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Abstract

We consider a general model of dynamic common agency with symmetric information. We focus on Markov perfect equilibria and characterize the equilibrium set for a refinement of the Markov perfect equilibria.

Particular attention is given to the existence of a marginal contribution equilibrium where each principal receives her contribution to the coalition of agent and remaining principals. The structure of the intertemporal payoffs is analyzed in terms of the flow marginal contribution. As a by-product, new results for the static common agency game are obtained.

The general characterization results are then applied to two dynamic bidding games for a common agent: (*i*) multi-task allocation and (*ii*) job matching under uncertainty.

KEYWORDS: Common agency, dynamic bidding, marginal contribution, markov perfect equilibrium, coalition-proof equilibrium, job matching, multi-task allocation.

JEL CLASSIFICATION: D81, D83

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1 Introduction

Common agency refers to a broad class of problems in which a single individual, the agent, controls a decision that has consequences for many individuals with distinct preferences. The other affected parties, the principals, may attempt to influence the agent with payments that are contingent on the action chosen by the agent. The static model of common agency under perfect information was introduced by Bernheim & Whinston (1986*a*) as a model of an auction where bidders are submitting a menu of offers to the auctioneer. Since then it gained prominence in many applications, such as procurement contracting, models of political economy (Dixit, Grossman & Helpman (1997) or Dixit (1996)), as well as strategic international trade (Grossman & Helpman (1994)).

The objective of this paper is to examine the structure of dynamic common agency problems. The extension of the model beyond the static version is of particular interest for the applications above. Political choices are rarely made only once, and the future implications of a current policy are often more important than its immediate repercussions. If the politician and the lobbyists cannot commit to future actions and transfers, a dynamic perspective is needed. Similarly, many procurement situations involve staged development with bidding occurring at each stage of the process.

The dynamic perspective also broadens the reach of common agency models. Consider for example a dynamic matching problem where the employee works in each period for at most a single employer, but may change employers over time. In the language of common agency, the employee has only one principal in each period, but with specific or general human capital the future employers may have preferences over the career path of the employee. Thus the intertemporal element introduces a more subtle aspect of common agency to the job matching problem.

We start our analysis with the static common agency model of Bernheim & Whinston (1986*a*), who concentrated on a refinement of Nash equilibrium, called *truthful equilibrium.* A strategy is said to be truthful relative to a given action if it reflects accurately the principals' willingness to pay for any other action relative to the given action. For the static game, we show that the truthful equilibrium is unique if and only if the marginal contributions of the principals to the value of the grand coalition are weakly superadditive. We show that in such equilibria, all principals receive their marginal contributions as payoffs. We call equilibria satisfying this property *marginal contribution equilibria.*

Marginal contribution equilibria are particularly attractive from a welfare point of view. Even if the actual game is preceded by an ex ante stage with actions that cannot be contracted upon, the resulting equilibrium in the overall game is socially efficient. As an example, one may think of a government agency choosing between two suppliers. The ex ante stage could consist of an opportunity for one of the suppliers to install a new cost reducing technology. If the common agency game fails to have such equilibria, then the incentives at the ex ante stage are misspecified, and the overall equilibrium need not be efficient. The second part of the paper derives conditions for the existence of a marginal contribution equilibrium in the dynamic framework. Since we assume that the players lack commitment power over periods, the interplay between the payoffs received at different stages of the game becomes important.

To illustrate the issues arising in the dynamic game, consider the following two period example. The agent can allocate an indivisible unit of time in each period to one of two potential projects. Each project is owned by a separate principal and yields a value of one to the principal if the agent works for the project in one of the periods. The project yields no additional value if the agent works in it for a second period. Assume that it is costless for the agent to work in either of the projects, but that he also has the option of refusing to work in a given period. With transferable utility, efficiency coincides with surplus maximization and therefore efficiency requires that the agent works for one period in each project. This yields an overall social value of 2. The marginal contribution of each principal (or her project) is easily seen to be 1 and therefore the sum of the marginal contributions does not exceed the marginal contribution of the union of the two principals. In other words, the marginal contribution is superadditive on the principals. This game, however, fails to have a marginal contribution equilibrium. To see this, notice that the agent would get a payoff of zero in any such equilibrium. But the agent can always guarantee a payoff of 1 by not working in the first period. In this case, the marginal contribution of both firms is zero in the second period, and therefore the agent will receive the entire surplus of 1 by standard Bertrand type arguments.

Notice that in this example, the agent can reduce the marginal contribution of both agents from 1 to zero by not working in the first period. This reduction in marginal contributions more than offsets the efficiency loss of 1 that deviating from the efficient path entails. We show that the intuition coming from this example is general in the sense that a marginal contribution equilibrium exists if and only if at each stage the welfare loss from choosing an inefficient action outweighs a possible reduction in future marginal contributions.

The dynamic model of the common agency is formulated with a general state space and can be equally well interpreted as a deterministic or a stochastic model. We concentrate on the Markov strategies since we want to study the effects of payoff relevant changes on the dynamics of the game in isolation from the effects created by conditioning on payoff irrelevant histories. In the spirit of Bernheim & Whinston (1986*a*), we are particularly interested in truthful Markovian policies. We also extend the concept of a coalition proof equilibrium to the dynamic context and note that the equivalence result between truthful and coalition proof equilibria in Bernheim & Whinston (1986*a*) carries over to the dynamic model: The set of thruthful Markov Perfect equilibria is outcome equivalent to the set of dynamic coalition-proof Markov Perfect equilibria.

Since we are focusing on a demanding concept, i.e. the marginal contribution equilibrium, it is important to know that the set of games that possesses such an equilibrium includes economically important models. To this end, we analyze a model of optimal scheduling of tasks when the reward from each task evolves according to a deterministic process. This model may represent the equilibrium utilization of an asset with finite capacity, such as a depleting natural resource, by competing users. The second example is a basic job matching problem à la Jovanovic (1979), where a number of firms learn about the firm specific human capital of a given worker. In each of these cases, the unique marginal contribution equilibrium is characterized.

These two examples also illustrate an additional reason to be interested in the existence of marginal contribution equilibria. In the job market example, for instance, one might argue that the present model allows for an unrealistically large number of possible transfers between the firms. In particular, wages paid to the worker when working in a different firm are rarely observed. We show that whenever a marginal contribution equilibrium exists in these models, a marginal contribution equilibrium also exists in a related game, where each principal's payment depends only on whether the agent worked for her. We show also that whenever a marginal contribution equilibrium fails to exist in the common agency model, the model with restricted transfer opportunities fails to have an efficient truthful equilibrium. This gives an additional reason for caution in interpreting the efficiency of equilibria that are not marginal contribution equilibria.

The papers that are the most closely related to the current paper are those by Bernheim & Whinston (1986*a*), Dixit, Grossman & Helpman (1997), and Laussel & LeBreton (1995), (1996). Our paper extends the static model of common agency in Bernheim & Whinston (1986*a*) to a dynamic setting. The second major point of departure relative is our insistence on marginal contributions equilibria as an interesting solution concept in this class of games. Another recent extension of the basic model of common agency may be found in Dixit, Grossman & Helpman (1997), where the assumption of quasi-linear preferences is dropped. Whereas the motivation for that extension is based on concerns relating to the distribution of the payoffs within the single period of analysis, our motivation is based on the distribution of payoffs between the players over time when commitment is precluded. The work by Laussel and LeBreton analyzes the payoffs received by the agent in a class of static common agency games.

The techniques developed in the paper are then applied to an intertemporal task allocation problem and to a job market matching model in the spirit of

Jovanovic (1979). We do not allow the firms to commit to future wages and as a result, we create a genuine surplus sharing problem between the firms and the worker. By considering a more general framework than Felli & Harris (1996) who treat the case of two potential employers, it becomes clear that marginal contributions equilibria are possible only if the job performance of the worker in one firm is essentially independent from that in others. In a related model, Bergemann & Välimäki (1996), we analyzed a duopolistic market for experience goods. Again, the general results in the current paper show the directions in which the earlier work may be extended fruitfully.

The paper is organized as follows. In Section 2 we introduce the common agency model in its dynamic version. The notion of marginal contribution is introduced here as well. Section 3 introduces the basic results for the static model of common agency. Section 4 presents the main results for the dynamic common agency. The characterization of the truthful Markov Perfect equilibrium is given here and necessary and sufficient conditions for its uniqueness are stated as well. In Section 5 the equivalence between truthful and dynamic coalition-proof equilibria is established. Section 6 illustrates the general results with two applications. The first is a model of dynamic task allocation and the second is a stochastic job matching model.

2 Model

2.1 Payoffs

We extend the common agency model of Bernheim & Whinston (1986a) to a dynamic setting. The set of players is the same in all periods, but actions available to them as well as payoffs resulting from the actions may change from period to period.

The principals are indexed by $i \in \mathcal{I} = \{1, ..., I\}$. Time is discrete and is denoted by $t = 0, 1, ..., T$, where T is finite or infinite. After each history h_t, the agent can select in each period an action $a_t \in A(h_t)$, where $A(h_t)$ is assumed to be a finite set for every h_t, and without loss of generality $n = |A(h_t)|$ for all h_t. Each principal i offers a reward scheme $r_i(a_t, h_t) \in \mathbb{R}_+^n$, which can depend on the history h_t and the action a_t chosen by the agent in period t. Let $r_t \triangleq (r_1(\cdot,\cdot), ..., r_I(\cdot,\cdot))$, $\mathbf{a} \triangleq (a_0, ..., a_t, ...)$ and $\mathbf{r} \triangleq (\mathbf{r}_0, ..., \mathbf{r}_t, ...)$. The future in period t is the sequence of future actions $(a^t; r^t) = (a_{t+1}, ..., r_{t+1}, ...)$. We denote by $H(h_t)$ the set of all possible histories h_{t+1} which are accessible from history h_t, and similarly $H(a_t, h_t)$ the set of all possible histories h_{t+1} generated by h_t and a_t.

We want the actions in all periods to be sequentially rational from all players' point of view. In other words, we do not allow the agent to commit to strings of actions and accordingly, current period payments depend only on current period actions.

The stage game is not necessarily stationary and the transition may be deterministic or stochastic. The payoff relevant state of the world (in the sense of Maskin & Tirole (1997)) in period t is θ_t. The cost of action a_t in period t to the agent is given by $c(a_t, \theta_t)$. The benefit to principal i is $v_i(a_t, \theta_t)$, which may again depend θ_t. The sum of the contributions by a subset of principals S$\subset \mathcal{I}$ is:

$$r_{\mathcal{S}}(a_t, h_t) \triangleq \sum_{i \in \mathcal{S}} r_i(a_t, h_t),$$

and the sum of the benefits is

$$v_{\mathcal{S}}(a_t, \theta_t) \triangleq \sum_{i \in \mathcal{S}} v_i(a_t, \theta_t).$$

The aggregate benefits are denoted by $v(a_t, \theta_t) \triangleq v_{\mathcal{I}}(a_t, \theta_t)$ and the aggregate rewards similarly by $r(a_t, h_t) \triangleq r_{\mathcal{I}}(a_t, h_t)$. Without loss of generality we shall assume that $v_i(a_t, \theta_t) \geq 0$ and $c(a_t, \theta_t) \geq 0$ for all a_t and θ_t. We also assume the existence of a (default) action $a_t \in A_t(\theta_t)$ such that $c(a_t, \theta_t) = 0$ for all θ_t.

The history of the game is $h_t \triangleq (a_0, ..., a_{t-1}, r_0, ..., r_{t-1}, \theta_0, ..., \theta_t)$. The transition function, $q(\theta_{t+1} | a_t, \theta_t)$ is assumed to be Markovian in the sense that the probability of the payoff relevant state being θ_{t+1} in period $t+1$ depends only on current actions, a_t and the current state θ_t. Let H_t be the set of all possible t period histories. All agents maximize expected discounted value and their common discount factor for future periods is δ.

2.2 Social Values

With transferable utility between the agent and the principals, Pareto efficiency coincides with surplus maximization. The value of the socially efficient program is denoted by

$$W(\theta_t) \triangleq W_{\mathcal{I}}(\theta_t),$$

and the value of the efficient program with a subset $\mathcal{S}$ of principals and the agent is denoted by $W_{\mathcal{S}}(\theta_t)$. These values are obtained from a familiar dynamic programming equation:

$$W_{\mathcal{S}}(\theta_t) = \max_{a_t \in A(\theta_t)} \mathbb{E}\left\{v_{\mathcal{S}}(a_t, \theta_t) - c(a_t, \theta_t) + \delta W_{\mathcal{S}}(\theta_{t+1})\right\}.$$

Similarly the value of a set of firms $\mathcal{I} \backslash \mathcal{S}$ is denoted by $W_{-\mathcal{S}}(\theta_t)$. In this game, it is relatively easy to assign values to coalitions other than the grand coalition. In all of the value calculations, we include the agent in the coalition of principals under study. The excluded set of principals cannot affect the value to the coalition under study and thus we avoid some of the usual problems in finding the characteristic function of a normal form game.

