automatically satisfied if a winner's payoff function is additively separable.

IC constraint in any scoring mechanism as a monotonicity condition, we are able obtain a mechanism optimal among scoring mechanisms through an extension of the "ironing" technique in Myerson [15, Section 6]. Remarkably, our optimal mechanism in this case is still the same scoring-rule auction as described above, except that the scoring rule is revised by the ironing procedure (Proposition 4.1). In this case, since our monotonicity representation of the IC constraint is valid only among scoring mechanisms, we only know that our mechanism is optimal among scoring mechanisms. An auction optimal among *all* mechanisms is still unknown in the case of binding IC constraints.

Our result of optimal auction implies that an optimizing seller should commit to evaluating bids by the scoring rule ρ^* instead of her own preferences; the latter would yield suboptimal outcomes (Proposition 4.2).[8] We further prove that the optimal scoring rule rewards the nonmonetary bundle x less than the seller's true preference would and the difference between them are calculated explicitly (Equation (31)). We have therefore extended the "downward distortion" result from unidimensional (Che [7]) to multidimensional settings, which says that an optimizing seller would induce downward distortion of nonmonetary bundles from the first-best configuration. In our health care example, this distortion result implies that even an employer cares as much about her employees' health care benefits as her employees do, she would commit herself to putting less weight on these provisions when selecting an insurance company.

Our result also yields an explicit optimal tariff in the special case of non-auction multidimensional screening (Corollary 4.2). This is new in that literature, because our solution covers the case of non-monotone hazard rate (binding IC constraint). Different from the cost-based tariff in Armstrong [1], the optimal tariff here need not be based on the monopolist's cost. Corresponding to the aforementioned downward distortion result, the monopolist would charge more for a nonmonetary bundle x than her cost of providing it.

In the enterprise of multidimensional optimal auction design, this paper provides an explicit solution for a class of environments, whether the incentive-compatibility is binding or not. The main message is that the common sense "auctioning the good to the highest bidder" in unidimensional settings can be restored in multidimensional settings, provided an optimal scoring rule and minimum score. The main restriction of the environments considered in this paper is the assumption of multiplicative separability. This assumption confines our search for optimal scoring rules to those based on a one-dimensional summary $L(x)$ of the nonmonetary attributes x of a bid. The multidimensional structure is thus compromised. Nevertheless, we still partially retain the multidimensional structure. The reason is that bidders having a same score in our scoring-rule auctions can have different transactions, depending on their actual multidimensional types. See Subsection 4.6 for an example. The optimal auction design without the dimension-compromising restriction is a

[8]In some actual auctions, the seller does not announce the scoring rule before the bidding. Such a practice occurred in the aforementioned Conservation Reserve Program (Osborn [16]) and school milk procurement auctions (Tichy [23]). Proposition 4.2 implies that such a "scoring-rule hiding" practice is not optimal. The reason is that bidders would expect that the seller would rank the bids according to her true preferences, without committing to a different scoring rule.

wide open question. I hope that this paper has provided results and techniques useful for further scientific exploration.

2 A Model

Consider an auction setting where a seller is to allocate at most one indivisible object to at most one of n competing bidders. The object is characterized by several attributes. One may think of the object as a multiple-provision contract between the seller and the winner. Depending on their agreement, the attributes of the object is configured as a vector x in a Euclidean space X. A bidder can also make a monetary payment $y \in R$ to the seller. Call such a pair (x, y) a *transaction*.

A bidder's privately known type is a vector in R^m. Bidder-types are independently and identically distributed across bidders, according to a known probability distribution with density function f and support Θ.

Given type $\vartheta \in R$ and transaction (x, y), a bidder's payoff is

$$u(x, \vartheta) - y$$

for some function $u : X \times R^m \to R$, and the seller's payoff is

$$v(x) + y$$

for some function $v : X \to R$. If a bidder does not win the object, then his payoff is $-y$.

For example, we may think of the object being auctioned as a weapon procurement contract between a government and a winning weapon firm. The term y, which may be negative, is a lump sum monetary transfer from the firm to the government. The vector x is a contingency reimbursement plan for the firm's overrun cost in the R&D phase for the weapon. The firm's valuation $u(x, \vartheta)$ of the contract depends on the x provision and its type. The government bears a cost $|v(x)|$ for the cost reimbursement plan x. Notice that the standard model of independent private value auction is a special case of the current setup, with the vector x degenerate to a constant.

By the Revelation Principle, we can denote an auction mechanism and its equilibrium by the corresponding direct revelation game $(q, \tilde{x}, \tilde{y})$, where $q(\vartheta, \theta^{(-i)})$ is the probability with which a type-ϑ bidder wins given his rivals' reported types $\theta^{(-i)}$, and $(\tilde{x}(\vartheta, \theta^{(-i)}), \tilde{y}(\vartheta, \theta^{(-i)}, \delta)$ his transaction with the seller, contingent on his winning status $\delta \in \{\text{win}, \text{lose}\}$. Under a mechanism $(q, \tilde{x}, \tilde{y})$, a type-$\vartheta$ bidder's expected payoff from mimicking type $\hat{\vartheta}$, expecting others abiding to the equilibrium, can be easily calculated as

$$\pi(\hat{\vartheta}, \vartheta) = \mathrm{E}_{\theta^{(-i)}} q(\hat{\vartheta}, \theta^{(-i)}) u \left[\tilde{x}(\hat{\vartheta}, \theta^{(-i)}), \vartheta \right] - \bar{y}(\hat{\vartheta}),$$

where $\mathrm{E}_{\boldsymbol{\theta}(-i)}$ denotes a bidder's expected value operator on functions of his rivals' types, and $\bar{y}(\hat{\vartheta})$ his expected monetary payment. Define for each bidder-type ϑ

$$U(\vartheta) := \pi(\vartheta, \vartheta).$$

Incentive-compatibility says that

$$U(\vartheta) = \max_{\hat{\vartheta} \in \Theta} \pi(\hat{\vartheta}, \vartheta), \ \forall \vartheta \in \Theta.$$

As usual, we call U the *surplus*, *indirect utility* or *equilibrium expected payoff* function of the underlying mechanism.

The following is a standard useful fact due to the quasi-linear structure of a bidder's payoff function. The proof is trivial and hence omitted.

Lemma 2.1 *Assume that $u(x, \cdot)$ ($\forall x \in X$) is a convex function of ϑ. Then the surplus function U of any incentive-compatible mechanism is convex.*

Throughout this paper, we will maintain the following assumption.

Assumption 1 *The functions $u(x, \cdot)$ ($\forall x \in X$) are convex, nondecreasing, linearly homogeneous, at least three times continuously differentiable, and strictly increasing in at least one dimension of R^m. The density function f is continuously differentiable on its support Θ and positive at all but finite points of Θ. The support Θ is compact and convex, contained by R^m_+ and containing $\mathbf{0}$, with full dimension in R^m, and its boundary consists of finitely many compact smooth $(m-1)$-manifolds.*[9]

For tractability, we will confine attention to *regular* mechanisms, i.e., those that meet the following Condition 1. As the rest of this section will explain, we need this regularity condition to calculate the surplus function U. We will prove that this condition is automatically guaranteed if a bidder's payoff is additively separable, as in the independent private value model. An impatient reader may skip to the next section.

As usual in mechanism design, we will need to calculate the gradient of U through the partial derivatives of $\pi(\hat{\vartheta}, \cdot)$. If the function π were continuously differentiable, we could do that by the Envelope Theorem. In an auction setting, however, the function π is usually not even continuous. The reason is that the winning probability q is usually not continuous in a bidder's type ϑ. That is because the seller may give zero winning probability to some bidder-types, through charging an entrance fee or committing to a reserve price.

[9] A *smooth k-manifold* is a metric space such that at every interior point there is a sufficiently small neighborhood diffeomorphic to R^k, and at every boundary point there is a sufficiently small neighborhood diffeomorphic to the half space of R^k. We will sometimes call a smooth k-manifold a *k-surface*, for convenience.

For a simple example, consider a first-price sealed-bid auction with independent unidimensional private type $\vartheta \in R$, so $\pi(\hat{\vartheta}, \vartheta) = \text{prob}(\text{win}|\hat{\vartheta})(\vartheta - \hat{\vartheta})$. Suppose the seller commits to a reserve price p_*. Then the winning probability $\text{prob}(\text{win}|\hat{\vartheta})$ is zero for all $\hat{\vartheta} < p_*$ and jumps up to a positive number at $\hat{\vartheta} = p_*$ (assuming continuous strictly increasing distribution of types). The function π is therefore discontinuous at all points (p_*, ϑ), except at (p_*, p_*), where π is not differentiable (Figure 1).

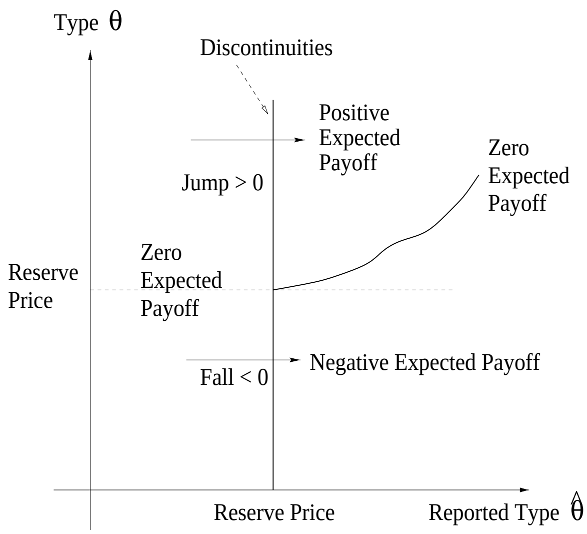

Figure 1: The Discontinuity of a Bidder's Expected Payoff π

Thus, it is not automatically valid to set the gradient of U equal to the gradient of $\pi(\hat{\vartheta}, \cdot)$. Fortunately, we can find a regularity condition under which the differentiation, except on a set of measure zero, is valid. This condition confines our attention to *regular* mechanisms, i.e., those satisfying the following condition. These mechanisms include standard auctions with reserve prices (e.g., the one depicted in Figure 1).

Condition 1 (Regularity) *For almost every type $\vartheta \in \Theta$ and for every $j = 1, \ldots, m$, the mapping $t \mapsto \frac{\partial}{\partial \vartheta_j} \pi(\hat{\vartheta}, \vartheta)\Big|_{\hat{\vartheta} = \vartheta + te_j}$ is continuous at the point $t = 0$, where e_j denotes the unit vector having the direction of the jth coordinate axis.*

When a bidder's payoff is additively separable between type and transaction (i.e., the vector x degenerates to a constant), such as the model in Myerson [15], an incentive-compatible mechanism is automatically regular:

Lemma 2.2 *If a bidder's payoff conditional on winning is additively separable in the sense that $u(\vartheta, x)$ is independent of x, and if $u(\vartheta, x)$, denoted by $u(\vartheta)$ with an abuse of notation, is a differentiable function of ϑ, then any incentive-compatible mechanism is regular.*

Proof: Incentive-compatibility is equivalent to

$$\pi(\vartheta', \vartheta) - \pi(\vartheta', \vartheta') \leq U(\vartheta) - U(\vartheta') \leq \pi(\vartheta, \vartheta) - \pi(\vartheta, \vartheta') \tag{1}$$

for any two types $\vartheta, \vartheta \in \Theta$. With additively separable preferences, this inequality implies

$$(u(\vartheta) - u(\vartheta'))(\bar{q}(\vartheta) - \bar{q}(\vartheta')) \geq 0 \tag{2}$$

for any two types ϑ, ϑ, where we denote $\bar{q}(\vartheta) := \mathrm{E}_{\theta^{(-i)}} q(\vartheta, \theta^{(-i)})$ for a bidder's expected winning probability conditional on his type ϑ. Thus, the probability $\bar{q}(\vartheta)$ is a nondecreasing function of $u(\vartheta)$. Consequently, the monotone function $u(\vartheta) \mapsto \bar{q}(\vartheta)$ has at most countably many discontinuous points in the range $u[\Theta]$ of u. If this mapping is discontinuous at a point $a \in u[\Theta]$, then the function $\bar{q}$ may be discontinuous at the boundary of the level set $u^{-1}(a)$ in R^m. Such a boundary is of measure zero in R^m, since $u(\cdot)$ is continuous by hypothesis. Therefore, we have deduced that the function $\bar{q}$ is continuous almost everywhere on the support Θ.

With the additive separability assumption and $u(\vartheta)$ assumed to be differentiable in ϑ, one easily calculates that $\frac{\partial}{\partial \vartheta_j} \pi(\hat{\vartheta}, \vartheta) = \bar{q}(\hat{\vartheta}) \frac{\partial}{\partial \vartheta_j} u(\vartheta)$. This partial derivative is continuous at $\hat{\vartheta}$ for almost all $\hat{\vartheta}$, since $\bar{q}(\hat{\vartheta})$ has been proved to be so. Thus, the Regularity Condition is satisfied. This proves the lemma. **Q.E.D.**

If an incentive-compatible mechanism is regular, then we can easily deduce

$$D_j U(\vartheta) = \mathrm{E}_{\theta^{(-i)}} \left[q(\vartheta, \theta^{(-i)}) \left. \frac{\partial}{\partial \vartheta_j} u(x, \vartheta) \right|_{x = \tilde{x}(\vartheta, \theta^{(-i)})} \right] \tag{3}$$

for almost every type $\vartheta \in \Theta$ and for all $j = 1, \ldots, m$. To see that, simply replace the ϑ' in Equation (1) with $\vartheta + te_j$ and use the regularity condition. For almost all type $\vartheta \in \Theta$ and for all vector $\mathbf{w} \in R^m$, the directional derivative $U'(\vartheta; \mathbf{w})$ at point ϑ along vector $\mathbf{w}$ can be easily calculated from Equation (3) as

$$U'(\vartheta; \mathbf{w}) = \mathrm{E}_{\theta^{(-i)}} \left[q(\vartheta, \theta^{(-i)}) \sum_{j=1}^{m} (\mathbf{w} \cdot e_j) \left. \frac{\partial}{\partial \vartheta_j} u(x, \vartheta) \right|_{x = \tilde{x}(\vartheta, \theta^{(-i)})} \right]. \tag{4}$$

3 An Exclusion Principle in Auctions

Our model is different from the usual model of independent private value auctions in only two aspects. One is that a bidder's private type is multidimensional. The other is that a bidder's type is not necessarily additively separable from his transaction with the seller. The first difference leads to an exclusion result, which says that an optimizing seller gives zero winning probability to a positive measure of bidder-types.

Proposition 3.1 (Exclusion Principle) *Suppose that $m \geq 2$, the support Θ of types is strictly convex,*[10] *and the function v is nonpositive. Then any optimal mechanism that is regular gives nonpositive expected payoffs to a positive measure of bidder-types.*

[10] A set is said to be *strictly convex* if any strictly convex combination of its elements is interior to the set.

This proposition will be proved in Appendix A.1. Its intuition is similar to the exclusion principle Armstrong [1] established for non-auction nonlinear pricing settings. In the following we will demonstrate the power of our exclusion result in a familiar special case—independent private value auctions.

Let us consider the usual independent private value auction with one indivisible good and n competing bidders. Here a bidder's payoff conditional on winning is $v - p$, where v is his private valuation of the good being auctioned and p his monetary payment to the seller. If the valuation were the underlying bidder-type, then the standard theory would predict, given sufficiently strong assumption of the distribution of bidder-types, the seller's expected revenue is the same across auction mechanisms; specifically, the Vickrey auction without reserve price would be seller-optimal (revenue-maximizing at bidding equilibrium).

