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I INTRODUCTION

Vickrey [7] and Dummet and Farquharson [1] conjectured and Gibbard [3], Satterthwaite [6] and Schmeidler and Sonnenschein [8] proved that when the number of social alternatives is at least three, any nonimposed and nondictatorial voting scheme is manipulable (in the sense of it being profitable for some voter at some profile to misrepresent his preferences in order to secure a social outcome preferred by him to that resulting in the event his vote reflects his true preferences). This manipulability (or noncheatproofness) result was obtained by the above mentioned authors under the (implicit) assumption that the number of individuals (voters) is finite. In the case of an infinite set of individuals it has been shown by Pazner and Wesley [5] that (much as Fishburn's [2] result on the possibility of an Arrovian social welfare function in the case of an infinite set of individuals) the impossibility of a ("democratic") cheatproof social choice function no longer holds; however, while the existence of a cheatproof social choice function is rigorously proved in [5], the essentially nonconstructive method of proof used there does not make it possible to actually present any concrete example of such a cheatproof method of social choice.

In this paper we turn to the constructive aspects of the problem of designing a cheatproof social choice function for large societies. In particular, it is shown in the next section that the simple (and intuitively appealing) plurality rule is cheatproof in a precise limiting sense. Namely, as the number

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of voters approaches infinity we show that the proportion of profiles at which the plurality rule is cheatproof approaches one. The concluding section of the paper is devoted to the interpretation of this result and to a discussion of its relationship to the general problem of incentive compatibility in social choice theory.

II A LIMIT THEOREM ON THE PLURALITY RULE

Let $\mathcal{A} = \{a_1,\dots,a_k\}$ be a set of alternatives and let $V_n = v_1,\dots,v_n$ be a set of individuals. Let Σ denote the set of all strong orderings (i.e., the set of total, asymmetric and transitive binary relations) on $\mathcal{A}$. Each element $p = (p_1,\dots,p_n)$ in Σ^{V_n} (the set of functions from V_n to Σ) where $p_i \in \Sigma$ for all $i, 1 \leqq i \leqq n$, is called a preference profile. A function f from Σ^{V_n} to $\mathcal{A}$ is called a social choice function (SCF). An SCF f is said to be manipulable at $p = (p_1,\dots,p_n) \in \Sigma^{V_n}$ if there exist $p_i' \in \Sigma$ and an i_o, $1 \leq i_o \leq n$ such that $f\,(p_1', p_2',\dots,p_n')\ p_{i_o}\ f\,(p_1,p_2,\dots,p_n)$ where $p_i' = p_i$ for all $i \neq i_o$. f is cheatproof at p if it is not manipulable at that profile.

For every $p = (p_1,\dots,p_n)$ in Σ^{V_n} and every i, $1 \leq i \leq k$, let $C(p,i) = \{v_j \mid v_j \in V_n$ and $a_i\ p_j\ a_\ell$ for all a_ℓ in $\mathcal{A}$ such that $a_\ell \neq a_i\}$, i.e. $C(p,i)$ in the set of individuals who most prefer a_i under the profile p. Let $|C(p,i)|$ be the number of individuals in $C(p,i)$. In conformance with this notation, let $|V_n|$ be the number of individuals in V_n, i.e. $|V_n| = n$. We define the plurality rule $F\colon \Sigma^{V_n} \to \mathcal{A}$ as follows:

Let $p = (p_1,\dots, p_n)$ be a preference profile in Σ^{V_n}. If for some i, $1 \leq i \leq k$, $|C(p,i)| > |C(p,\ell)|$ for all ℓ, $i \leq \ell \leq k$, $\ell \neq i$, then let $F(p) = a_i$. If not, let $F(p) = a_{j_1}$, where j_1 is the smallest index such that $|C(p,j_1)| \geq |C(p,\ell)|$ for all ℓ, $1 \leq \ell \leq k$.

Let $\mathcal{D}_n$ be the set of preference profiles in Σ^{V_n} for which F is cheatproof. Let $|\mathcal{D}_n|$, $|\Sigma^{V_n}|$ be the number of elements in $\mathcal{D}_n$ and Σ^{V_n}, respectively.

<u>THEOREM</u>:

$$\lim_{n\to\infty} \frac{|\mathcal{D}_n|}{|\Sigma^{V_n}|} = 1$$

To prove the theorem, two auxiliary lemmas will be utilized. In order to formulate them, we introduce some additional notation.