The marginal contribution of principal i is given by

$$M_i(\theta_t) \triangleq W(\theta_t) - W_{-i}(\theta_t). \tag{1}$$

The marginal contribution of a subset of principals $\mathcal{S} \subset \mathcal{I}$ to the value of the program is defined by:

$$M_{\mathcal{S}}(\theta_t) \triangleq W(\theta_t) - W_{-\mathcal{S}}(\theta_t). \tag{2}$$

In words, the marginal contributions of an individual principal or of a coalition of principals measure the increase in the total value of the grand coalition from adding a particular principal or a coalition of principals respectively.

3 Static Common Agency

This section presents the equilibrium concept and new characterization results for the static common agency game. The basic model and the equilibrium notions were first introduced by Bernheim & Whinston (1986*a*).

A strategy for principal i is a reward function $r_i : A \rightarrow \mathbb{R}_+$ by which the principal offers a reward to the agent contingent on the action chosen by him. The net benefit from action a to principal i is $n_i(a) \triangleq v_i(a) - r_i(a)$. The vector of net benefits is $\mathbf{n}(a) = (n_1(a), ..., n_I(a))$ and the aggregate benefits for a subset S is $n_S(a) = \sum_{i \in S} n_i(a)$. The net benefit to the agent is given by $r(a) - c(a)$.

Definition 1 (Best response)

1. *An action $\bar{a}$ is a best response to the rewards $r(\cdot)$ if*

$$\bar{a} \in \arg\max_{a \in A} r(a) - c(a).$$

2. *A reward function $r_i(\cdot)$ is best response to the rewards $r_{-i}(\cdot)$, if there does not exist another reward function $\hat{r}_i(\cdot)$ and action $\hat{a}$ such that*

$$v_i(\hat{a}) - \hat{r}_i(\hat{a}) > v_i(a) - r_i(a)$$

where a and $\hat{a}$ are best responses to $(r_i(\cdot), r_{-i}(\cdot))$ and $(\hat{r}_i(\cdot), r_{-i}(\cdot))$ respectively.

Definition 2 (Nash equilibrium) *A* Nash Equilibrium *of the common agency game is an n-tuple of reward functions $\{r_i^*(\cdot)\}_{i=1}^I$ and an action a^* s. th. $r_i^*(\cdot)$ and a^* are best responses.*

Definition 3 (Marginal Contribution equilibrium) *A* Marginal Contribution Equilibrium *of the common agency game is a Nash Equilibrium where $n_i(a^*) = M_i$ for all i.*

Bernheim and Whinston suggested that the focus be put on a subset of the Nash equilibria where all strategies satisfy an additional restriction, called truthfulness in Bernheim & Whinston (1986*a*). In addition, they showed that the set of thruthful equilibria is outcome equivalent with the set of coalition-proof equilibria. In this section, we shall restrict ourselves to the discussion of the thruthful equilibria and refer the reader to Bernheim & Whinston (1986*b*) and Bernheim, Peleg & Whinston (1987) for results on the equivalence between the two solution concepts. In section 5, we take up this equivalence again in the dynamic model.

Definition 4 (Truthful strategy)

1. *A reward function $r_i(\cdot)$ is said to be thruthful relative to a_0 if and only if for all $a \in A$, either*

   (a) $n_i(a) = n_i(a_0)$, *or,*

   (b) $n_i(a) < n_i(a_0)$, *and* $r_i(a) = 0$.

2. *The strategies* $\left\{ \{r_i^*(\cdot)\}_{i=1}^{I}, a^* \right\}$ *are said to be a Truthful Nash Equilibrium if and only if it is a Nash Equilibrium and* $\{r_i^*(\cdot)\}_{i=1}^{I}$ *are truthful strategies relative to* a^*.

A truthful strategy by player i reflects accurately the relative value of two actions to i unless the nonnegativity constraint on the reward function is binding. The set of Truthful Nash Equilibrium can then be characterized by a set of inequalities, relating the social value of the grand coalition to the social value of smaller coalitions. Bernheim & Whinston (1986*b*) show that the set $\mathcal{E}_{\mathcal{I}}$ of equilibrium net payoffs for the principals is described by

$$\mathcal{E}_{\mathcal{I}} = \left\{ \mathbf{n} \in \mathbb{R}^I \;\middle|\; \begin{array}{c} (i)\ \forall \mathcal{S} \subset \mathcal{I},\ n_{\mathcal{S}} \le W - W_{-\mathcal{S}} \\ (ii)\ \mathbf{n} \ge \mathbf{n}',\ \text{for all } \mathbf{n}' \text{ satisfying } (i). \end{array} \right\}. \tag{3}$$

Note that we are not including the state variable θ_t as an argument for W and $W_{-\mathcal{S}}$ since we are analyzing the static model for the moment. Next we present two equivalent conditions for the uniqueness of the truthful equilibrium.

Theorem 1 (Uniqueness)

1. *The truthful equilibrium payoff vector is unique iff*

$$\forall \mathcal{S} \subset \mathcal{I},\ \sum_{i \in \mathcal{S}} M_i \leq M_{\mathcal{S}}. \tag{4}$$

2. *The unique truthful equilibrium is a marginal contribution equilibrium.*

3. *A sufficient condition for uniqueness is that $M_{\mathcal{S}}$ is superadditive:*

$$\forall \mathcal{S}, \mathcal{T}, \mathcal{S} \cap \mathcal{T} = \emptyset,\ M_{\mathcal{S}} + M_{\mathcal{T}} \leq M_{\mathcal{S} \cup \mathcal{T}}. \tag{5}$$

Proof. See Appendix. ■

Condition (4) requires that the sum of the marginal contributions of each firm $i \in \mathcal{S}$ to $\mathcal{I}$ is less than the marginal contribution of the entire set $\mathcal{S}$ to $\mathcal{I}$. Condition (4) is referred to as *weakly superadditive*. The superadditivity condition (5) is a sufficient condition for (4) and it agrees with (4) if $|\mathcal{I}| = 2$. We have the following corollary relating truthful equilibria to strong equilibria.[1]

Corollary 1 *If the truthful equilibrium is unique, then it is also a strong equilibrium.*

Laussel & LeBreton (1995) also consider the structure of the equilibrium payoffs in the static common agency. The main objective of their paper is to determine conditions under which the agent does not receive positive rents in equilibrium. They also present the following sufficient condition for the uniqueness of the truthful equilibrium: If the social values are strongly subadditive, i.e. $\forall \mathcal{S}, \mathcal{T} \subset \mathcal{I}$, such that $\mathcal{S} \cap \mathcal{T} = \emptyset$,

$$W \leq W_{-\mathcal{S}} + W_{-\mathcal{T}} - W_{-(\mathcal{S} \cup \mathcal{T})},$$

then the truthful equilibrium is unique. It can be verified that the strong subadditivity of the social values is identical to the superadditivity of the marginal contributions as stated in Proposition 1. A sufficient condition for both in turn

[1] See Aumann (1959) for the definition of strong equilibrium.

is the concavity of the social values, which is defined as follows: $\forall \mathcal{S}, T \subset \mathcal{I}$ such that $\mathcal{S} \subset T$ and $i \notin T$,

$$W_{T \cup i} - W_T \leq W_{\mathcal{S} \cup i} - W_{\mathcal{S}}.$$

The following example shows that strongly subadditive is a stronger condition than weak superadditivity of the marginal contributions:

$$\begin{array}{cccc} & a_1 & a_2 & a_3 \\ c(\cdot) & -3 & -3 & -2 \\ v_1(\cdot) & 0 & 3 & 3 \\ v_2(\cdot) & 4 & 3 & 0 \\ v_3(\cdot) & 4 & 3 & 3 \end{array} \tag{6}$$

The example satisfies weak superadditivity of the marginal contributions, but fails superadditivity as $M_2 + M_{13} > W$, as well as $M_3 + M_{12} > W$.

4 Dynamic Common Agency

The equilibrium of the dynamic common agency game is defined in Subsection 4.1. The general characterization is given in Subsection 4.2, where the notion of marginal contribution in the dynamic setting is explored in some detail. Necessary and sufficient conditions for the uniqueness of the truthful equilibrium are given in Subsection 4.3.

4.1 Truthful Equilibrium

In the dynamic game, a reward strategy for principal i is a sequence of mappings

$$r_i : A_t \times H_t \to \mathbb{R}^n_+$$

which assigns to every possible action $a_t \in A_t$ of the agent a nonnegative reward, possible contingent on the entire past history of the game. A strategy by the agent is a sequence of actions over time

$$a : \mathbb{R}^n_+ \times H_t \to A_t$$

which depend on the aggregate reward in period t and history until period t. If the strategies do not depend on the entire history of the game, but only on the state θ_t, then the strategies are called Markov strategies. Indeed in the following the equilibrium analysis is restricted to strategies which are contingent only on the payoff-relevant history of the common agency game. The reason for this modeling choice is twofold. First, we want to study the effects of changes in the payoff relevant states of the world in isolation from the purely strategic effects that arise from conditioning on a payoff irrelevant past. Second, if the transfers between each principal and the agent are not observable to outsiders, it may be hard to find continuation payoffs that allow for a richer dependence on the past. In Bergemann & Valimaki (1998), we show that in a repeated dynamic common agency game with transfers that are not observable to outsiders, only Markovian equilibria, i.e. repetitions of stage game survive as truthful equilibria in the repeated game. Hence the restriction to Markovian strategies may be seen as a simple model yielding the same restrictions as a more complicated model with imperfect observability.

The expected discounted payoff with a history h_t for a given sequence of reward policies $\mathbf{r}$ and action profiles $\mathbf{a}$ is denoted by $V_0(h_t)$ for the agent and $V_i(h_t)$ for principal i. When $\mathbf{a}$ and $\mathbf{r}$ are Markov policies, then the values are given $V_0(\theta_t)$ and $V_i(\theta_t)$ if the state is θ_t in period t. In this context $\mathbb{E}V_i(a_t, \theta_t)$ represents the expectation of the continuation value in period $t+1$ if in period t the action was a_t and the state was θ_t. While the transition from θ_t to θ_{t+1} may be stochastic, we shall omit the expectations operator $\mathbb{E}[\cdot]$ for simplicity and all values are henceforth understood to represent expected values.

Definition 5 (Markov Perfect Equilibrium)
The strategies $\{r_i^*(a_t,\theta_t)\}_{i\in\mathcal{I}}$ *and* $a^*(r(\cdot),\theta_t)$ *form a Markov Perfect Equilibrium (MPE) if*

1. $\forall\theta_t, \forall r(\cdot)$, $a^*(r(\cdot),\theta_t)$ *is a solution to*

$$\max_{a_t\in A_t}\{r(a_t,\theta_t)-c(a_t,\theta_t)+\delta V_0(a_t,\theta_t)\},$$

2. $\forall i, \forall\theta_t$, *there is no other reward function* $\hat{r}_i(a_t,\theta_t)$ *such that*

$$v_i(\hat{a}_t,\theta_t)-\hat{r}_i(\hat{a}_t,\theta_t)+\delta V_i(\hat{a}_t,\theta_t)>v_i(a_t^*,\theta_t)-r_i^*(a_t^*,\theta_t)+\delta V_i(a_t^*,\theta_t),$$

where a^* *and* $\hat{a}$ *are best response actions to* $(r_i^*(\cdot),r_{-i}^*(\cdot))$ *and* $(\hat{r}_i(\cdot),r_{-i}^*(\cdot))$ *respectively.*

Truthful strategies are defined as in the static game by the property that they reflect correctly each principal's net willingness to pay. The major difference to the static definition is that the allocation relative to which truthfulness is defined is now an action a_t *and* a state θ_t. The intertemporal net benefit $n_i(a_t,\theta_t)$ of an allocation a_t in the state θ_t is the flow benefit $v_i(a_t,\theta_t)-r_i(a_t,\theta_t)$ and the continuation benefit $\delta V_i(a_t,\theta_t)$:

$$n_i(a_t,\theta_t)\triangleq v_i(a_t,\theta_t)-r_i(a_t,\theta_t)+\delta V_i(a_t,\theta_t). \tag{7}$$

With this extension to the dynamic framework, the definition of a truthful (Markov) strategy and an associated MPE in truthful strategies is immediate.

Definition 6 (Truthful (Markov) strategy)

1. *A reward function* $r_i(a_t,\theta_t)$ *is said to be thruthful relative to* (a,θ_t) *if and only if for all* $a_t\in A(\theta_t)$, *either*

   (a) $n_i(a_t,\theta_t)=n_i(a,\theta_t)$, *or,*

   (b) $n_i(a_t,\theta_t)<n_i(a,\theta_t)$, *and* $r_i(a_t,\theta_t)=0$.

2. *The strategies* $\{r_i^*(\cdot)\}_{i=1}^I$ *and* $a^*(r(\cdot),\theta_t)$ *are said to be a Markov Perfect Equilibrium in truthful strategies if and only if it is a Markov Perfect equilibrium and* $\{r_i^*(\cdot)\}_{i=1}^I$ *are truthful strategies relative to* $a^*(\cdot)$.

