On-the-Job Signalling and Self-Confidence.*

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Abstract

The labour economics literature on signalling assumes workers know their own abilities. Well-settled experimental evidence contradicts that assumption: in the absence of hard facts, subjects are on average overconfident. First we show that in any equilibrium of any signalling model, overconfidence cannot make players better off. In order to obtain more detailed predictions, we then introduce a specific on-the-job signalling model. We show that at fully-separating equilibrium, overconfident workers choose tasks that are too onerous, fail them, and, dejected by such a failure, settle down for a position inferior to their potential. Such a pattern leads to permanent underemployment of workers, and inefficiency of the economy. For the case of unbiased workers uncertain about their own value, we determine a necessary and sufficient condition for the existence of fully-separating equilibrium

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"...and that's the news from Lake Wobegon, Minnesota, where all the women are strong, and all the man are good-looking, and all the children are above average !"

Garrison Keillor
"A Prairie Home Companion"

1 Introduction

In the signalling literature of labour economics, workers may disclose their ability, which is private-information, by producing adequate signals. It is customary to assume each worker precisely assess her own ability. This paper relaxes that assumption, by explicitly looking at the issue of self-confidence. As well as considering the case of workers who are just uncertain about their abilities, we shall devote attention to the case of overconfident workers. Well-settled experimental evidence suggests that, in the absence of hard facts to base their judgements upon, subjects are on average overconfident.[1]

Common wisdom holds confidence advantageous, so we may believe that overconfident workers would be successful. On the contrary, *at any equilibrium of any signalling game,* workers cannot be made ex-ante better-off by being overconfident. In fact, each overconfident worker signals *as if* she were a higher type. If, at that equilibrium, her true type is pooled with her perceived type, the payoff is the same. If the two types are separated, the incentive compatibility condition implies that the worker will not be better-off when signalling any type different from the true one, regardless of whether she does so in good faith, or with fraudulent intentions.[2]

Beyond this observation, one-shot analysis is insufficient to settle the issue. While we

[1] The typical experiment protocol (cf. Lichtenstein, Fischoff, and Phillips (1997), or Einhorn and Hogarth (1978)) asks each subject in a sample in which percentile she belongs to, with respect to intelligence, ability etc. The derived aggregate cumulative distribution stochastically dominates (it lies below) the identity distribution. This is possible only if the subjects are overconfident. See also Thaler (1991), and Camerer (1997) for a general overview of related topics.

[2] This observation is not valid when comparing ex-ante utilities of different games, or across different equilibria of the same game. Also, it is valid as long as overconfidence consists of incorrect private information in a signalling game, and the comparizon is in terms of ex-ante utility. In the search theoretic literature, Flam and Risa (1998) show that overconfident players may achieve a better final position than unbiased ones, and Dubra (1999) shows that an optimistic prior on the distribution of offers may avoid the searcher forego valuable sampling.

postulate that workers are initially uncertain about their abilities, we need to ask ourselves whether overconfidence will be relevant in the long-run, and so we explicitly introduce a learning dimension into the problem. By analyzing a specific model of on-the-job promotion we are able to establish whether overconfidence is relevant in the long-run, how it affects workers' learning behavior, and also what is its welfare impact on the population. At the same time, we study the robustness of separating equilibrium, and determine a necessary and sufficient condition for its existence in scenarios where workers are uncertain about their own abilities, but not necessarily overconfident.

In our model, an employed worker freely chooses to participate in costless screening programs, that upgrade her qualification if successfully completed. For simplicity, we represent them as pass-fail error-free tests.[3] On the basis of the test outcome, the firm decides whether to promote the worker or not. If the worker is not promoted, she may resort to an outside option, whose value is a function of the qualification achieved, and in particular coincides with the value of the highest test passed by the worker. Self-confidence is easily incorporated in the model by focusing on the worker's *prior belief* about her own ability.

This model belongs to the class of repeated bargaining games with incomplete information, whose solution is usually rather involved.[4] However, the assumption that the value of the outside option is function of the qualification achieved allows us for a (conceptually) simple solution technique. Whenever the firm chooses to hire the worker, it will offer her the value of the outside option. Thus we may solve first the worker's optimal stopping problem in absence of the firm, where the worker decides whether to continue testing, or to accept the outside option, and if choosing to test, she must select the optimal test. The solution to this problem gives us the value of the outside option in all possible histories, and allows us to calculate the firm's equilibrium belief, and thus its optimal policy.

In the benchmark case in which workers know their own abilities, we show that each

[3]Our results are robust in the testing technology: they would hold also with tests allowing small mistakes.

[4]See for example Gul, Sonnenschein, and Wilson (1986), Ausubel and Deneckere (1989), and Aumann and Maschler (1995).

of them will choose the highest-qualification program she can successfully complete. The only solution for the model is the Riley outcome: a second-best efficient, *fair* separating equilibrium in which each worker's ability is revealed to both firm and worker, and coincides with her qualification and wage.

In order to analyze uncertainty, we first consider the case in which workers are not overconfident, but they are uncertain about their own abilities, so that each worker's private information consists of a distribution over possible abilities (we call that distribution, her *confidence).* The analysis is trivial if the solution is a pooling equilibrium: as all workers obtain the same treatment, self-confidence and worker's ability are irrelevant with respect to fairness. In order to be meaningful, the model should allow for the existence of a separating Perfect Bayesian Equilibrium.

It turns out, however, that a fully-separating equilibrium exists if and only if the model satisfies the following requirement. When a worker has lower mean confidence than another worker, we must allow the first one to have smaller variance, or else she will pool with the second one. Say for instance, that two workers with confidence distributed uniformly respectively on $[\theta - s_\theta, \theta + s_\theta]$ and $[\theta' - s_{\theta'}, \theta' + s_{\theta'}]$ will separate if and only if s_θ □ $s_{\theta'}/2$ and $\theta \geq \theta' - s_{\theta'}/2$. In such an equilibrium workers are hired after one period of screening, and they may fail their first test. We show that the outcome is ex-ante and interim fair, even though it does not fully distinguish between workers of different ability, and so is not ex-post fair.[5]

In order to analyze the learning processes of overconfident workers, we need to restrict attention to models where workers are uncertain. If an overconfident worker with degenerate priors fails a test that she believed she was able to pass, then the Bayes rule is not well-defined, and it is unclear how the worker will revise her beliefs. We define a worker as overconfident if she holds a confidence higher than her actual distribution of ability, and study a model with uncertain workers, some overconfident, and some unbiased. The firm's

[5]The formal distinction between ex-ante, interim, and ex-post results was first introduced by Holmstrom and Myerson (1983).

behavior depends on its beliefs over the worker's confidence. We think that the firm should be aware that, on average, workers are overconfident.[6]

At least when training programs are of small cost, we may believe that workers will eventually learn their own abilities, and their overconfident priors will be irrelevant. On the contrary, we will show that fully-separating equilibrium implies a pattern of failure, dejection, and under-employment. In equilibrium, the firm does not have any initial information with respect to the worker. For fear of hiring the worker at a wage above her ability, it does not make any offer until the worker has taken a test. Since workers with different confidences take different tests, the firm finds out the worker's confidence upon observing the first test choice, and thus it may calculate the permanent wage that the worker would require in order to stop testing.

In the class of models selected above, if an overconfident worker has failed the first test, she revises her confidence so as to become dejected, and *underconfident.* Now, she may accept underpayment: in order to increase her qualification, in fact, she needs to pass an appropriate test, but, as she believes such a test too difficult, she will not even try to take it. While the firm does not know the worker's ability, it believes that it is most likely above the worker's current confidence. It is not in the firm's best interest to have the worker take one more test, as she will most likely pass it, achieve a higher qualification, and thus require a higher permanent wage. Therefore, while unbiased workers will be hired for the interim-fair wage, dejected workers will be underpaid.

If a worker has passed her first test, her current confidence is above her ability, and she will not accept the firm's offer, trying instead more difficult tests. Eventually she will fail and revise her confidence, so as to become unbiased. Interestingly, the presence of overconfident workers in the population makes it impossible for uncertain workers to be hired after one successful test. The firm will not be able to distinguish them from overconfident workers, and so will refuse to hire any worker before they fail at least one

[6]In order to show our result, we do not need to assume that the firm knows the distribution of confidence across workers. Nevertheless, we shall impose that assumption in the model, as it allows to construct an equilibrium in the spirit of the Harsanyi doctrine.

test. Nevertheless, all the workers who pass the first test will eventually be hired for a fair wage.

The last question we consider is efficiency. With costless screening programs that do not affect productivity, the worker's choice does not matter for welfare. In order to assess the efficiency issue, it is necessary to introduce training programs that bear a non-negligible cost, and that increase worker productivity if successfully completed. Repeating the analysis summarised above, we show the equilibrium to be inefficient. In fact, overconfident workers will not accept the firm's offer, they will take difficult training programs they will not be able to complete, and thus produce a social cost. At the same time, dejected workers will not take training programs they could accomplish, and will not increase their productivity up to the limit imposed by their personal ability, reducing social efficiency.

A critical assumption of our model is that firms enjoy some monopsony power in the promotion decision of the workers they employ.[7] We might conjecture that each firm screens each single worker, and engages in a bidding war for her, even if she is currently working for a different employer. In such a case, self-confidence may be irrelevant. In fact, the competing firms would settle to offer the wages that coincide with their assessment of the worker's ability. Since firms are aware of workers' overconfidence, they may be able to fully adjust for the workers' biases, and find out their actual abilities.[8]

The paper is presented as follows. The second section shows that in any signalling model, overconfidence is not beneficial. The third section introduces our on-the-job signalling model and studies the case in which each worker knows her ability. The fourth section extends the model to consider uncertain, unbiased workers. The fifth section introduces overconfident workers in the model of the fourth section. The sixth section covers efficiency. The last section discusses our model and relates it with the literature. Appendix A presents

[7] In the class of labour models of repeated bargaining with incomplete information, some models require perfect competition, others, like Hosios and Peters (1993) stipulate monopsony.

[8] We should also stress that if firms do not know that workers are overconfident, underpayment will occur even with perfectly competitive firms. That is contrary to the intuition that underpayment obtains mainly because the firm has an informational advantage on the worker.

the worker's optimal stopping problem. The reader will find all the proofs in Appendix B.

2 Overconfidence and Signalling

In this section we present a minimal (and thus unrestrictive) signalling model, and show that overconfidence cannot be ex-ante beneficial at equilibrium. Let a player with individual characteristic $\theta \in \Theta$ choose a strategy $s \in S$. After observing s, her opponent takes a strategy f, that influences the player's payoff $u_\theta(s, f(s))$, the opponent does not know θ. Note that θ need not be a number, it may indicate a distribution of personal abilities, or any individual characteristic one may consider appropriate, and s need not be a single action, but may be a complicated strategy, or an infinite horizon policy. Say that the player's actual characteristic is θ, but she believes it to be θ'. If one assigns an appropriate order on the set Θ, she can define that player *overconfident,* and contrast her with an individual who knows that her characteristic is θ.

Consider any equilibrium (s^*, f^*). While the overconfident player's choice s^* depends on her belief θ', her actual utility depends on her actual characteristic θ.

If θ and θ' pool, $s^*(\theta) = s^*(\theta')$, then clearly

$$u_\theta(s^*(\theta'), f^*(s^*(\theta'))) = u_\theta(s^*(\theta), f^*(s^*(\theta)))$$

so that the overconfident player achieves the same ex-ante utility as the unbiased one.

If θ and θ' separate, $s^*(\theta) \neq s^*(\theta')$, then Incentive Compatibility requires

$$u_\theta(s^*(\theta'), f^*(s^*(\theta'))) \square u_\theta(s^*(\theta), f^*(s^*(\theta))),$$

thus the overconfident player cannot achieve higher ex-ante utility than the unbiased one.