A crucial element of the above setup is that a bidder's type is modeled as a unidimensional valuation v. The implicit assumption is that all aspects of a bidder's private information can be summarized to a one-dimension variable. Let us take this implicit assumption seriously. Thus, assume that a bidder's valuation v is a function u of the bidder's m-dimensional private information $\vartheta \in R^m$, with $m \geq 2$:

$$v = u(\vartheta), \ \forall \vartheta \in R^m. \tag{5}$$

Let F_* denote the marginal distribution function of bidders' valuation induced by the underlying density function f of bidder-types. That is,

$$F_*(v) := \text{Prob}\{\vartheta \in \Theta : u(\vartheta) \leq v\}, \ \forall v \in R.$$

Denote f_* for the marginal density function of F_*, if it exists. It follows immediately from Assumption 1 and Lemma A.1 that f_* does exist, and it is continuous on its support $u[\Theta]$ and positive almost everywhere on $u[\Theta]$.

A special case is that the bidders differ in their valuations v and another dimension ϑ_2 that has no effect on their valuations, i.e., $u(\vartheta) = \vartheta_1$ $(\forall \vartheta)$, and the support of the distribution of types is a narrow horizontal band (Figure 2). At first glance, one may think that the ϑ_2

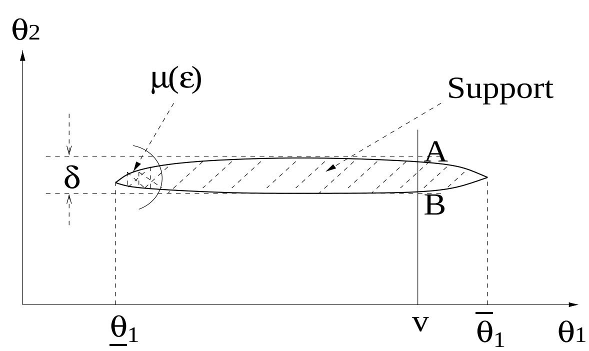

Figure 2: Can we neglect ϑ_2 when δ is small?

dimension would not change the prediction of the unidimensional model. After all, a bidder's valuation is simply the horizontal projection of his type, and the bidders' heterogeneity along the vertical dimension is bounded by the small width δ.

It turns out, however, that such a small "tremble" of the seemingly irrelevant ϑ_2 dimension changes our prediction significantly. No matter how small the tremble δ is, our exclusion principle implies that the Vickrey auction, as well as other auctions where almost all bidders stand a positive probability to win, is not seller-optimal:

Corollary 3.1 (Suboptimality of Vickrey Auction) *If the support Θ of bidder-types is strictly convex, then the first- and second-price sealed-bid auctions without reserve price are not seller-optimal.*

Proof: By the Exclusion Principle (Proposition 3.1), an auction is suboptimal if (i) almost every bidder-type gets a positive expected payoff at equilibrium and (ii) the auction is regular (Condition 1). We will prove that for both the Vickrey (second-price) and first-price auctions.

Let us first notice the obvious fact that the dominant-strategy equilibrium of the Vickrey auction is that each bidder bids truthfully his valuation $u(\vartheta)$ if ϑ is his type. It then follows from Lemma 2.2 that the mechanism is regular.

We next calculate a bidder's expected payoff in this mechanism. To do that, recall a fact from previous remark that the density function f_* of the a bidder's valuation $u(\vartheta)$ exists, and it is positive almost everywhere on its support $u[\Theta]$. Thus, the density function of the highest valuation among a bidder's rivals exists and is almost everywhere positive on $u[\Theta]$. Denote this density function by $f_{*,n-1}$. Let $\underline{v} := \min_\Theta u$. A type-$\vartheta$ bidder's expected payoff $\pi(\hat{\vartheta}, \vartheta)$ from bidding $u(\hat{\vartheta})$ is

$$\pi(\hat{\vartheta}, \vartheta) = \int_{\underline{v}}^{u(\hat{\vartheta})} [u(\vartheta) - v] f_{*,n-1}(v) dv.$$

By the Exclusion Principle, if the mechanism were optimal, then it must give nonnegative expected payoff to a set of bidder-types of positive measure. Let ϑ be such a type. Then the equilibrium expected payoff for this type is zero, i.e.,

$$\int_{\underline{v}}^{u(\vartheta)} [u(\vartheta) - v] f_{*,n-1}(v) dv = 0.$$

Since the integrand is nonnegative and $f_{*,n-1}$ almost everywhere positive on its support, we are forced to deduce that $u(\vartheta) = v$ for almost every $v \in [\underline{v}, u(\vartheta)]$, which is impossible unless $u(\vartheta) = \underline{v}$. Since u is assumed to be strictly increasing in at least one dimension of the type, the set of such ϑ is of measure zero. It follows that the Vickrey auction gives positive expected payoff to almost all bidder-types. Thus, this mechanism is not seller-optimal.

The case for the first-price sealed-bid auction is similar. As in the standard auction model, the symmetric Bayes-Nash equilibrium where the bid is differentiable in the valuation exists and is unique. At this equilibrium, a type-ϑ bidder submits a bid $\beta(u(\vartheta))$ below his valuation $u(\vartheta)$, and the bid is strictly increasing in the valuation $u(\vartheta)$. Consequently, a type-ϑ bidder's expected payoff is

$$\pi(\hat{\vartheta}, \vartheta) = [u(\vartheta) - \beta(u(\hat{\vartheta}))] F_{*,n-1}(u(\hat{\vartheta}))$$

from bidding $\beta(u(\hat{\vartheta}))$. Since $F_{*,n-1}$ is strictly increasing and $\beta(u(\vartheta)) < u(\vartheta)$ except for those types whose valuation is the minimum level, almost all bidder-types get positive equilibrium expected payoff. Since Lemma 2.2 implies that the mechanism is regular, it follows from the Exclusion Principle that this mechanism is suboptimal. Thus, we have proved the corollary. **Q.E.D.**

A reader may be puzzled by the suboptimality of the Vickrey auction, because here the other dimension ϑ_2 of a bidder's type can have no effect on his valuation of the good. The reader may ask: What is wrong with the standard argument of optimal auctions in the unidimensional theory?

The answer is that a crucial assumption—the hazard rate condition—in that standard proof no longer holds when bidder-types are multidimensional. To explain this answer, let us recall the essence of the standard derivation of optimal auctions. That derivation assigns to each bidder-valuation v a real number $\mathrm{MR}(v)$, which measures the seller's *marginal revenue* from raising the probability of winning for that type:[11]

$$\mathrm{MR}(v) = v - \frac{1 - F_*(v)}{f_*(v)}, \ \forall v \in u[\Theta], \tag{6}$$

Thus, a crucial condition for an auction without reserve price to be seller-optimal is that the marginal revenue $\mathrm{MR}(u(\vartheta))$ of any bidder-type ϑ is nonnegative. If this condition is violated, the seller would rather withhold the good from some bidder-types. When bidder-types are unidimensional, one can guarantee this nonnegativity condition by assumptions of the type distributions. When bidder-types are multidimensional, in contrast, one can prove that this condition is violated. To see the reason, Let us look at the example in Figure 2 and assume that the two-dimension bidder-type $(\vartheta_1, \vartheta_2)$ is uniformly distributed on the support. The marginal density $f_*(v)$ is then the length of the segment AB (Figure 2), which is the intersection between the support and the vertical line $\{(\vartheta_1, \vartheta_2) : \vartheta_1 = v\}$. Obviously, when the valuation v moves to its infimum $\underline{\vartheta}_1$, the length of AB shrinks to zero. Thus, $f_*(v) \to 0$ and $\mathrm{MR}(v) \to -\infty$, no matter how large this infimum is. That is why the multidimensionality of types necessitates the exclusion of a positive measure of bidder-types.

4 Optimal Auctions

This section will take on the problem of finding a mechanism that maximizes the seller's expected payoff at equilibrium. This problem is challenging because of the multidimensional

[11]The term marginal revenue here are also called virtual surplus, virtual utility, or virtual welfare.

bidder-types, which make the incentive-compatibility condition hard to characterize. We are able to obtain a solution for optimal auctions in a subset of environments that satisfy an assumption of multiplicative separability in Armstrong [1]. We will consider both binding and non-binding incentive-compatibility constraints.

4.1 The Usual Beginning Steps

Let $(\tilde{x}, \tilde{y}, q)$ be a regular (Condition 1) and incentive-compatible direct-revelation game. Denote U for the corresponding surplus function. Recall from Assumption 1 that the support Θ of bidder-types is contained in R^m_+ and contains the point $\mathbf{0}$.

As usual in optimal auction theory, we start by calculating U. The obstacle of multi-dimensional types in this step is resolved by the technique of "integration along the ray" in Armstrong [1]. For each type $\vartheta \in \Theta$, define a function $\tilde{U}_\vartheta : [0,1] \to R$ by $t \mapsto U(t\vartheta)$ from the unit interval to the reals. This function is well-defined because Θ is assumed to be convex. We want to calculate $U(\vartheta)$ via $\tilde{U}_\vartheta(1)$. Since U is a convex function (Lemma 2.1), so is $\tilde{U}_\vartheta$. Thus, $\tilde{U}_\vartheta$ is absolutely continuous and so

$$U(\vartheta) - U(\mathbf{0}) = \int_0^1 \tilde{U}'_\vartheta(t)dt.$$

By the definition of $\tilde{U}_\vartheta$, the derivative $\tilde{U}'_\vartheta(t)$ is the directional derivative of the scalar field U at the point $t\vartheta$ along the vector ϑ. Since the mechanism is assumed to be regular and incentive-compatible, this directional derivative can be calculated according to Equation (4). Thus, for almost all $t \in [0,1]$,

$$\begin{aligned} \tilde{U}'_\vartheta(t) &= \mathrm{E}_{\theta^{(-i)}} \left[q(t\vartheta, \theta^{(-i)}) \sum_{j=1}^m \vartheta_j \left. \frac{\partial}{\partial \vartheta_j} u(x, t\vartheta) \right|_{x=\tilde{x}(t\vartheta, \theta^{(-i)})} \right] \\ &= \mathrm{E}_{\theta^{(-i)}} \left[q(t\vartheta, \theta^{(-i)}) u(\tilde{x}(t\vartheta, \theta^{(-i)}), t\vartheta) \right] / t, \end{aligned}$$

where the second equality follows from the linear homogeneity of $u(x, \cdot)$ (Assumption 1). We have therefore obtained

$$U(\vartheta) = U(\mathbf{0}) + \int_0^1 \mathrm{E}_{\theta^{(-i)}} \left\{ q(t\vartheta, \theta^{(-i)}) u(\tilde{x}(t\vartheta, \theta^{(-i)}), t\vartheta)/t \right\} dt, \ \forall \vartheta \in \Theta. \tag{7}$$

We next calculate the seller's expected payoff in the above mechanism. Let $\theta^{(i)} := (\theta_j^{(i)})_{j=1}^m$ denote the type of bidder i, and $\theta^{(-i)} := (\theta^{(k)})_{k \neq i}$ the type profile of the other bidders. Since the types are independent across bidders, the seller's expected payoff is

$$\sum_{i=1}^n \left\{ \mathrm{E}_{\theta^{(i)}, \theta^{(-i)}} \left(q(\theta^{(i)}, \theta^{(-i)}) [v(\tilde{x}(\theta^{(i)}, \theta^{(-i)})) + u(\tilde{x}(\theta^{(i)}, \theta^{(-i)}), \theta^{(i)})] \right) - \mathrm{E}_{\theta^{(i)}} U(\theta^{(i)}) \right\}.$$

To calculate this quantity explicitly, define for each type $\vartheta \in \Theta$

$$g(\vartheta) := \int_1^\infty t^{m-1} f(t\vartheta) dt. \tag{8}$$

For each attribute bundle $x \in X$ and each type $\vartheta \in \Theta$, define the *virtual utility* as

$$V(x, \vartheta) := v(x) + u(x, \vartheta) \left(1 - \frac{g(\vartheta)}{f(\vartheta)}\right). \tag{9}$$

Using Equation (7) and the Tonelli's Theorem (Royden[21, p. 309]), one can calculate the seller's expected payoff as

$$\mathrm{E}_{\theta^{(i)},\theta^{(-i)}} \left\{ \sum_{i=1}^{n} q(\theta^{(i)}, \theta^{(-i)}) V \left(\tilde{x}(\theta^{(i)}, \theta^{(-i)}), \theta^{(i)} \right) \right\} - nU(\mathbf{0}). \tag{10}$$

We omit the details, which can be found from Armstrong [1, p. 62].

As standard in optimal auction theory, there is no loss of generality to let $U(\mathbf{0}) = 0$. This equation implies individual rationality, because $U(\vartheta) \geq U(\mathbf{0})$ for all type $\vartheta \in \Theta$, which results from the assumption that $u(x, \cdot)$ is nondecreasing and $\Theta \subseteq R^m_+$. There is no need to consider mechanisms where $U(\mathbf{0}) > 0$, which are obviously suboptimal for the seller.

We have therefore derived the following fact, similar to its counterpart in unidimensional models:

Lemma 4.1 *The problem of maximizing the seller's expected payoff subject to the constraints of regularity, incentive-compatibility and individual rationality is equivalent to maximizing*

$$\mathrm{E}_{\theta^{(i)},\theta^{(-i)}} \left\{ \sum_{i=1}^{n} q(\theta^{(i)}, \theta^{(-i)}) V \left(\tilde{x}(\theta^{(i)}, \theta^{(-i)}), \theta^{(i)} \right) \right\}$$

among all $(\tilde{x}, q)$ subject to the conditions of incentive-compatibility (Equation (1)), resource feasibility

$$\sum_{i=1}^{n} q(\theta^{(i)}, \theta^{(-i)}) \leq 1 \quad \textit{and} \quad 0 \leq q(\theta^{(i)}, \theta^{(-i)}) \leq 1, \ \forall i = 1, \ldots, n, \ \forall (\theta^{(i)}, \theta^{(-i)}) \in \Theta^n, \tag{11}$$

and regularity (Condition 1).

Therefore, if the incentive-compatibility constraint is non-binding, the seller would follow a *greedy algorithm* in descending order of the virtual utility V: sell the good to a bidder whose $\max_x V(x, \theta^{(i)})$ is highest among all bidders if that amount is positive, and withhold the good if otherwise; in addition, configure the attributes x of the good to attain to the maximum $\max_x V(x, \theta^{(i)})$.

4.2 Incentive-Compatibility in Scoring Mechanisms

The main difficulty in our model is how to characterize incentive-compatibility (Equation (1)). When bidder-types are unidimensional, it is well-known that incentive-compatibility is equivalent to a monotonicity condition, which says that a bidder's winning probability given his reported type is nondecreasing in that type. This equivalence result would allow us to apply the technique in Myerson [15] to obtain optimal auctions.[12] With multidimensional bidder-types, although it remains valid when bidders' payoff functions are additively separable (i.e., $u(x, \vartheta)$ depends only on ϑ), the equivalence result does not hold in general. Researchers in this field have found the condition of incentive-compatibility quite intractable.