4.2 Characterization

The characterization of the set of truthful equilibria relies as in the static model on the marginal contribution of each principal. The marginal contribution of principal i is, as defined earlier,

$$M_i(\theta_t) = W(\theta_t) - W_{-i}(\theta_t). \tag{8}$$

It can be decomposed into the flow and the future contribution along the *efficient path*:

$$M_i(\theta_t) = m_i(\theta_t) + \delta M_i(a, \theta_t). \tag{9}$$

Here $M_i(a, \theta_t)$ is the marginal contribution in period $t+1$, which depends on the state θ_t and the *efficient action* a in period t. The socially optimal allocation in state θ_t is denoted by $a \triangleq a_{\mathcal{I}}$ if the set $\mathcal{I}$ of principals is present, and by $a_{-\mathcal{S}}$ if only a subset $\mathcal{I}\backslash\mathcal{S}$ of principals participate in the game. As the future marginal contribution is given by

$$M_i(a, \theta_t) = W(a, \theta_t) - W_{-i}(a, \theta_t),$$

we can identify the flow contribution by (8) and (9) as:

$$\begin{aligned} m_i(\theta_t) \quad = \quad & (v(a, \theta_t) - c(a, \theta_t)) - (v_{-i}(a_{-i}, \theta_t) - c(a_{-i}, \theta_t)) \\ & + \delta W_{-i}(a, \theta_t) - \delta W_{-i}(a_{-i}, \theta_t). \end{aligned} \tag{10}$$

The flow contribution $m_i(\theta_t)$ can be decomposed into two elements. The first component is the difference in the social flows generated by the intertemporally optimal action relative to $\mathcal{I}$ and $\mathcal{I}\backslash\{i\}$, respectively.. The second component reflects the intertemporal aspect. Action a by the agent today precludes action a_{-i} which is more favorable to $\mathcal{I}\backslash\{i\}$. The extent to which this affects the future payoffs to $\mathcal{I}\backslash\{i\}$ is reflected by $\delta W_{-i}(a_{-i}, \theta_t) - \delta W_{-i}(a, \theta_t)$ which has to be attributed to the flow contribution of i today, since the coalition $\mathcal{I}\backslash\{i\}$ will not be able to recover this difference in the future. The flow contribution $m_i(\theta_t)$ is not identical to the contribution of i were we to consider to add or remove i only in period t.

If the optimal action in state θ_t is independent of the presence of principal i in the *entire* game and $a = a_{-i}$, then $m_i(\theta_t)$ reduces to

$$m_i(\theta_t) = v_i(a, \theta_t). \tag{11}$$

In contrast, if the addition of i influences the efficient allocation and $a \neq a_{-i}$, then

$$\begin{aligned}(v_{-i}(a, \theta_t) - c(a, \theta_t)) - (v_{-i}(a_{-i}, \theta_t) - c(a_{-i}, \theta_t)) \\ +\delta W_{-i}(a, \theta_t) - \delta W_{-i}(a_{-i}, \theta_t) < 0,\end{aligned}$$

and it follows that

$$m_i(\theta_t) < v_i(a, \theta_t). \tag{12}$$

Inequality (12) states that if the addition of principal i leads to a change in the action relative to the optimal action with the set of principals $\mathcal{I} \backslash \{i\}$, then the flow contribution of i is less than the gross benefit to principal i, $v_i(a, \theta_t)$, from action a. It is easy to extend the characterization of the flow marginal contribution in (10) for $\mathcal{S} = \{i\}$ to any larger subset $\mathcal{S}$ of principals, or

$$\begin{aligned} m_{\mathcal{S}}(\theta_t) &= (v(a, \theta_t) - c(a, \theta_t)) - (v_{-\mathcal{S}}(a_{-\mathcal{S}}, \theta_t) - c(a_{-\mathcal{S}}, \theta_t)) \\ &\quad +\delta W_{-\mathcal{S}}(a, \theta_t) - \delta W_{-\mathcal{S}}(a_{-\mathcal{S}}, \theta_t). \end{aligned} \tag{13}$$

The relationship between the flow marginal contribution $m_{\mathcal{S}}(\theta_t)$ and the flow payoff $v_{\mathcal{S}}(a, \theta_t)$ is analogous to the one derived for the individual principal i in (11) and (12):

$$\begin{aligned} m_{\mathcal{S}}(\theta_t) = v_{\mathcal{S}}(a, \theta_t) &\Leftrightarrow a = a_{-\mathcal{S}}, \\ m_{\mathcal{S}}(\theta_t) < v_{\mathcal{S}}(a, \theta_t) &\Leftrightarrow a \neq a_{-\mathcal{S}}. \end{aligned}$$

The social value which a set $\mathcal{S}$ of principals can achieve in equilibrium is limited by the value which the complementary set $\mathcal{I} \backslash \mathcal{S}$ can realize by itself. In the static world this value was achieved by selecting the action $a_{-\mathcal{S}}$ which maximizes the flow payoffs $v_{-\mathcal{S}}(a) - c(a)$ to the remaining principals $\mathcal{I} \backslash \mathcal{S}$ in conjunction with the agent. In the dynamic setting, the set $\mathcal{I} \backslash \mathcal{S}$ cannot commit itself to exclude the participation of the set $\mathcal{S}$ forever. But the set $\mathcal{I}\backslash\mathcal{S}$ can seek to maximize the joint payoff

$$\sum_{i\notin\mathcal{S}}\left(v_i\left(a_t,\theta_t\right)+\delta V_i\left(a_t,\theta_t\right)\right)-c\left(a_t,\theta_t\right)+\delta V_0\left(a_t,\theta_t\right)$$

for given equilibrium continuation payoffs $V_i\left(a_t,\theta_t\right)$ induced by selecting action a_t in period t. To this effect, we define $W\left(\theta_t\left|a_t\right.\right)$ to be the social value of the program which starts with an arbitrary and not necessarily efficient action a_t, but thereafter chooses an intertemporally optimal action profile. Similarly, let $M_i\left(\theta_t\left|a_t\right.\right)=W\left(\theta_t\left|a_t\right.\right)-W_{-i}\left(\theta_t\left|a_t\right.\right)$.

The next theorem shows that any truthful equilibrium has to be efficient. Since the equilibrium continuation play is also efficient, the value of the subset $\mathcal{I}\backslash\mathcal{S}$ is maximized along the equilibrium path by selecting a_t so as to solve

$$\max_{a_t}\left\{W\left(\theta_t\left|a_t\right.\right)-v_{\mathcal{S}}\left(a_t,\theta_t\right)-\sum_{i\in\mathcal{S}}\delta V_i\left(a_t,\theta_t\right)\right\}. \tag{14}$$

As a result, the maximal value the set $\mathcal{S}$ of principals can extract from the remaining players is given by the difference between $W\left(\theta_t\right)$ and the maximand of (14). If the members of $\mathcal{S}$ were receiving more than the difference, there would be a coalitional deviation available for $\mathcal{I}\backslash\mathcal{S}$. Since we have fixed the continuation play, we can appeal to the equivalence result between coalition proof equilibria and truthful equilibria in Bernheim & Whinston (1986*a*) to obtain the same restriction for truthful equilibria. The net value $n_{\mathcal{S}}\left(\theta_t\right)$ of the set $\mathcal{S}$ of principals in truthful equilibrium must then satisfy:

$$n_{\mathcal{S}}\left(\theta_t\right)\leq W\left(\theta_t\right)-\max_{a_t}\left\{W\left(\theta_t\left|a_t\right.\right)-v_{\mathcal{S}}\left(a_t,\theta_t\right)-\sum_{i\in\mathcal{S}}\delta V_i\left(a_t,\theta_t\right)\right\}.$$

The maximal value $n_{\mathcal{S}}\left(\theta_t\right)$ which can be secured by coalition $\mathcal{S}$ can be related to its marginal contribution in period t and its equilibrium continuation values in period $t+1$. By relating the equilibrium continuation values $V_i\left(\theta_t\right)$ recursively to the marginal contributions $M_i\left(\theta_t\right)$, we obtain the following:

Theorem 2 (Efficiency)

1. *All Markov equilibria in truthful strategies are efficient.*

2. *For all i and $\mathcal{S}$,*

$$\sum_{i\in\mathcal{S}} V_i(\theta_t) \leq M_{\mathcal{S}}(\theta_t). \tag{15}$$

Proof. See Appendix. ■

The equilibrium characterization in the static game permitted an additional result relating the equilibrium payoffs to the marginal contributions. Namely, for all i, there is a set $\mathcal{S}$, with $i \in \mathcal{S}$ such that the joint equilibrium payoff of the set $\mathcal{S}$ of principals equals their marginal contribution, or

$$\sum_{i\in\mathcal{S}} V_i = M_{\mathcal{S}}.$$

This lower bound on the equilibrium payoffs for any set $\mathcal{S}$ of principals is no longer valid in the dynamic environment as the example given in the introduction illustrates.

4.3 Marginal Contribution Equilibrium

The dynamic common agency game preserves the efficiency of the static game. The intertemporal aspects of the game weaken the position of the principals as neither an individual principal nor any group of principals can receive their marginal contribution in general. In this section, we give necessary and sufficient conditions for a marginal contribution equilibrium to exist. It is useful to note first that the equivalence between the uniqueness of the truthful equilibrium and the marginal contribution equilibrium is still valid in the dynamic game.

Theorem 3 (Uniqueness) *The truthful equilibrium payoff vector is unique if and only if the equilibrium is a marginal contribution equilibrium.*

Proof. See Appendix. ■

The equivalence result is particularly interesting for the equilibrium analysis as the description of the marginal contributions is in many settings a much simpler task than the explicit derivation of the equilibrium strategies which support the marginal contribution payoffs. This will be illustrated further with the applications in Section 6. There are two obvious candidates for necessary conditions for the marginal contribution equilibrium. The first is the weak superadditivity of the marginal contribution $M_i(\theta_t)$ and the second is the weak superadditivity of the flow marginal contribution $m_i(\theta_t)$:

$$\sum_{i \in \mathcal{S}} m_i(\theta_t) \leq m_{\mathcal{S}}(\theta_t), \quad \forall \mathcal{S}, \forall \theta_t.$$

The second condition is obviously stronger as it implies the weak superadditivity of the marginal contribution since

$$M_{\mathcal{S}}(\theta_t) = \sum_{\tau=t}^{\infty} \delta^{\tau} m_{\mathcal{S}}(\theta_\tau)$$

holds for all $\mathcal{S} \subset \mathcal{I}$.

Theorem 4 (Necessary Conditions)
If a marginal contribution equilibrium exists, then

$$\sum_{i \in \mathcal{S}} M_i(\theta_t) \leq M_{\mathcal{S}}(\theta_t), \quad \forall \mathcal{S}, \forall \theta_t.$$

Proof. See Appendix. ■

The weak superadditivity of $M_i(\theta_t)$ is however not a sufficient condition for the marginal contribution equilibrium as the two period example in the introduction illustrates. The marginal contribution $M_i(\theta_t)$ does not track sufficiently precise how the distribution of the future surplus is affected by the current decision of the agent. This suggest that if a marginal contribution equilibrium is to exist, then the current decision by the agent should not be motivated too strongly by his interest to depress the future shares of the principals relative to the shares along the efficient path. Since the agent is the residual claimant after the principals receive their marginal contribution, a formal statement of this requirement is that the social loss from a deviation from the efficient policy exceeds the loss in the marginal contributions of the principals, or

$$W(\theta_t | a_t) - W(\theta_t) \leq \sum_{i \in \mathcal{S}} (M_i(\theta_t | a_t) - M_i(\theta_t)), \; \forall a_t \in A_t, \; \forall \mathcal{S}.$$

Recall that $W(\theta_t | a_t)$ is the social value of the program which starts with an arbitrary action a_t, but thereafter chooses an intertemporally optimal action profile, and if follows that $W(\theta_t | a_t) - W(\theta_t) < 0$ for all $a_t \neq a$.

Theorem 5 (Existence)

The marginal contributions equilibrium exists if and only if

$$\sum_{i \in \mathcal{S}} (M_i(\theta_t) - M_i(\theta_t | a_t)) \leq W(\theta_t) - W(\theta_t | a_t), \ \forall a_t \in A_t, \ \forall \mathcal{S}. \qquad (16)$$

Proof. See Appendix. ■

The equilibrium characterization by the inequality (16) is quite powerful in applications. Since all values entering the inequality can be obtained from appropriate efficient (continuation) programs, the inequality can be established independetly of any equilibrium considerations. As efficient programs are in general easier to analyze than dynamic equilibrium conditions, the technique suggested here may be usefully applied to a wide class of dynamic bidding models. The applications presented in section 6, while of interest in their own, also serve to illustrate this point. In the case of a repeated common agency game, condition (16) reduces to the condition of weak superadditivity of the marginal contributions in the static game as the transition from period t to $t+1$ is of course independent of the action chosen in period t. It should also be pointed out that the weak superadditivity of the flow contributions as obtained from the efficient program:

$$\sum_{i \in \mathcal{S}} m_i(\theta_t) \leq m_{\mathcal{S}}(\theta_t)$$

is neither a necessary nor sufficient condition for the marginal contribution equilibrium. The example in the introduction provides an example where the flow marginal contributions are additive, but a marginal contribution equilibrium fails to exist. To see that the weak superadditivity of flow marginal contributions is not necessary either, consider the following example. The agent can choose in each period between two actions, which have zero cost to the agent. If the agent chooses a_1 in the first period, then the continuation payoff is given by the gross payoff matrix (17), if he chooses a_2 it is (18). For simplicity, the action in the first period doesn't provide any direct payoffs and only determines the payoff matrix in the next period. The gross payoffs to the principals are:

$$\begin{array}{cccc} & v_1(\cdot) & v_2(\cdot) & v_3(\cdot) \\ a_{11} & 2 & 2 & 1 \\ a_{12} & 0 & 0 & 0 \end{array} \tag{17}$$

and

$$\begin{array}{cccc} & v_1(\cdot) & v_2(\cdot) & v_3(\cdot) \\ a_{21} & 5 & 0 & 0 \\ a_{22} & 0 & 5 & 0 \end{array} \tag{18}$$

The flow contributions in the first period are $m_{\mathcal{S}} = 0$ for all $\mathcal{S} = \{i\}$, but $m_{12} = -1$. However, there is now a marginal contribution equilibrium where the agent chooses action a_1 in the first period. Thus the weak subadditivity of the flow contributions is not a necessary condition either.