3 On-the-Job Signalling

3.1 Basic Model

An employed worker holds private information with respect to her own ability q, we define this information as *confidence*, and denote it by $\theta \in [\underline{\theta}, \bar{\theta})$, with $\underline{\theta} > 0, \bar{\theta} < \infty$. While in the fourth section, the worker is uncertain, and in the fifth section she is overconfident, here she knows her ability, and we may simply assume $q = \theta$. Confidence (and thus ability) is distributed in the population of workers with a continuously differentiable and strictly increasing cumulative distribution Φ. The worker's productivity is defined as the net product contributed to the firm (for simplicity, we assume it independent of other workers' contribution, and of different factors of production) for the moment we set it equal to the worker's ability q. In the sixth section, we will extend the model to allow productivity to depend also on training.

At each period t, the worker is offered a wage w_t by the firm. The worker may accept the offer (we denote that choice by $D_\theta^t = A$), choose an outside option ($D_\theta^t = R$), or take an error-free pass-fail test indexed by $x, x \geq 0$. The workers may freely choose the test's index x, we denote such a choice by $D_\theta^t = x$. The firm is not legally allowed to walk away from a contract with a worker, that is $\forall t$ if $D_\theta^t = A$, then $w_{t+1} \geq w_t$.

For any q, we denote the outcome of the test x by $f(x, q)$ and assume that:

$$f(x,q) = \begin{cases} 0 & \text{if } \; x < q \\ 1 & \text{if } \; x \geq q \end{cases}$$

For any time t, we define the set X_θ^t as the set of tests taken by the θ worker before t. Formally, $X_\theta^t = \{x | \exists \tau < t, D_\theta^\tau = x\}$. Because of the error-free assumption, the set of successfully taken tests for a worker of ability q at time t is $Y_\theta^t(q) := X_\theta^t \cap [\underline{\theta}, q]$. Note that when q is unknown, $Y_\theta^t(q)$ is the realization of a random set we denote by Y_θ^t. In this section, in the eyes of the worker, Y_θ^t coincide with $Y_\theta^t(q)$.

Also, define $a_\theta^t = \max Y_\theta^t$ (with $\max_\emptyset = \underline{\theta}$) and $b_\theta^t = \min[X_\theta^t \setminus Y_\theta^t]$ (with $\min_\emptyset = \bar{\theta}$). Each test allows workers and the firm to update their beliefs according to the Bayes rule whenever possible, that is, whenever the test yields an outcome prior believed to occur with non-null probability.[9]

The institutional setting is such that at each time t, if a worker with ability q chooses the

[9] Experimental findings show that economic agents do not follow the Bayes rule. In particular, all experiments prove that the prior is understated when the updating is conducted... Our results are strengthened with any updating rule that understates the prior.

outside option $D_\theta^t = R$, she may report the outcome of her tests and obtain the permanent wage $r(Y_\theta^t) = a_\theta^t$.

At time T, the firm observes the worker's decision, the tests taken and their results. So its information is (H_T, Z_T) where $H_T := \{(D_\theta^t, w_t)\}_{t<T}$ is the action path and $Z_T = (X_\theta^T, Y_\theta^T)$ is the test path. Let $w(Z_T, H_T)$ be its behavioural strategy. Given the prior Φ, the firm will use (H_T, Z_T) to update belief its belief over the worker's ability q. At time T, the worker's information consists of (H_T, w_T, Z_T), and $D_\theta(H_T, w_T, Z_T)$ is the worker's strategy. The solution concept is (pure-strategy) Perfect Bayesian Equilibrium. Proceeding according to sequential rationality, each player will its maximise its expected payoff from T onwards.

For any event E, let $\chi_{(E)}$ denote the indicator function, assuming value of 1 if E occurs and value of 0 otherwise. Each worker of type θ continuation payoff at time T, after history (H_T, w_T, Z_T) is:

$$U_\theta^T = (1-\delta)\sum_{t=T}^{\infty} \delta^{t-T}[w_t \chi_{(D_\theta^t = A)} + r(Y_\theta^t)\chi_{(D_\theta^t = R)}]$$

In the eyes of the firm, for any time T, Y_θ^{T+1} is a random set depending on the random variable q. Thus the firm's beliefs on q determine also its beliefs on the continuation game it will enter at time $T+1$. The firm's continuation payoff at time T after history (H_T, Z_T) is thus:

$$U_F^T = (1-\delta)E_F\left[\sum_{t=T}^{\infty} \delta^{t-T}[q - w_t]\chi_{(D_\theta^t = A)} \middle| H_T, Z_T\right]$$

We define an equilibrium as *ex-post fair* if the firm does not make any rents, and any worker is paid what she is worth. Formally, for any T, $U_F^T = 0$, and $U_\theta^T = q$ when $\theta = q$. In sections three and four we will need to weaken this definition of fairness and introduce the concepts of ex-ante and interim fair.

3.2 Correct Judgement

For δ close to 1, in a Perfect Bayesian Equilibrium, almost all workers are separated already at time 0. Even though workers may take any number of almost costless tests, they will pick the correct test at time 0, and be employed with a fair wage at time 1. Specifically, the equilibrium path is such that, at time 0, each worker θ takes test θ, pass it, and gets hired by the firm for a permanent wage of $w_t = \theta = q$. Since the firm is not making any profit in this equilibrium, there also exists separating PBE in which the firm does not hire workers. We rule these equilibria out as they are meaningless for the analysis. In this and in the following propositions, for brevity, we present only the equilibrium path, rather than the entire equilibrium strategy, which is always presented in the proofs.

Proposition 1 *For $\delta \approx 1$, at the unique Perfect Bayes Equilibrium, almost any worker θ play $D^0_\theta = \theta$, to obtain $w_t = \theta$ for any $t > 0$, so that the equilibrium is almost ex-post fair.*

Specifically, the firm sets $w_0 \to \underline{\theta}$, each worker with confidence $\theta \geq w_0/\delta$ plays $D^0_\theta = \theta$, to obtain $w_t = \theta$ for any $t > 0$, and each worker of confidence $\theta < w_0/\delta$ accepts wage w_0 forever. The intuition is as follows. Each type of workers knows her type and can reveal it by taking and passing the appropriate test. If one chooses too difficult a test, she fails it, and she is punished by a zero wage offer; if one takes too easy a test, her wage is lowered. The firm, finally, knows that if it offers a wage higher than the reservation wage, it will get the worker to stop testing only if her ability is lower than the offered wage. For fear of hiring the worker for a wage above her ability, the firm will wait until she takes a test, and then make her an offer. Its initial offer will attract only those workers whose ability is too low to bother pursuing qualification through testing.

4 Uncertain Judgement

In this section, we modify the model of the third section to allow workers to be uncertain about their own ability. It is common knowledge that the ability q of each worker with confidence $\theta \in \Theta$ is distributed according to the uniform probability measure[10] on

[10] Such a specification allows for simple calculations and also for a more intuitive presentation of the results. The model Uniform-Delta is just an extreme case of the Beta-Binomial model. It would be interesting to generalize our result to the Beta-Binomial Bayesian model.

$[\theta - s_\theta, \theta + s_\theta]$, for some given s_θ that satisfies $\theta - s_\theta > 0$. Now, also the worker will update her belief over her ability q. Denote by $E_\theta(\cdot|H_T, w_T, Z_T)$ the expected value given information (H_T, w_T, Z_T).[11] While the expected utility of the firm is unchanged, worker θ's utility is now expressed as follows.

$$U_\theta^T = (1-\delta)E_\theta\left[\sum_{t=T}^{\infty}\delta^{t-T}[w_t\chi_{(D_\theta^t=A)} + r(Y_\theta^t)\chi_{(D_\theta^t=R)}]\middle| H_T, w_T, Z_T\right]$$

We finally extend our definition of fairness. An equilibrium is *ex-ante fair* if $U_F^0 = 0$, and each worker θ obtains a utility $U_\theta^0 = \theta$. An equilibrium is *interim fair* if for any T, $U_F^T = 0$ and $U_\theta^T = E_\theta[q|Z_T]$ for any θ.

In order to approach the problem, the first question to consider is the optimal decision rule for the θ worker, at time T, given test information Z_T, when the firm does not exist, so that $\forall t \geq T, D_\theta^T \neq A$. Let χ be solution of that problem, and $U_\theta(\chi|Z_T)$ the associated continuation value, such value corresponds to the interim value of the outside option, for any θ and Z_T. For δ close to 1, Lemma 1 in Appendix A shows that the continuation value coincides with the value of testing for a long time, approximately resolving the uncertainty about her own ability, and then settle down for the permanent wage granted by the qualification achieved. Specifically, for any T, Z_T and for high discount values, as $r(Y_\theta^t) = a_\theta^t, \forall t \geq T$, the discounted value of being tested $U_\theta(\chi|Z_T)$ approximates from below the random value q's expectation $[a_\theta^T + b_\theta^T]/2$, and $U_\theta(\chi|Z_T)$ is a sub-martingale. Moreover, the time T optimal test x_θ^T approximates $[a_\theta^T + b_\theta^T]/2$ from above. We finally show that, fixing T and δ, $\forall t \geq T, x_\theta^t > [a_\theta^t + b_\theta^t]/2$ and $x_\theta^t - [a_\theta^t + b_\theta^t]/2$ is a strict sub-martingale. Instead of using recursive techniques, we proved our results by formulating the working hypothesis that the worker always chooses tests equal to $[a_\theta^t + b_\theta^t]/2$, so as to calculate optimal stopping times. Then we can characterize the effective optimal test policy proceeding backwards from the stopping times, and conclude that for $\delta \to 1$, the optimal test converges to the working hypothesis in the metric that discounts future discrepancies. The derivation of the

[11] Since $Y_\theta(q)$ subsumes all the information relative to q in the worker's history (H^T, w^T, Z^T), we do not need to extend the definition of a worker's strategy to account for uncertain q.

equilibrium is then concluded by considering the firm's problem. Since it makes a take-it-or-leave-it offer, it may stop the worker from testing by offering her the interim value of the outside option. It will stop the worker whenever that maximises its future profits.

The analysis focuses on fully-separating equilibria. Proposition 2 below characterises all such PBE; unexpectedly it restricts attention to only one equilibrium. Let x_θ be the first test in the optimal policy χ when $Z = \emptyset$.

Proposition 2 *For δ sufficiently close to* 1, *there is a unique separating PBE. Almost all workers θ play $D_\theta^0 = x_\theta \approx \theta$. For any $t \geq 1$, $D_\theta^0 = A$: if $f(\theta) = 0$, $w_t \approx \theta - s_\theta/2$, if $f(\theta) = 1$, $w_t \approx \theta + s_\theta/2$. Such an equilibrium is almost ex-ante fair and almost interim fair.*

The main intuition is as follows: after the first test, the firm recognizes the worker's type. As it discounts future payoffs, it prefers to hire the worker now, rather than to wait for further testing. To hire her, it offers the wage that makes the worker indifferent between further testing and accepting the job. As that value is maximised for $D_\theta^0 = x_\theta \approx \theta$, the worker will never choose any other $x, x \neq x_\theta$ at time 0. The separating PBE is thus unique. In the benchmark-case of the second section, we let workers privately know their own ability, and thus we obtain an ex-post fair outcome, as the firm may recognize the ability of each worker by the test chosen in equilibrium. Now the private information consists of a distribution over possible ability values. So the firm cannot perfectly separate the different workers' ability, but only the distributions, and ex-post fairness is unattainable. The unique fully-separating PBE, however, is shown to be ex-ante and interim fair.

In Theorem 1 below, we find necessary and sufficient conditions for the existence of that separating PBE, and we show that under such conditions, it is the unique PBE. Somewhat unexpectedly, the existence of a fully-separating equilibrium requires the low worker to be much more stubborn that the high type.