This paper bypasses the above obstacle in the following way. First, we temporarily look at a subset of mechanisms, which we will call scoring mechanisms. Such a mechanism makes bidders' payoffs additively separable, thereby salvaging the monotonicity condition for incentive-compatibility. We will then mimic Myerson's technique to characterize the optimal auctions within this subset of mechanisms, for both binding and non-binding incentive-compatibility constraints. Finally, we will prove that, when incentive-compatibility is non-binding, which is guaranteed by a hazard rate assumption, the optimal auction we obtain is also optimal among the entire class of mechanisms.

By a *scoring mechanism* we mean a *scoring rule* $\rho : X \times R \to R$ given by

$$\rho(x, y) = y + \omega(x)$$

for some function $\omega : X \to R$ and the following rule: if a bidder wins, he is assigned a score s in the range of ρ and he chooses a transaction (x, y) such that $\rho(x, y) = s$. Denote such a mechanism by its scoring rule ρ.

The class of scoring mechanisms contains the *scoring-rule auctions* we often observe in industries. By a scoring-rule auction we mean the following mechanism:

1. The seller commits to a *minimum score* $\underline{s}$, a scoring rule ρ, and an integer $k = 1, 2$.
2. Each bidder independently pledges a score $s \in R$. The seller awards the good to a highest-score bidder if his score is above $\underline{s}$, and otherwise withholds the good.
3. The winner carries out a transaction $(x, y) \in X \times R$ subject to the condition that $\rho(x, y)$ is equal to the maximum of $\underline{s}$ and the kth highest score pledged by the bidders.

We say the auction is *first-score* if $k = 1$, and *second-score* if $k = 2$.

Compared to a general mechanism, a scoring mechanism has the special feature of delegating the determination of a winner's transaction to the winner himself, subject to a scoring rule. Compared to a general scoring mechanism, a scoring-rule auction has the special feature that the scoring rule also selects a winner.

[12] Specifically, this equivalent representation allows the seller to replace the incentive-compatibility constraint with the tractable monotonicity condition. If the virtual utility V is nondecreasing in the bidder-type, she simply follows the greedy algorithm in descending order of V (so the monotonicity condition is automatically guaranteed). Otherwise, she first "irons" out the non-monotone parts of the virtual utility V (Myerson [15, Section 6]), so the revised V becomes monotone, and then follows the greedy algorithm in descending order of this revised V.

In a scoring mechanism ρ, once a type-ϑ bidder wins, he chooses a payoff-maximizing transaction (x, y) to fulfill his score. Since a winner's payoff u and the scoring rule ρ are both additively separable between x and y, a winner's payoff, given type ϑ and score s, is

$$u_\rho(s, \vartheta) := -s + \max_{x \in X} \{u(x, \vartheta) + \omega(x)\}. \tag{12}$$

Thus, a winner's payoff is an additively separable function of his score and a one-dimensional statistic $\max_{x \in X} \{u(x, \vartheta) + \omega(x)\}$ induced by the scoring rule and his type. Therefore, we can regard a scoring mechanism as a direct-revelation game, where the message space is the range of this induced statistic, and the allocation outcome consists of the winning status and the score assigned to the winner. Consequently, by the standard argument in Myerson [15, Lemma 2], the incentive-compatibility of such a mechanism can be shown to be equivalent to the monotonicity condition that a bidder's winning probability is a nondecreasing function of his reported message.

A scoring mechanism therefore partially salvages the monotonicity representation of incentive-compatibility. The only obstacle in our way is that this monotonicity condition depends on the scoring mechanism itself. To bail out this obstacle, we need stronger assumptions about the fundamentals. We want to have an assumption strong enough to give us a mechanism-independent (unidimensional) statistic of the bidder-type, yet not too strong to allow the multidimensional structure of our model. We found such an assumption from Armstrong [1]:

Assumption 2 (Multiplicative Separability) *There exist functions $\zeta : \Theta \to R$, $L : X \to R$, and $\nu : \mathrm{range}\, L \to R$ such that, for all $\vartheta \in \Theta$ and all $b \in \mathrm{range}\, L$,*

$$\max\{u(x, \vartheta) : L(x) = b\} = \zeta(\vartheta)\nu(b), \tag{13}$$

where: (i) the function ζ is nonnegative, linearly homogeneous, and three-times continuously differentiable with nonzero gradient at every point $\vartheta \neq \mathbf{0}$, and (ii) the function L is continuous and linear, and $L(x) > 0$ unless $x = \mathbf{0}$. There exist continuous functions $f_\zeta : R_+ \to R_+$ and $f_0 : \Theta \to R_+$ such that

$$f(\vartheta) = f_\zeta(\zeta(\vartheta)) \times f_0(\vartheta), \ \forall \vartheta \in \Theta, \tag{14}$$

where f_ζ is positive over the interior of $\mathrm{range}\, \zeta$, and f_0 is positive and homogeneous of degree zero.

Essentially, this assumption yields a mechanism-independent statistic $\zeta(\vartheta)$ of the bidder-type ϑ. For convenience, we need one more assumption:

Assumption 3 *For any $\vartheta \neq \mathbf{0}$, $u(\cdot, \vartheta)$ is strictly concave, differentiable, and satisfies the Inada Condition.[13] If $x \neq \mathbf{0}$ and $\vartheta \neq \mathbf{0}$, $u(x, \vartheta) > 0$ and $u(\lambda x, \vartheta)$ is strictly increasing in $\lambda \geq 0$. There is a function $\tilde{v} : R \to R$ such that*

$$v(x) = \tilde{v}(L(x)), \ \forall x \in X,$$

[13] That is, each partial derivative of $u(\cdot, \vartheta)$ goes to positive infinity as x goes to $\mathbf{0}$, and goes to zero as $\|x\|$ goes to infinity.

and the function $\tilde{v}$ is concave, continuous, strictly decreasing, nonpositive, and $\tilde{v}(0) = 0$.

For example, let the seller's payoff be $y - c\sum_{j=1}^{m} x_j$, a type-ϑ winning bidder's payoff be $\sum_{j=1}^{m} \vartheta_j x_j^{1/2} - y$, and the distribution of bidder-types be $F(\vartheta) = \|\vartheta\|^{\alpha}$ on the support $\{\vartheta \in R_+^m : \|\vartheta\| \leq 1\}$. Then one can calculate that $L(x) = \sum_{j=1}^{m} x_j$, $\tilde{v}(b) = -bc$, $\zeta(\vartheta) = \|\vartheta\|$, $\nu(b) = \sqrt{b}$, $f_{\zeta}(z) = \alpha z^{\alpha-1}$, and $f_0 \equiv 1$. It is easily to check that the above assumptions are satisfied.

Assumptions 2 and 3 imply nice properties of the functions ζ and ν, stated in the following lemma and proved in Appendix A.2.

Lemma 4.2 *(i) For any $\vartheta \neq \mathbf{0}$, $\zeta(\vartheta) > 0$.*
(ii) The function ν is continuous, differentiable, strictly concave, strictly increasing, and strictly positive except at the point zero, where $\nu(0) = 0$.

As intended, Assumptions 2 and 3 turn the incentive-compatibility constraint of a scoring mechanism into a monotonicity condition with respect to the mechanism-independent statistic $\zeta(\vartheta)$. This we will show in the following.

Let us consider a scoring mechanism with scoring rule $\rho(x, y) = y + \omega_{\rho}(L(x))$ $(\forall x, y)$ for some function ω_{ρ}, where the function L is given by Assumption 2. Denote $L[X]$ for the range of function L. Assumptions 2 and 3 imply

$$\max_{x\in X} \{u(x, \vartheta) + \omega_{\rho}(L(x))\} = \max_{b\in L[X]} \max\{u(x, \vartheta)+\omega_{\rho}(b) : L(x) = b\} = \max_{b\in L[X]} \{\zeta(\vartheta)\nu(b)+\omega_{\rho}(b)\}.$$

For each z in the range of the statistic ζ, define the *induced type* as

$$\tau_{\rho}(z) := \max_{b\in \text{range } L} \{z\nu(b) + \omega_{\rho}(b)\}. \tag{15}$$

By Equation (12), a type-ϑ winner's payoff in the scoring mechanism ρ is $\tau_{\rho}(\zeta(\vartheta) - s$. Therefore, the mechanism is equivalent to an independent private value auction, where a bidder's private valuation is his induced type $\tau_{\rho}(\zeta(\vartheta))$. Formally, a scoring mechanism $\rho(x, y) = y + \omega_{\rho}(L(x))$ is equivalent to the following direct-revelation game, denoted by the triple $(\rho, q, \tilde{s})$:

> Each bidder's message space is the range of the induced type τ_{ρ}. Given any reported message profile $(t_i, \mathbf{t}_{-i})$, the probability with which bidder i wins is $q(t_i, \mathbf{t}_{-i})$, the score assigned to i if he wins is $\tilde{s}((t_i, \mathbf{t}_{-i})$, and a bidder's payoff upon winning is his induced type minus his score.

For any possible induced type t, denote $\bar{q}(t) := \mathrm{E}_{\mathbf{t}_{-i}} q(t, \mathbf{t}_{-i})$ for a bidder's winning probability if he reports t, and denote $\bar{s}(t) := \mathrm{E}_{\mathbf{t}_{-i}} \tilde{s}(t, \mathbf{t}_{-i})$ for his expected score. A bidder's expected payoff from reporting induced type $\hat{t}$, with true induced type t, is

$$\pi(\hat{t}, t) = t\bar{q}(\hat{t}) - \bar{s}(\hat{t}). \tag{16}$$

To characterize incentive-compatibility as a monotonicity condition with respect to the mechanism-independent statistic ζ, consider a scoring mechanism $\rho(x, y) = y + \omega_\rho(L(x))$ such that the function $z\nu(b) + \omega_\rho(b)$ has a maximum $b_\rho(z)$ in the range $L[X]$, for each z in the range of ζ. The Envelope Theorem gives the derivative

$$\tau'_\rho(z) = \nu(b_\rho(z)),$$

which is nonnegative by Lemma 4.2 and the assumption that $L[X] \subseteq R_+$ (Assumption 2). Thus, the induced type $\tau_\rho(z)$ is nondecreasing in the statistic z. Consequently, a monotonicity condition with respect to the induced type becomes a monotonicity condition with respect to the mechanism-independent ζ. The next lemma states this result.

Lemma 4.3 (Monotonicity Condition) *If a scoring mechanism $(\rho, q, \tilde{s})$ satisfies $\rho(x, y) = y + \omega_\rho(L(x))$ $(\forall x, y)$ and allows a maximum of $z\nu(b) + \omega_\rho(b)$ for each z in the range of the statistic ζ, then the mechanism is incentive-compatible iff*

a. *the winning probability $\bar{q}(\tau_\rho(z))$ $(\forall z \in \mathrm{range}\,\zeta)$ is nondecreasing in z, and*

b. *the assignment of scores satisfies $\pi(t, t) - \pi(\hat{t}, \hat{t}) = \int_{\hat{t}}^{t} \bar{q}(t')dt'$ for all $t, \hat{t} \in \mathrm{range}\,\tau_\rho$.*

Proof: In such a scoring mechanism $(\rho, q, \tilde{s})$, a winner's payoff is additively separable (Equation (16)). Thus, one can easily mimic the proof in Myerson [15, Lemma 2] to show that incentive-compatibility is equivalent to condition (b) and the nondecreasing monotonicity of $\bar{q}$. The proof will be complete if this nondecreasing monotonicity condition is equivalent to that of $\bar{q} \circ \tau_\rho$. By the proved fact that τ_ρ is nondecreasing, "$\bar{q}$ is nondecreasing" implies "$\bar{q} \circ \tau_\rho$ is nondecreasing." To prove the converse, suppose that $\bar{q}$ is nondecreasing. For each t in the range of τ_ρ, pick any element in the inverse image $\tau_\rho^{-1}(t)$ and denote it by $Z(t)$. Then the function Z is nondecreasing, since τ_ρ has been proved to be so. Thus, $\bar{q} = \bar{q} \circ \tau_\rho \circ Z$ is nondecreasing. Thus, the nondecreasing monotonicity of $\bar{q}$ is equivalent to that of $\bar{q} \circ \tau_\rho$, as desired. This proves the lemma. **Q.E.D.**

We will say that a scoring mechanism $(\rho, q, \tilde{s})$ is *well-behaved* if it satisfies the hypothesis of the above lemma, i.e., $\rho(x, y) = y + \omega_\rho(L(x))$ $(\forall x, y)$ and the function $z\nu(\cdot) + \omega_\rho(\cdot)$ has a maximum for each z in the range of ζ. Although Lemma 4.3 reduces the incentive-compatibility constraint of well-behaved scoring mechanisms to a monotonicity condition with respect to the unidimensional variable $\zeta(\vartheta)$, the multidimensional structure in our model is intact. The reason is that a scoring mechanism delegates the choice of a multidimensional transaction bundle (x, y) to the winning bidder; thus, we do not assume away the question how to induce the winner to choose the transaction optimal to the seller.

4.3 Optimal Auctions among Scoring Mechanisms

Using the convenient representation of incentive-compatibility delivered by Lemma 4.3, this subsection will characterize the optimal auction among scoring mechanisms. Given a hazard rate assumption, the next subsection will show that this optimal auction is also optimal among the entire class of regular mechanisms.

Let Φ denote the distribution function of $\zeta(\vartheta)$ induced by the underlying density function f of bidder-types. That is,

$$\Phi(z) := \text{Prob}\{\vartheta \in \Theta : \zeta(\vartheta) \leq z\},\ \forall z \in \text{range}\,\zeta.$$

Denote ϕ for the density function of Φ, if it exists. By Corollary A.1, the density function exists, has finite value and is continuous on the range $\zeta[\Theta]$ of ζ; it is positive over the interior of $\zeta[\Theta]$. Furthermore,

$$\phi(z) = kz^{m-1} f_\zeta(z),\ \forall z \in \zeta[\Theta], \tag{17}$$

where k is a positive constant. Hence the cdf Φ is continuously differentiable.

Recall from Lemma 4.1 that the seller would follow the greedy algorithm in descending order of $\max_x V(x,\vartheta)$ if the incentive-compatibility constraint were not binding. Due to Assumptions 2 and 3, $\max_x V(x,\vartheta)$ is collapsed to a function of the statistic $\zeta(\vartheta)$: For each type $\vartheta \in \Theta$,

$$\max_{x\in X} V(x,\vartheta) = \max_{b\in \text{range}\, L} \left\{ \nu(b)\left(z - \frac{1-\Phi(z)}{\phi(z)}\right) + \tilde{v}(b)\right\},\quad \text{with } z := \zeta(\vartheta). \tag{18}$$

To prove this equation, notice that $\max_x V(x,\vartheta) = \max_{b\in L[X]} \max\{V(x,\vartheta) : L(x) = b\}$. By the definition of the virtual utility V (Equation (9)), Assumptions 2 and 3, we have

$$\max\{V(x,\vartheta) : L(x) = b\} = \nu(b)\zeta(\vartheta)\left(1 - \frac{g(\vartheta)}{f(\vartheta)}\right) + \tilde{v}(b).$$

Equation (18) then follows from a fact proved by Armstrong [1, Section 4.4]: for each $\vartheta \in \Theta$, $\frac{\zeta(\vartheta)g(\vartheta)}{f(\vartheta)} = \frac{1-\Phi(\zeta(\vartheta))}{\phi(\zeta(\vartheta))}$.[14]

Let $\underline{z} := \min \zeta[\Theta]$. For each $z \in \zeta[\Theta]$ and each $b \in L[X]$, define

$$W(b,z) := R(z)\nu(b) + \tilde{v}(b),\ \text{with } R(z) := z - \frac{1-\Phi(z)}{\phi(z)}. \tag{19}$$

Given any profile $(\theta^{(i)})_{i=1}^n$ of types across bidders, let $z_i := \zeta(\theta^{(i)})$ and let $z_{-i} := (z_j)_{j\neq i}$. The problem of optimizing auctions among well-behaved scoring mechanisms (defined in previous subsection) is stated by the following lemma.