The inequality (16) which is both a necessary and a sufficient condition suggests however an interpretation in terms of a modified notion of the flow marginal contribution. For any state θ_t and for all allocations a_t in period t, define $\forall \mathcal{S}, \forall \theta_t$:

$$\hat{m}_{\mathcal{S}}(\theta_t) \triangleq \left\{ W(\theta_t) - \sum_{i \in \mathcal{S}} \delta M_i(a, \theta_t) \right\} - \max_{a_t} \left\{ W(\theta_t | a_t) - v_{\mathcal{S}}(a_t, \theta_t) - \sum_{i \in \mathcal{S}} \delta M_i(a_t, \theta_t) \right\}.$$

as the *dynamic flow contribution* of a subset $\mathcal{S}$ of principals. Notice that for $\mathcal{S} = \{i\}$, we have $m_i(\theta_t) = \hat{m}_i(\theta_t)$. However for a set $\mathcal{S}$ with $|\mathcal{S}| > 1$, the equality fails. Since

$$\begin{aligned} m_{\mathcal{S}}(\theta_t) &= (v(a, \theta_t) - c(a, \theta_t)) - (v_{-\mathcal{S}}(a_{-\mathcal{S}}, \theta_t) - c(a_{-\mathcal{S}}, \theta_t)) \\ &\quad + \delta W_{-\mathcal{S}}(a, \theta_t) - \delta W_{-\mathcal{S}}(a_{-\mathcal{S}}, \theta_t), \end{aligned}$$

we have in general

$$m_{\mathcal{S}}(\theta_t) \neq \hat{m}_{\mathcal{S}}(\theta_t).$$

The example above illustrates how $m_{\mathcal{S}}$ and $\hat{m}_{\mathcal{S}}$ may differ, but since $m_i(\theta_t) = \hat{m}_i(\theta_t)$, we know that in the special case of two principals, the weak superadditivity of the flow contributions is in fact a necessary condition. Notice that the main difference between the flow marginal contribution, $m_{\mathcal{S}}(\theta_t)$, and the dynamic flow contribution, $\hat{m}_{\mathcal{S}}(\theta_t)$ lies in the differential treatment of the continuation values to the coalition. While $m_{\mathcal{S}}(\theta_t)$ attributes the entire future marginal contribution of coalition $\mathcal{S}$ to its members, $\hat{m}_{\mathcal{S}}(\theta_t)$ attributes only the sum of individual marginal contributions. In general these two will be different, and the latter will be relevant for the characterization of a marginal contribution equilibrium. With the dynamic flow contribution, we can formulate an alternative condition for the uniqueness of the truthful equilibrium more in the spirit of the static condition.

Corollary 2 *A truthful equilibrium is a marginal contribution equilibrium if and only if* $\forall \theta_t, \forall \mathcal{S}$:

$$\sum_{i \in \mathcal{S}} \hat{m}_i(\theta_t) \leq \hat{m}_{\mathcal{S}}(\theta_t). \tag{19}$$

Proof. See Appendix. ■

5 Coalition-Proof Equilibrium

In this section we show the equivalence between the thruthful equilibria characterized in Section 4 and coalition-proof equilibria. The static notion of coalition-proof Nash equilibrium is introduced in Bernheim, Peleg & Whinston (1987), who also suggest a dynamic extension, called perfectly coalition-proof Nash equilibrium. We adopt a slightly different notion which captures the idea of coalition-formation over time in a perhaps more natural way. We briefly comment on the differences after stating the definitions. Naturally, a dynamic notion of coalition-proofness faces similar challenges as any notion of renegotiation-proofness. As renegotiation-proofness, it is relatively unambiguous in finite horizon games, and more open for challenge in infinite horizon games. The finite horizon version of the concept was introduced by Ferreira (1996) under the name of communication-proof equilibrium.

In the static common agency game, consider any subset $\mathcal{S}$ of players, including the agent, and a collection of strategies, $\{r_i(\cdot)\}_{i \notin \mathcal{S}}$. We define the component game $\Gamma_{\mathcal{S}}$ relative to $\{r_i(\cdot)\}_{i\notin\mathcal{S}}$ as:

$$\Gamma_{\mathcal{S}} \triangleq \Gamma/\{r_i(\cdot)\}_{i\notin\mathcal{S}} = \left\{\mathcal{S}, \{\{v_i(\cdot)\}_{i\in\mathcal{S}}, c(\cdot)+r_{-\mathcal{S}}(\cdot)\}, \left\{\{r_i(\cdot)\}_{i\in\mathcal{S}\setminus 0}, a\right\}\right\}.$$

The component game is the restriction of the original game to the principals in $\mathcal{S}$ and the agent, holding the strategies $\{r_i(\cdot)\}_{i\notin\mathcal{S}}$ of the remaining principals fixed.

Definition 7 (Coalition-Proof Nash Equilibrium)

1. *In a common agency game with a single principal,* $\{r_1(\cdot), a\}$ *is a coalition proof Nash equilibrium if and only if it is a Nash equilibrium.*

2. *In a common agency game with* $|\mathcal{I}| > 1$ *assume that a coalition proof Nash equilibrium has been defined with fewer than* $|\mathcal{I}|$ *principals. Then,*

   (a) $\left\{\{r_i(\cdot)\}_{i\in\mathcal{I}}, a\right\}$ *is self-enforcing if for all* $\mathcal{S} \subsetneq \mathcal{I}$, $\left\{\{r_i(\cdot)\}_{i\in\mathcal{S}}, a\right\}$ *is a coalition-proof equilibrium in the component game* $\Gamma_{\mathcal{S}}$

   (b) $\left\{\{r_i(\cdot)\}_{i\in\mathcal{I}}, a\right\}$ *is a coalition-proof equilibrium if it is a self-enforcing Nash equilibrium and the equilibrium net payoffs* $\mathbf{n}_{\mathcal{S}}$ *are not Pareto dominated by any other self-enforcing Nash equilibrium.*

Denote by $\Gamma(h_t)$ the finite horizon game starting with history h_t. The dynamic coalition-proof equilibrium is defined by induction on τ with $t = T - \tau$. Continuation strategies in period t are denoted by (a^t, r^t).

Definition 8 (Dynamic Coalition-Proof Equilibrium: $T < \infty$)

1. *Define the set* $\mathcal{E}(h_T)$ *as the set of all coalition-proof equilibrium profiles of the stage game* $\Gamma(h_T)$.

2. *Suppose the set* $\mathcal{E}(h_s)$ *has been defined for all* $h_s \in H_s$ *with* $s > t$. *The set of dynamic coalition-proof equilibria at* h_t *in period* t *is the set of strategy profiles* $\{a_t, r_t(\cdot)\}$ *such that* $\{a_t, r_t(\cdot);\ a^t, r^t(\cdot)\}$ *is a coalition proof equilibrium when* $(a^t, r^t) \in \mathcal{E}(h_{t+1})$.

3. *A strategy profile* $\{a^0, r^0(\cdot)\}$ *is a dynamic coalition-proof equilibrium if* $\{a^0, r^0(\cdot)\} \in \mathcal{E} = \mathcal{E}(h_0)$.

The dynamic extension defines coalition-proofness recursively by an induction on the number of remaining periods τ. In period $t = T - \tau$, the strategy space for any coalition of players is formed by the current actions and any sequence of future actions which in itself has to be part of a coalition-proof equilibrium in the continuation game. Any coalitional deviation is hence required to pick continuation strategies which are immune to deviations in the continuation game with all players. In contrast the extension suggested by Bernheim, Peleg & Whinston (1987) is based on an induction across coalitions and time periods simultaneously. It implies in particular that the self-enforceability of a coalition is verified in the entire subgame without allowing the subset to join after one period. In finite horizon games of perfect information a dynamic coalition-proof equilibrium always exists and the equilibrium set is a subset of the set of subgame perfect equilibria. In contrast, Peleg (1992) shows that a perfectly coalition-proof equilibrium does not always exists even in games of perfect information.

The extension of dynamic coalition-proofness to games with an infinite horizon is based on recursive consistency as well. Consider a subgame perfect equilibrium profile $(\mathbf{a}, \mathbf{r})$. The set $\mathcal{E}$ now denotes the set of continuation profiles of $(\mathbf{a}, \mathbf{r})$ for all possible histories $h_t \in H_t$ for all t. We denote by $\mathcal{E}(\theta_t)$ the set of continuation profiles in state θ_t.

Definition 9 (Dynamic Coalition-Proof Equilibrium: $T = \infty$)
A strategy profile $(\mathbf{a}, \mathbf{r})$ *is a dynamic coalition-proof equilibrium if for all continuation profiles* $(\mathbf{a}, \mathbf{r}) = \{(a_0, r_0), (\mathbf{a}^0, \mathbf{r}^0)\}$ *there exists no other profile* $(\hat{\mathbf{a}}, \hat{\mathbf{r}}) = \{(\hat{a}_0, \hat{r}_0), (\hat{\mathbf{a}}^0, \hat{\mathbf{r}}^0)\}$ *such that*

1. $(\hat{\mathbf{a}}^0, \hat{\mathbf{r}}^0) \in \mathcal{E}$,

2. $(\hat{\mathbf{a}}, \hat{\mathbf{r}})$ *is a coalition-proof equilibrium with* $(\mathbf{a}^0, \mathbf{r}^0) \in \mathcal{E}$,

3. $V_i(\hat{\mathbf{a}}, \hat{\mathbf{r}}) \geq V_i(\mathbf{a}, \mathbf{r})$ *for all* i *with at least one strict inequality.*

The notion of dynamic coalition-proof for the infinite horizon is weak in the sense that it only refers to the set generated by an equilibrium profile itself, and may be regarded as an *internal consistency* requirement. For games with two players, the notion is almost equivalent to the notion of weak renegotiation proof equilibrium by Farell & Maskin (1989) and consistency by Bernheim & Ray (1989). For two players, our notion is slightly stronger as we require that the set of continuation profiles $\mathcal{E}$ should not allow the players to generate an equilibrium which is Pareto improving and only uses the stage game strategies and continuation strategies from the set $\mathcal{E}$ itself.[2] In two player, finite horizon games it is equivalent to the notion of consistency by Bernheim & Ray (1989) and similar notions for finitely repeated games.

Theorem 6 (Equivalence in Markov Strategies)

1. *For every truthful equilibrium, there exists an equivalent dynamic coalition-proof equilibrium.*

2. *Every dynamic coalition-proof equilibrium in Markov strategies is payoff-equivalent (in every state θ_t) to a truthful equilibrium.*

Proof. See Appendix. ■

The qualification to payoff-equivalent is necessary as there are coalition-proof equilibria in Markov strategies in which the strategies are not equivalent to any truthful equilibrium, but the difference is not payoff relevant as it concerns contribution to actions off the equilibrium path. In the earlier section we chose to define truthful strategies only with respect to the current allocation and the payoff relevant state of the world. If we were to generalize the notion of truthful with respect to entire history, then the equivalence statement in Theorem 6 could be made generally and without the Markovian restriction. The proof would proceed similarly.

[2] Ray (1989) suggests a stricter condition of internally renegotiation-proof by going beyond the consideration of current and continuation payoffs, to set of all self-generating payoffs.

6 Applications

In this section we present two applications of the dynamic common agency framework. In Subsection 6.1 an agent can process a finite number of tasks in each period. The principals compete for the services of the agent and their aggregate demand exceeds the total number of tasks the agent can perform in a single period. The principals then try to influence the scheduling of the task by the agent through the transfer payments. In Subsection 6.2 we consider job matching under uncertainty and show how common agency arises from intertemporal considerations.

6.1 Multi-Task Allocation

Consider an agent who offers his services over time to many principals. Each principal has a finite or infinite number of projects she wishes to complete. The realization of each project requires the services of the agent. The agent can supervise at most n projects in every period. The supervision of any one of the projects is costless for him, as long as the total number of projects under supervision does not exceed n per period. The agent in the model may be thought to have an asset with a capacity constraint in his possession. As a consequence the agent has to decide the order in which he services the competing principals. Such a situation arises frequently in the context of outsourcing of services. Consider for example an agent with a computing facility, such as a supercomputer, and several principals are bidding for computing time on the facilities. Another example arises in deregulated electricity markets where the distributors bid in spot markets for electricity by a given supplier.