Theorem 1 *For δ sufficiently close to* 1, *the separating PBE exists if $s_\theta < s_{\theta'}/2$, $\theta > \theta' - s_{\theta'}/2$, and only if $s_\theta \square s_{\theta'}/2$, $\theta \geq \theta' - s_{\theta'}/2$, whenever $\theta < \theta'$. Under such conditions, the separating PBE is the unique PBE.*

The intuition is as follows. When a confident worker fails a test, the choice of the test nevertheless reveals her private information. If the resulting wage is high, less confident workers will copy her decision. To allow for a separating PBE is thus necessary to have a low enough wage after failure of a high test. This is possible only if the ability distribution associated with confident workers has a "tail" to the left of the distribution associated to less confident ones. That results in the two requirements in the Theorem. In contrast with the model of the second section, the two requirements imply that the space Θ be discrete. Again we assume that Θ has a minimum, that we denote by $\underline{\theta}$. The requirement of $\theta - s_\theta$ not to be negative implies that Θ is finite.

Remark 1 When allowing for partially-separating equilibria, the analysis yields different results. It is in fact possible to construct an equilibrium of that sort when the space Θ consists of two disjoint intervals (the low type and the high type). The equilibrium is constructed as follows. Let the high type take the test path derived in Lemma 1 that solves her problem in absence of the firm. Require the low type to mix between taking the test path that solves her problem and copy the high type. At the same time, require the firm to mix when she observes the worker fail the tests on the high type solution, and hire the worker immediately otherwise. The most interesting feature of this equilibrium is that the probability that the firm hires the worker who fails the high type test path is *increasing* over time, and yet the probability that low type keeps on mimicking the high type is *decreasing* over time.

5 Overconfidence

In this section, we introduce overconfident workers into the model developed in the fourth section. We distinguish between the *actual* distribution over a worker's ability and her belief. While the ability of a worker with parameter $(\alpha, \theta) \in \Theta^2$ is distributed uniformly on $[\alpha - s_\alpha, \alpha + s_\alpha]$, she believes it to be distributed uniformly on $[\theta - s_\theta, \theta + s_\theta]$, where $\theta \geq \alpha$. The worker is overconfident if $\theta > \alpha$.

While, overconfident workers are by definition unaware of being mistaken, the firm instead is aware that workers are, on average, overconfident. In order to define an equilibrium for the scenario, let us imagine a continuum of workers. As in the previous sections, their parameter θ is drawn from a commonly known distribution. Since the two requirements of Theorem 1 imply that the space Θ be discrete, we cannot refer to the distribution of the second section, and we introduce the distribution of confidence ϕ. Conditional on θ, their parameter α is distributed according to $\gamma|\theta$. While each worker believes that her ability is distributed according to θ, it also knows that the *other* workers may be overconfident, and it knows the distributions ϕ and γ. One worker out of the population is extracted randomly and assigned to play against the firm, which also knows the distributions ϕ and γ. Equilibrium beliefs are never refuted on the path because the population is continuous, so that each worker's self-confidence bears no impact on the expectations she places on the firm's behavior. Even though we assume that each player is almost correct with respect to the distribution of types in the population, our results follow even if requiring only that each player is aware that (other) workers are on average overconfident.

As in the previous section, we restrict attention to fully-separating equilibria. In Proposition 3 we show that for δ sufficiently close to 1, there is a unique fully-separating PBE, and we describe its path.

Proposition 3 *For any δ close enough to 1, there exists a unique separating PBE. Almost all workers (α, θ) play $D_\theta^0 = x_\theta \approx \theta$. If $f(x_\theta) = 0$, $\forall t \geq 1, w_t = w_1 \approx \theta - s_\theta/2$, and $D_\theta^t = A$. If $f(x_\theta) = 1$, then a.s. there exists a finite time $\tau(\delta) > 2$ such that $w_t = 0, D_\theta^t = x_\theta^t, \forall t < \tau$, and $w_t = U_\theta(\chi|Z_t)$, $D_\theta^t = A, \forall t \geq \tau$.*

At time 0, each worker with confidence θ would choose the test $x_\theta \approx \theta$, regardless of her actual distribution of ability. At the beginning of the game, each worker's private information is completely summarized by the parameter θ. Thus the addition of the parameter α is irrelevant with respect to the worker's decision whether to reveal her private information through the optimal testing choice.

At time 1, the firm knows the worker's confidence θ, and thus it assigns the probability $\gamma|\theta$ to her parameter α. The firm will offer the wage $w_1 \approx \theta - s_\theta/2$ to the worker that fails the test. It knows in fact that $\alpha \,\square\, \theta$. When $\alpha = \theta$, the worker will be correct in her judgement, and will choose to work only for the fair wage (approximately $\alpha - s_\alpha/2$). When $\alpha < \theta$, the worker will be underconfident and accept to work for approximately $\theta - s_\theta/2$. Since $\theta - s_\theta/2 < \alpha - s_\alpha/2$, the firm makes a positive profit. Moreover, if the worker chooses to test more, she picks $x^1_\theta \approx \theta - s_\theta/2$, she most likely passes the test, and becomes correct in her judgement. Since the tests are error-free, once a worker has passed one test and failed another one, her initial confidence is irrelevant.

When the worker passes her first test, the firm will not hire the worker until she has failed one test. With positive probability, she may be overconfident, and so the firm would incur an expected loss by hiring her. The worker will eventually fail some tests. When the firm believes that she is not likely to be overconfident, it will hire her with an approximately interim-fair wage.

The second result shows that Theorem 1 is valid also when injecting overconfident workers in the population, according to the distribution system γ. The intuition is that, regardless of their actual distribution of ability, workers believe it to coincide with their confidence, so they follow the same incentives as in the framework of the previous section, and thus Theorem 1 extends.

Corollary 2 *For any δ sufficiently close to* 1*, the separating PBE of Proposition 3 exists and is the unique PBE, under the conditions of Theorem 1.*

The main result of this section is Theorem 3 below. It demonstrates that in a fully-separating equilibrium, while non-overconfident workers achieve an ex-ante, interim fair wage, overconfident workers will be dejected and under-paid. Such results hold ex-ante for all overconfident workers, and hold interim for most of them. Most of the overconfident workers will fail and accept a wage lower than their actual expected ability conditional on failing the test. Those who pass the test, instead, will eventually accept an approximately interim fair wage. The time-0 ex-ante utility of each worker with $\alpha < \theta$ turns out to be bounded below α even for δ close to 1.

Theorem 3 *For any α, $\lim_{\delta\to 1} U^0_{\alpha\theta}$ is decreasing in θ, and $\lim_{\delta\to 1} U^0_{\alpha\alpha} = \alpha$. When δ is sufficiently close to 1, for any $\theta > \alpha$, $\Pr(f(x_\theta) = 0|\alpha,\theta) > 1/2$, and there exist strictly positive bounds M, uniform in δ, such that $U^0_{\alpha\theta} < \alpha - M_1$, and $U_{\alpha\theta}[f(x_\theta) = 0] < E_\alpha [q|f(x_\theta) = 0] - M_2$.*

While workers who pass the first test will be hired for an approximately interim-fair wage, all overconfident workers who fail the first test will be under-paid. Specifically, for $\delta \approx 1$, for almost all (α, θ), the interim-utility after failing the first test is $U_{\alpha\theta}[f(x_\theta) = 0] \approx \theta - s_\theta/2$. The interim-fair wage, instead is $E_\alpha[q|f(x_\theta) = 0] = [\theta + \alpha - s_\alpha]/2$. So that $U_{\alpha\theta}[f(x_\theta) = 0] = E_\alpha[q|f(x_\theta) = 0]$ only if $\theta = \alpha$, and $U_{\alpha\theta}[f(x_\theta) = 0] < E_\alpha[q|f(x_\theta) = 0]$ when the worker is overconfident. The probability to fail the first test is $\Pr(f(x_\theta) = 0|\alpha, \theta) \approx [\theta - \alpha + s_\alpha]/4s_\alpha$, larger than 1/2 when $\theta > \alpha$. Combining the outcome after passing the first test and after failing it, the worker's ex-ante utility results

$$U^0_{\alpha\theta} \approx \frac{(\alpha + s_\alpha)^2 - \theta^2 + (2\theta - s_\theta)[\theta - \alpha + s_\alpha]}{4s_\alpha}.$$

So that overconfident workers will be ex-ante less successful than non-overconfident ones.

6 Efficiency

The analyses of the previous sections concerned the efficacy of costless screening programs, chosen by the workers, to achieve a fair outcome. Now we investigate whether the players will achieve efficient production using costly training programs. To that purpose, we will compare the equilibrium efficiency when workers know their ability, with the case in which workers are uncertain but unbiased, and with the instance in which some workers are overconfident.

In this section, we let each worker's on-the-job productivity be a function of her training x and of her ability q. Formally we denote the productivity as $\pi(x, q) = \pi x + (1 - \pi)q$.

For simplicity, we assume that it does not matter how many training programs the worker takes, but only the most difficult she passes:[12] so that $\pi(x, q) := \pi(a, q)$. The institutional setting is such that at each time t, the worker θ is granted an outside option with permanent wage of $r(Y_\theta^t) = \pi(a_\theta^t, a_\theta^t) = a_\theta^t$. In fact, a worker qualification a_θ^t is easily observable by third parties, but on-the-job productivity $\pi(a_\theta^t, q)$ is not.

Now, the firm's continuation payoff at time T is

$$U_F^T = (1-\delta)E_F\left[\sum_{t=T}^{\infty}\delta^{t-T}[(1-\pi)q + \pi a_\theta^t - w_t]\chi_{(D_\theta^t=A)}\middle| H_T, Z_T\right]$$

The worker's continuation payoff is unchanged.

In the previous sections, we assumed screening programs to take a short period of time and to be costless, so that we could conduct the analysis for δ approaching 1. Such assumption is not realistic anymore when introducing training programs that have non-negligible benefits in terms of productivity. The analysis of this section will be conducted for any δ smaller than a bound below 1. In fact, the discount factor fully summarises the cost of the training programs, in terms of foregone wages and profits.

Remark 2 As δ approaches 1, and π is fixed and strictly positive, Proposition 2 does not hold. There would exist a separating PBE where the firm, after recovering the worker's private information, keeps on training her for a time that approaches infinity, hires them only at the end of time, to gain extra productivity at no cost. That modifies the Incentive Compatibility condition, and allows for a larger set of spaces Θ than the one characterized in Theorem 1. In particular, $\delta \to 1$ and $\pi > 0$ imply that training programs yield large benefits at no-cost, so that the firm would choose to keep on training workers forever, even though it knows their ability, and to get them to work only at the end of time. $\diamond$

In the previous section, an equilibrium was defined as fair if for any worker θ, $U_\theta^0 = \theta$. With δ bounded below 1, that outcome may not be achieved as the worker must pay the

[12] Such assumption could be relaxed allowing the productivity to be raised also in case of failed programs, albeit less than for accomplished ones.

cost of training at least once, in order to signal her ability. Thus we shall modify our definition, and call an equilibrium *ex-ante fair* if $U_\theta^1 = \theta$, and $U_F^0 = 0$. The definitions of interim and ex-post fairness are similarly modified.

Since we are interested in productive efficiency, we define the welfare function $W(H)$ as the average discounted net product when each worker θ plays strategy $\mathcal{D}_\theta$:

$$W(H) = (1-\delta)E\left[\sum_{t=0}^{\infty}\delta^t[(1-\pi)q + \pi a_\theta^t]\chi_{(D_\theta^t = A)}\right] \qquad (1)$$

As $a_\theta^0 = 0, \forall\theta$ and $f(x,q) = 0$ if and only if $x > q$, integrating Equation 1 one readily realizes that *first best efficiency* requires: $D_q^0 = q$ $D_q^t = A, \forall t > 0$ if $\delta \geq 1-\pi$, and $D_q^t = A, \forall t$ otherwise. The welfare is equal to $W^*(H) = \max\{\delta, 1-\pi\}$, consistently with the interpretation of $1-\delta$ as the cost of the training programs, and π as its benefit.