Lemma 4.4 *Suppose that the probabilities $q(z_i, z_{-i})$ ($\forall i, z_i, z_{-i}$) and the functions $\beta : \zeta[\Theta] \to L[X]$ and $\omega : L[X] \to R$ jointly maximize*

$$\mathrm{E}_{z_i,z_{-i}} \sum_{i=1}^n [q(z_i, z_{-i}) W(\beta(z_i), z_i)] \tag{20}$$

[14]This fact results from Equation (14) and the differentiability and linear homogeneity of the function ζ (Assumption 2).

subject to three constraints: "$\bar{q}(z_i) := \mathrm{E}_{z_{-i}} q(z_i, z_{-i})$ is nondecreasing in z_i," resource feasibility (Equation (11)), and

$$\max_{b \in L[X]} \{z\nu(b) + \omega(b)\} = \underbrace{z\nu(\beta(z)) + \omega(\beta(z))}_{:=\tau(z)}, \ \forall z \in \zeta[\Theta]. \tag{21}$$

Let $\rho(x, y) := y + \omega(L(x))$ ($\forall x, y$) and $\tilde{s}(z) := \tau(z)\bar{q}(z) - \int_{\underline{z}}^{z} \bar{q}(t)dt$ ($\forall z \in \zeta[\Theta]$). Then the scoring mechanism $(\rho, q \circ \tau^{-1}, \tilde{s} \circ \tau^{-1})$ maximizes the seller's expected payoff among incentive-compatible, individually rational and well-behaved scoring mechanisms.

Proof: One can easily prove this from Equation (18), and Lemmas 4.1 and 4.3, by mimicking the proof in Myerson [15, Lemma 3]. There are only two details worth mentioning. One is that Equation (21) is needed as a constraint because a bidder's valuation $\tau(z)$ of the good being auctioned results from his optimal choice of a transaction bundle. The other is that the regularity condition in Lemma 4.1 is automatically guaranteed by a well-behaved scoring mechanism, by Lemma 2.2. **Q.E.D.**

We want to calculate optimal mechanisms from Lemma 4.4. Unlike the usual optimal auction problems, where the only choice variables are the probabilities q, our problem here contains another choice variable, the function $\beta(\cdot)$, which is needed here to induce a winner to carry out a seller-optimal transaction.

To solve the constrained optimization problem, we need to "iron" out the non-monotone parts of R. This procedure is needed to satisfy the incentive-compatibility constraint, which may be binding. Specifically, with the objective (20), the seller would wish to maximize $W(\cdot, z)$ and assign the highest winning probabilities q to the highest $\max_b W(b, z)$, so a bidder's winning probability would be increasing in $\max_b W(b, z)$. On the other hand, incentive-compatibility requires that the winning probability increase in z. Thus, the seller needs to maximize $W(\cdot, z)$ subject to the constraint that the constrained maximum of $W(\cdot, z)$ is increasing in z. This is done by extending the technique of Myerson [15, Section 6].

We first notice that the distribution function Φ of the statistic ζ is strictly increasing. The reason is that its density function ϕ is positive almost everywhere, as pointed out at Equation (17). Thus, the inverse Φ^{-1} exists.

For each probability $p \in [0, 1]$, define $h(p) := R \circ \Phi^{-1}(p)$. Pick an $a \in (0, 1)$. Then $\Phi^{-1}(a)$ is an interior point of $\zeta[\Theta]$, so $\phi(\Phi^{-1}(a)) > 0$ (Equation (17)) and $h(a)$ is finite. Thus, we can define $H(p) := \int_a^p h(r)dr$ for each $p \in (0, 1)$ and continuously extend H to the boundary points 0 and 1. Let $G : [0, 1] \to R$ be the convex hull of the function H (Myerson [15, Eq. (6.3)]). For each $z \in \zeta[\Theta]$ and each $b \in L[X]$, define

$$\begin{aligned}
\overline{R}(z) &:= G'(\Phi(z)); && (22)\\
\overline{W}(b, z) &:= \overline{R}(z)\nu(b) + \tilde{v}(b); && (23)\\
\overline{\beta}(z) &:= \arg\max_{b \in L[X]} \overline{W}(b, z); && (24)\\
\overline{W}_*(z) &:= \overline{W}(\overline{\beta}(z), z). && (25)
\end{aligned}$$

The following lemma says that $\overline{\beta}$ and $\overline{W}_*$ are well-defined and nondecreasing.

Lemma 4.5 *The functions $\overline{\beta}$ and $\overline{W}_*$ are continuous, nonnegative, and nondecreasing, with the following properties:*

a. If $\overline{R}(z) \le 0$ then $\overline{\beta}(z) = 0$ and $\overline{W}_*(z) = 0$.

b. If $\overline{R}(z) > 0$, then $\overline{\beta}(z)$ is unique and positive, and $\overline{W}_*(z) > 0$.

c. If $\overline{R}(z) = \overline{R}(z')$, then $\overline{\beta}(z) = \overline{\beta}(z')$ and $\overline{W}_*(z) = \overline{W}_*(z')$. If $\overline{R}(z), \overline{R}(z') > 0$ and $\overline{R}(z) > \overline{R}(z')$, then $\overline{\beta}(z) > \overline{\beta}(z')$ and $\overline{W}_*(z) > \overline{W}_*(z')$.

d. The monotonicity of $\overline{\beta}$ implies that it is continuous.

e. Equation (21) is satisfied by $\omega^* : L[X] \to R$ defined by

$$\omega^*(b) := \begin{cases} -\int_0^b \overline{\beta}_*^{-1}(t)\nu'(t)dt & \text{if } b \in \operatorname{range}\overline{\beta} \\ -\infty & \text{otherwise,} \end{cases} \tag{26}$$

where, for each $b \in \operatorname{range}\overline{\beta}$,

$$\overline{\beta}_*^{-1}(b) := \text{the maximum of the inverse image } \overline{\beta}^{-1}(b). \tag{27}$$

This lemma will be proved in Appendix A.3.

For any vector $\mathbf{z} := (z_i)_{i=1}^n \in \zeta[\Theta]^n$ of statistics ζ indexed by bidders, let $M(\mathbf{z})$ be the set of bidders for whom $\overline{W}_*(z_i)$ is maximal among all bidders and is positive:

$$M(\mathbf{z}) := \{i = 1, \ldots, n : 0 < \overline{W}_*(z_i) = \max_{k=1,\ldots,n} \overline{W}_*(z_k)\}.$$

We can now state our first main result: in an auction optimal among all scoring mechanisms, (i) the good is sold to the bidder with the highest $\overline{W}_*(z_i)$, provided this is positive; further, (ii) the winner is assigned to honor a score according to a scoring rule ρ^* specified below.

Proposition 4.1 *Suppose Assumptions 1, 2, and 3. Define ω^* by Equation (26). For each bidder i and each $(z_i, z_{-i}) \in \zeta[\Theta]^n$, let*

$$q^*(z_i, z_{-i}) := \begin{cases} 1/\#M(z_i, z_{-i}) & \text{if } i \in M(z_i, z_{-i}) \\ 0 & \text{if } i \notin M(z_i, z_{-i}), \end{cases} \tag{28}$$

$$\rho^*(x, y) := y + \omega^*(L(x)), \text{ and} \tag{29}$$

$$\bar{s}^*(z_i) := \left(z_i\nu(\overline{\beta}(z_i)) + \omega^*(\overline{\beta}(z_i))\right) \mathrm{E}_{z_{-i}} q^*(z_i, z_{-i}) - \int_{\underline{z}}^{z_i} \mathrm{E}_{z_{-i}} q^*(t, z_{-i})dt. \tag{30}$$

Denote $(q^*, \bar{s}^*, \rho^*)$ for the mechanism that selects winners by the probabilities q^* and, for a winner with "type" z_i, requires the winner carry out a transaction whose score equals to a number $s^*(z_i, z_{-i})$ according to the rule ρ^*, where the number $s^*(z_i, z_{-i})$ is in average $\mathrm{E}_{z_{-i}} s^*(z_i, z_{-i}) = \bar{s}^*(z_i)$. Then (ρ^*, q^*, s^*) is seller-optimal among all well-behaved scoring mechanisms.

Proof: The proof is similar to Myerson's proof of the theorem in [15]. Recall the seller's expected payoff (20). By the definition of $\overline{R}$ and $\overline{W}$ and mimicking the calculation in Eqs. (6.9) and (6.10) of Myerson [15], we have

$$\sum_{i=1}^{n} \mathrm{E}_{z_i,z_{-i}}\left[q(z_i,z_{-i})W(\beta(z_i),z_i)\right] =$$
$$\underbrace{\sum_{i=1}^{n} \mathrm{E}_{z_i,z_{-i}}\left[q(z_i,z_{-i})\overline{W}(\beta(z_i),z_i)\right]}_{:=A(q,\beta)} - \underbrace{\sum_{i=1}^{n} \nu(\beta(z_i)) \int_{\zeta[\Theta]} (H(\Phi(t)) - G(\Phi(t)))\, d\bar{q}(t)}_{:=B(q,\beta)}.$$

Consider $(q^*, \overline{\beta})$, with q^* defined in the theorem and $\overline{\beta}$ define previously. By definition, $\overline{\beta}(z_i)$ maximizes $\overline{W}(\cdot, z_i)$, and q^* puts all probability on bidders for whom $\overline{W}(\overline{\beta}(z_i), z_i)$ is positive and maximal. Thus, for any function $\beta : \zeta[\Theta] \to L[X]$ and any probability assignment $q(z_i, z_{-i})$,

$$A(q^*, \overline{\beta}) \geq A(q, \beta).$$

On the other hand,

$$B(q^*, \overline{\beta}) = 0.$$

To see this, notice that "$H(\Phi(t)) \neq G(\Phi(t))$" implies that $G' \circ \Phi$ is flat over a neighborhood of point t (G is the convex hull of H). Thus, $\overline{R}$ is constant over that neighborhood. By Lemma 4.5 (c), $\overline{W}_*$ and $\bar{q}^*$ are also constant there. Consequently, the integrals in the term $B(q^*, \overline{\beta})$ are all zero.

For any well-behaved scoring mechanism satisfying incentive-compatibility, its winning probability $\bar{q}(z_i) := \mathrm{E}_{z_{-i}} q(z_i, z_{-i})$ is nondecreasing in z_i (Lemma 4.3). Consequently, with $H \geq G$ (by construction) and $\nu \geq 0$ (Lemma 4.2), we have $B(q, \beta) \geq 0$.

Therefore, for any incentive-compatible well behaved scoring mechanism $(q, \tilde{s}, \rho)$ and any associated function $\beta : \zeta[\Theta] \to L[X]$,

$$\sum_{i=1}^{n} \mathrm{E}_{z_i,z_{-i}}\left[q^*(z_i,z_{-i})\overline{W}_*(\overline{\beta}(z_i),z_i)\right] \geq \sum_{i=1}^{n} \mathrm{E}_{z_i,z_{-i}}\left[q(z_i,z_{-i})W(\beta(z_i),z_i)\right].$$

Thus, the proof will be complete if the mechanism (ρ^*, q^*, s^*) satisfies the sufficient conditions in Lemma 4.4. Equation (21) in that lemma is satisfied by ω^*, by the proof of Lemma 4.5 (e). We hence need only to prove that the winning probability $\bar{q}^*(z_i)$ is nondecreasing in z_i. That directly follows from the fact that $\bar{q}^*(z_i)$ is nondecreasing in $\overline{W}_*(z_i)$ (by construction) and the fact that $\overline{W}_*$ is nondecreasing (Lemma 4.5). Thus, Lemma 4.4 implies that the mechanism (ρ^*, q^*, s^*) is optimal among all well-behaved scoring mechanisms. This proves the proposition. **Q.E.D.**

The above result is similar to the general solution in Myerson [15, Section 6] in the sense that both select winners in descending order of some ironed virtual utilities ($\overline{W}_*$ in our case). The new element in our solution is a scoring rule, which is needed to resolve the obstacle of incentive-compatibility for multidimensional bidder-types.

4.4 Optimal Auctions among All Mechanisms

The optimal mechanism obtained above turns out to be simpler than it appears. In the following we will prove that the mechanism is almost equivalent to a Vickrey auction, except that the bids are ranked by our scoring rule ρ^* and the minimum score is zero. Furthermore, this mechanism is optimal among all mechanism when the function $R(\cdot)$ (Equation (19)) is increasing. This monotonicity condition of $R(\cdot)$ corresponds to the monotone hazard rate condition in the unidimensional optimal auction theory.

Theorem 4.1 *Suppose Assumptions 1, 2, and 3.*

a. *The mechanism (ρ^*, q^*, s^*) constructed by Equations (28)–(30) is equivalent to the second-score scoring-rule auction using ρ^* (Equation (29)) as the scoring rule and zero as the minimum score.*

b. *If the function $R(\cdot)$ (Equation (19)) is increasing, then the scoring-rule auction maximizes the seller's equilibrium expected payoff among all regular mechanisms. If the function $R(\cdot)$ is not increasing, then the scoring-rule auction maximizes the seller's equilibrium expected payoff among all well-behaved scoring mechanisms.*

Proof: We shall prove Claim (b) first. Proposition 4.1 has proved the case when the function $R(\cdot)$ is not increasing. We thus need only to consider the case where $R(\cdot)$ is increasing. In that case, it is obvious that $\overline{R} \equiv R$. By the definition of $\overline{\beta}$ (Equation (24)), $\overline{W} \equiv W$ and $\overline{\beta}(z) \equiv \arg\max_b W(b, z)$. Thus, for any profile $(z_i)_{i=1}^n \in \zeta[\Theta]^n$ indexed by bidders, $\overline{\beta}(z_i)$ maximizes $W(\cdot, z_i)$ and $q^*(z_i, z_{-i})$ puts all probability on bidders for whom $W(\overline{\beta}(z_i), z_i)$ is maximal and positive. Consequently, the mechanism (ρ^*, q^*, s^*) maximizes the weighted sum (20). By Lemma 4.1, this weighted sum is equal to the seller's expected payoff from any incentive-compatible, individually rational and regular mechanisms.

As proved in Proposition 4.1, the mechanism (ρ^*, q^*, s^*) is incentive-compatible. It is individually rational by Equations (28) and (30). Being a well-behaved scoring mechanism, (ρ^*, q^*, s^*) is regular (Condition 1), as observed in the proof of Lemma 4.4. Therefore, we have proved that the mechanism (ρ^*, q^*, s^*) is seller-optimal among all incentive-compatible, individually rational and regular mechanisms, whenever $R(\cdot)$ is increasing. This proves Claim (b).

We next prove Claim (a). To prove that the second-score scoring-rule auction is equivalent to the mechanism (ρ^*, q^*, s^*), we need only to show that the auction game (i) generates the same winning probabilities as q^* and (ii) induces any winner to choose transactions in such a way that yields the same expected payoff for the seller as the winner does in the mechanism (ρ^*, q^*, s^*).