The undiscounted value of any particular project of principal i is denoted by $v_i(T_i) \geq 0$, where T_i is indexing the projects of each principal. The projects are ordered inversely by the index T_i, i.e. $v_i(T_i) \geq v_i(T_i + 1)$ for all i and all T_i. We refer to this set-up as the decreasing returns model. Denote by $T_i(t)$ the number of services provided to principal i until and including period t. The state θ_t in period t is then simply the vector of counting times

$$\theta_t = (T_1(t-1), T_2(t-1), ..., T_I(t-1)).^3 \tag{20}$$

An allocation policy is given by

$$a : \Theta \to A,$$

where the action associates to every state θ a subset of principals whose projects are realized in t, which in turn generates a new vector $\theta_{t+1} = (T_1(t), T_2(t), ..., T_I(t))$. The agent is free to supervise more than one project of any given principal at any given period. An allocation policy $a(\cdot)$ is socially efficient if it maximizes the discounted payoff over time:

$$\max_{a(\cdot)} \sum_{t=0}^{\infty} \delta^t \left(\sum_{i=1}^{I} \sum_{T_i = T_i(t-1)+1}^{T_i(t)} v_i(T_i) \right).$$

The socially optimal policy in this environment is simply to select in every period among the remaining tasks the n tasks which yield the highest payoff.

We characterize the equilibrium of the scheduling game using techniques suggested by Theorem 3 and 5. The optimal policy assigns every project T_i by principal i its rank τ in the sequence of all realized projects:

$$\tau_i : \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \tag{21}$$

where $\tau_i(T_i) = \infty$ means that project T_i of principal i is never realized by the optimal policy. The assignment τ_i for every i implies that the value v_τ of a project of rank τ is equal to $v(T_i)$ if and only if $\tau_i(T_i) = \tau$:

$$v_\tau = v_i(T_i) \Leftrightarrow \tau_i(T_i) = \tau.$$

In the following it will be more convenient to use the one-dimensional order τ and the associated value v_τ induced by the optimal path, rather than the I

[3] The identification is unique if, as we assume, principal i can only realize the projects in the order specified earlier. This is without loss of generality as it is optimal, both privately and socially to first realize the project with lowest remaining T_i.

dimensional vector of counting times $(T_1, ..., T_I)$. Since the agent can employ up to n projects in each period, the index τ runs at n times the speed of real time. To write the value functions using the ranks given by τ but respecting the true discounting, we need to introduce the function $t(\tau)$ that gives the real time period in which the project with rank τ is employed under the allocation policy that always chooses the n projects with the lowest rank. To this end, let

$$t(\tau) = \max\left\{t \middle| t \in \mathbb{N},\ t < \frac{\tau}{n}\right\}.$$

The social value under the new clock τ is then given by

$$W(\tau) = \sum_{s=\tau}^{\infty} \delta^{t(s)-t(\tau)} v_s. \tag{22}$$

Define the counter $T_i(\tau)$ to keep track of how many times principal i has obtained the services of the agent under this new and faster clock τ:

$$T_i(\tau) \triangleq \{\#s \,|v_s = v_i(\cdot),\ s \leq \tau.\ \}.$$

Associated with each counter $T_i(\tau)$, is an accelerated counter:

$$\sigma_i(\tau) \triangleq \min\{s\,|s - T_i(s) = \tau\}.$$

The counter $\sigma_i(\tau)$ associates with to τ and v_τ another time $\sigma_i(\tau)$ and corresponding allocation $v_{\sigma_i(\tau)}$. The counter $\sigma_i(\tau)$ accelerates τ by excluding principal i from the set of alternatives. It is needed when calculating the marginal contribution of principal i. The counters $T_i(\tau)$ and $\sigma_i(\tau)$ can be localized by starting them at an arbitrary $\gamma > 0$:

$$T_i(\tau\,|\gamma) \triangleq \{\#s\,|v_s = v_i(\cdot),\ \gamma \leq s \leq \tau.\ \},$$

and

$$\sigma_i(\tau\,|\gamma) \triangleq \min\{s\,|s - T_i(s\,|\gamma) = \tau\}. \tag{23}$$

Consider then the optimal policy in the absence of principal i. As the rewards are decreasing over time, the optimal policy is simply to accelerate the original policy. By scheduling alternative $\sigma_i(\tau)$ instead of τ, the value of the remaining program is

$$W^{-i}(\tau) = \sum_{s=\tau}^{\infty} \beta^{t(s)-t(\tau)} v_{\sigma_i(s|\tau)}.$$

The marginal contribution of principal i is given by

$$M_i(\tau) = \sum_{s=\tau}^{\infty} \beta^{t(s)-t(\tau)} \left(v_s - v_{\sigma_i(s|\tau)}\right), \tag{24}$$

and the flow marginal contribution is

$$m_i(\tau) = \sum_{s=\tau}^{\infty} \beta^{t(s)-t(\tau)} \left(v_{\sigma_i(s|\tau+1)} - v_{\sigma_i(s|\tau)}\right), \tag{25}$$

with the convention that

$$v_{\sigma_i(\tau|\tau+1)} = v_\tau.$$

Consider first the situation where $n = 1$ and each principal has a single project requiring the assistance of the agent. With $n = 1$, we have $t(\tau) = \tau$ for all τ. The socially optimal arrangement is to order the principals according to the value of their project. We identify principal i with the time τ_i at which the project is realized optimally. Since each principal has only one project we have $\sigma_i(s|\tau+1) = s$ and $\sigma_i(s|\tau) = s+1$ for all $s > \tau$ if $\tau = \tau_i$. The flow marginal contribution can then simply be written as

$$m_i(\tau) = \sum_{s=\tau_i}^{\infty} \beta^{s-\tau_i} (v_s - v_{s+1}),$$

if $\tau = \tau_i$, and zero otherwise. The immediate contribution of principal i is the difference in value between v_{τ_i} and the next best project v_{τ_i+1}. But the availability of project v_{τ_i} also allows to postpone all future realizations by exactly one period. As these benefits only occur in the future, they are appropriately discounted, but they have to be attributed to principal i in τ_i, since after its realization in τ_i its benefit is sunk. The marginal contribution of principal i is then simply $M_i(\tau) = \beta^{\tau_i-\tau} m_i(\tau_i)$ for $\tau \leq \tau_i$, $M_i(\tau) = 0$ for $\tau > \tau_i$.

The question is then whether principal i is able to realize her marginal contribution in equilibrium. By Theorem 5, this is equivalent to satisfying the inequality

$$W(\tau) - W(\tau\,|k) \geq \sum_{i \in \mathcal{S}} (M_i(\tau) - M_i(\tau\,|k)). \tag{26}$$

Without loss of generality, assume $\tau = 0$. The realization of task k in period 0 entails delaying the efficient plan by one period until period k . After k, the two plans are identical. The net change in the social value is thus:

$$W(0) - W(0\,|k) = (1-\beta) \sum_{s=0}^{k-1} \beta^s v_s - \left(1 - \beta^k\right) v_k \tag{27}$$

The change in the marginal contribution for principal i is due to the postponement of its realization, and we have

$$M_i(0) - M_i(0\,|k) = (1-\beta)\,\beta^{\tau_i} \sum_{s=\tau_i}^{k-1} \beta^s (v_s - v_{s+1}). \tag{28}$$

The early realization of v_k has no private benefit to principal i. Notice that the marginal contribution of principal i with $\tau_i > k$ is not affected by the change from the optimal policy to the modification by k, as the time of its realization remains unchanged. Since $M_i(0) - M_i(0\,|k) \geq 0$, the inequality (26) is most demanding for $\mathcal{S} = \{0, 1, ..., k-1\}$. After inserting (27) and (28) and dividing by $(1-\beta)$ we can write (26) as:

$$\sum_{s=0}^{k-1} \left(\beta^s v_s - \beta^k v_k\right) \geq \sum_{\tau_i=0}^{k-1} \sum_{s=\tau_i}^{k-1} \beta^s (v_s - v_{s+1}). \tag{29}$$

The lhs of (29) expresses for every v_s with $s < k$, the value difference between v_s and v_k in the optimal program:

$$\beta^s v_s - \beta^k v_k \geq 0.$$

The rhs also presents for every v_s a differential expression between v_s and v_k, but it proceeds in steps $v_\tau - v_{\tau+1}$ which accumulate less value as they are increasingly discounted. This reflects the value difference between v_s and v_k in terms of the marginal contribution. Since the marginal contribution only picks up the inframarginal differences in every period, it follows directly that the inequality (26) holds.

The equilibrium payoffs are computed with the assistance of the marginal contribution. Principal i receives in τ_i his marginal contribution $m_i(\tau_i)$. The agent receives in τ_i the residual:

$$\begin{aligned} v_{\tau_i} - m_i(\tau_i) &= v_{\tau_i+1} - \sum_{s=\tau_i+1}^{\infty} \delta^{s-\tau_i}(v_s - v_{s+1}) \\ &= (1-\delta)\sum_{s=\tau_i+1}^{\infty} \delta^{s-\tau_i} v_s, \end{aligned}$$

which is the value of the next best project minus the future marginal contribution of this alternative. Equivalently, it is the average flow value of the sequence of all future projects. The truthful equilibrium strategies are also represented with the assistance of the marginal contribution. We restrict ourselves to the on-the equilibrium-path strategies and the state θ_t can simply be represented by time t itself. Consider first principal i who already realized her project along the equilibrium path and $\tau_i < t$. Her future payoff is $M_i(t+1\,|u) = 0$ for all current allocations u, not necessarily equal to t, and hence the requirement for a truthful strategy is $r_i(t,t) = r_i(u,t)$ and the only best response is to set the rewards $r_i(t,t) = r_i(u,t) = 0$. Consider then any principal who along the equilibrium path realizes their project after t, or $\tau_i > t$. A truthful strategy requires that

$$-r_i(u,t) + \beta M_i(t+1\,|u) \le -r_i(t,t) + \beta M_i(t+1\,|t), \text{ for } u \neq \tau_i,$$

and

$$v_{\tau_i} - r_i(\tau_i,t) \le -r_i(t,t) + \beta M_i(t+1\,|t), \text{ for } u = \tau_i.$$

For all $u < \tau_i$, we showed earlier that $M_i(t+1\,|u) = M_i(t+1\,|t)$, and hence $r_i(u,t) = r_i(t,t)$, and the only best response is again to set $r_i(u,t) = r_i(t,t) = 0$. Finally, the direct reward offered to the agent for the realization of project i is given by

$$r_i(\tau_i,t) = v_{\tau_i} - \beta^{\tau_i - t} m_i(\tau_i), \tag{30}$$

in which principal i offers the agent the entire difference between her marginal contribution $m_i(\tau_i)$ which she receives in τ_i and the value of the project today. It follows that in equilibrium, each principal offers transfers only for the realization of her own project, and no other transfers. The rewards offered by (30) allow us to verify that the agent is in equilibrium indifferent only between projects t and $t+1$, and the indifference is resolved in equilibrium in favor of project t. Thus, while the principals can offer general reward schemes, in equilibrium they offer non-trivial rewards only for the realization of their own projects. It follows that the marginal contribution equilibrium is also an equilibrium in a more restricted bidding game where each principal can only bid for services provided directly to her. We summarize the results for the general model.

Theorem 7

1. *The dynamic scheduling game has a unique truthful equilibrium.*
2. *Each principal i receives in period t:*

$$\sum_{\forall\{\tau|t(\tau)=t,T_i(\tau)\neq T_i(\tau-1)\}} m_i(\tau)$$

3. *The agent receives in period t:*

$$\sum_{\{\tau|t(\tau)=t\}}\left(v_{\tau}-\sum_{i\in\mathcal{I}}\sum_{\{\tau|T_i(\tau)\neq T_i(\tau-1)\}} m_i(\tau)\right)$$

Proof. See Appendix. ■

The extension to many principals with many tasks faces at least one potential difficulty, which we briefly illustrate. With a single project, each individual principal receives the marginal contribution of the project as her payoff. When one principal has multiple tasks, the marginal contribution of all her tasks generally exceeds the sum of the marginal contributions of the individual tasks. Consider, e.g. a model with two principals, one with a single task of value 1, the other with two task of values 2 and 1, respectively. The marginal contributions of the projects are then δ^2, $1+\delta^2$ and δ^2 respectively, but the marginal contribution of the second principal is $1+\delta+\delta^2>1+2\delta^2$. The first principal is willing to pay the agent $1-\delta^2$ in the first period. In the unique truthful equilibrium of this game, the second firm pays only $1-\delta$, but the agent accepts this lower payment since scheduling the first firm's project would make it impossible to obtain any future payments as firm 2 would face no competing projects and thus extracts all surplus. By scheduling the second firm's task first, the agent makes sure that she receives a payment of $1 - \delta$ in the next period as well. Thus, even though the marginal contributions of the projects of principal i do not satisfy weak superadditivity, a marginal contributions equilibrium exists.

We conjecture that the theorem can be generalized to an arbitrary and not necessarily decreasing return model, as long as the returns $v_i(T_i)$ for principal i depend only the history of her own projects. The extension would rely on techniques to be introduced in the next example.

6.2 Job Matching

The second application we consider is a model of job matching under uncertainty as first introduced by Jovanovic (1979). Each principal i has an employment opportunity for the agent. The productivity of the agent in the job offered by principal i is $v_i(\theta_t)$. The state variable θ_t may represent the information in period t about the agent's productivity in different jobs. While the agent is employed by at most one principal at any point of time, he may change his employers over time as more information about the quality of the various matches becomes available. We refer to the assignments over time as the career path of the agent. The agent may have a variety of talents, and for this reason future employers may have preferences over his current employer. This may in turn prompt future employers to influence the agent's current assignment by offering rewards conditional on his current choice. The following example may illustrate the interaction between current and future employers.