Whenever $\delta < 1-\pi$, fairness is incompatible with first best. The latter requires players not to be trained, and participation in a training program is the only way to reveal the workers ability, and thus allow the firm to offer a fair wage scheme. When $\delta > 1-\pi$, the relative benefits of the training programs are high enough to make it optimal to get trained, so that fairness is reconciled with first best: in the remainder of the section we focus on that latter case.

When workers know their own ability, the PBE is separating (and thus fair), and achieves the first best outcome.[13] In fact, repeating the argument in the proof of Proposition 1, we obtain that any worker θ will accept the time 0 wage w if and only if $\theta \,\square\, \delta/w_0$, and that at time t, the firm makes a strictly positive profit only when hiring workers with $X^t = \emptyset$. Thus the time-0 firm problem is:

$$\max_{w_0\in[\underline{\theta},\bar{\theta}]} -w_0 + \int_{[\underline{\theta},w_0/\delta]}(1-\pi)\theta d\Phi(\theta) \qquad (2)$$

If $\delta \geq 1-\pi$, then $-w_0 + \int_{[\underline{\theta},w_0/\delta]}(1-\pi)\theta d\Phi(\theta) < -w_0(1-\pi)w_0/\delta \,\square\, 0$ so that $w_0 = \underline{\theta}$ solves the problem (2).

[13] If $\delta < 1-\pi$, the PBE achieves First Best only if it is pooling, that requires $\delta < \frac{1-\pi}{\phi(1)}$ and $\delta < (1-\pi)E_\phi(\theta)$. A semi-pooling equilibrium may occur when $\delta < 1-\pi$, and there is a local interior maximum, that is, there exists a $w_0 \in (\underline{\theta},\bar{\theta})$ s.t. $\phi(w_0/\delta)w_0(1-\pi)/\delta^2 = 0$ and $\frac{\partial\phi(w_0/\delta)/\phi(w_0/\delta)}{\partial(w_0/\delta)/(w_0/\delta)} < -1$.

We now consider uncertain priors and overconfident ones. Given the finite confidence space Θ, with spreads $\{s_\theta\}_{(\theta\in\Theta)}$ denote by W^s the equilibrium welfare function for the case of uncertain but unbiased priors. On the same state space Θ, given the overconfidence distribution system γ, denote by W^γ the equilibrium welfare function. We will show that uncertainty reduces welfare with respect to the first-best, and that overconfidence reduces economic efficiency even further.[14]

Theorem 4 *For $\delta \in (1-\pi, b)$, the equilibrium welfare functions are ranked as follows: $W^\gamma < W^s < W^*$, for any Θ, and γ.*

The main intuition is as follows. The firm wants to keep the reservation value down, to offer a wage lower than productivity and make a profit. Thus the firm needs to avoid workers to achieve their highest possible qualification. It will stop them when they are underestimating their true ability. That will prevent them from passing the next test, become unbiased, and increase their productivity. When workers are overconfident, the firm cannot hire them or it will incur a negative profit. As it will hire them only after they have failed some tests and become unbiased, their net product will not contribute to the economy, and no benefit will be generated in terms of increased productivity. These two effects will both result in less than efficient training.

7 Discussion and Related Literature

As is well known, the first seminal contribution in the signalling literature is by Spence (1973). Later on, the literature divided itself with respect to the identification of the signal that allows separation at equilibrium. While Salop and Salop (1976), and Guasch and Weiss (1980) focused on the worker's preferred contractual arrangements, and Burdett and Mortensen (1981) considered tests prior to market participation, we consider training

[14]Unlike the analysis of the fifth section, however, Theorem 4 compares efficiency across *different* games. In principle, there could exist different models of signalling with overconfidence in which an analogous result would not hold.

programs entered while on the job. The original static results have then been motivated in a repeated game setting by Noldeke and Van Damme (1990).

Unlike the above contributions, but following Hosios and Peters (1993), we restrict attention to a monopsonistic firm that makes a take-it-or-leave-it offer. Clearly such an assumption is not to be taken literally: it is just a simplification to represent some amount of monopsony power enjoyed by the firm when dealing with the promotions of specialised workers. As pointed out for example by Waldman (1990), the current employer holds a relevant competitive advantage with respect to other potential employers. It is then possible that, because of complexity costs, market frictions, or strategic commitment by the firms, employed workers receive a negligible number of offers from competitors. Burdett and Mortensen (1998) show that, in such a case, Diamond's (1971) result is approximated, and firms can almost make a take-it-or-leave-it offer to each employed worker.[15]

In seeking promotions, as in Killingsworth (1982) or Brown (1989), we assume that workers freely choose to stop working and participate in training programs. If they successfully complete the programs, their qualification are upgraded. We suppose that a higher qualification translates into a higher reservation wage. That directly represents scenarios where legislative devices, or trade-unionised national contracts, guarantee appropriate wages. In such a case our model may be interpreted as a "regulated monopsony". In the US, for example this is the case of civil servants.[16] But even absent a regulated monopsony, it is plausible to suppose that workers will not accept any wage inappropriate for their qualification, as she knows that she can actively search outside the firm and easily obtain an appropriately remunerated position.

While the firm may not pay a wage inappropriate for the worker's qualification, the actual worker's productivity on the job depends also on her ability, which is unobservable

[15] Pissarides (1990) and Mortensen (1998), however, point out that the "Diamond" equilibrium is upset by free entry. Our understanding is that the very reasons which make negligible the flow of on-the-job offer may also preclude entry.

[16] I thank Dale Mortensen for this observation. To our knowledge a formal model of public officers' promotion has never been written.

to third parties. In our simple bargaining model, the firm makes a profit only if it hires workers with on-the-job productivity above their qualifications.[17] Again that assumption is not to be taken literally, as surely there are instances in which the firm may increase its profit by increasing their workers' qualification. One may complicate our model to differentiate workers in terms of relative advantages (as in Burdett and Mortensen 1981) instead of absolute ones, and obtain essentially the same results.

The heart of our formal analysis consists of the worker's problem in absence of the firm, where the worker must decide whether to continue testing, or to accept the outside option; and if choosing to continue testing, she must select the optimal test. The problem belongs to the class of optimal stopping models with a continuous set of alternatives. We found that workers choose relatively more difficult tests, the closer they are to the stopping time. Such result is related to the analysis of Moscarini and Smith (1998). They consider testing decisions with convex costs in a discrete state-space, and show that the amount of testing increases when the uncertainty is soon to be resolved. We show that workers become more daring when the uncertainty becomes small.

While we analyze self-confidence in a signalling framework, that phenomenon has been studied by Flam and Risa (1998), and by Dubra (1999) in search theoretic frameworks. Dubra (1999) studies a searcher sampling offers from an unknown distribution, and defines her beliefs as a distribution over possible distributions of offers. He defines the searcher 'overconfident' if the resulting mixture dominates the actual distribution, and shows that if searchers are not patient, a slightly overconfident one may fare better than unbiased ones. When a searcher becomes pessimistic about her distribution, she may accept too low an extraction, a slightly optimistic prior may counteract such a bias, and make the searcher better off. While our paper is concerned with ex-ante utility comparison, Flam and Risa (1998) are interested in the eventual position achieved by the players. In their model, an individual chooses to take tests whose outcome depends on her own ability, and she is

[17] Take for example a secretary who is assigned some management functions. She holds the qualification and wage of a secretary, yet is able to fulfill management functions. By utilising the secretary, the company saves the differential wage it would pay a management-trained person.

allowed to override failed tests. Thus overconfident players will eventually hold a higher status that unbiased ones.

Our model of on-the-job promotion may be of particular interest for the European Union, because of the institutional characteristics of its labour markets. First, many industries are still monopsonistic[18] at the national level, so all the workers who acquire an industry-related expertise will be able to sell their labour to one company only. Secondly, professional training is administered in state-sponsored programs.[19] In most European countries, the choice of whether to be trained more is a voluntary small-cost decision by the worker. Finally, there generally exist legislative devices[20] making the outcome of the qualification tests legally binding. Those workers who successfully pass the test will get promoted and paid more, even in a labour-monopsonistic industry. Despite the existence of national labour monopsonists, at prima-facie, the labour system seems fair and efficient. Our paper suggests that not to be the case, by pointing out the unfair and inefficient outcome resulting from overconfidence and dejection. That naturally yields the consideration that a regulated monopsony will not be as fair as a perfectly competitive industry.

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[18] Among some examples we quote the car industry in France, Italy, Spain, the television industry in Germany, the telecommunications in Germany, the chemical industry in Italy, the oil industry in the Netherlands etc... Whereas the market for goods is effectively European, after the Single Act 1984 and the Maastricht Treaty 1993, linguistic barriers and many physical or sociological frictions to mobility keep the labour markets essentially national.

[19] For instance, the "Corsi di Formazione Professionale - Obbiettivo 4" in Italy, or the "Ausbildung Kurse" in Germany are mainly subsidized by the local governments; whereas the national level takes a larger share of the costs in France. On-the-job training programs are almost never paid by the workers.

[20] In most countries, there exist trade-unionized national contracts that guarantee a base-wage differentiated in the professional qualification achieved by the worker. In Germany, Austria and Denmark, moreover, unemployment benefits correspond to a high percentage of the (first-job) average wage of the members of the profession the unemployed worker belongs to.

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A The Worker's Optimal Stopping Problem

Consider the problem of a worker of type θ, given test information Z_T, and discount factor δ. Define by $U_\theta(\chi|Z_T,\delta)$ the continuation value of her optimal choice restricted in such a way that $\forall t \geq T, D_\theta^T \neq A$. Let $\chi_\theta(Z_T,\delta)$ be the optimal sequence of tests, and $x_\theta(Z_t,\delta)$ the test taken give information Z_t and any $t \geq T$. Let $\mathcal{D}_\theta^T = \{D_\theta^t\}_{t\geq T}$ denote any arbitrary sequence of decisions. We shall isolate the analysis of that problem in the following Lemma. For any pair (y,z), y function of δ, z independent of δ, let the notation $y \uparrow z$ mean that $\forall \delta, y(\delta) < z$ and $y(\delta) \to z$ for $\delta \to 1$. Analogously interpret $y \downarrow z$. For simplicity, we drop the subscript θ, and use interchangeably the information Z_t, and the interval $[a^t, b^t)$ it induces.

Lemma 1 *Suppose q is distributed uniformly on the interval $[a_T, b_T)$ induced by Z_T. Denote as χ the solution of Problem*

$$\max_{D_T} U(D_T|Z_T,\delta) := (1-\delta)E\left[\sum_{t=T}^{\infty} E\left[\delta^{t-T} a_t \chi_{(D_t=R)}|Z_{t-1}\right]\middle| Z_T\right] \tag{3}$$

1. *For $\delta \to 1$, $U(\chi|Z_T,\delta) \uparrow [a_T+b_T]/2$, and $D_T = x_T \downarrow [a_T+b_T]/2$.*

2. *For any $\delta < 1$, and any $t > T$, $E[U(\chi|Z_t,\delta)|Z_T] > U(\chi|Z_T,\delta)$, and $E[x_t - [a_t + b_t]/2|Z_T] > x_T - [a_T+b_T]/2$ (strict sub-martingale property).*

3. *For any D_T, $E[E[q|Z_t]|Z_T] = E[q|Z_t], \forall t > T$ (martingale property).*

Proof. The third claim is obvious.

To prove the first part of the first claim, we first show that the Problem (3) is an optimal stopping problem: let's rewrite it in recursive representation.

$$U(a,[a,b)) = \max\{\frac{a}{1-\delta}, \delta[\max_x U(x,[x,b))\frac{b-x}{b-a} + U(a,[a,x))\frac{x-a}{b-a}]\} \tag{4}$$

Because of stationarity, if $D_t(a,b) = R$ then also $D_{t+1}(a,b) = R$, therefore, any optimal D_T is such that for any test path $Z = \{Z_t\}_{t\geq 0}$, $D_t(a,b) = R$ if and only if $t > \tau$, for some $\tau(Z) < \infty$.