Let us start our proof of Claim (a) by looking at a winner's choice of transactions. Given a score s and type ϑ, a winner in the scoring-rule auction chooses a nonmonetary bundle x

to maximize $u(x,\vartheta)$ subject to $L(x) = b_*(\zeta(\vartheta))$, where $b_*(\zeta(\vartheta))$ maximizes $\zeta(\vartheta)\nu(b) + \omega^*(b)$ for all b; he picks a money payment y as $s - \omega^*(b_*)$. By Lemma 4.5 (e), $b_* = \overline{\beta}$ pointwise. Consequently,

$$\max_{b\in L[X]}\{z\nu(b) + \omega^*(b)\} = z\nu(\overline{\beta}(z)) + \omega^*(\overline{\beta}(z)) := \tau_{\rho^*}(z),\ \forall z \in \zeta[\Theta].$$

By the properties of ν (Lemma 4.2) and $\overline{\beta}$ (Lemma 4.5), we know that the induced type $\tau_{\rho^*}(z) > 0$ iff $\overline{R}(z) > 0$ and τ_{ρ^*} is strictly increasing over the region where $\overline{R}(z) > 0$. By the properties of $\overline{W}_*$ (Lemma 4.5), we know that $\overline{W}_*(z) > 0$ iff $\overline{R}(z) > 0$ and $\overline{W}_*$ is strictly increasing over the region where $\overline{R}(z) > 0$. In the scoring-rule auction, therefore, *bidders with nonpositive induced types are exactly those with nonpositive $\overline{W}_*$, and bidders with higher positive induced types have higher positive $W_*(z_i)$.*

We can now prove condition (i), namely, the scoring-rule auction has the same winner-selection criterion as the mechanism (ρ^*, q^*, s^*). Recall from Equation (12) that a winner's payoff in the scoring-rule auction is the additively separable form $\tau_{\rho^*}(z) - s$, with $\tau_{\rho^*}(z)$ being his induced type and s the score he needs to fulfill. Since the dominant-strategy equilibrium of our *second-score* scoring-rule auction is that every bidder submits his true induced type $\tau_{\rho^*}(z)$ as his score, the auction game puts all winning probability on bidders whose induced type is maximal across bidders and is positive (since the minimum score is zero). The italic claim in the previous paragraph then implies that the scoring-rule auction generates the same winning probabilities as q^*, by the definition of q^*.

Finally, we prove condition (ii), namely, a winner's choice of transactions in the scoring-rule auction yields the same expected payoff for the seller as in the mechanism (ρ^*, q^*, s^*). From our previous analysis of a winner's decision, a winner's transaction in the scoring-rule auction will be equivalent to that of the mechanism (ρ^*, q^*, s^*) if *the scoring-rule auction assigns scores so that a bidder's expected score depends on his type in the same way as the function $\bar{s}^*$ in the mechanism (ρ^*, q^*, s^*).* One can prove this by mimicking the standard revenue equivalence argument in unidimensional settings, with "revenue" and "type" there replaced by "score" and "induced type" here, respectively. The reasons why we can apply the revenue equivalence argument are: First, a winner's payoff in the scoring-rule auction is the additively separable form $\tau_{\rho^*}(z) - s$, with the induced type $\tau_{\rho^*}(z)$ independent across bidders; second, the equilibrium bidding function in the auction game is strictly increasing in the induced type, as shown previously; third, the bidders getting zero equilibrium payoff in the auction game are exactly those getting zero equilibrium in the mechanism (ρ^*, q^*, s^*) (i.e., those with nonpositive $\overline{W}_*(z)$). Thus, condition (ii) follows. This proves Claim (a) of the theorem. The proof of the theorem is therefore complete. **Q.E.D.**

A convenient feature of our optimal auction is that the seller does not need to select winners or determine transactions on a case-by-case basis. The auction delegates both of these tasks to the bidders by having them compete according to the scoring rule ρ^*. Remarkably, this convenient feature remains whether the hazard rate $R(z)$ is increasing (non-binding IC constraint) or not (binding IC constraint).

The intuition by which we constructed the scoring-rule auction is the following. From the usual steps in optimal auction design, we know that the seller's equilibrium expected payoff cannot exceed the weighted sum

$$\sum_{i=1}^{n} [q(z_i, z_{-i}) W(\beta(z_i), z_i)]$$

at each possible state. Thus, the best she could do is to (i) maximize the virtual utilities $W(\cdot, z_i)$ for each z_i and (ii) assign all the winning probabilities q to bidders with the maximal and positive $\max_b W(b, z_i)$.

When the "hazard rate" $R(\cdot)$ is increasing, our scoring-rule auction implements both operations and satisfies the incentive-compatibility (IC) constraint, as explained in the Introduction. When $R(z)$ is not increasing in z, however, no scoring mechanism can accomplish the two maximization steps without violating the IC constraint. In this case, the highest-score bidder need not be the one with whom the seller most desires to trade. We need to revise the scoring rule on one hand, and revise the seller's winner-selection criterion $\max_b W(b, z_i)$ on the other, so that both can move in the same direction. Among scoring mechanisms, the best the seller can do is to maximize the weighted sum

$$\sum_{i=1}^{n} \left[q(z_i, z_{-i}) \overline{W}(\beta(z_i), z_i)\right]$$

at each possible state, where $\overline{W}$ is the revised (ironed) criterion for winner-selection. Remarkably, the maximization in this seemingly messy case can still be implemented by a scoring-rule auction. We design the scoring rule so that a winner would choose a transaction to maximize $\overline{W}(\cdot, z_i)$, and a bidder's pledged score moves in the same direction as $\max \overline{W}(\cdot, z_i)$. Here the optimal auction retains the convenient feature of delegating both tasks of winner-selection and transaction determination to the bidders. Due to the binding constraint of incentive-compatibility, our optimal auction in this case does not achieve the upper bound $\max_q \max_\beta \sum_{i=1}^{n} [q(z_i, z_{-i}) W(\beta(z_i), z_i)]$. Nevertheless, the mechanism maximizes the seller's payoff among a class of scoring mechanisms, containing scoring-rule auctions (Proposition 4.1). It is still unknown whether our mechanism is optimal among all mechanisms.

Let us close the circle by noting that the optimal auction constructed in Proposition 4.1 gives zero winning probability to a positive measure of bidder-types, as an instance of the exclusion principle in Section 3.

Corollary 4.1 *Suppose Assumptions 1-3 and that $R(z)$ is increasing in z. Then there is a unique constant $z_0 \in \zeta[\Theta]$ such that bidders whose types belong to the set $\{\vartheta \in \Theta : \zeta(\vartheta) \leq z_0\}$ have zero winning probability, and this set is of measure $\Phi(z_0) > 0$.*

Proof: Recall the fact that bidders with types $\vartheta \in \Theta$ such that $\overline{W}_*(\zeta) \leq 0$ have zero winning probability. By the monotonicity of $R(\cdot)$ and the fact that $\overline{W}_*(\zeta) \leq 0$ if $R(z) \leq 0$, we need only to prove that the supremum z_0 of the set $\{z \in \zeta[\Theta] : R(z) \leq 0\}$ is greater than $\underline{z} := \min \zeta[\Theta]$, for then $\Phi(z_0) > 0$ by the fact that Φ is strictly increasing (Equation 17)). To show that $z_0 > \underline{z}$, we need only $\underline{z} < (1 - \Phi(\underline{z})/\phi(\underline{z})$. That is equivalent to $\underline{z} < 1/\phi(\underline{z})$, with $\Phi(\underline{z}) = 0$. This inequality will be true if $\underline{z} = \zeta(\mathbf{0})$, because $\zeta(\mathbf{0}) = 0$ by the homogeneity of ζ (Assumption 2). To prove that $\underline{z} = \zeta(\mathbf{0})$, pick any nonzero $b \in L[X]$ and any $\vartheta \in \Theta$. Since $u(x, \cdot)$ is increasing (Assumption 1), $u(x, \vartheta) \geq u(x, \mathbf{0})$ for any attribute bundle x such that $L(x) = b$. Thus, Equation (13) implies that $\zeta(\vartheta)\nu(b) \geq \zeta(\mathbf{0})\nu(b)$ and, with $\nu(b)$ positive (Lemma 4.2), $\zeta(\vartheta) \geq \zeta(\mathbf{0})$. Thus, $\underline{z} = \zeta(\mathbf{0})$, as desired. This proves the corollary. **Q.E.D.**

4.5 Some Implications

Our formula of optimal auctions have several implications. One of them is that a seller would rather commit to a bid-ranking criterion different than her own preferences. Our result also contributes to the literature of non-auction multidimensional screening by providing an optimal mechanism for both binding and non-binding incentive-compatibility constraints.

4.5.1 Downward Distortion of Nonmonetary Attributes

Let us recall from our model that the seller's preferences on the transactions (x, y) are given by her utility function $v(x) + y$. The optimal scoring rule $\rho(x, y)$ in Theorem 4.1 is another ranking criteria on the transactions. A question is how the two ranking criteria are different. The next proposition answers the question. It says that an optimizing seller would commit to ranking bids by the optimal scoring rule instead of her true preferences. Furthermore, the optimal scoring rule rewards the nonmonetary provisions x less than her true preference would do. Here we calculate the explicit amount by which the optimal scoring rule distorts the seller's preferences.

Proposition 4.2 *Suppose Assumptions 1–3. Suppose also that $R(\cdot)$ is strictly increasing and the function $\tilde{v}$ is differentiable on $L[X]$. Then:*

a. *It is suboptimal to use the seller's utility function $v(x) + y$ as the scoring rule, whether the auction is first-score or second-score.*

b. *For any nonmonetary bundle $x \in X$, the optimal scoring rule $\rho^*(x, y)$ ranks x lower than the seller's true preferences $u(x, y)$: if $L(x)$ is interior to the range of $\overline{\beta}$ (Equation (27)), then the difference is*

$$\frac{\partial}{\partial L(x)} u(x, y) - \frac{\partial}{\partial L(x)} \rho^*(x, y) = \frac{1 - \Phi(\overline{\beta}_*^{-1}(L(x)))}{\phi(\overline{\beta}_*^{-1}(L(x)))} \nu'(L(x)) > 0. \tag{31}$$

c. Let $z_0 := \sup\{z \in \zeta[\Theta] : R(z) \leq 0\} > 0$. *Then*

$$\lim_{\tilde{v} \longrightarrow_{\text{pointwise}} 0} \rho^*(x, y) = y - z_0 \nu(L(x)), \ \forall (x, y) \in X \times R. \tag{32}$$

Thus, for any transaction (x, y), $\rho^*(x, y) \approx y - z_0 \nu(L(x)) \neq y$ *when the seller's utility from* (x, y) *is approximately* y.

Appendix A.4 will prove this proposition. Part (b) of this result implies that an optimizing seller would give less credit to bidders' nonmonetary provisions x than her true preferences would. This generalizes the result of Che [7] in the unidimensional setting, and the intuition here is similar to that given by Che. Part (c) of our proposition implies that the amount by which the optimal scoring rule distorts the seller's true preferences is bounded away from zero. The reason is intuitively obvious. Since the bundle x is related to a bidder's type ϑ in his valuation function $u(x, \vartheta)$, an optimizing seller would try to exploit this relationship, whether x affects her own utility or not.

4.5.2 Optimal Multidimensional Screening

As mentioned in the Introduction, the environment of non-auction multidimensional screenings (Armstrong [1], Rochet and Choné [20], etc.) corresponds to the special case in our model where the number of bidders is one. Applying Theorem 4.1, we obtain an explicit formula for the optimal nonlinear pricing mechanism in this setting, whether the hazard rate is increasing or not.

Corollary 4.2 *Suppose Assumptions 1–3 and that there is only one bidder. For each nonmonetary bundle* $x \in X$, *let*

$$p(x) := \begin{cases} \int_0^{L(x)} \overline{\beta}_*^{-1}(t) \nu'(t) dt & \textit{if } L(x) \in \text{range}\, \overline{\beta} \\ \infty & \textit{otherwise}, \end{cases} \tag{33}$$

where the function $\overline{\beta}_*^{-1}$ *is defined by Equation (27). Then the nonlinear pricing mechanism—the bidder carries out a transaction* $(x, p(x))$ *with the seller—is optimal among all regular mechanisms if* $R(\cdot)$ *is increasing, and optimal among all* $L(x)$*-based tariffs (mechanisms of the form* $(x, p(L(x)))$*) if* $R(\cdot)$ *is not increasing.*

Proof: By Theorem 4.1, our optimal mechanism is the second-score scoring-rule auction using ρ^* as the scoring rule and zero as the minimum score. Since the auction is second-score and there is only one bidder, the score assigned to the bidder is zero. The definition of ρ^* (Equation (29)) implies that the money transfer y paid to the seller is $y = -\omega^*(L(x))$ for any bundle $x \in X$. Equation (33)) then follows from the definition of ω^* (Equation (26)). It is trivial to check that the nonlinear pricing mechanism $(x, p(x))$ is equivalent to the second-score scoring-rule auction. (Note that the mechanism allows bidders not to participate by choosing the transaction $(\mathbf{0}, 0)$.) The optimality of the mechanism, for both cases of $R(\cdot)$, follows from Theorem 4.1; the only detail worth mentioning is that a well-behaved scoring mechanism in the one-bidder setting corresponds to an $L(x)$-based tariff. The corollary is proved. **Q.E.D.**

The above optimal nonlinear pricing function is a new result in the non-auction multidimensional screening literature. The reason is that the pricing function has an explicit formula, the optimality remains even if the hazard rate $R(\cdot)$ is not monotone, and the pricing function need not be a cost-based tariff. Let us expand the last point here. Our optimal pricing function is a tariff based on a function $L(x)$ of the attribute bundle x, and $L(x)$ need not be the cost $|v(x)|$ for the seller. In particular, even when the seller's cost $v(x)$ from attribute bundles x goes to zero, the seller's price would still vary with x: $p(x) \approx z_0 \nu(L(x))$, with the weight z_0 bounded away from zero (Proposition (4.2). This feature is absent in the cost-based tariff of Armstrong [1]. The reason is that his assumption of multiplicative separability is cost-based, and mine is $L(x)$-based.

When the hazard rate $R(\cdot)$ is strictly increasing, Proposition 4.2 implies that the monopolist would overcharge the attribute bundle x by an amount given by Equation (31). The intuition is that the seller separates the market according to $L(x)$ and becomes the monopolist in each of them. The right-hand side of Equation (31) can then be viewed as the monopolist's markup for a market where consumers demand attribute bundles x have a common $L(x)$.

4.6 An Example

In our auction setting, let the seller's payoff be $y - c\sum_{j=1}^{m} x_j$, a type-ϑ winning bidder's payoff be $\sum_{j=1}^{m} \vartheta_j x_j^{1/2} - y$, and the distribution of bidder-types be $F(\vartheta) = \|\vartheta\|^\alpha$ on the support $\{\vartheta \in R_+^m : \|\vartheta\| \le 1\}$, for some $\alpha > 0$. Hence the density function is $f(\vartheta) = \alpha\|\vartheta\|^{\alpha-1}$. Notice that Assumption 1 is satisfied. Let $m \ge 2$.