There are three principals with whom the agent can work and two states of the world. In ω_1 the productivity of the agent is $(5, 4, 0)$ and in state ω_2 it is $(5, 5, 7)$. State ω_1 is as likely as state ω_2 and suppose that $\delta = 1/2$. The productivity of the agent is thus initially known with principal 1, but unknown with principal 2 and 3. However, after the first match with either principal

2 or 3, the true state of the world is revealed and the productivity with both principals becomes known. The efficient career path is for the agent to first work with principal 2 and then switch either to principal 1 or principal 3 conditional on the information arising with the experience of the match in the first period. The match with principal 2 is as informative as the match with principal 3, but as the expected return with principal 2 is higher, the optimal match in the initial period is with principal 2. The marginal contribution is $M_i(0) = \frac{1}{2}$, but due to the correlation $M_{23}(0) = \frac{1}{2}$ as well. Thus a marginal contribution equilibrium cannot exist. The failure of the marginal contributions to be weakly superadditive leads to multiple equilibria. The following contributions form the set of truthful equilibrium schedules in period 0:

$$\begin{aligned} r_1(\cdot, 0) &= \left(4\frac{3}{4}, 0, 0\right) \\ r_2(\cdot, 0) &= (0, 4 + x, 0) \\ r_3(\cdot, 0) &= \left(0, 1 - x, 4\frac{1}{2} - x\right) \end{aligned}$$

for any $x \in \left[0, \frac{1}{2}\right]$. For every x, the principals 2 and 3 achieve their joint marginal contribution, and for the extreme points of the interval, either principal 2 or 3, respectively also obtain their individual marginal contribution, but at the expense of squeezing the surplus of the other principal to 0.

Suppose next that we would restrict attention to contribution schemes in which transfers only occur in concurrence with match formation. We refer to transfers satisfying this restriction as spot wages. Then the only equilibrium using truthful strategies is the match formed between the agent and the first principal. In this equilibrium all principals offer wages $r_i(i, t) = 4\frac{1}{2}$ and the agent selects the first principal in every period.

This example suggests that the existence of a marginal contribution equilibrium requires independence in the information structure across the different principals. We shall call jobs *independent* if (i) the state variable θ_t can be represented by a vector $\theta_t = \left(\theta_t^1, ..., \theta_t^I\right)$, (ii) the productivity in job i satisfies $v_i\left(\hat{\theta}_t\right) = v_i(\theta_t)$ for all $\hat{\theta}_t$ with $\hat{\theta}_t^i = \theta_t^i$ and (iii) $\theta_t^j = \theta_{t+1}^j$ if the agent works with principal i in period t for all $j \neq i$. The optimal solution of the matching problem with independent jobs can be characterized by dynamic allocation indices developed in the theory of multi-armed bandits (Whittle (1982)). The independence structure implies in particular that the order in which the agent should pass through the (remaining) jobs is not affected by the removal of any particular alternative j. As a consequence, the flow contribution of all but the matched alternative are zero.

Theorem 8

1. *The stochastic job matching model has a unique truthful equilibrium.*
2. *The employing principal i receives: $m_i(\theta_t) = v_i(\theta_t) - r_i(i, \theta_t)$ and all other principals receive $m_j(\theta_t) = 0$.*
3. *The agent receives only spot wages: $r(i, \theta_t) = r_i(i, \theta_t)$.*

Proof. See Appendix. ■

With independent jobs a marginal contribution equilibrium exists, and due to the structure of the flow marginal contribution, it can even be sustained by spot wages. The flow marginal contribution of the efficient employer i is

$$m_i(\theta_t) = v_i(\theta_t) - v_{j_{-i}}(\theta_t) + \delta W_{-i}(i, \theta_t) - \delta W_{-i}(j_{-i}, \theta_t),$$

where j_{-i} is the efficient match in the absence of i. The flow contribution of any inefficient employer j is

$$m_j(\theta_t) = v_i(\theta_t) - v_i(\theta_t) + \delta W_{-j}(i, \theta_t) - \delta W_{-j}(i, \theta_t) = 0,$$

since the absence of j doesn't change the optimality of the match with i in θ_t. The payoff for employer i therefore consists of two parts: (i) the productivity difference between the job offered by i and the next best job j_{-i}, and (ii) the intertemporal component: $\delta W_{-i}(i, \theta_t) - \delta W_{-i}(j_{-i}, \theta_t)$. With independent jobs $W_{-i}(i, \theta_t) = v_{j_{-i}}(\theta_t) + \delta W_{-i}(j_{-i}, \theta_t)$, since more information about i is without value when considering the optimal match structure for all jobs but i. Since the value functions $W(\theta_t)$ and $W_{-i}(\theta_t)$ are convex in θ_t due to the value of information, it follows that $\delta W_{-i}(i, \theta_t) - \delta W_{-i}(j_{-i}, \theta_t) < 0$ and hence the intertemporal component represents the opportunity cost for the coalition $\mathcal{I} \backslash i$ to match with i rather than with j_{-i}. If the set $\mathcal{I}$ contains only two alternatives $\{i, j\}$, $\delta W_{-i}(i, \theta_t) - \delta W_{-i}(j, \theta_t) = 0$ for all θ_t, and the flow contribution reduces to the static productivity differences.[4] We can conclude that the only transfers arising in equilibrium are the direct payments from current employer to the agent. The marginal contribution equilibrium is therefore also a subgame perfect equilibrium of a game where no other transfers but wage payments are permitted.

In this context, the common agency framework can be regarded as technical device to analyze dynamic games with a Bertrand type pricing structure. The analysis of the job matching model with I alternatives is rather intractable as pricing policies (here wage offers) depend on the continuation values over which very little can be said in general. In contrast, the detour via general reward schemes allows us to first establish efficiency. The intertemporal structure of the marginal contributions leads in a second step to the conclusion that simple Bertrand prices (here spot wages) are sufficient to form the equilibrium with the structure established in the first step.

In the context of the job matching model, we conjecture that the order independence in the match formation, which lead to at most one nontrivial flow marginal contribution being different from zero, is also a necessary condition for a marginal contribution equilibrium to exist. Similarly, we expect any equilibrium in spot wages to be inefficient without order independence in the alternatives.

[4] This result is first stated in Bergemann & Välimäki (1996) and for a continuous model in Felli & Harris (1996).

7 Conclusion

This paper considered common agency in a general class of dynamic games with symmetric information. By focusing on Markovian equilibria, a detailed characterization of the equilibrium strategies and payoffs was possible for this class of games. As in the static analysis by Bernheim & Whinston (1986*a*), the link between truthful strategies and the social value of various coalitions was central in obtaining the results. In the dynamic context the link is even more valuable. The continuation payoffs which determine the current bidding strategies, are themselves endogenous to the equilibrium and hence of little help in determining the equilibrium strategies. In contrast, the marginal contributions are defined independently of equilibrium considerations.

As the "first price" menu auction represents a model of competitive bidding over a general set of possible allocations, the model presented here is sufficiently general to accommodate various dynamic bidding games. Even if the set of feasible transfers permitted by the menu offers is considered too large for some applications, the menu offers may provide an essential tool for solve for an equilibrium with transfers of considerably smaller dimension, as exemplified with the job matching model.

The analysis is less complete in investigating the notion of coalition-proofness. In the static environment, the class of coalition-proof equilibria coincides with the notion of truthful strategy. In a dynamic setting, coalition-proofness is basically an extension of renegotiation-proofness to more than two players by requiring the equilibrium to be renegotiation-proof to stable coalitions. As there is no general agreement on the appropriate notion of renegotiation-proofness even in the two player case, we use an extension of the weakest notion of renegotiation-proofness as defined by Farell & Maskin (1989). An open question in this context is how different non-Markovian equilibria are from Markovian, when we insist on coalition-proofness. Since common agency games have transferable utility, all intertemporal utility transfers can take place in a single period. As coalition-proofness leads to efficient allocations relative to the set of feasible allocations, the two facts together may point to an equivalence result between Markovian and non-Markovian coalition-proof equilibria in dynamic games with transferable utility.

We restricted our analysis to symmetric information environments. Bernheim & Whinston (1986*a*), however, observed that in the static context with two bidders for a single good, the principals net payoffs are equivalent to the equilibrium net payoffs of the Groves-Clarke-Vickrey mechanism with incomplete information. Recent work by Dasgupta & Maskin (1998) showed how the Groves-Clarke-Vickrey mechanism can be extended to many goods. It is an open question for future research to what extent the equivalence in the payoffs between the first price menu auction and the auction mechanism under incomplete information can be extended to sequential bidding problems with incomplete information. In this context, it should be noted that the asymmetry of information between the principals is to be distinguished from the analysis of Bernheim & Whinston (1986*b*) or Martimort (1996), where moral hazard or adverse selection is due to a better informed agent.

8 Appendix

This section contains proofs to all theorems and propositions stated in the main body of the paper.

Proof of Theorem 1.1. ($\Leftarrow$) Suppose that $\sum_{i\in\mathcal{S}} M_i \leq M_{\mathcal{S}}$. Set $n_i = M_i$ for all i, and by hypothesis $n_{\mathcal{S}} \leq M_{\mathcal{S}}$. Moreover, lowering n_i for a subset $\mathcal{S}$, with $i \in \mathcal{S}$ doesn't permit the increase of any other n_j, $j \notin \mathcal{S}$, as $n_j = M_j$ is a binding constraint and hence uniqueness follows.

($\Rightarrow$) We prove the contrapositive: If for some $\mathcal{S} \subset \mathcal{I}$, $\sum_i M_i > M_{\mathcal{S}}$, then the equilibrium allocation is not unique. It is convenient to distinguish two different cases: (i) for all i, $\mathcal{S}$ with $i \in \mathcal{S}$, $M_i \leq M_{\mathcal{S}}$, and (ii) for some i, $M_i > M_{\mathcal{S}}$. We start with (i) and construct two distinct equilibria with the greedy algorithm. Define $n_1 \triangleq M_1$, and in general

$$n_i \triangleq \min_{\{\mathcal{S};i\in\mathcal{S}\}} M^i_{\mathcal{S}},$$

where

$$M^i_{\mathcal{S}} = M_{\mathcal{S}} - \sum_{j\in\mathcal{S},j<i} n_j,$$

then one can verify that the induced allocation $\{n_1, n_2, ..., n_I\}$ is an equilibrium allocation, with, by hypothesis, $n_{i'} < M_{i'}$, for some $i' > 1$. Consider next a permutation $\sigma : \mathcal{I} \rightarrow \mathcal{I}$ such that $i' \mapsto 1$. By applying the greedy algorithm to the new ordering, we again obtain an equilibrium allocation, but clearly $n_{\sigma(i')} = M_{\sigma(i')}$ which is distinct from the previous allocation. The case of (ii) is even easier. Suppose without loss of generality that for $i = 1$, $M_i > M_{\mathcal{S}}$. Then set

$$n_1 = \min_{\{\mathcal{S},1\in\mathcal{S}\}} M_{\mathcal{S}},$$

and for the remaining allocations apply the greedy algorithm as before. Then there necessarily exists some $i' > 1$ with $n_{i'} = 0$. By a similar permutation σ as before, $\sigma : \mathcal{I} \rightarrow \mathcal{I}$, and $\sigma(i') = 1$, we obtain $r_{\sigma(i')} > 0$, and have hence again obtained a distinct equilibrium allocation.

2. The characterization $n_i = M_i$ follows directly from the proof of 1.

3. It is sufficient to prove that (5) is a sufficient condition for (4). Consider sets $\mathcal{S}, \mathcal{T} \subset \mathcal{I}$, and suppose that (5) holds, then we have for any $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, $\mathcal{S} \cap \mathcal{T} = \emptyset$,

$$M_{\mathcal{S}_1} + M_{\mathcal{S}_2} \leq M_{\mathcal{S}} \text{ and thus}$$

$$M_{\mathcal{S}_1} + M_{\mathcal{S}_2} + M_{\mathcal{T}} \leq M_{\mathcal{S} \cup \mathcal{T}}.$$

As we continue to split up $\mathcal{S}$ and $\mathcal{T}$ until we have coalitions consisting of single principals on the left hand side, we obtain

$$\sum_{i \in \mathcal{S} \cup \mathcal{T}} M_i \leq M_{\mathcal{S} \cup \mathcal{T}},$$

which completes the claim. ■

Proof of Theorem 2.1. By the assumption of Markovian strategies, the continuation values for the agent and the principals depend only on the action a_t inducing the transition from θ_t to θ_{t+1}. This implies by Theorem 2 and 3 of Bernheim & Whinston (1986*a*) efficiency.