Now we reformulate Problem (3) with limit payoff average representation. Consider the problem

$$\max_{D_T} U(D_T,Z_T) := E\left[\lim_{\tau\to\infty}\sum_{t=T}^{\tau} E\left[\frac{a_t\chi_{(D_t=R)}}{\tau - T}\middle| Z_{t-1}\right]\middle| Z_T\right] \tag{5}$$

In this form, continuity at infinity breaks down, and the problem does not admit a stopping solution. Nevertheless, for any ε, the Problem (5) admits an ε-solution with stopping representation. In fact, consider the sequence D_T s.t. for some τ $D_t = x_t$, $x_t = [a_t + b_t]/2$, $\forall t < \tau$ and $D_t = R$, $\forall t \geq \tau$. Then

$$U(D_T, Z_T) = [a_T + b_T]/2 - 1/2^{\tau - T}.$$

Yet, for any D_T,

$$\begin{aligned} U(D_T, Z_T) &< E\left[\lim_{\tau\to\infty}\sum_{t=T}^{\tau} E\left[\frac{a_t \chi_{(D_t=R)}}{\tau - T}\middle| Z_{t-1}\right]\middle| Z_T\right] = E\left[\lim_{\tau\to\infty}\sum_{t=T}^{\tau}\frac{a_t \chi_{(D_t=R)}}{\tau - T}\middle| Z_T\right] \\ &< E\left[\lim_{\tau\to\infty}\sum_{t=T}^{\tau}\frac{q}{\tau - T}\middle| Z_T\right] = E[q|Z_T] = \frac{a_T + b_T}{2}. \end{aligned}$$

So set $\tau : 1/2^{\tau - T} < \varepsilon$ and the ε-solution is found. It is well known that $U(D_T|Z_T, \delta) \to U(D_T, Z_T)$ for $\delta \to 1$. Moreover, $\forall \delta < 1$, $U(\chi|Z_T, \delta) < U(D_T, Z_T)$ because with discounting the weight of stage-utility is decreasing over time, and χ is a stopping strategy. We have proved that $U(\chi|Z_T, \delta) \uparrow [a_T + b_T]/2$ for $\delta \to 1$.

To show that for any δ, and any $t > T$, $E[U(\chi|Z_t, \delta)|Z_T] > U(\chi|Z_T, \delta)$, recall that for any test path Z of an optimal D_T, there exist $\tau(Z) > T$, such that $D_t(Z) = R$ if and only if $t \geq \tau$. $(1-\delta)U(D_T|Z^\tau, \delta) = a_\tau$. Thus the utility at time t, if the path Z is taken is:

$$(1-\delta)U^t(Z) = \delta^{t-\tau(Z)} a_{\tau(Z)}, \quad \forall t < \tau(Z).$$

As, for any Z, $U^t(Z)$ is increasing from 0 to $\tau(Z)$ and constant thereafter, integrating over each path Z, and noting that $\sup_Z\{\tau(Z)\} \to \infty$ for $\delta \to 1$, we have shown the claim.

Now we prove that, for δ close enough to 1, $D_T = x_T$ where $x_T \downarrow [a_T + b_T]/2$. Consider Problem (4), we already know that its solution is a stopping strategy. Let us formulate the working hypothesis that $\forall a_t, b_t$, $\forall t$,

$$[a_t + b_t]/2 = \arg\max_x U(x, [x, b_t])\frac{b_t - x}{b_t - a_t} + U(a_t, [a_t, x])\frac{x - a_t}{b_t - a_t}.$$

Under that working hypothesis, $\forall t$, the test choices X^t and outcomes Y^t imply that $\forall a_t, b_t$,

$$b_t = a_t + [b_T - a_T]/2^{t-T}.$$

Thus the pair (a,t) is isomorphic to (a_t,b_t). Let $(a,\cdot)$ denote the class of choices for Problem (4) summarised by (a,t) for fixed a and free t. Given any a_t,b_t, it is optimal to stop whenever $a_t > \delta(a_t+b_t)/2$ i.e. iff

$$a_t > \frac{\delta[b_T - a_T]}{(1-\delta)2^{t-T+2}}$$

Denote by $t(a,\delta)$ the optimal stopping time for the class $(a,\cdot)$. That is, using the above passage,

$$t(a,\delta) := \min\{t : t+2-T > \log_2 \delta/(1-\delta) + \log_2 [b_T - a_T]/a\}$$

Consider the subsequence of $\delta \to 1$ defined as follows: $\delta_n = 2^n/(1+2^n)$. Along that subsequence, $\log_2 \delta_n/(1-\delta_n) = n$. Thus,

$$t(a,\delta_n) := \min\{t : t+2-T > n + \log_2 [b_T - a_T]/a\}$$

Consider the map $\xi : a \mapsto t(a,\delta_n), a \in [a_T, b_T]$. Note that fixed δ_n, ξ is non-increasing in a. For any n, denote by $R(\xi,\delta_n)$ its range: note that $R(\xi,\delta_n) = R(\xi,\delta_0)+n$. The cardinality of the range is invariant, but the value of the stopping times increasing to infinity.

Fix δ. For any t, consider the set of reservation values possible under the working hypothesis: $A_t = \{a_t = k[b_T - a_T]/2^t + a_T,\ k = 0,\cdots,2^t-1\}$. For any $a \in [a_T,b_T]$, the pair (a,t) is a stopping choice iff $a \in A_t$ and $t = t(a,\delta)$. The collection of intervals $(a_\tau, a_\tau + 1/2^\tau)$ s.t. (a,τ) is a stopping choice, partitions $[a_T,b_T]$. For $a \in [0,1)$, the range $R(\xi,\delta_n)$ is unbounded, so take an $\varepsilon > 0$ and restrict ξ on $[\varepsilon, 0)$. Now $\max R(\xi,\delta_n)$ exists finite.

Proceeding from $\max R(\xi,\delta_n)$ backwards, for any t, let $\mathcal{G}$ (good set) denote the set of all choices (a_t,t) s.t. $a_t \in A_t$, $(a_t,t+1)$ is a stopping choice and $(a_t+1/2^{t+1},t+1)$ is a stopping choice. Similarly let $\mathcal{B}$ (bad set), the set of all choices (a_t,t) s.t. $a_t \in A_t$, $(a_t+1/2^{t+1},t+1)$ is a stopping choice, but $(a_t,t+1)$ is not a stopping choice. Finally, call $\mathcal{G}^2$ the set of all choices (a_t,t) s.t. $a_t \in A_t$, $(a_t+1/2^{t+1},t+1) \in \mathcal{G}$, and $(a_t,t+1) \in \mathcal{G}$. Iteratively, define the sets $\mathcal{G}^{2^k}$ for any k.

The result of the above paragraph implies that with $\delta \to 1$, for each $(a_t,t) \in \mathcal{B}$ there exist a K s.t $\forall k < K$, $(a_t + k/2^t, t) \in \mathcal{G}$, and $K \to \infty$, with $\delta \to 1$. At the same there exist a K_2 s.t $\forall k < K_2$, $(a_t - k/2^t, t) \in \mathcal{G}^2$, and $K_2 \to \infty$.

On each choice $(a,t) \in \mathcal{G}$, with corresponding interval (a,b), the Problem 4 is as follows:

$$\max_x U(a,b) = x\frac{b-x}{b-a} + a\frac{x-a}{b-a}.$$

Let $\gamma_0 = 1$. It yields solution $x = [a+b]/2$ and $U(a,b) = [3a+b]/4$, let $\gamma_1 = 3/4$ Moreover, notice that for any Problem

$$U(a,b) = \max_x U(x,b)\frac{b-x}{b-a} + U(a,x)\frac{x-a}{b-a},$$

if $\exists\gamma \in [0,1]$ s.t. $U(y,z) = \gamma y + (1-\gamma)z$, then $x = [a+b]/2$. That shows that for any problem in $\mathcal{G}^{2^k}$, $x = [a+b]/2$, and $U(a,b) = \gamma_{k+1}a + (1-\gamma_{k+1})b$, where $\gamma_{k+1} = (2\gamma_k + 1)/4$. Note that $\gamma_K \downarrow 1/2$ with $K \to \infty$ which is implied by $\delta \to 1$. On each $(a,t) \in \mathcal{B}$ with interval (a,b), the Problem 4 is as follows:

$$\max_x U(a,b) = x\frac{b-x}{b-a} + \frac{3a+x}{4}\frac{x-a}{b-a}.$$

It yields solution $x = [a+2b]/3$ and $U(a,b) = [2a+b]/3$. At time $t-1$, either $(a,t-1) \in A_{t-1}$ or $(a_1/2^{t-1}, t-1) \in A_{t-1}$. Let $\mathcal{B} * \mathcal{G}$ denote the (bad-good) set that includes $(a,t-1)$, and $\mathcal{G}^2 * \mathcal{B}$ the set that includes $(a_1/2^{t-1}, t-1)$. For any $(a,t) \in \mathcal{B} * \mathcal{G}$, with interval (a,b), $x^* = [2a+3b]/5$ and $U(a,b) = [3a+2b]/5$. For any $(a,,t) \in \mathcal{G}^2 * \mathcal{B}$, with interval (a,b), $x = [3a+4b]/7$ and $U(a,b) = [3b+4a]/7$.

Now we can iterate the procedure by introducing the sets $(\mathcal{B} * \mathcal{G}) * \mathcal{G}^2$ and $\mathcal{G}^4 * (\mathcal{B} * \mathcal{G})$ sets and $\mathcal{G}^4 * (\mathcal{G}^2 * \mathcal{B})$ and $(\mathcal{G}^2 * \mathcal{B}) * \mathcal{G}$ and so on for higher iterations: let the union of these sets be denoted by $\mathcal{R}$. Since

$$\max_x \frac{b-x}{b-a}[\gamma x + (1-\gamma)b] + \frac{x-a}{b-a}[\zeta a + (1-\zeta)x],$$

yields

$$x = [(2\gamma-1)a + (2\zeta-1)b]/2[\zeta+\gamma-1]$$

It follows that for any $(a,t) \in \mathcal{R}$, with corresponding interval (a,b), $\gamma_K \downarrow 1/2$ with $K \to \infty$ (which is implied by $\delta \to 1$). To conclude the step, take $\varepsilon \to 0$.

For any pair of sequences of decisions (D_T, D'_T), denote by

$$d(D_T, D'_T) = \lim_{\delta\to 1}(1-\delta)E\left[\sum_{t=T}^{\infty}\delta^{t-T}|x(D_t) - x(D'_t)|\middle| Z_T\right]$$

where $x(D_t)$ is the test choice corresponding to D_T (let $x(D_t) = 0$ if $D_t = R$). Note that d is a metric up to equivalence classes. We have demonstrated that under the working hypothesis D_T we obtain a solution D'_T that for $\delta \to 1$, converges to D_T in the metric d. As the stopping times $t(a, D_T)$ construction is continuous in d, the approximation D'_T is correct. ■

B Proofs

Proof of Proposition 1. In the proposed equilibrium, the firm sets $w_0 \downarrow \underline{\theta}$, $w_t = r(Y^t_\theta), \forall t > 0$, and each worker θ plays $D^t_\theta = \theta$ if $\delta\theta > \max\{w_t, r(Y^t_\theta)\}$, $D^t_\theta = A$ if $w_t \geq \max\{r(Y^t_\theta), \delta\theta\}$ and $D^t_\theta = A$, otherwise.