For any $\varepsilon > 0$ and any natural number n, and any i,j $1 \leq i \leq k$, $1 \leq j \leq k$, $i \neq j$, let

$$T(n,\varepsilon,i,j) = \{p \mid p \in \Sigma^{V_n}, \left| \frac{|C(p,i)| - |C(p,j)|}{\frac{2n}{k}} \right| < \varepsilon\}.$$

Let $|T(n,\varepsilon,i,j)|$ be the number of preference profiles in $T(n,\varepsilon,i,j)$. Then

<u>Lemma 1</u>:

$$\lim_{n\to\infty} \frac{|T(n,\varepsilon,i,j)|}{|\Sigma^{V_n}|} \leq \frac{2\varepsilon}{\sqrt{2\pi}}$$

for any i,j, $1 \leq i \leq k$, $1 \leq j \leq k$, $i \neq j$.

<u>Proof</u>: In proving the Lemma we make use of the central limit theorem in probability theory ([4], p. 290). We assume that a probability measure P is defined over Σ so that all preference orders in Σ are equally likely to occur when random choices are made.*

Suppose that each $v_m \in V_n$ randomly chooses a preference ordering p_m in accordance with the probability measure P. A randomly selected preference profile $p = (p_1,\ldots,p_n)$ is thereby obtained. The probabilities are then such that

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* Note that throughout, the lower-case p and p_i stand for preference profiles and orderings respectively while capital P denotes the probability measure.

given any $a_{i_1}, a_{i_2} \in \mathcal{A}$, where $a_{i_1} \neq a_{i_2}$, we have $P(a_{i_1} p_m a_{i_2}) = P(a_{i_2} p_m a_{i_1})$. Let i, j, $i \neq j$, be any fixed natural number between 1 and k, inclusive. Define in the following manner the random variables $\xi_1, \xi_2, \ldots$ over the set of infinite sequences of preference orders:

$$\xi_\ell \overset{\text{def}}{=} \xi_\ell(p_1, p_2 \ldots) = \begin{cases} 1 \text{ if under } p_\ell,\ a_i \text{ is preferred over all other elements in } \mathcal{A} \\ -1 \text{ if under } p_\ell,\ a_j \text{ is preferred over all other elements in } \mathcal{A} \\ 0 \text{ otherwise} \end{cases}$$

Each ξ_ℓ then depends only on the ℓ^{th} element p_ℓ of the infinite sequence, i.e. $\xi_\ell(p) = \bar{\xi}_\ell(p_\ell)$ where $\bar{\xi}_\ell$ is defined in obvious fashion. It is thus easily seen that

$$\{p = (p_1, \ldots, p_n) \mid p \in \Sigma^{V_n},\ \frac{|\sum_{\ell=1}^{n} \bar{\xi}_\ell(p_\ell)|}{\frac{2n}{k}} < \epsilon\} = T(n, \epsilon, i, j)$$

and that consequently

$$P\left(\frac{|\sum_{\ell=1}^{n} \xi_\ell|}{\frac{2n}{k}} < \epsilon\right) = \frac{|T(n, \epsilon, i, j)|}{|\Sigma^{V_n}|}$$

Therefore, it is sufficient to prove that

$$\lim_{n \to \infty} P\left(\frac{|\sum_{\ell=1}^{n} \xi_\ell|}{\frac{2n}{k}} < \epsilon\right) \leq \frac{2\epsilon}{\sqrt{2\pi}}$$

For each random variable ξ_ℓ, let $\bar{a}_\ell = E(\xi_\ell)$, the expectation of ξ_ℓ. Then $\bar{a}_\ell = 0$ for all ℓ. Let $\bar{b}_\ell^2 = E((\xi_\ell - E(\xi_\ell))^2) = \frac{2}{k}$.
Let $B_n^2 = \sum_{\ell=1}^{n} \bar{b}_\ell^2 = \frac{2n}{k}$.

Let F_ℓ be the distribution function of the random variable ξ_ℓ.

Then for any $\tau > 0$

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{\ell=1}^{n} \int_{|x-\bar{a}_\ell|>\tau B_n} (x-\bar{a}_\ell)^2 \, d\,F_\ell(x) = \lim_{n\to\infty} \frac{k}{2n} \sum_{\ell=1}^{n} \int_{x>\tau B_n} x^2 \, d\,F_\ell(x) = 0$$

Hence, the Lindeberg condition (Gnedenko [4], p.289) is satisfied. Then by the central limit theorem (Gnedenko [4], p.290), as $n \to \infty$

$$P\{ \frac{1}{B_n} \sum_{k=1}^{n} (\xi_k - \bar{a}_k) < x\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz$$

uniformly in x. Thus

$$P(- \varepsilon < \frac{1}{B_n} \sum_{\ell=1}^{n} \xi_\ell < \varepsilon) \to \frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} e^{\frac{-z^2}{2}} \, dz < \frac{2\varepsilon}{\sqrt{2\pi}}$$

or $\lim_{n\to\infty} P\left(\frac{|\Sigma\, \xi_\ell|}{\frac{2n}{k}} < \varepsilon\right) \leq \frac{2\varepsilon}{\pi}$, from which the truth of the lemma follows.