2. The equilibrium value function $V_i(\theta_t)$ of principal i are required to satisfy the following set of equalities, $\forall i$,

$$\begin{aligned} V_i(\theta_t) \quad \leq \quad & \max_{a_t} \left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\} \\ & - \max_{a_t} \left\{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \neq i} \delta V_k(a_t, \theta_t) \right\} \end{aligned} \tag{31}$$

and inequalities $\forall \mathcal{S} \subset \mathcal{I}$,

$$\begin{aligned} \sum_{i \in \mathcal{S}} V_i(\theta_t) \quad \leq \quad & \max_{a_t} \left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\} \\ & - \max_{a_t} \left\{ v_{-\mathcal{S}}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \notin \mathcal{S}} \delta V_k(a_t, \theta_t) \right\}, \end{aligned} \tag{32}$$

Since all thruthful equilibria are efficient by 1. we have the identity:

$$W(\theta_t) = \max_{a_t} \left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\}.$$

Next we argue by contradiction. Suppose the inequality (15) doesn't hold for some $\mathcal{S}$, but the inequalities (31) and (32) are still satisfied. Then there $\exists \varepsilon > 0$ such that

$$\sum_{i \in \mathcal{S}} V_i(\theta_t) - M_{\mathcal{S}}(\theta_t) > \varepsilon, \tag{33}$$

and a fortiori

$$W(\theta_t) - \max_{a_t} \left\{ v_{-\mathcal{S}}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \notin \mathcal{S}} \delta V_k(a_t, \theta_t) \right\} - M_{\mathcal{S}}(\theta_t) > \varepsilon,$$

or equivalently

$$W(\theta_t) - \max_{a_t} \left\{ v_{-\mathcal{S}}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \notin \mathcal{S}} \delta V_k(a_t, \theta_t) \right\} - m_{\mathcal{S}}(\theta_t) - \delta M_{\mathcal{S}}(a, \theta_t) > \varepsilon, \tag{34}$$

with $m_{\mathcal{S}}(\theta_t)$ as defined in (13). Since the inequality in (34) holds for the maximizing a_t in (34), it has to hold for $a_{-\mathcal{S}}$ as well, so that (34) may be rewritten in this instance as

$$\delta W(a, \theta_t) - \sum_{k \notin \mathcal{S}} \delta V_k(a_{-\mathcal{S}}, \theta_t) - \delta W_{-\mathcal{S}}(a, \theta_t) + \delta W_{-\mathcal{S}}(a_{-\mathcal{S}}, \theta_t) - \delta M_{\mathcal{S}}(a, \theta_t) > \varepsilon, \tag{35}$$

and since

$$\delta W(a, \theta_t) = \delta W_{-\mathcal{S}}(a, \theta_t) + \delta M_{\mathcal{S}}(a, \theta_t),$$

it follows from (35) that

$$W_{-\mathcal{S}}(a_{-\mathcal{S}}, \theta_t) - \sum_{k \notin \mathcal{S}} V_k(a_{-\mathcal{S}}, \theta_t) > \frac{\varepsilon}{\delta},$$

which is equivalent to

$$\sum_{i \in \mathcal{S}} V_i(a_{-\mathcal{S}}, \theta_t) - M_{\mathcal{S}}(a_{-\mathcal{S}}, \theta_t) > \frac{\varepsilon}{\delta}. \tag{36}$$

But by repeating the argument, which we started at (33), it then follows that the equilibrium value for the set $\mathcal{S}$ of principals increases without bound along some path $(\theta_t, \theta_{t+1}, ...)$ which delivers the contradiction as the value of the game is finite. ■

Proof of Theorem 3. ($\Rightarrow$) If the truthful equilibrium is unique, then the inequalities which present an upper bound on the equilibrium value of each individual principals must be satisfied as equalities for all i and all θ_t, or

$$V_i(\theta_t) = W(\theta_t) - \max_{a_t} \left\{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \neq i} \delta V_k(a_t, \theta_t) \right\}. \quad (37)$$

Suppose then in equilibrium at least one principal receives a value less than his marginal contribution, or $\exists \varepsilon > 0$ such that

$$M_i(\theta_t) - V_i(\theta_t) > \varepsilon. \quad (38)$$

We now argue that there must be some action a_t such that the discrepancy between marginal contribution and equilibrium value increases such that

$$M_i(a_t, \theta_t) - V_i(a_t, \theta_t) > \frac{\varepsilon}{\delta}.$$

By the hypothesis (38), it must be that

$$\max_{a_t} \left\{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \neq i} \delta V_k(a_t, \theta_t) \right\} - V_{-i}(\theta_t) > \varepsilon.$$

Suppose in all continuation games following θ_t and an action a_t we have

$$V_i(a_t, \theta_t) = M_i(a_t, \theta_t) - \frac{\varepsilon}{\delta},$$

then

$$a_{-i} \in \arg\max_{a_t} \left\{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \neq i} \delta V_k(a_t, \theta_t) \right\}$$

and the following equality holds:

$$v_{-i}(a_{-i}, \theta_t) - c(a_{-i}, \theta_t) + \sum_{k \neq i} \delta V_k(a_{-i}, \theta_t) - V_{-i}(\theta_t) = \varepsilon.$$

It then follows immediately, that there must be at least one action a_t such that

$$M_i(a_t, \theta_t) - V_i(a_t, \theta_t) > \frac{\varepsilon}{\delta}.$$

By repeating the argument, we then come to the conclusion that the marginal contributions of principal i grows without bounds, since $V_i(a_t, \theta_t) \geq 0$. But this is a contradiction to the fact that the value of the game is bounded.

($\Leftarrow$) Suppose the marginal contribution equilibrium is a truthful equilibrium, but it is not the unique truthful equilibrium. It then follows that for some θ_t and some $\mathcal{S}$, the inequality

$$\sum_{i\in\mathcal{S}} V_i(\theta_t) \leq W(\theta_t) - \max_{a_t}\{v_{-S}(a_t,\theta_t) - c(a_t,\theta_t) + \delta W(a_t,\theta_t) - \delta V_i(a_t,\theta_t)\}, \tag{39}$$

holds as an equality and $\exists i \in S$ such that $V_i(\theta_t) < M_i(\theta_t)$. By Theorem 2, $V_i(\theta_t) > M_i(\theta_t)$ is impossible. By hypothesis a marginal contribution equilibrium exists and hence continuation values $V_i(a_t,\theta_t) = M_i(a_t,\theta_t)$ would satisfy (39) as a strict inequality:

$$\sum_{i\in\mathcal{S}} V_i(\theta_t) < W(\theta_t) - \max_{a_t}\left\{v_{-S}(a_t,\theta_t) - c(a_t,\theta_t) + \delta W(a_t,\theta_t) - \delta\sum_{i\in\mathcal{S}} M_i(a_t,\theta_t)\right\},$$

It then follows that $\exists a_t \in A_t$ and $\varepsilon > 0$ such that the continuation value of the principals $V_i(a_t,\theta_t)$ for $i \in \mathcal{S}$ satisfy

$$\sum_{i\in\mathcal{S}} M_i(a_t,\theta_t) - \sum_{i\in\mathcal{S}} V_i(a_t,\theta_t) > \varepsilon/\delta. \tag{40}$$

It follows that there is at least one state θ_{t+1} which is reached from θ_t and a_t with strictly positive probability so that

$$\sum_{i\in\mathcal{S}} V_i(\theta_{t+1}) \leq W(\theta_{t+1}) -$$
$$\max_{a_{t+1}}\left\{v_{-S}(a_{t+1},\theta_{t+1}) - c(a_{t+1},\theta_{t+1}) + \delta W(a_{t+1},\theta_{t+1}) - \delta\sum_{i\in\mathcal{S}} V_i(a_{t+1},\theta_{t+1})\right\},$$

but by (40) and the existence of a marginal contribution equilibrium, it follows that $\exists a_{t+1} \in A_{t+1}$ such that

$$\sum_{i\in\mathcal{S}} M_i(a_t,\theta_t) - \sum_{i\in\mathcal{S}} V_i(a_t,\theta_t) > \varepsilon/\delta^2,$$

and by repeatedly applying this argument, we conclude that the hypothesis of multiple truthful equilibria leads to a contradiction, since marginal contribution and equilibrium value are both necessarily finite. ■

Proof of Theorem 4. The proof is by contradiction. Suppose for some $\mathcal{S}$ and θ_t, we have

$$\sum_{i\in\mathcal{S}} M_i(\theta_t) > M_{\mathcal{S}}(\theta_t) = W(\theta_t) - W_{-\mathcal{S}}(\theta_t),$$

but the inequality

$$\sum_{i\in\mathcal{S}} M_i(\theta_t) \leq W(\theta_t) - \max_{a_t}\left\{v_{-\mathcal{S}}(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k\notin\mathcal{S}} \delta V_k(a_t,\theta_t)\right\}, \quad (41)$$

still holds. It then follows that for some $\varepsilon > 0$

$$W_{-\mathcal{S}}(\theta_t) - \max_{a_t}\left\{v_{-\mathcal{S}}(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k\notin\mathcal{S}} \delta V_k(a_t,\theta_t)\right\} > \varepsilon,$$

and in particular for $a_t = a_{-\mathcal{S}}$,

$$\delta W_{-\mathcal{S}}(\theta_{t+1}) - \delta\left(W(\theta_{t+1}) - \sum_{i\in\mathcal{S}} M_i(\theta_{t+1})\right) > \varepsilon,$$

or

$$\sum_{i\in\mathcal{S}} M_i(\theta_{t+1}) - M_{\mathcal{S}}(\theta_{t+1}) > \frac{\varepsilon}{\delta}.$$

But if the inequality (41) is still to hold in period $t+1$, then it necessarily follows that

$$W_{-\mathcal{S}}(\theta_{t+1}) - \max_{a_{t+1}}\left\{v_{-\mathcal{S}}(a_{t+1},\theta_{t+1}) - c(a_{t+1},\theta_{t+1}) + \sum_{k\notin\mathcal{S}} \delta V_k(a_{t+1},\theta_{t+1})\right\} > \frac{\varepsilon}{\delta},$$

and as before this in turn implies that

$$\sum_{i\in\mathcal{S}} M_i(\theta_{t+1}) - M_{\mathcal{S}}(\theta_{t+1}) > \frac{\varepsilon}{\delta^2}.$$

By repeating this argument we come to the conclusion that the marginal contributions of a fixed subset $\mathcal{S}$ of principals grows without bound which is a contradiction to the fact that the value of the game is bounded. ■

Proof of Theorem 5. By Theorem 3 it suffices to show that $\{M_i(\theta_t)\}_{i\in\mathcal{I}}$ satisfy the following set of equalities, $\forall i$,

$$\begin{aligned} M_i(\theta_t) &= \max_{a_t}\left\{v(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k=0}^{I} \delta V_k(a_t,\theta_t)\right\} \quad (42) \\ &\quad - \max_{a_t}\left\{v_{-i}(a_t,\theta_t) - c(a_t,\theta_t) + \sum_{k\neq i} \delta V_k(a_t,\theta_t)\right\} \end{aligned}$$

and inequalities $\forall \mathcal{S} \subset \mathcal{I}$,

$$\sum_{i \in \mathcal{S}} M_i(\theta_t) \leq \max_{a_t} \left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\} - \max_{a_t} \left\{ v_{-\mathcal{S}}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \notin \mathcal{S}} \delta V_k(a_t, \theta_t) \right\}, \tag{43}$$

if and only if the conditions in (16) are met. By hypothesis $V_k(a_t, \theta_t) = M_k(a_t, \theta_t)$ for all $k > 0$. For notational ease, we omit that a_t is restricted to $a_t \in A(\theta_t)$. We start with the set of equalities (42). Since all thruthful equilibria are efficient by Theorem 2 we have the identity:

$$W(\theta_t) = \max_{a_t} \left\{ v(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k=0}^{I} \delta V_k(a_t, \theta_t) \right\}.$$

Consider next the term

$$\max_{a_t} \left\{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \neq i} \delta V_k(a_t, \theta_t) \right\}$$

which can be written as

$$\max_{a_t} \{ v_{-i}(a_t, \theta_t) - c(a_t, \theta_t) + \delta V(a_t, \theta_t) - \delta M_i(a_t, \theta_t) \} = W_{-i}(\theta_t),$$

where the equality follows from the definition of the marginal contribution in (1) and hence the equality in (42) is satisfied. Consider next the set of inequalities (43):

$$\sum_{i \in \mathcal{S}} M_i(\theta_t) \leq W(\theta_t) - \max_{a_t} \left\{ v_{-\mathcal{S}}(a_t, \theta_t) - c(a_t, \theta_t) + \sum_{k \notin \mathcal{S}} \delta V_k(a_t, \theta_t) \right\}. \tag{44}$$

We denote the socially optimal action $a_t = a$ and an action a_t maximizing the program in (44) by $a_{-\mathcal{S}}$. If for any group $\mathcal{S}$, $a_{-\mathcal{S}} = a$, it follows that

$$\sum_{i \in \mathcal{S}} m_i(\theta_t) = v_{\mathcal{S}}(a, \theta_t),$$

and hence the group as an aggregate is not making any net contributions to $\mathcal{I} \backslash \mathcal{S}$, and (44) is satisfied. Suppose next that $a_{-\mathcal{S}} \neq a$, then (44) is equivalent to

$$\sum_{i \in \mathcal{S}} M_i(\theta_t) \leq W(\theta_t) - \max_{a_t} \left\{ W(\theta_t | a_t) - v_{\mathcal{S}}(a_t, \theta_t) - \sum_{i \in \mathcal{S}} \delta M_i(a_t, \theta_t) \right\}. \tag{45}$$

Since the inequality has to hold for the action $a_{-\mathcal{S}}$ which maximizes the payoff inside the bracket, it follows a fortiori that the inequality has to hold for an arbitrary action a_t. Then we may write (45) as:

$$\sum_{i\in\mathcal{S}} M_i(\theta_t) - v_{\mathcal{S}}(a_t,\theta_t) - \sum_{i\in\mathcal{S}} \delta M_i(a_t,\theta_t) \leq W(\theta_t) - W(\theta_t | a_t),$$

or equivalently

$$\sum_{i\in\mathcal{S}} \left(M_i(\theta_t) - M_i(\theta_t | a_t)\right) \leq W(\theta_t) - W(\theta_t | a_t),$$

which completes the proof. ■

Proof of Corollary 2. It suffices to show the equivalency of (16) and (19). The inequality (19) can be written explicitly as:

$$\sum_{i\in\mathcal{S}} m_i(\theta_t) \leq \left(W(\theta_t) - \sum_{i\in\mathcal{S}} \delta M_i(a_t,\theta_t)\right) - \max_{a_t}\left\{W(\theta_t | a_t) - \sum_{i\in\mathcal{S}} \delta M_i(a_t,\theta_t)\right\},$$

which is equivalent to

$$\max_{a}\left(W(\theta_t | a_t) - \sum_{i\in\mathcal{S}} \delta M_i(a,\theta_t)\right) \leq W(\theta_t) - \sum_{i\in\mathcal{S}} M_i(a_t,\theta_t),$$

and since the equality holds for the maximizing action $a_{-\mathcal{S}}$, it holds a fortiori for all a_t. ■

Proof of Theorem 6. Fix a strategy profile $(\mathbf{a},\mathbf{r})$ and the associated set of continuation profiles $\mathcal{E}$. The vector of continuation payoffs for a given history h_t is $\mathbf{V}(h_t) = (V_0(h_t), V_1(h_t), ..., V_I(h_t))$. We define the set of all continuation payoff profiles in the state θ by $\mathcal{E}_V(\theta)$. The vector $\mathbf{V}(h_t) \in \mathcal{E}_V(\theta)$ if $\theta(h_t) = \theta$.