First we prove that the above is a Perfect Bayes Equilibrium for δ close to 1. As $q = \theta$, $f(x, q) = 1$ if $x \square \theta$ and $f(x, q) = 0$ otherwise. Given the above firm's strategy, sequential rationality implies that when offered w, and holding reservation r, worker θ problem is

$$U_\theta(w) = \max\{\frac{w}{1-\delta}; 0 + \delta[U_\theta(x)\chi_{(\theta \geq x)} + U_\theta(r)\chi_{(\theta < x)}]\}.$$

She chooses $D_\theta = A$ whenever $w \geq \max\{r, \delta\theta\}$, and $D_\theta = \theta$ if $\delta\theta \geq \max\{r, w\}$, and $D_\theta = R$ otherwise. Given those strategies by the workers, if $Y^t_\theta \neq \emptyset$, the firm is indifferent between setting $w_t < r(Y^t_\theta)$, for any Y^t_θ or $w_t = r(Y^t_\theta), \forall Y^t_\theta$.

The former is a trivial strategy as it implies that the firm will not hire the worker: if $w_t < r(Y^t_\theta)$, the worker holding Y^t_θ will choose $D_\theta = R$. We rule out such an outcome as it is meaningless.

At time 0, $Y^0_\theta = \emptyset$, given the continuation strategies, the firm will offer some wage w_0 and get all workers to choose $D_\theta = \theta$, for any $\delta\theta \geq w_0$ and $D_\theta = A$ otherwise. Yet, for any $w_0 > \underline{\theta} + \varepsilon$, there exists a δ large enough such that

$$\int_{[\underline{\theta}, w_0/\delta]} \theta d\Phi(\theta) < w_0.$$

So that $\lim_{\delta \to 1} w_0 = \underline{\theta}$.

To prove uniqueness, first we show that for any information set (Z_T, H_T) of the firm, its continuation payoff vanishes for $\delta \to 1$.

By definition of Y^t_θ, it follows that $a^t \square \theta$, $\forall t$, thus the firm continuation payoff follows:

$$U^T_F \square (1-\delta)E_F\left[\sum_{t=T}^{\infty} \delta^{t-T}[\theta - w_t]\chi_{(D^t_\theta = A)} \middle| H_T, Z_T\right]$$

Now, take any continuation strategy $w_T = \{w_t(Z_T, H_T)\}_{t \geq T}$. Any worker of type θ at time T knows that if she takes test $D^T_\theta = \theta$ she will pass it and thus $r(Y^T_\theta) = \theta$, guaranteeing herself a continuation utility of at least θ.

Thus she will prefer $D_\theta^T = A$ over $D_\theta^T = \theta$ only if $U_\theta(A|w_T)$, the continuation payoff for $D_\theta^T = A$, satisfies:

$$U_\theta(A|w_T) = (1-\delta)\left(w_T\chi_{(D_\theta^T=A)} + \delta\left[\sum_{t=T+1}^{\infty}\delta^t[w_t\chi_{(D_\theta^t=A)} + r(Y_\theta^T)\chi_{(D_\theta^t=R)}]\right]\right) \geq \delta\theta.$$

Therefore:

$$U_F^T \square (1-\delta)E_F\left[\sum_{t=T}^{\infty}\delta^{t-T}[\theta - w_t]\chi_{(D_\theta^t=A)}\middle|(H_T, Z_T), \delta\theta \square U_\theta(A|w_T)\right]. \tag{6}$$

Now consider all information sets (H_t, w_t, Z_t), reached with positive probability given Φ, and (H_T, Z_T), such that $D_\theta(H_t, w_t, Z_t) \neq A$.

In case $D_\theta(H_t, w_t, Z_t) = x$ for some test x, the worker receives stage-payoffs $u_\theta^t = 0$, and the firm receives $u_F^t = 0$. In case $D_\theta(H_t, w_t, Z_t) = R, u_\theta^t \square \theta$, and $u_F^t = 0$. It follows that $\delta\theta \square U_\theta(A|w_T)$ implies $\delta\theta \square U_\theta(\{D_\theta^t = A\}_{t\geq T}|w_T) = \sum_{t=T}^{\infty}\delta^{t-T}w_t\chi_{(D_\theta^t=A)}$, the continuation payoff given by the periods in which the worker is hired by the firm.

Equation (6) implies that:

$$U_F^T \square (1-\delta)E_F\left[\sum_{t=T}^{\infty}\delta^{t-T}[\theta - w_t]\chi_{(D_\theta^t=A)}\middle|(H_T, Z_T), \delta\theta \square \sum_{t=T}^{\infty}\delta^{t-T}w_t\chi_{(D_\theta^t=A)}\right].$$

Thus, for $\delta \to 1$, $U_F^T \square 0$.

At the same time, it must be that $U_F^T \geq 0$, as the firm can always offer $w_t = 0, \forall t$, guaranteeing herself $U_F^T = 0$. Therefore, $U_F^T \downarrow 0$.

Now we prove that $w_t = r(Y_\theta^t), \forall(H_T, Z_T), \forall\Phi$.

First of all, by assumption, we rule out the case in which $w_t < r(Y_\theta^t)$.

The firm offers a continuation wage strategy (w_T, w_{T+1}) such that, by construction,

$$\begin{aligned}(w_t, w_{T+1}) &\in \arg\max_w E_F[U_F(w_T)|H_T, Z_T] \\ &= \int_{\underline{\theta}}^{\bar{\theta}}[\theta - w]\chi_{(D_\theta=A)}d\Phi(\theta) + E_F[U_F(w_{T+1})|H_{T+1}, Z_{T+1}]\end{aligned}$$

We know that $D_\theta = A$ only if $w + \delta w_{T+1} \geq \max\{\delta\theta, r(Y_\theta^T)\}/(1-\delta)$. Thus, if $w > r(Y_\theta^T)$, it follows that $D_\theta = A$ only if $w + \delta w_{T+1} > \delta\theta/(1-\delta)$.

Say that $\delta w_{T+1} \square \delta^2\theta/(1-\delta)$, then, if $w > r(Y_\theta^T) + \varepsilon$, for δ close enough to 1,

$$\int_{\underline{\theta}}^{\bar{\theta}}\theta\chi_{(D_\theta=A)}d\Phi(\theta) < \theta,$$

which contracts

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta - w]\chi_{(D_\theta = A)} d\Phi(\theta) \downarrow 0,$$

which is required by

$$\max_w E_F\left[U_F(w)|H_T, Z_T\right] \downarrow 0, \qquad E_F\left[U_F(w_{T+1})|H_{T+1}, Z_{T+1}\right] \downarrow 0.$$

If, instead $\delta w_{T+1} > \delta^2\theta/(1-\delta)$, then, for $\delta \to 1$, we contradict the requirement that the expected value of $U_F(w_{T+1})$ vanishes on information sets following $D_\theta^T = A$.

So, it must be that $w_T \uparrow r(Y_\theta^T)$. ■

Proof of Proposition 2. First consider the PBE s.t. $D_\theta^0 = x_\theta$ $\forall\theta$ s.t. $U_\theta^0(\chi|\emptyset) \geq w_0$, $w_0 \downarrow \underline{\theta}$ and $U_\theta^0(\chi|\emptyset) \uparrow \theta$, $x_\theta \downarrow \theta$ in $\delta \to 1$, where x_θ is the first optimal test in the policy χ of Lemma 1.

As in the Proof of Proposition 1, since each worker θ plays $D_\theta^0 = x_\theta$, the firm, upon observing D_θ^0 recovers the worker confidence θ. Given $D_\theta^0 = x_\theta$, it thus believe q to be distributed uniformly on $[\theta - s_\theta, \theta + s_\theta]$: the worker has revealed her private information. Both players observe the outcome $f(x_\theta)$. By the Bayes rule, if $f(x_\theta) = 0$, they believe q to be distributed uniformly on $[\theta - s_\theta, x_\theta)$, and if $f(x_\theta) = 1$, they believe q to be distributed uniformly on $[x_\theta, \theta + s_\theta)$. By Lemma 1, at time 1, the worker's value for entering a sequence of testing and then settle for the reservation wage is

$$\begin{aligned} U_\theta[\chi|f(x_\theta) &= 0] \approx \frac{\theta - s_\theta + x_\theta}{2} \approx \theta - \frac{s_\theta}{2} \\ U_\theta[\chi|f(x_\theta) &= 1] \approx \frac{\theta + s_\theta + x_\theta}{2} \approx \theta + \frac{s_\theta}{2} \end{aligned}$$

for $\delta \approx 1$.

At any time T, given test history Z_T, the firm must offer a continuation utility $U_\theta(A|w_T) \geq U_\theta(\chi|Z_T)$ to have the worker accept the offer.

Suppose the firm wants to hire the worker: since it is not allowed to walk away from a contract, its offer w_T must be equal to the constant wage $U_\theta(\chi|Z_T)$, as can be shown repeating part of the Proof of Proposition 1. Thus the firm maximal profit for hiring at time T is

$$U_F^T[w_T = U_\theta(\chi|Z_T)] = (1-\delta)E\left[\sum_{t=T}^{\infty} \delta^{t-T}[q - U_\theta(\chi|Z_T)] \middle| Z_T\right] > 0.$$

Whereas its profit for waiting is

$$U_F^T[w_T < U_\theta(\chi|Z_T)] = (1-\delta)\max_{t>T} E\left[E\left[\sum_{\tau=t}^{\infty} \delta^{\tau-T}[q - U_\theta(\chi|Z_t)] \middle| Z_t \right] \middle| Z_T \right].$$

For any $t > T$, by Lemma 1, since $\delta < 1$,

$$E\left[U_\theta(\chi|Z_t)|Z_T\right] < U_\theta(\chi|Z_T), \quad E[E[q|Z_t]|Z_T] = E[q|Z_T].$$

that implies that

$$\{E[q|Z_t] - U_\theta(\chi|Z_t)\}_{t\geq T}$$

is a strict supermartingale. Therefore,

$$U_F^T[w_T = U_\theta(\chi|Z_T)] > U_F^T[w_T < U_\theta(\chi|Z_T)],$$

the firm prefers to hire the worker at time $t = 1$, instead of waiting and get her to test more.

We then conclude that

$$\text{for } \delta \to 1, \quad w_1 \to \begin{cases} \theta - s_\theta/2 & \text{if } f(x_\theta = 0) \\ \theta + s_\theta/2 & \text{if } f(x_\theta = 1) \end{cases}$$

$$U_\theta^0 \to [\theta - \frac{s_\theta}{2}]\frac{[\theta - (\theta - s_\theta)]}{2s_\theta} + [\theta + \frac{s_\theta}{2}]\frac{[(\theta + s_\theta) - \theta]}{2s_\theta} = \theta$$

So that ex-ante fairness holds.

To show that any separating equilibrium must be such that $D_\theta^0 = x_\theta \; \forall\theta$ s.t. $U_\theta^0(\chi|\emptyset) \geq w_0$, $w_0 \downarrow \underline{\theta}$ for $\delta \to 1$, first note that separation requires different testing choices by different workers.

Secondly, given that separation occurs, because of discounting, high-ability workers prefer it to occur at time 0. Then as in the Proof Proposition 1, the choice to signal by high-ability workers' forces all workers to reveal themselves at the same time.

Therefore at any separating PBE, $\forall\theta, D_\theta^0 = x$ and x is different across θ.

Now contradict the thesis by assuming the existence of a separating PBE different from the one claimed above. Again, the firm, upon observing $D_\theta^0 = x$ recovers worker's prior confidence θ. Again, it will hire the worker at time 1 both after $f(x) = 0$ and $f(x) = 1$, by offering the worker respectively $U_\theta[\chi|f(x) = 0]$, and $U_\theta[\chi|f(x) = 1]$. So that

$$w_1 = \begin{cases} U_\theta(\chi|f(x) = 0) & \text{if } f(x) = 0 \\ U_\theta(\chi|f(x) = 1) & \text{if } f(x) = 1 \end{cases}$$

For any x define

$$E^0[U^1_\theta(\chi|x)] := U_\theta[\chi|f(x)=0]\Pr[f(x)=0|\theta] + U_\theta[\chi|f(x)=1]\Pr[f(x)=1|\theta]$$

By Lemma 1,

$$E^0[U^1_\theta(\chi|x)] < U^0_\theta(\chi|\emptyset) = E^0[U^1_\theta(\chi|x_\theta)]$$

with $x_\theta \downarrow \theta$ for $\delta \to 1$.