We first calculate the virtual utility by Equation (9):

$$V(x, \vartheta) = -c\sum_{j=1}^{m} x_j + \left(1 - \frac{g(\vartheta)}{f(\vartheta)}\right) \sum_{j=1}^{m} \vartheta_j x_j^{1/2}, \tag{34}$$

where the function g is, by Equation (8),

$$g(\vartheta) = \int_1^\infty t^{m-1} f(t\|\vartheta\|)dt = \|\vartheta\|^{-m} \int_{\|\vartheta\|}^1 t^{m-1} f(t)dt$$

and so

$$\frac{g(\vartheta)}{f(\vartheta)} = \|\vartheta\|^{-(m+\alpha-1)} \int_{\|\vartheta\|}^1 t^{m-\alpha-2} dt.$$

Therefore, the expression $\left(1-\frac{g(\vartheta)}{f(\vartheta)}\right)$ is a function of the Euclidean norm $\|\vartheta\|$ of the type ϑ. Denote this function by $h : [0,1] \to R$. Notice that for all $z \in [0,1]$,

$$h(z) = 1 - z^{-(m+\alpha-1)} \int_z^1 t^{m-\alpha-2} dt = \frac{m+\alpha - z^{1-m-\alpha}}{m+\alpha-1}. \tag{35}$$

Notice that

$$h(z) > \text{ (resp. } \geq) \; 0 \Longleftrightarrow (m+\alpha) z^{m+\alpha-1} > \text{(resp. } \geq) \; 1. \tag{36}$$

Note that the function h is strictly increasing on $[0,1]$ and the equation $h(z)=0$ has a unique root in $(0,1)$. Denote this root by z_0. Thus, the set of types $\vartheta \in \Theta$ such that $h(\|\vartheta\|) < 0$ is the interior of the set $\{\vartheta \in R^m_+ : \|\vartheta\| < z_0\}$. Note that this z_0 corresponds to the one defined in Corollary 4.1.

By Lemma 4.1, the best the seller could do is to maximize $V(\cdot, \vartheta)$ for each ϑ and put all probabilities to the bidders for whom $V_*(\vartheta) := \max_x V(x, \vartheta)$ is maximal across bidders and is positive. Let $\tilde{x}(\vartheta) := \arg\max_x V(x, \vartheta)$. Note that the function is concave iff $h(\|\vartheta\|) \geq 0$. Thus, one can easily calculate that

$$\tilde{x}(\vartheta) \;\; = \;\; \begin{cases} \mathbf{0} & \text{if } h(\|\vartheta\|) \leq 0 \\ \frac{h(\|\vartheta\|)^2}{4c^2} (\vartheta_j^2)_{j=1}^m & \text{if } h(\|\vartheta\|) \geq 0; \end{cases} \tag{37}$$

$$V_*(\vartheta) \;\; = \;\; \begin{cases} 0 & \text{if } h(\|\vartheta\|) \leq 0 \\ \frac{1}{4c} \|\vartheta\|^2 h(\|\vartheta\|)^2 & \text{if } h(\|\vartheta\|) \geq 0. \end{cases} \tag{38}$$

Let us pause and notice that Equations (37) and (38) illustrate some features we do not see in unidimensional auction settings. One is that the equations require that bidders of different types ϑ get different attribute bundles $\tilde{x}(\vartheta)$, *even if their ranks $V_*(\vartheta)$, and hence their probabilities of being a winner, are identical.* This requirement is absent in unidimensional frameworks, where implementability is only a matter of preventing bidders from manipulating the probabilities of being a winner. Another feature is that the virtual utility can be negative when the norm of the bidder-type is sufficiently small. (Also look at Equation (35).) Thus, from the seller's viewpoint, those bidders having such types should stand no chance to win. This is an instance of the exclusion principle in Section 3.

We now construct an optimal mechanism by the formulas in Propositions 4.1 and 4.1. One can calculate that $L(x) = \sum_{j=1}^m x_j$, $\tilde{v}(b) = -bc$, $\zeta(\vartheta) = \|\vartheta\|$, $\nu(b) = \sqrt{b}$, $f_\zeta(z) = \alpha z^{\alpha-1}$, $f_0 \equiv 1$, and $\zeta[\Theta] = [0,1]$. It is easily to check that Assumptions 2 and 3 are satisfied. Furthermore,

$$R'(z) = \frac{d}{dz}\left(z - \frac{1-\Phi(z)}{\phi(z)}\right) = h(z) + z h'(z) = \left[m + \alpha + (m+\alpha-2) z^{1-m-\alpha}\right] / (m+\alpha-1)$$

by Equation (35). Since $m \geq 2$, the above quantity is greater than zero. Thus, the function $R(\cdot)$ is increasing, so Proposition 4.1 implies that our optimal mechanism is a scoring-rule auction.

To calculate the optimal scoring rule, we first solve the maximization problem $\max_b W(b, z)$. As before, denote the solution by $\beta(z)$. It is easy to calculate that, for every $z \in \zeta[\Theta]$,

$$h(z) \leq 0 \implies \beta(z) = 0;$$
$$h(z) > 0 \implies zh(z) = 2c\sqrt{\beta(z)}.$$

Note that $zh(z)$ is equal to 1 when $z = 1$, goes to $-\infty$ as $z \to 0_+$, and is strictly increasing on $(0, 1)$. Thus, the equation $zh(z) = a$ has a unique root in $(0, 1]$ for each $a \leq 1$, so the root, denoted by $z_*(a)$, exists and is unique for any such a. As one can easily show that $2c\sqrt{\beta(z)} \leq 1$, we obtain the inverse $\beta^{-1} : (\infty, 0] \to \zeta[\Theta]$:

$$\beta^{-1}(b) = z_*\left(2c\sqrt{b}\right), \ \forall b \leq 0.$$

Thus, we obtain the scoring rule ρ^* in our optimal auction:

$$\rho^*(x, y) := y - \frac{1}{2}\int_0^{\sum_{j=1}^m x_j} z_*\left(2ct^{1/2}\right) t^{-1/2} dt, \ \forall (x, y) \in X \times R, \tag{39}$$

where $z_*(a)$ ($\forall a \leq 1$) denotes the unique root of the equation $zh(z) = a$ in the interval $(0, 1]$.

The seller would do worse if she uses her utility function $\rho(x, y) = y - c\sum_{j=1}^m x_j$ instead of our ρ^* as the scoring rule. As reasoned in previous subsection, a type-ϑ winner in an auction using ρ chooses the attribute bundle x such that $L(x)$ maximizes $\|\vartheta\|\sqrt{b} - bc$ among all $b \in L[X]$. Denoting the solution by $\beta_\rho(\|\vartheta\|)$, we have

$$\beta_\rho(\|\vartheta\|) = \frac{\|\vartheta\|^2}{4c^2}, \ \forall \vartheta \in \Theta. \tag{40}$$

In contrast, to maximize the seller's equilibrium expected payoff would require, by Equation (37), that a type-ϑ winner choose a bundle $\tilde{x}(\vartheta)$ such that

$$\sum_{j=1}^m \tilde{x}_j(\vartheta) = \frac{\|\vartheta\|^2 h(\|\vartheta\|)^2}{4c^2} \quad \text{a.e. } [f]. \tag{41}$$

This is violated in the auction, because Equation (40) implies that a type-ϑ winner chooses $\tilde{x}(\vartheta)$ such that

$$\sum_{j=1}^m \tilde{x}_j(\vartheta) = \frac{\|\vartheta\|^2}{4c^2}, \ \forall \vartheta \in \Theta.$$

This violates Equation (41), because $h(\|\vartheta\|)^2 \neq 1$ unless $\|\vartheta\| = 1$ (Equation (35)).

Finally, let us look at the optimal scoring rule ρ^* when the seller cares almost only about the money payment y. Notice from the definition of $z_*(\cdot)$ that $z_*(2ct^{1/2}) \to z_0$ as $c \to 0$. (Recall that z_0 is the unique root of $zh(z) = 0$ on $(0, 1]$.) Consequently, By Equation (39),

$$\lim_{c \to 0_+} \rho^*(x, y) = y - z_0 \left(\sum_{j=1}^m x_j\right)^{1/2}, \ \forall (x, y) \in X \times R.$$

Thus, the optimal weight z_0 on the nonmonetary bundle x is bounded away from zero, even when the nonmonetary provision is almost costless to the seller.

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A Appendix

A.1 The Proof of the Exclusion Principle

The proof parallels the proof in Armstrong [1] for non-auction cases, except for the more careful usage of the Divergence Theorem here (Footnote 15).

Suppose, on the contrary of the proposition, that almost all bidder-types have positive expected payoff in an optimal mechanism. Let $(q, \tilde{x}, \tilde{y})$ denote the corresponding direct-revelation game of this mechanism, and $U : \Theta \to R$ the corresponding indirect utility function. For each $\epsilon \geq 0$ define

$$\Theta(\epsilon) := \{\vartheta \in \Theta : \ U(\vartheta) \leq \epsilon\}.$$

Since Θ is convex and U is convex (Lemma 2.1) and continuous, each $\Theta(\epsilon)$ is compact and convex. By the optimality of the mechanism, the set $\Theta(0)$ must be nonempty, for otherwise the seller would benefit by increasing the monetary payment $\tilde{y}(\theta)$ for every type profile $\theta \in \Theta^n$ by the amount $\min U$ (which exists by the compactness of Θ and the continuity of U).

Now suppose that the nonempty set $\Theta(0)$ contains two distinct bidder-types ϑ' and ϑ''. Since U is convex and nonnegative (individual rationality), the point $\lambda\vartheta' + (1-\lambda)\vartheta''$, with $0 < \lambda < 1$, also receives zero expected payoff. With Θ assumed to be strictly convex, this point lies in the interior of Θ. Thus, the set $\{\theta \in \Theta : \ \vartheta \leqq \lambda\vartheta' + (1-\lambda)\vartheta''\}$ has positive Lebesgue measure and so has positive measure with respect to the density function f. But since U is increasing with respect to the coordinate-wise relation $\geqq$ (because $u(x, \cdot)$ is so), this set is also contained in $\Theta(0)$ and we reach the conclusion of the proposition, a contradiction, that $\Theta(0)$ has positive measure. We have thus deduced that $\Theta(0)$ is a singleton. Then, letting $V(\Theta(\epsilon))$ and $S(\Theta(\epsilon))$ denote respectively the volume and surface area of the set $\Theta(\epsilon)$, we have

$$\lim_{\epsilon \to 0} V(\Theta(\epsilon)) = V(\Theta(0)) = 0; \ \lim_{\epsilon \to 0} S(\Theta(\epsilon)) = S(\Theta(0)) = 0. \tag{42}$$

Here the last equality is due to $m \geq 2$.

We hope to show that the seller's expected payoff $\pi(\epsilon)$ gained from a bidder of types in the set $\Theta(\epsilon)$ is in smaller order than ϵ. The seller's payoff obtained from such a bidder, say i, is

$$\int_{\Theta(\epsilon)} \left\{ \mathrm{E}_{\theta^{(-i)}} \left(q(\vartheta, \theta^{(-i)})[v(\tilde{x}(\vartheta, \theta^{(-i)})) + u(\tilde{x}(\vartheta, \theta^{(-i)}), \vartheta)] \right) - U(\vartheta) \right\} f(\vartheta) d(\vartheta).$$

Because $U \geq 0$ and v is assumed to be nonpositive, we have

$$\pi(\epsilon) \leq \int_{\Theta(\epsilon)} \mathrm{E}_{\theta^{(-i)}} q(\vartheta, \theta^{(-i)}) u(\tilde{x}(\vartheta, \theta^{(-i)}), \vartheta) f(\vartheta) d\vartheta.$$

Since $u(x,\vartheta)$ is homogeneous of degree one in ϑ, we have $u(x,\vartheta) \equiv \vartheta \cdot D_2 u(x,\vartheta)$; since Equation (3) holds almost everywhere in $\Theta(\epsilon)$, we obtain

$$\pi(\epsilon) \le \int_{\Theta(\epsilon)} \vartheta \cdot \nabla U(\vartheta) f(\vartheta) d\vartheta. \tag{43}$$

Next we use divergence theorem in multivariate calculus, which states that

$$\int_A \operatorname{div}(\mathbf{w}) d\vartheta = \int_{\partial A} \mathbf{w} \cdot \mathbf{n} \, dS,$$

where A is a closed convex set in R^m, ∂A is the surface of this set, $\mathbf{n}$ the outward-pointing unit normal vector at a point on the surface ∂A, $\mathbf{w} = (w_1, \ldots, w_m)$ is an m-dimensional vector-valued function defined on A, $\operatorname{div}(\mathbf{w}) := \sum_{j=1}^m \partial w_j / \partial \vartheta_j$ is the *divergence* of the vector field $\mathbf{w}$, and dS denotes integration over the surface ∂A. The only assumption required by the theorem is that the vector field $\mathbf{w}$ is continuous and each $w_j(\vartheta)$ ($\forall j = 1, \ldots, m$) is an absolutely continuous function of ϑ_j (so the divergence is defined almost everywhere).[15] To apply the divergence theorem in our case, let $\mathbf{w}(\vartheta) := U(\vartheta) f(\vartheta) \vartheta$ for all $\vartheta \in \Theta$. This vector $\mathbf{w}$ is obviously continuous. Furthermore, each component $w_j(\vartheta) = U(\vartheta) f(\vartheta) \vartheta_j$ is an absolutely continuous function of ϑ_j, since U is convex and f is assumed to be continuously differentiable. Thus, the divergence theorem applies and so Inequality (43) becomes

$$\pi(\epsilon) \le \int_{\partial\Theta(\epsilon)} U(\vartheta) f(\vartheta) \vartheta \cdot \mathbf{n} \, dS - \int_{\Theta(\epsilon)} U(\vartheta) \operatorname{div}(\vartheta f(\vartheta)) d\vartheta.$$

With f continuously differentiable on the compact set $\Theta(\epsilon)$, there is some positive number B such that each of $f(\vartheta)$, $f(\vartheta)\vartheta \cdot \mathbf{n}$, and $\operatorname{div}(\vartheta f(\vartheta))$ is bounded in $\Theta(\epsilon)$ in absolute value by B, for all sufficiently small ϵ. Since $U(\vartheta) \le \epsilon$ in $\Theta(\epsilon)$, the above inequality implies

$$\pi(\epsilon) \le \epsilon B [S(\Theta(\epsilon)) + V(\Theta(\epsilon))].$$

[15] The usual version of the divergence theorem assumes the differentiability of the vector field $\mathbf{w}$. However, this assumption can be weakened to our assumption stated above. To see that, we simply walk through the standard proof of the theorem (e.g., Kaplan [10, pp. 329-331]). As in such a proof, it suffices to prove that

$$\int_A \frac{\partial w_j}{\partial \vartheta_j} d\vartheta = \int_{\partial A} w_j dS, \ \forall j = 1, \ldots, m. \tag{44}$$

Let us do that for $j = 1$ and $m = 3$. Since the space A is convex, its projection on the plane for all the points $(\vartheta_2, \vartheta_3)$ is a convex region A_{23}. Thus, we can represent the space A as the set of vectors $(\vartheta_1, \vartheta_2, \vartheta_3)$ such that

$$k_1(\vartheta_2, \vartheta_3) \le \vartheta_1 \le k_2(\vartheta_2, \vartheta_3), \ \ (\vartheta_2, \vartheta_3) \in A_{23},$$

for some real functions k_1 and k_2. Then the left-hand side of Equation (44) is equal to

$$\int_{A_{23}} \left[\int_{k_1(\vartheta_2,\vartheta_3)}^{k_2(\vartheta_2,\vartheta_3)} \frac{\partial w_1}{\partial \vartheta_1} d\vartheta_1 \right] d\vartheta_2 d\vartheta_3 = \int_{A_{23}} [w_1(k_2(\vartheta_2,\vartheta_3), \vartheta_2, \vartheta_3) - w_1(k_1(\vartheta_2,\vartheta_3), \vartheta_2, \vartheta_3)] \, d\vartheta_2 d\vartheta_3,$$

where the equality holds because the function $w_1(\cdot, \vartheta_2, \vartheta_3)$ is absolutely continuous. The rest of the proof is the same as the standard proof. □

Thus, the seller's expected payoff $\pi(\epsilon)$ obtained from a bidder of types in the set $\Theta(\epsilon)$ is in smaller order than ϵ, by Equation (42).