<u>Lemma 2</u>: Given $\varepsilon > 0$ and any positive integer n, let

$T(n,\varepsilon) = \bigcup_{\substack{1\leq i,j\leq k \\ i\neq j}} T(n,\varepsilon,i,j)$. Then $\lim_{n\to\infty} \frac{|T(n,\varepsilon)|}{|\Sigma^{Vn}|} \leq k^2 \frac{2\varepsilon}{\sqrt{2\pi}}$

<u>Proof</u>: Let $P(T(n,\varepsilon))$ be the probability that a randomly chosen preference profile in Σ^{Vn} occur in $T(n,\varepsilon)$. Then $P(T(n,\varepsilon)) = \frac{|T(n,\varepsilon)|}{|\Sigma^{Vn}|}$. However,

$$P(T(n,\varepsilon)) = P(\bigcup_{\substack{1\leq i,j\leq k \\ i\neq j}} T(n,\varepsilon,i,j)) \leq \sum_{\substack{1\leq i,j\leq k \\ i\neq j}} P(T(n,\varepsilon,i,j))$$

which for sufficiently large $n \leq k^2 \frac{2\varepsilon}{\sqrt{2\pi}}$.

Then $\lim_{n\to\infty} \frac{|T(n,\varepsilon)|}{|\Sigma^{V_n}|} = \lim_{n\to\infty} P(T(n,\varepsilon)) \leq k^2 \frac{2\varepsilon}{\sqrt{2\pi}}$,

which completes the proof of the lemma.

Now, consider $\Sigma^{V_n} \setminus T(n,\varepsilon)$. It follows from Lemma 2 that

$$\lim_{n\to\infty} \frac{|\Sigma^{V_n} \setminus T(n,\varepsilon)|}{|\Sigma^{V_n}|} \geq 1 - \frac{k^2 2\varepsilon}{\sqrt{2\pi}} .$$

However, given any fixed $\varepsilon > 0$, for all sufficiently large n

$$\Sigma^{V_n} \setminus T(n,\varepsilon) \subset \mathcal{D}_n .$$

Thus, given any fixed $\varepsilon > 0$

$\lim_{n\to\infty} \frac{|\mathcal{D}_n|}{|\Sigma^{V_n}|} \geq 1 - \frac{k^2 2\varepsilon}{\sqrt{2\pi}}$, from which the theorem follows.

III CONCLUDING REMARKS

The theorem proved here indicates that the plurality rule (cum a reasonable tie-breaking device) is approximately cheatproof in large finite societies. Specifically, we have shown that as the number of individuals tends to infinity the proportion of profiles at which the plurality rule is cheatproof tends to one. This means that the issue of preference misrepresentation by any single voter in a large society can be ignored for all practical purposes when social choices are made according to the plurality rule. This of course should have been expected on purely intuitive grounds as any isolated individual does not really count when society is sizeable.

This brings us however to a more fundamental issue of incentive compatibility in social choice theory. Namely that the fact that a given method of social choice might be cheatproof under certain circumstances should not automatically be taken to imply that every individual can be expected to vote sincerely. Rather, the meaning of cheatproofness results in general should be taken to indicate that under most circumstances (profiles) the expression of preferences by any single individual makes really no difference at all. The general conclusion therefore ought to be that under any cheatproof social decision rule the only real incentive that the single individual might have is not to vote at all since his vote does not really count (independently of how many other voters there are).

This of course is obvious for imposed or dictatorial social choice rules, which are cheatproof by definition. Slightly less obvious, under democratic decision rules, at all profiles where a difference of one vote does not alter the outcome, the single voter can really have no incentive at all to vote. In general, for societies or committees, in which there are no less than three individuals, any majority or plurality rule will run into profiles at which nobody has an incentive to vote. While consistent with the definition of the cheatproofness requirement (as the absence of any incentive to vote implies in particular the absence of any incentive to cheat), such a phenomenon raises a serious difficulty from the viewpoint of the logical consistency of incentive compatibility in the theory of social choice. No attempt is made here to cope with this particular issue, but it seems to us that future efforts in the area of incentive compatible methods of social choice ought to be directed to this troublesome problem. Only after this issue is satisfactorily resolved will it be possible to assess the existing cheatproofness results at their true value.

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