Given the firm's best reply, the worker prefers to pick $D^0_\theta = x_0$, and then keeping on testing for a long time to eventually accept the outside option, instead of picking $D^0_\theta = x$, and being recognized and hired by the firm. Therefore there cannot exist any PBE s.t $D^0_\theta = x$ and $x \neq x_\theta$ for some θ. ■

Proof of Theorem 1. Consider any pair of types $\theta < \theta'$. Incentive Compatibility for θ requires

$$U_\theta(D^0_\theta = x_\theta) \geq U_\theta(D^0_\theta = x_{\theta'}),$$

and analogously for θ'. For $\delta \to 1$, by Lemma 1

$$U_\theta(D^0_\theta = x_\theta) \to \Pr[f(x_\theta)=0|\theta]\frac{[\theta + x_\theta - s_\theta]}{2} + \Pr[f(x_\theta)=1|\theta]\frac{[\theta + x_\theta + s_\theta]}{2} \to \theta$$

whereas

$$\begin{aligned} U_\theta\left(D^0_\theta = x_{\theta'}\right) &\approx \Pr[f(x_{\theta'})=0|\theta]\frac{\theta' + x_{\theta'} - s_{\theta'}}{2} + \Pr[f(x_{\theta'})=1|\theta]\frac{\theta' + x_{\theta'} + s_{\theta'}}{2} \\ &\approx \frac{[\theta' - (\theta - s_\theta)]}{2s_\theta}[\theta' - \frac{s_{\theta'}}{2}] + \frac{[\theta + s_\theta - \theta']}{2s_\theta}[\theta' + \frac{s_{\theta'}}{2}] \end{aligned}$$

Thus Incentive Compatibility translates as:

$$\begin{aligned} \theta &\geq \frac{[\theta' - (\theta - s_\theta)]}{2s_\theta}[\theta' - \frac{s_{\theta'}}{2}] + \frac{[\theta + s_\theta - \theta']}{2s_\theta}[\theta' + \frac{s_{\theta'}}{2}] \\ \theta' &\geq \frac{[\theta - (\theta' - s_{\theta'})]}{2s_{\theta'}}[\theta - \frac{s_\theta}{2}] + \frac{[\theta' + s_{\theta'} - \theta]}{2s_{\theta'}}[\theta + \frac{s_\theta}{2}]. \end{aligned} \tag{7}$$

Solving out both conditions, one obtains two subcases: for $\theta' - \theta \square s_\theta$ it must $s_\theta \square s_{\theta'}/2$, and for $\theta' - \theta \geq s_\theta$ it must $\theta \geq \theta' - s_{\theta'}/2$. The conditions of the two subcases are satisfied if and only if

$$s_\theta \square \frac{s_{\theta'}}{2}, \qquad \theta \geq \theta' - \frac{s_{\theta'}}{2}.$$

The first requirement directly implies that Θ must be at most discrete and ordered.

Since we are dealing with limit argument, the if part of the Proof requires strict inequalities in Equations (7).

To show uniqueness, by Proposition 2 we only need to rule out semi-separating equilibria. By contradiction, assume that $\exists \theta < \theta'$ s.t. $D^0_\theta = D^0_{\theta'} = x$ (if $D^0_\theta = D^0_{\theta'} = A$ or R the Proof is analogous as the case in which $D^0_\theta = D^0_{\theta'} = \theta - s_\theta$). Then

$$\begin{aligned} E_\theta[q|Z^1] \;&\square\; E_F[q|Z^1] \,\square\, E_{\theta'}[q|Z^1] \\ w_1 \;&\square\; E_F[q|Z^1], \end{aligned}$$

or else the firm would make negative profits. But, by Lemma 1,

$$U_{\theta'}(\chi|Z^1) \uparrow E_{\theta'}[q|Z^1], \quad \text{for } \delta \to 1.$$

So $D^1_{\theta'}(w_1) \neq A$. Proceeding as in the Proof of Proposition 1, it is concluded that the optimal choice by the firm is to set $w_1 = U_\theta(\chi|Z^1)$. Then by choosing $D^0 = x$ (for any x) worker θ obtains

$$U_\theta(D^0 = x) = U_\theta[\chi|f(x) = 0]\Pr[f(x) = 0|\theta] + U_\theta[\chi|f(x) = 1]\Pr[f(x) = 1|\theta]$$

and type θ' obtains

$$U_{\theta'}(D^0 = x) = U_{\theta'}[\chi|f(x) = 0]\Pr[f(x) = 0|\theta'] + U_{\theta'}[\chi|f(x) = 1]\Pr[f(x) = 1|\theta'].$$

Therefore, by Lemma 1 their optimal choice is really $D^0_\theta = x_\theta$ and $D^0_\theta = x_{\theta'}$. So the considered non-separating strategy profile cannot be a PBE. ■

Proof of Theorem 3. As in the proof of Propositions 1, and 2, the event that the firm hires the worker before she reveals her private information with the first test is of negligible probability. Also, the firm will hire a worker who has revealed private information by offering her a constant wage $\mathbf{w}_T = U_\theta(\chi|Z_T)$, whenever it maintains it profitable. Because of that, and since each (α, θ) worker believes her ability to be distributed according to the parameter θ, the same argument presented in the proof of Proposition 2 implies that there is a unique separating PBE, and that at such a PBE, each type (α, θ) s.t. $U_\theta(\chi|\emptyset) \geq w_0$, chooses $D_{\alpha\theta} = x_\theta$ where $w_0 \downarrow \underline{\theta}$ and $U_\theta(\chi|\emptyset) \approx \theta$, for $\delta \approx 1$.

Unlike Proposition 2, however the worker may be incorrect in her beliefs, we shall condition probability assessment on α, or θ, to identify the parameter according to which the beliefs are formed.

When worker (α, θ) takes test x_θ, the outcomes are as follows:

$$\begin{aligned}
\Pr[f(x_\theta) &= 0|\alpha] = \frac{x_\theta - \alpha + s_\alpha}{2s_\alpha} \\
\Pr[f(x_\theta) &= 1|\alpha] = \frac{\alpha + s_\alpha - x_\theta}{2s_\alpha}.
\end{aligned}$$

Consider first the continuation of $f(x_\theta) = 0$. For notational ease, we drop the reference to the worker type in the variables $a^T_{\alpha\theta}, b^T_{\alpha\theta}$, and $x^T_{\alpha\theta}$.

The worker (α, θ) believes q to be uniformly distributed on $[\theta - s_\theta, \theta)$, whereas it is in fact distributed uniformly on $[\alpha - s_\alpha, \theta)$.

Consider any T,and Z_T following $f(x_\theta) = 0$. If $a_T \geq \alpha - s_\alpha$ then the distribution of q conditional on Z_T is independent of the parameters α and θ: the worker correctly believes q to be distributed uniformly on $[a_T, b_T)$, where b_T □ x_θ. If the firm knew the value of α, as in the proof of Proposition 2, it would stop the worker from further testing by offering her the constant wage

$$w_T = U_\theta(\chi|Z_T) \approx \frac{a_T + b_T}{2}, \quad \text{for } \delta \approx 1.$$

In case that $a_T < \alpha - s_\alpha$, all players with $\theta > \alpha$ are underconfident. If they test more, they most likely become less underconfident, and increase their reservation value from the outside option. In fact, $\theta > \alpha$, together with the conditions of Theorem 1 imply:

$$\begin{aligned}
& E[U_\theta(\chi|Z_{T+1})|Z_T, \alpha] \qquad (8) \\
= \; & \Pr\left[f(x_T) = 0|Z_T, \alpha\right] U_\theta[\chi|Z_T, f(x_T) = 0] + \Pr\left[f(x_T) = 1|Z_T, \alpha\right] U_\theta[\chi|Z_T, f(x_T) = 1] \\
= \; & \frac{x_T - a_T}{b_T - a_T} U_\theta[\chi|Z_T, f(x_T) = 0] + \frac{b_T - x_T}{b_T - a_T} U_\theta(\chi|Z_T, f(x_T) = 1) \\
> \; & \frac{x_T - \theta + s_\theta}{b_T - \theta + s_\theta} U_\theta[\chi|Z_T, f(x_T) = 0] + \frac{b_T - x_T}{b_T - \theta + s_\theta} U_\theta[\chi|Z_T, f(x_T) = 1] \\
= \; & \Pr\left[f(x_T) = 0|Z_T, \theta\right] U_\theta[\chi|Z_T, f(x_T) = 0] + \Pr[f(x_T) = 1|Z_T, \theta] U_\theta[\chi|Z_T, f(x_T) = 1] \\
= \; & E[U_\theta(\chi|Z_{T+1})|Z_T, \theta] > U_\theta(\chi|Z_T),
\end{aligned}$$

where the latter inequality is valid because the process $\{U_\theta(\chi|Z_t)\}_{t \geq T}$ is a strict submartingale, by Lemma 1. Proceeding as in Equation (8), moreover, one shows that

$$\begin{aligned}
E[q|Z_T, \alpha] &> E[q|Z_T, \theta] \qquad (9) \\
&> U_\theta(\chi|Z_T),
\end{aligned}$$

where the latter inequality follows by Lemma 1.

Putting together Equations (8) and (9) with Lemma 1, we obtain that for any $\theta \geq \alpha$,

$$\begin{aligned} U_\theta(\chi|Z_T) &< E[q|Z_T,\alpha] = E[E[q|Z_{T+1},\alpha]|Z_T,\alpha] \\ U_\theta(\chi|Z_T) &< E[U_\theta(\chi|Z_{T+1})|Z_T,\alpha] \end{aligned}$$

where the last equality in the first line is because $\{E[q|Z_t,\alpha]\}_{t\geq T}$ is a martingale, for any α.

The firm does not know α, it assesses its evaluation using $\gamma|\theta$, whose support is $\{\alpha \square \theta\}$. Putting together the case in which $a_T < \alpha - s_\alpha$, and the case in which $a_T \geq \alpha - s_\alpha$ thus we obtain:

$$\begin{aligned} U_\theta(\chi|Z_T) &< E[E[q|Z_T,\alpha]|\theta] = E_F[q|Z_T] = E_F[E_F[q|Z_{T+1}]|Z_T] \\ U_\theta(\chi|Z_T) &< E[E[U_\theta(\chi|Z_{T+1})|Z_T,\alpha]|\theta] = E_F[U_\theta(\chi|Z_{T+1})|Z_T]. \end{aligned}$$

We conclude that for any T,and Z_T following $f(x_\theta) = 0$, the firm will stop the worker from further testing by offering her the constant wage $\mathbf{w}_T$=$U_\theta(\chi|Z_T)$.

The above result allows us to find the equilibrium path after $f(x_\theta) = 0$. At time 1, $a_1 = \theta - s_\theta < \alpha - s_\alpha$, under the conditions of Theorem 1, therefore the firm stops the worker by offering her $\mathbf{w}_1$=$U_\theta(\chi|Z_1) \approx \theta - s_\theta/2$.

Now consider the continuation of $f(x_\theta) = 1$, again pick any time T, and test history Z_T.

If $b_T \square \alpha + s_\alpha$, the worker correctly believes q to be distributed uniformly on (a_T, b_T) where $a_T > x_\theta$. If it knew α, as in the proof of Proposition 2, the firm would stop the worker from testing by offering her the constant wage $w_T = U_\theta(\chi|Z_T) \approx [a_T + b_T]/2$ for $\delta \approx 1$.