Finally, consider the change $\Delta\pi$ of the seller's expected payoff in raising the bidders' monetary payment $\tilde{y}$ uniformly by $\epsilon > 0$. This will cause a bidder of types in $\Theta(\epsilon)$ not to participate; the seller will lose $\pi(\epsilon)$ for that. Because a uniform change of the monetary payment does not distort incentive-compatibility, the seller will gain ϵ more payoff from each remaining bidder-type; these remaining types are of measure at least $1 - V(\Theta(\epsilon))B$. Thus,

$$\Delta\pi \geq \epsilon[1 - V(\Theta(\epsilon))B] - \epsilon B[S(\Theta(\epsilon)) + V(\Theta(\epsilon))] = \epsilon[1 - B(2V(\Theta(\epsilon)) + S(\Theta(\epsilon)))],$$

which is positive for all sufficiently small ϵ, by Equation (42). This contradicts the supposed optimality of the mechanism $(q, \tilde{x}, \tilde{y})$. This proposition is hence proved. **Q.E.D.**

A.2 The Proof of Lemma 4.2

By Equation (13) and the linearity of L (Assumption 2), one easily shows that $\zeta(\vartheta)\nu(b) > 0$ for all $\vartheta \neq \mathbf{0}$ and all $b \neq 0$. Consequently, since the function ζ is nonnegative (Assumption 2), $\zeta(\vartheta) > 0$ unless $\vartheta = \mathbf{0}$. Thus, Claim (i) follows. It also follows that $\nu(b) > 0$ for all $b \neq 0$ and $\nu(0) = 0$. For any nonzero ϑ, from the linearity of L, Equation (13) and the strict concavity of $u_1(\cdot, \vartheta)$ (Assumption 2), one can easily prove that $\zeta(\vartheta)\nu(\cdot)$ is strictly concave. Consequently, by the proved Claim (ii), the function ν is strictly concave. Thus, it is continuous.

We prove the rest of Claim (ii). First, let us show that ν is strictly decreasing. To do that, pick any $b, b' \in L[X]$. Then $b, b' \geq 0$ (Assumption 2). Thus, $b' = \lambda b$ for some $\lambda \geq 0$. Suppose $b < b'$, then $\lambda > 1$. Pick any $\vartheta \neq \mathbf{0}$. Let $x_*(b, \vartheta)$ be a maximum for the constrained maximization problem in Equation (13). By the linearity of L (Assumption 2), $L(\lambda x_*(b, \vartheta)) = b'$, so

$$\zeta(\vartheta)\nu(b') \geq u_1(\lambda x_*(b, \vartheta), \vartheta) > u_1(x_*(b, \vartheta), \vartheta) = \zeta(\vartheta)\nu(b),$$

where the strict inequality follows from the assumption that $u_1(\lambda x, \vartheta)$ is strictly increasing in λ (Assumption 3). Thus, we have $\nu(b') > \nu(b)$, since $\zeta(\vartheta) > 0$ by the first paragraph of the proof.

To prove that the function ν is differentiable, we use the Benveniste-Scheinkman Theorem (Stokey and Lucas [22, p. 84]). By Equation (13), we need only to prove the differentiability of the function $\zeta(\vartheta)\nu(\cdot)$, for some $\vartheta \neq \mathbf{0}$ (hence $\zeta(\vartheta) > 0$). Hence pick any such ϑ. We have proved that $\zeta(\vartheta)\nu(\cdot)$ is concave (Claim (i)). By the Benveniste-Scheinkman Theorem, we still need, for each interior point b_0 of $L[X]$, to construct a concave differentiable function A on a neighborhood of b_0 such that A is below $\zeta(\vartheta)\nu$ and touches the latter at the point b_0. Thus, pick any interior point b_0 of $L[X]$. The solution for the maximization problem in Equation (13) exists by Assumption 2. Since $u_1(\cdot, \vartheta)$ is strictly concave, L linear (Assumption 2), and X convex, the solution exists and is unique. Let $\hat{x}(b_0)$ denote this solution.

Being a Euclidean space, X is open, so it has a neighborhood N of b_0 in $L[X]$ such that $(b/b_0)\hat{x}(b_0) \in X$ for all $b \in N$. Define

$$A(b) := u_1\left(\frac{b}{b_0}\hat{x}(b_0), \vartheta\right), \ \forall b \in N.$$

Notice that A is concave and differentiable over N, since $u_1(\cdot, \vartheta)$ is concave and differentiable. Note also $A(b_0) = \zeta(\vartheta)\nu(b_0)$. Furthermore, for any $b \in N$, $A(b) \leq \zeta(\vartheta)\nu(b)$, because the linearity of L implies $L((b/b_0)\hat{x}(b_0)) = (b/b_0)L(\hat{x}(b_0)) = b$, so $(b/b_0)\hat{x}(b_0)$ is feasible for the maximization problem in Equation (13) given the parameter b. Thus, the Benveniste-Scheinkman Theorem implies that $\zeta(\vartheta)\nu(\cdot)$ is differentiable at b_0. Since b_0 can be any interior point of $L[X]$, we have proved that the function $\zeta(\vartheta)\nu(\cdot)$, and hence ν, is differentiable. This proves Claim (ii). Thus, we have proved the lemma. **Q.E.D.**

A.3 The Proof of Lemma 4.5

The nonnegativity of $\overline{\beta}$ and $\overline{W}_*$ will immediately follows from Claims (a) and (b) of this lemma. Since the function $\overline{R}$ is nondecreasing by construction (Equation (22)), Claim (c) of the lemma will imply that $\overline{\beta}$ and $\overline{W}_*$ are nondecreasing. Consequently, Claim (d) will imply the continuity of $\overline{\beta}$, which in turn will imply that $\overline{W}_*$ is continuous. Thus, we need only to prove each itemized claim of the lemma.

For Claim (a), $\overline{R}(z) \leq 0$ implies that $\overline{W}(\cdot, z)$ is strictly decreasing, because ν is nonnegative and strictly increasing (Lemma 4.2 (i)), and because the function $\tilde{v}$ is strictly decreasing (Assumption 3). Thus, the maximum $\overline{\beta}(z)$ of $\overline{W}(\cdot, z)$ is $\min L[X]$, and $\min L[X] = 0$ by the nonnegativity and linearity of L. Since $\tilde{v}(0) = 0$ (Assumption 3), we have $\overline{W}_*(z) = 0$. Thus, Claim (a) follows.

We now prove Claim (b). When $\overline{R}(z) > 0$, $\overline{W}(\cdot, z)$ is strictly concave, since ν is strictly concave (Lemma 4.2) and $\tilde{v}$ is assumed to be concave (Assumption 3). Thus, the maximum $\overline{\beta}(z)$ of $\overline{W}(\cdot, z)$ is unique. Furthermore, the maximum exists and is nonzero, because ν satisfies the Inada Condition (as $u(\cdot, \vartheta)$ does, by Assumption 3). Since L is nonnegative, we have $\overline{\beta}(z) > 0$. Consequently, $\overline{W}_*(z) > 0$. The reason is that $0 \in L[X]$ and $\overline{\beta}(z)$ is the unique maximum, so $0 = \overline{W}(0, z) < \overline{W}(\beta(z), z) = \overline{W}_*(z)$. This proves Claim (b).

The first half of Claim (c) follows directly from Claims (a) and (b). To prove its second half, pick any $z, z' \in \zeta[\Theta]$ such that $\overline{R}(z)$ and $\overline{R}(z')$ are both positive. By the proved uniqueness of the maximum $\overline{\beta}(z)$ for z (and $\overline{\beta}(z')$ for z'),

$$[\overline{R}(z) - \overline{R}(z')][\nu(\overline{\beta}(z)) - \nu(\overline{\beta}(z'))] > 0.$$

Since ν is strictly increasing (Lemma 4.2), $\overline{R}(z) > \overline{R}(z')$ implies $\overline{\beta}(z) > \overline{\beta}(z')$. With $\overline{\beta}$ being positive at z and z' (Claim (b)), ν is positive at these points (Lemma 4.2). Consequently, $\overline{R}(z) > \overline{R}(z')$ implies

$$\overline{W}_*(z') = \overline{W}(\overline{\beta}(z'), z') < \overline{W}(\overline{\beta}(z'), z) < \overline{W}_*(z).$$

Claim (c) is thus proved.

We next prove Claim (d). The function $\overline{W}$ is continuous, by the continuity of the functions ν (Lemma 4.2), ϕ (Equation (17)), and $\tilde{v}$ (Assumption 3). To prove that $\overline{\beta}$ is continuous, pick any $z \in \zeta[\Theta]$ and any infinite sequence $(z_j)_j$ that converges to z. (Such sequence exists, since the set $\zeta[\Theta]$ is an interval by the convexity of Θ and continuity of ζ.) Consider the sequence $(b_j)_j$ defined by $b_j := \overline{\beta}(z_j)$ $(\forall j)$. Since $\overline{\beta}$ is well-defined and assumed to be monotone on the compact set $\zeta[\Theta]$, the infinite sequence $(b_j)_j$ is bounded, and so has a cluster point b, so $b_{j_i} \to_i b$ for some subsequence $(b_{j_i})_i$. We claim that $b = \overline{\beta}(z)$. Suppose not, then $\overline{W}(b', z) > \overline{W}(b, z)$ for some $b' \neq b$. With $\overline{W}$ continuous, there are open disks O_1 and O_2 in R^2 such that $(b', z) \in O_1$, $(b, z) \in O_2$, and

$$\left[(\bar{b}, \bar{z}) \in O_1 \text{ and } (\hat{b}, \hat{z}) \in O_2\right] \Longrightarrow \overline{W}(\bar{b}, \bar{z}) > \overline{W}((\hat{b}, \hat{z}).$$

But, for all i sufficiently large, $(b', z_{j_i}) \in O_1$ and $(b_{j_i}, z_{j_i}) \in O_2$, so $\overline{W}(b', z_{j_i}) > \overline{W}(b_{j_i}, z_{j_i})$ for any such i, contradicting the fact that $b_{j_i} = \overline{\beta}(z_{j_i})$ $(\forall i)$. Thus, we have proved that $b = \overline{\beta}(z)$. Since z was chosen arbitrarily from $\zeta[\Theta]$, the function $\overline{\beta}$ is continuous. Thus, Claim (d) is proved.

Finally, we prove Claim (e). Notice that the function $\overline{\beta}_*^{-1}$ is well-defined by Equation (27), since the maximum of the inverse image $\overline{\beta}^{-1}(b)$ exists because $\overline{\beta}$ is continuous (Claim (d)). Notice that $\overline{\beta}_*^{-1}$ is strictly increasing. By the definition of ω^* in Equation (26), the objective in the maximization problem of Equation (21) becomes

$$J(b) := z\nu(b) - \int_0^b \overline{\beta}_*^{-1}(t)\nu'(t)dt.$$

We need only to show that $b = \overline{\beta}(z)$ maximizes $J(b)$. This is trivial when the range of $\overline{\beta}$ is a singleton. We hence focus on the other case. In that case, the range of $\overline{\beta}$ is a nondegenerate interval, because $\overline{\beta}$ is continuous (Claim (d)) and the domain $\zeta[\Theta]$ of $\overline{\beta}$ is a nondegenerate interval. Consequently, with ν differentiable (Lemma 4.2), $J(b)$ is a differentiable function of b. Thus, for any $b \in L[X]$, the derivative of J at b is

$$J'(b) = \nu'(b)(z - \overline{\beta}_*^{-1}(b)) = \text{positive term} \times (z - \overline{\beta}_*^{-1}(b)),$$

where the second equality follows from the fact that ν is strictly increasing (Lemma 4.2). Since $\overline{\beta}_*^{-1}$ is strictly increasing, the derivative $J'(b)$ positive for all $b < \overline{\beta}(z)$ and nonpositive for all $b \geq \overline{\beta}(z)$. Thus, $\overline{\beta}(z)$ is a maximum of $\overline{W}(\cdot, z)$ on $L[X]$. This proves Claim (e). We have therefore completed the proof of the lemma. **Q.E.D.**

A.4 The Proof of Proposition 4.2

Since the optimal auction $(\rho^*, 0)$ constructed in Proposition 4.1 gives the seller an expected payoff equal to the maximum value of expression (20), it suffices Claim (a) to show that using the seller's true utility function $v(x) + y$ as the scoring rule does not give the seller an expected payoff as high as that maximal level. Denote this scoring rule by ρ. Since $v(x) = \tilde{v}(L(x))$ (Assumption 3), we have

$$\rho(x, y) = y + \tilde{v}(L(x)),\ \forall x \forall y.$$

Whether the scoring-rule auction using ρ is first- or second-score, a winner of type ϑ in this game chooses his attribute bundle x such that $L(x)$ solves

$$\tau_\rho(z) := \max_{b \in L[X]} \left\{ z\nu(b) + \tilde{v}(b) \right\}, \quad \text{with } z := \zeta(\vartheta). \tag{45}$$

Let $\beta_\rho(z)$ denote a solution of this problem. Comparing this problem with the problem $\max_b \overline{W}(b, z) = \max_b W(b, z)$ (since $R(\cdot)$ is assumed to be strictly increasing), which is solved by $\overline{\beta}(z)$, we know

$$\beta_\rho(z) = \overline{\beta}(R^{-1}(z)),\ \forall z \in \zeta[\Theta],$$

where the inverse R^{-1} exists since $R(\cdot)$ is assumed to be strict monotone. Since $R(\cdot)$ is negative up to the point $z_0 > \min \zeta[\Theta]$ (the proof of Corollary 4.1), we have

$$\beta_\rho(z) \neq \overline{\beta}(z),\ \forall z \in \zeta[\Theta].$$

For each $z \in \zeta[\Theta]$, since $\overline{\beta}(z)$ is the unique maximizer of $W(\cdot, z)$ (Lemma 4.5 (a) and (b)), $\beta_\rho(z)$ does not maximize $W(\cdot, z)$. We have therefore deduced that using ρ as the scoring rule does not maximize the seller's equilibrium expected payoff (20). This proves Claim (a).

We now prove Claim (b). Pick any $x \in X$ such that $L(x)$ is interior to the range of $\overline{\beta}$. By Assumption 3 and Equations (29),

$$\frac{\partial}{\partial L(x)} u(x, y) - \frac{\partial}{\partial L(x)} \rho^*(x, y) = \nu'(L(x)) \left(\frac{\tilde{v}'(L(x))}{\nu'(L(x))} + \overline{\beta}^{-1}(L(x)) \right).$$

Since the function $\overline{\beta}$ is positive and strictly increasing over $(z_0, \infty) \cap \zeta[\Theta]$, and is constantly zero for $z \leq z_0$ (Lemma 4.5 and Corollary 4.1). Thus, the inverse $\overline{\beta}^{-1}(L(x))$ is an interior solution for the problem $\max W(\cdot, \beta^{-1}(L(x)))$, so the first-order necessary condition gives

$$R(\overline{\beta}^{-1}(L(x))) = -\frac{\tilde{v}'(L(x))}{\nu'(L(x))}. \tag{46}$$

Equation (31) then follows from the definition of $R(\cdot)$ and the fact that ν' is positive. This proves Claim (b).