In case $b_T > \alpha + s_\alpha$, the worker believes q to be uniformly distributed on $[a_T, b_T)$,whereas it is in fact uniformly distributed on $[a_T, \alpha + s_\alpha)$. So, for $\delta \approx 1$, by Lemma 1,

$$U_\theta(\chi|Z_T) \approx \frac{a_T + b_T}{2} > \frac{a_T + \alpha + s_\alpha}{2} = E[q|Z_t,\alpha].$$

A worker with $\theta > \alpha$ will accept to work only for a wage above her interim expected ability.

In order to show that the firm will eventually hire the worker, we need to show inspect the realized optimal testing path in absence of the firm.

We want to show that for any α, there exist almost surely a $T(\alpha)$ such that $b_{T(\alpha)} \square \alpha+s_\alpha$, and $b_t > \alpha + s_\alpha$, for any $t < T(\alpha)$. At time $t = 1$, $b_1 = \theta + s_\theta$. For $\alpha = \theta$, we are done, so consider the case in which $\alpha < \theta$. Under the condition of Theorem 1, $\theta + s_\theta > \alpha + s_\alpha$.

By Lemma 1, for any $t, (a_t, b_t)$ the optimal test is $x_t \approx (a_t, b_t)/2$. By construction $x_1 \approx \theta + s_\theta/2 > \alpha + s_\alpha$ so $f(x_1) = 0$, and $b_2 = \theta + s_\theta/2 > \alpha + s_\alpha$.

Consider any $T > 2$. If for any $0 < t < T$, $f(x_T) = 0$, then, for $\delta \approx 1$,

$$x_T \approx \theta + \frac{s_\theta}{2^{T-1}} \to \theta \text{ for } T \to \infty.$$

Moreover $\theta < \alpha + s_\alpha$,or else, since $x_\theta > \theta$, $f(x_\theta) = 0$. Therefore, there exist a T_1 such that $x_{T_1} < \alpha + s_\alpha$. By construction, for any $0 < t < T_1$, $x_t > \alpha + s_\alpha$, so that $\forall t \square T_1$, $b_t > \alpha + s_\alpha$.

If $f(x_{T_1}) = 0$, then $b_{T_1+1} \square \alpha + s_\alpha$, and we are done. If $f(x_{T_1}) = 1$, then $\forall t \square T_1 + 1$, $b_t > \alpha + s_\alpha$. Such event occurs with probability

$$\Pr[f(x_{T_1}) = 1] = \frac{\alpha + s_\alpha - x_{T_1}}{\alpha + s_\alpha - \theta}$$

In such a case, repeating the argument used for T_1, there exist a T_2 s.t. $x_{T_2} \square \alpha + s_\alpha$, and $\forall t < T_2$, $x_t > \alpha + s_\alpha$, so that $\forall t \square T_2$, $b_t > \alpha + s_\alpha$.

If $f(x_{T_2}) = 0$, then $b_{T_2+1} \square \alpha + s_\alpha$. The complementary event occurs with probability

$$\Pr[f(x_{T_2}) = 1] = \frac{\alpha + s_\alpha - x_{T_2}}{\alpha + s_\alpha - x_{T_1}}$$

Iterating the argument we construct a sequence $\{T_k\}_{k \geq 1}$ where for each k,

$$x_{T_k-1} - x_{T_k} \approx \frac{1}{2^{T_k}} s_\theta, \quad \text{and} \quad x_{T_k} < \alpha + s_\alpha < x_{T_k-1}$$

$$\text{so that :} \quad \Pr[f(x_{T_k}) = 1, \forall k < K] = \frac{\alpha + s_\alpha - x_{T_K}}{\alpha + s_\alpha - \theta} \to 0, \text{ for } K \to \infty.$$

Thus we proved that a.s. there exist a $T(\alpha)$ s.t. $b_t > \alpha - s_\alpha$, for any $t < T(\alpha)$, and $b_{T(\alpha)} < \alpha - s_\alpha$.

Now we consider the firm's decision. For any θ the set $\{\alpha \square \theta\}$ is finite. Thus there exists a finite time

$$\tau := \sup_{\{\alpha \square \theta\}} T(\alpha)$$

after which all workers (α, θ), with $\{\alpha \, \square \, \theta\}$ would achieve correct judgement. As in the proof of Proposition 2, and in the first part of the current proof, in such case the firm would stop from testing and hire them with a constant wage. The value τ is an upper bound for the values T, such that there exist a Z_T after which the firm hires the worker with $w_T = U_\theta(\chi|Z_T)$.

The actual expression for such critical (T, Z_T),is

$$\begin{aligned} E_F[q|Z_T] - U_\theta(\chi|Z_T) &\geq 0, \quad \text{and} \\ E_F[q|Z_T] - U_\theta(\chi|Z_T) &\geq E_F[E_F[q|Z_t] - U_\theta(\chi|Z_t)|Z_T], \quad \forall t > T : \end{aligned}$$

it is function of $\gamma|\theta$, and of the discount factor δ.

While τ escapes to infinity for $\delta \to 1$, it is finite for any $\delta < 1$, regardless of how close it is to 1. ■

Proof of Corollary 2. Since the worker (θ, α) receives continuation payoff equal to $U_\theta(\chi|Z_T)$ for any test history, Z_T, the proof of Theorem 1 extends without any change. ■

Proof of Theorem 3. The proof of the Theorem consists in the calculation of expected utility at the moment in which workers are hired, and in the composition across different test paths.

In case $f(x_\theta) = 0$, Proposition 3, shows that all workers (α, θ) will be immediately hired with constant wage $\mathbf{w}_1 = U_\theta[\chi|f(x_\theta) = 0] \uparrow \theta - s_\theta/2$, for $\delta \to 1$. Thus whenever $\theta > \alpha$, under the conditions of Theorem 1, they are interim underpaid:

$$U_\theta(\chi|f(x_\theta) = 0) \uparrow \theta - \frac{s_\theta}{2} < \frac{\alpha - s_\alpha + \theta}{2} = E[q|f(x_\theta) = 0, \alpha],$$

Under the conditions of Theorem 1, $\theta - \frac{s_\theta}{2}$ is decreasing in α. The interim bound is:

$$M_1 = \frac{(\alpha - s_\alpha) - (\theta - s_\theta)}{2} > 0$$

Clearly when $\theta = \alpha$, $U_\theta(\chi|f(x_\theta) = 0) \uparrow E[q|f(x_\theta) = 0, \alpha]$, for $\delta \to 1$.

The probability to fail the first test is

$$\Pr[f(x_\theta) = 0|\alpha] = \frac{x_\theta - \alpha + s_\alpha}{2s_\alpha} > \frac{1}{2},$$

when $\theta > \alpha$.

In case $f(x_\theta) = 1$, the firm will hire the worker (α, θ), after (T, Z_T) defined in equation (**??**), and receive a permanent wage $w_T \to U_\theta(\chi|Z_T)$. By Lemma **??**, for $\delta \to 1$, $U_\theta(\chi|Z_T) -$

$E[q|Z_T,\theta] \to 0$. Since $supp(\gamma|\theta) = \{\alpha \square \theta\}$, $E_F[q|Z_T] < E[q|Z_T,\theta]$, unless $E[q|Z_T,\theta] = E[q|Z_T,\alpha] = E_F[q|Z_T]$. Thus the firm will wait to hire the worker until it is approximately sure about her α, so that $w_T \to E(q|Z_T,\alpha)$, for $\delta \to 1$. The worker will almost surely be hired with an approximately interim-fair wage. Also, clearly $E[E(q|Z_T,\alpha)|f(x_\theta = 1,\alpha)] = E[q|f(x_\theta = 1,\alpha)]$, so that the worker approximately obtains the interim fair wage after $f(x_\theta) = 1$.

Now we can find the ex-ante utility of worker (α, θ):

$$\begin{aligned} U^0_{\alpha\theta} &= \Pr[f(x_\theta) = 0|\alpha]U_{\alpha\theta}[f(x_\theta) = 0] + \Pr[f(x_\theta) = 1|\alpha]U_{\alpha\theta}[f(x_\theta) = 1] \\ &\to \frac{[\theta + \alpha + s_\alpha][\alpha + s_\alpha - \theta]}{4s_\alpha} + \frac{[\theta - s_\theta/2][\theta - \alpha + s_\alpha]}{2s_\alpha}. \end{aligned}$$

Finally, we need to find the bound for the ex-ante utility, set

$$M_2 = \frac{[(\alpha - s_\alpha) - (\theta - s_\theta)][x_\theta - \alpha + s_\alpha]}{4s_\alpha}.$$

■

Proof of Theorem 4. We have previously shown that first-best efficiency requires each worker with ability q to take the test q at time 0, and then accept to work in the firm. That is impossible when $s > 0$. In fact, each type θ may hold any ability $q \in [\theta - s_\theta, \theta + s_\theta]$. Since the strategy $\mathcal{D}$ is a function of the type θ and not of the ability q, all the workers with ability in $[\theta - s_\theta, \theta + s_\theta]$, will take the same strategy at equilibrium, violating first-best efficiency.

In particular, if players with ability $q > x$ take test x, they yield a loss of at least $\delta(1-\delta)(1-\pi)q$, because, if in that first period they worked, their continuation productivity would be the same as after taking the test x. If they take test $x : x < q$, they incur a loss of at least $\delta(1-\delta)\pi(q - x)$, because if they took test q, at next period they would produce $(1-\pi)q + \pi q$, by taking test x, at next period they will produce less than $(1-\pi)q + \pi x$.

To show that $W^\gamma < W^s$ we need to compare the equilibrium strategies $(\mathcal{D}^s, \mathbf{w}^s)$ in the game Γ^s, with uncertain non-overconfident workers , and those $(\mathcal{D}, \mathbf{w}^\gamma)$ in the game Γ^γ that includes also overconfident workers, on the same Θ space.

If $\theta = \alpha$, it can be shown in much the same way as in Theorem 2, that for any test history (X, Y), with $X \neq \emptyset$, the firm and the worker always share the same belief with respect to q. When $\theta \neq \alpha$, as in the proof of Proposition 3, if $Y_t \neq \emptyset$ and $X_t \setminus Y_t \neq \emptyset$, then the firm and the worker share the same belief with respect to q.

Therefore, in order to compare W^γ and W^s, we can restrict attention to histories Z_t where either $Y_t = \emptyset$ or $X_t \setminus Y_t = \emptyset$. In the latter case, the worker has never failed any test, and she will be overconfident. In the former one, she has never passed any test, and if $\theta > \alpha$, she will be underconfident.

In the game Γ^s, the firm will not hire the worker if and only if $b-a$ is too large, and such uncertainty about the worker ability makes it optimal, given π and δ, to try and increase the worker productivity.

In the game Γ^γ, on the other hand, the firm will never hire overconfident workers, as they require too high a wage, and yield negative profits, as in the proof of Proposition 3. At equilibrium those workers will take difficult tests, and will never be hired until they fail one of them.

If the firm hires a worker with test history (X, Y) s.t. $Y = \emptyset$ in game Γ^s, it will also hire a worker with the same history in game Γ^γ. As in the Proof of Proposition 3, it will be able to hire her with a lower wage. If not hired, that underconfident worker immediately takes an easy test and pass it with very high probability, increasing her productivity.

In sum, for any history (a, b) the probability that the worker will fail a test, or will not pass a test, in the continuation game is strictly larger in game Γ^γ.

At time 0, the private information θ is not revealed to the firm yet, and so the above argument does not hold. Nevertheless, the choice of worker (α, θ) is $D^0_{\alpha\theta} = x_\theta$ increasing in θ. So that the amount of workers who fail a test at time 0 in game Γ^γ is larger than in Γ^s, as in the Proof of Theorem 3. The workers who pass the test do not matter, as they will not be hired by the firm anyway.

In conclusion, for any type θ, and any history, the probability to get an optimal qualification in the continuation is strictly smaller in game Γ^γ than in game Γ^s. That translates in a strictly positive welfare loss $W^s - W^g$. ■