For Claim (c), it suffices to prove Equation (32). Since $\tilde{v}$ is continuous, differentiable, and $\tilde{v}(0) = 0$ by assumptions, we have

$$\tilde{v}(b) = \int_0^b \tilde{v}'(t) dt,\ \forall b \in L[X].$$

Thus, "$\tilde{v} \longrightarrow 0$ pointwise" implies

$$\tilde{v}'(b) \longrightarrow 0 \text{ a.e. } b \in L[X].$$

This equation, coupled with Equation (46), implies that $R(\overline{\beta}^{-1}(b)) \to 0$ for almost all b in the range of $\overline{\beta}$. With $z = z_0$ being the unique root of $R(z) = 0$ ($R(\cdot)$ is strictly increasing) and R continuous (ϕ is continuous by Equation (17)), we have $\beta^{-1}(b)) \to z_0$ for almost all b in the range of $\overline{\beta}$. Equation (32) then follows from Equations (26) and (29). Thus, Claim (c) is proved. This proves the corollary. **Q.E.D.**

A.5 The Density of a Statistic of Random Vectors

If $u(\mathbf{x})$ is a statistic of the m-dimensional random vector $\mathbf{x}$, which is distributed according to a density function $f(\mathbf{x})$, how do we calculate the density function f_* of the statistic $u(\mathbf{x})$? The following lemma answers this question. It was used in Corollary 3.1 and was the basis of Equation (17). The answer says that the density $f_*(v)$ at a point v is the "surface" integral of f on the level set $u^{-1}(v)$. This fact was heuristically derived in Courant [8, pp. 300-302] and Armstrong [1]. (In the latter, the function u is subject to a stronger condition (homogeneity) than here.) For the convenience of the reader, we prove it here, following their intuition.

Lemma A.1 *Let $u : R^m \to R$, $f : R^m \to R$, and $K \subseteq R^m$. Suppose:*

a. *Except for finitely many points x in the boundary of K such that $u(x)$ is not interior to the range $u[K]$, the function u is three-times continuously differentiable and its gradient is nonzero everywhere.*

b. *Each level set of u is a smooth $(m-1)$-manifold in R^m and cuts R^m into two disconnected sets.*

c. *The set K is compact and convex, with full dimension in R^m, and its boundary consists of finitely many smooth $(m-1)$-manifolds.*

d. *The function f is continuous on K.*

Then for any v in the interior of the range $u[K]$,

$$\frac{d}{dv}\int_{\{x\in K: u(x)\le v\}} f(x)dx_1\cdots dx_m = \int_{\{x\in K: u(x)=v\}} \frac{f(x)}{\|\nabla u(x)\|}dS. \tag{47}$$

Proof: Since we can approximate the set K by compact sets K' excluding the finite singular boundary points, there is no loss of generality to assume that the function u is three-times differentiable and has nonzero gradient everywhere. Pick any v in the interior of the range $u[K]$. For each vector $x \in u^{-1}(a) \cap K$, construct a *gradient path* $\gamma_x : [0, \infty) \to R^m$ by

$$\begin{aligned} \gamma_x'(t) &= \frac{\nabla u(\gamma_x(t))}{\|\nabla u(\gamma_x(t))\|},\ \forall t \in (0, \infty); \\ \gamma_x(0) &= x. \end{aligned}$$

Since u is assumed to be at least twice continuously differentiable, with nonzero gradient everywhere, one can easily show that the above system of differential equations has a unique solution, so γ_x is well-defined. By the chain rule, $(u \circ \gamma_x)'(t) = \|\nabla u(\gamma_x(t))\|$. Thus, $u \circ \gamma_x(t)$ is strictly increasing in the parameter t, so the path γ_x starts at the point x and moves along the gradients of u.

Since K is compact and u is at least twice continuously differentiable, we can pick an $\eta > 0$ so small that for each $x \in u^{-1}(v)$ the gradient path γ_x intersects the level set $u^{-1}(v+\eta)$ at some point. Since the path is continuous, it intersects each level set $u^{-1}(a+\epsilon)$, with $0 < \epsilon < \eta$. Pick any such ϵ. We have

$$\forall x \in u^{-1}(v) \; \exists \text{ unique } \tau(x,\epsilon) > 0 \text{ s.t. } u \circ \gamma_x(\tau(x,\epsilon)) = a + \epsilon.$$

Here the uniqueness of the intersection point $\gamma_x(\tau(x,\epsilon))$ follows from the proved fact that the value of the function u is strictly increasing along the gradient path γ_x.

We want to calculate the integral

$$\int_{\{y \in K : v \leq u(y) \leq v+\epsilon\}} f(y) dy_1 \cdots dy_m. \tag{48}$$

To do that, we will "parameterize" the region $\{y \in K : v \leq u(y) \leq v+\epsilon\}$, on which we integrate f, as follows. The intuition is that a point in this region can be viewed as a point on some gradient path starting at some point on the level set $u^{-1}(v)$. Look at a gradient path that starts at an intersection points x between the level set $u^{-1}(v)$ and the boundary ∂K of the space K. We know that such a path reaches the level set $u^{-1}(v+\epsilon)$ at a unique point $\gamma_x(\tau(x,\epsilon))$. The set of all such arcs $x \rightsquigarrow \gamma_x(\tau(x,\epsilon))$, with x ranging over the intersection $u^{-1}(v) \cap \partial K$, comprises a cylinder-like smooth $m-1$ surface. Now look at the region circumscribed by this "cylinder" and bounded between the level sets $u^{-1}(v)$ and $u^{-1}(v+\epsilon)$. Denote this region by V_ϵ. We claim that

$$V_\epsilon = \{y \in R^m : \, y = \gamma_x(\tau(x,\epsilon)) \text{ for some } x \in u^{-1}(v) \cap K\}. \tag{49}$$

The "$\supseteq$" part of this equation is trivial. For the "$\subseteq$" part, from any point $y \in V_\epsilon$, we can construct a "reversed gradient path" that reaches the level set $u^{-1}(v)$ at a unique point x, so that the reverse of the path is the gradient path starting from the point x. Thus, the point y belongs to the set on the right-hand side of the equation. This proves Equation (49).

Compare the two regions V_ϵ and $\{y \in K : v \leq u(y) \leq v+\epsilon\}$. The latter is simply the region bounded between the level sets $u^{-1}(v)$ and $u^{-1}(v+\epsilon)$ and circumscribed by the boundary of K. Let ΔV_ϵ be the closure of the symmetric difference between the two regions. Clearly, ΔV_ϵ is compact, so the continuous function f has maximum and minimum values on ΔV_ϵ. Thus,

$$\int_{\Delta V_\epsilon} \min_K f \leq \int_{\{y \in K : v \leq u(y) \leq v+\epsilon\}} f(y) dy - \int_{V_\epsilon} f(y) dy \leq \int_{\Delta V_\epsilon} \max_K f. \tag{50}$$

We will show later that the both sides of this sandwich inequality are in smaller order than ϵ, so the integral (48) converges to $\int_{V_\epsilon} f$ in faster order than ϵ goes to zero.

We first calculate the integral $\int_{V_\epsilon} f$. By Equation (49), we can integrate in two steps: (i) for each point x on the level set $u^{-1}(v) \cap K$, integrate along the gradient path $x \rightsquigarrow \gamma_x(\tau(x,\epsilon))$; (ii) integrate on the level set $u^{-1}(v) \cap K$. That is,

$$\int_{V_\epsilon} f(y)dy = \int_{u^{-1}(v)\cap K} \int_0^{\tau(x,\epsilon)} f(\gamma_x(t))\|\gamma_x'(t)\|dtdS = \int_{u^{-1}(v)\cap K} \int_0^{\tau(x,\epsilon)} f(\gamma_x(t))dtdS,$$

where the last equality follow from the construction of the gradient path γ_x. By the Mean-Value Theorem, the inner integral is equal to $f(\gamma_x(\xi(x,\epsilon)))\tau(x,\epsilon)$ for some $\xi(x,\epsilon) \in (0, \tau(x,\epsilon))$. Thus,

$$\int_{V_\epsilon} f(y)dy = \int_{u^{-1}(v)\cap K} f(\gamma_x(\xi(x,\epsilon)))\tau(x,\epsilon)dS. \tag{51}$$

Since K is compact and u continuous, one can easily show that

$$\tau(\cdot,\epsilon) \to 0 \text{ uniformly on } K \cap u^{-1}(v) \text{ as } \epsilon \to 0. \tag{52}$$

Consequently, $\xi(\cdot,\epsilon) \to 0$ uniformly, so $f(\gamma_x(\xi(\cdot,\epsilon))) \to f(x)$ uniformly on $K \cap u^{-1}(v)$. We now calculate $\tau(x,\epsilon)$ for small ϵ. Since the gradient path γ_x is twice continuously differentiable (u is thrice continuously differentiable), Taylor's formula gives

$$\gamma_x(\tau(x,\epsilon)) = \gamma_x(0) + \tau(x,\epsilon)\gamma_x'(0) + o(\tau(x,\epsilon)) = x + \tau(x,\epsilon)\frac{\nabla u(x)}{\|\nabla u(x)\|} + o(\epsilon),$$

where the second equality uses the fact (52).[16] With u at least twice continuously differentiable, Taylor's formula gives

$$a + \epsilon = u(\gamma_x(\tau(x,\epsilon))) = u(x) + \tau(x,\epsilon)\|\nabla u(x)\| + o(\epsilon).$$

Thus,

$$\tau(x,\epsilon) = \epsilon\left(1 - \frac{o(\epsilon)}{\epsilon}\right)/\|\nabla u(x)\|.$$

It follows that $\tau(\cdot,\epsilon) \to \epsilon/\|\nabla u(\cdot)\|$ uniformly on $u^{-1}(v) \cap K$ as $\epsilon \to 0$. Equation (51) thus gives

$$\int_{V_\epsilon} f(y)dy \to \epsilon \int_{u^{-1}(v)\cap K} \frac{f(x)}{\|\nabla u(x)\|}dS \text{ as } \epsilon \to 0.$$

Consequently, by Equation (50), we will be done if $\int_{\Delta V_\epsilon} \max_K f$ and $\int_{\Delta V_\epsilon} \min_K f$ are both $o(\epsilon)$. (Notice that being $O(\epsilon)$ does not suffice.) We show that for $\int_{\Delta V_\epsilon} \max_K f$. The case for $\int_{\Delta V_\epsilon} \min_K f$ is similar. As for the case of V_ϵ, we can parameterize the set ΔV_ϵ by the bounded $m-1$ surface $u^{-1}(a+\epsilon) \cap \Delta V_\epsilon$ and the reversed gradient paths starting from points lying on $u^{-1}(a+\epsilon) \cap \Delta V_\epsilon$. (Here we need to start from the level set $u^{-1}(a+\epsilon)$ because the level set $u^{-1}(a)$'s intersection with ΔV_ϵ has only dimension $m-2$.) Thus, we can calculate the integral $\int_{\Delta V_\epsilon} \max_K f$ in two steps: (i) for each point in $u^{-1}(a+\epsilon) \cap \Delta V_\epsilon$, integrate along the reversed gradient path from that point up to the boundary ∂K; (ii) integrate the quantity

[16] An expression $a(\delta)$ is said to be $o(\delta)$, i.e., in smaller order than δ, if $\frac{a(\delta)}{\delta} \to 0$ as $\delta \to 0$. An expression $b(\delta)$ is said to be $O(\delta)$, i.e., in the same order as δ, if there is a finite number k such that $b(\delta) \to k\delta$ as $\delta \to 0$.

resulting from step (i) over the bounded surface $u^{-1}(a+\epsilon)\cap\Delta V_\epsilon$. Similar to the convergence fact (52), the arc lengths of the arcs on these reverse gradient paths converge uniformly to zero when $\epsilon\to 0$. Thus,

$$\int_{\Delta V_\epsilon}\max_K f \le O(\epsilon)S(u^{-1}(a+\epsilon)\cap\Delta V_\epsilon)\max_K f$$

for sufficiently small ϵ, where $S(u^{-1}(a+\epsilon)\cap\Delta V_\epsilon)$ denotes the "area" (Lebesgue measure in R^{m-1}) of the $m-1$ surface $u^{-1}(a+\epsilon)\cap\Delta V_\epsilon$. As $\epsilon\to 0$, this surface converges to $u^{-1}(a)\cap\Delta V_\epsilon$, which is of Lebesgue measure zero in R^{m-1}. Consequently, $S(u^{-1}(a+\epsilon)\cap\Delta V_\epsilon)$ converges to zero as $\epsilon\to 0$. Thus, $\int_{\Delta V_\epsilon}\max_K f = O(\epsilon)O(\epsilon)=o(\epsilon)$.

Finally, we have deduced that, when $\epsilon\to 0$,

$$\int_{\{y\in K: v\le u(y)\le v+\epsilon\}} f(y)dy = \int_{V_\epsilon} f(y)dy + o(\epsilon) \to \epsilon\left(\int_{u^{-1}(v)\cap K}\frac{f(x)}{\|\nabla u(x)\|}dS + \frac{o(\epsilon)}{\epsilon}\right),$$

hence

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\{y\in K: v\le u(y)\le v+\epsilon\}} f(y)dy = \int_{\{x\in K: u(x)=v\}}\frac{f(x)}{\|\nabla u(x)\|}dS,$$

as desired. This completes the proof of the lemma. **Q.E.D.**

By the above lemma, we can calculate the induced density function ϕ in Section 4.

Corollary A.1 *Suppose Assumptions 1–3. The density function ϕ of the statistic ζ (Assumption 2) has finite value and is continuous on the range $\zeta[\Theta]$ of ζ, and it is positive over the interior of $\zeta[\Theta]$. Furthermore, Equation (17) holds.*

Proof: Pick any interior point z of the range $\zeta[\Theta]$. Lemma A.1 and Equation (14) imply that

$$\phi(z) = \int_{\{\vartheta\in\Theta:\zeta(\vartheta)=z\}}\frac{f_\zeta(\zeta(\vartheta))\times f_0(\vartheta)}{\|\nabla\zeta(\vartheta)\|}dS.$$

By assumption, ζ is homogeneous of degree one, and f_0 is homogeneous of degree zero. Thus, by changing the variable $\vartheta/z\mapsto\vartheta$, we have

$$\phi(z) = z^{m-1}f_\zeta(z)\int_{\{\vartheta\in\Theta:\zeta(\vartheta)=1\}}\frac{f_0(\vartheta)}{\|\nabla\zeta(\vartheta)\|}dS.$$

This proves Equation (17) for all interior points z of $\zeta[\Theta]$, with the surface integral here being the constant k there. The equation can be continuously extended to the boundary of $\zeta[\Theta]$, because f_ζ is continuous (Assumption 2) and the integral is finite ("$\zeta(\vartheta)=1$" implies "$\nabla\zeta(\vartheta)\ne\mathbf{0}$" by Assumption 2). Thus, Equation (17) holds throughout $\zeta[\Theta]$, and ϕ is finite and continuous. Since f_0 is positive on $\zeta[\Theta]$ and f_ζ is positive over the interior of $\zeta[\Theta]$ (Assumption 2), the density function ϕ is positive over the interior of $\zeta[\Theta]$. This proves the corollary. **Q.E.D.**