1. Consider an MPE $(\mathbf{a},\mathbf{r})$ in truthful strategies. By the Markovian restriction, the set $\mathcal{E}_V(\theta)$ consists of a singleton for every θ. The equilibrium $(\mathbf{a},\mathbf{r})$ is then a dynamic coalition-proof equilibrium if it is a coalition proof equilibrium for every θ_t and associated (intertemporal) payoffs $V_i(\theta_t)$ which are uniquely defined by the singleton property of $\mathcal{E}_V(\theta)$. As the equilibrium is truthful at θ_t with continuation payoffs $V_i(a_t,\theta_t)$, it follows by Theorem 3 of Bernheim & Whinston (1986a) that $(\mathbf{a},\mathbf{r})$ is also a dynamic coalition equilibrium.

2. Consider a dynamic coalition-proof equilibrium $(\mathbf{a}, \mathbf{r})$ in Markov strategies. The set $\mathcal{E}_V(\theta)$ is a singleton for every θ by the Markov property of the strategies. It follows that the dynamic coalition-proof equilibrium is also a coalition-proof equilibrium for every θ_t and associated continuation payoffs $V_i(a_t, \theta_t)$. By Theorem 2 and 3 of Bernheim & Whinston (1986a) there is a truthful equilibrium at θ_t and relative to gross payoffs $v_i(a_t, \theta_t) + \delta V_i(a_t, \theta_t)$ and $-c(a_t, \theta_t) + \delta V_0(a_t, \theta_t)$ which is payoff-equivalent. And since there is a truthful equilibrium at every θ_t, it follows that there is an MPE in truthful strategies. ■

Proof of Theorem 7.1. By Theorem 5, it is sufficient to show that the inequalities

$$\sum_{i \in \mathcal{S}} (M_i(t) - M_i(t\,|a_t)) \leq W(t) - W(t\,|a_t), \; \forall t, \forall \mathcal{S}, \; \forall a_t \tag{46}$$

are satisfied. The action a_t is any suboptimal task allocation $a_t = (t_1, t_2, .., t_n)$ where t_i is the time of appearance of action component i along the original efficient sequence. Let $t_1 < t_2 < ... < t_n$ and $t(t_k) \geq t$ for all $k = 1, ..., n$ and at least one t_k with $t(t_k) > t$. Notice that $\mathcal{S} \cap \{i\,|T_i(t_k) \neq T_i(t_k - 1)\} = \emptyset$. Without loss of generality, let $t = 0$. Choosing task v_{t_k} earlier than optimal changes the timing of the remaining tasks and we introduce a new set of counters to track the modified sequence starting with a_0 and followed by an optimal sequence. Let

$$\hat{T}(\tau) \triangleq \{\#k\,|t_k \leq \tau\}$$

and define the accelerated time after using up $(\tau_1, \tau_2, .., \tau_n)$ in $t = 0$ by

$$\hat{\tau}(\tau) \triangleq \min\left\{s \,\middle|\, s - \hat{T}(s) = \tau\right\},$$

and define also for all i:

$$\hat{\sigma}_i(\tau) \triangleq \min\left\{s \,\middle|\, s - \hat{T}(s) - T_i(s) = \tau\right\}.$$

The value $W(0\,|a_0)$ is given by

$$W(0\,|a_0) = \sum_{k=1}^{n} v_{t_k} + \sum_{\tau=0}^{t_n - n} \delta^{t(\tau+n)} v_{\hat{\tau}(\tau)} + \sum_{\tau=t_n+1}^{\infty} \delta^{t(\tau)} v_\tau, \tag{47}$$

The marginal contribution of principal $i \in \mathcal{S}$ along the path that starts with a_0 is:

$$M_i(0|a_0) = \sum_{\tau=0}^{t_n-n} \delta^{t(\tau+n)} \left(v_{\hat{\tau}(\tau)} - v_{\hat{\sigma}_i(\tau)}\right) + \sum_{\tau=t_n+1}^{\infty} \delta^{t(\tau)} \left(v_\tau - v_{\sigma_i(\tau)}\right). \quad (48)$$

Using (22)-(24) and (47)-(48), inequality (46) can be written as:

$$\begin{aligned} & \sum_{\tau=0}^{t_n} \delta^{t(\tau)} v_\tau - \sum_{k=1}^{n} v_{\tau_k} - \sum_{\tau=0}^{t_n-n} \delta^{t(\tau+n)} v_{\hat{\tau}(\tau)} \\ \geq & \quad (49) \\ & \sum_{i \in \mathcal{S}} \left(\sum_{\tau=0}^{t_n} \delta^{t(\tau)} \left(v_\tau - v_{\sigma_i(\tau)}\right) - \sum_{\tau=0}^{t_n-n} \delta^{t(\tau+n)} \left(v_{\hat{\tau}(\tau)} - v_{\hat{\sigma}_i(\tau)}\right) \right). \end{aligned}$$

Notice first that

$$T_i(t_n) = 0 \Rightarrow M_i(0) = M_i(0|a_0),$$

which allows us to truncate the problem at τ_n. Moreover for all i with $T_i(t_n) > 0$, it follows from (24) and (48) that $M_i(0) - M_i(0|a_0) \geq 0$. We can then choose $\mathcal{S}$ without loss of generality to be maximal, or $\mathcal{S} = \mathcal{I} \backslash \{i \,|T_i(t_k) \neq T_i(t_k - 1)\}$. Define the truncated value function by:

$$\hat{W}(0) \triangleq \sum_{\tau=0}^{t_n - \hat{T}(t_n)} \delta^{t(\tau)} \hat{v}_{\hat{\tau}(\tau)}, \quad (50)$$

where

$$\hat{v}_{\hat{\tau}(\tau)} \triangleq \begin{cases} v_\tau & \text{if } \; t(\tau) = t(\hat{\tau}(\tau)), \\ 0, & \text{if } \; t(\tau) \neq t(\hat{\tau}(\tau)). \end{cases} \quad (51)$$

In the truncated program, we need to track only the contribution of principal i which is not equal to zero according to (51) as all other elements cancel out. The truncated value $\hat{W}_{-i}(0)$ after excluding the principal i is defined similarly:

$$\hat{W}_{-i}(0) \triangleq \sum_{\tau=0}^{\tau_n - T_i(\tau_n) - \hat{T}(\tau_n)} \delta^{t(\tau)} \hat{v}_{\hat{\sigma}_i(\tau)}, \quad (52)$$

where

$$\hat{v}_{\hat{\sigma}_i(\tau)} \triangleq \begin{cases} v_{\hat{\sigma}_i(\tau)} & \text{if } \; t(\tau) = t\left(\sigma_i^{-1}(\hat{\sigma}_i)\right), \\ 0, & \text{if } \; t(\tau) \neq t\left(\sigma_i^{-1}(\hat{\sigma}_i)\right), \end{cases}$$

where the inverse function $\sigma_i^{-1}(\hat{\sigma}_i)$ identifies the time at which the alternative $\hat{\sigma}_i$ is realized in the modified program relative to $\tau = \hat{\sigma}_i^{-1}(\hat{\sigma}_i)$. Using the truncated value functions introduced in (50) and (52), we can rewrite inequality (49) as:

$$
\begin{aligned}
&(1-\delta)\hat{W}(0) - \sum_{k=1}^{n}\left(1-\delta^{t(t_k)}\right)v_{t_k} \\
\geq\ & \\
&\sum_{i\in\mathcal{S}}\left((1-\delta)\hat{W}(0) + \sum_{k=1}^{n}\delta^{t(t_k)}v_{t_k} - (1-\delta)\hat{W}_{-i}(0) - \sum_{k=1}^{n}\delta^{t(t_k - T_i(t_k))}v_{t_k}\right)
\end{aligned}
$$

and after dividing by $(1-\delta)$ and collecting terms further we get

$$
\begin{aligned}
&\hat{W}(0) - \sum_{k=1}^{n}\frac{1-\delta^{t(t_k)}}{1-\delta}v_{t_k} \\
\geq\ & \\
&\sum_{i\in\mathcal{S}}\left(\hat{W}(0) - \hat{W}_{-i}(0) - \sum_{k=1}^{n}\frac{\left(1-\delta^{t(T_i(t_k))}\right)\delta^{t(\tau_k - T_i(t_k))}}{1-\delta}v_{\tau_k}\right).
\end{aligned}
\tag{53}
$$

The term $\hat{W}(0) - \hat{W}_{-i}(0)$ is simply the truncated marginal contribution $\hat{M}_i(0)$ which can be represented by the truncated flow marginal contributions:

$$
\hat{M}_i(0) = \sum_{\tau=0}^{t_n - \hat{T}(t_n)} \delta^{t(\tau)}\hat{m}_i(\hat{\tau}(\tau)) \tag{54}
$$

which are of the following form:

$$
\hat{m}_i(\hat{\tau}(\tau)) = \sum_{s=\hat{\tau}(\tau)}^{t_n - T_i(t_n|\hat{\tau}(\tau))} \delta^{t(s)-t(\hat{\tau}(\tau))}\left(\tilde{v}_{\hat{\sigma}_i(s|\hat{\tau}(\tau)+1)} - \hat{v}_{\hat{\sigma}_i(s|\hat{\tau}(\tau))}\right) \tag{55}
$$

with the localized counter:

$$
\hat{\sigma}_i(\gamma|\tau) \triangleq \min\left\{s \,\middle|\, s - T_i(s|\tau) - \hat{T}(s|\tau) = \gamma\right\},
$$

and the flow values defined by

$$
\tilde{v}_{\hat{\sigma}_i(\gamma|\hat{\tau}(\tau))} \triangleq \begin{cases} v_{\hat{\sigma}_i(\gamma|\hat{\tau}(\tau))} & \text{if } \ t(\gamma) = t(\hat{\tau}(\gamma)), \\ 0, & \text{if } \ t(\gamma) \neq t(\hat{\tau}(\gamma)), \end{cases}
$$

and

$$\hat{v}_{\hat{\sigma}_i(\gamma|\hat{\tau}(\tau))} = \begin{cases} v_{\hat{\sigma}_i(\gamma|\hat{\tau}(\tau))} & \text{if } t(\gamma) = t\left(\sigma_i^{-1}(\hat{\sigma}_i)\right), \\ 0, & \text{if } t(\gamma) \neq t\left(\sigma_i^{-1}(\hat{\sigma}_i)\right). \end{cases}$$

The difference between $\tilde{v}_{\hat{\sigma}_i(\cdot)}$ and $\hat{v}_{\hat{\sigma}_i(\cdot)}$ is that the cancelled terms arise in the first expression from the program with all principals and in the second term from the program with all principals but i. For any given $\hat{\tau}(\tau)$ with $\hat{\tau}(\tau) = \hat{\sigma}_i(\hat{\tau}(\tau)|\hat{\tau}(\tau))$, it is verified that $\hat{m}_i(\hat{\tau}(\tau)) = 0$. Hence for an arbitrary set $\mathcal{S}$, any $v_{\hat{\tau}(\tau)}$ makes at most one nontrivial contribution. On the other hand, for any τ with $\hat{\tau}(\tau) < \hat{\sigma}_i(\hat{\tau}(\tau)|\hat{\tau}(\tau))$, $\hat{m}_i(\hat{\tau}(\tau))$ can be represented by a sum of the form

$$\sum_{l=0}^{L} \delta^{n_l}(y_l - y_{l+1}) \tag{56}$$

with the following properties: (i) $y_l \geq y_{l+1}$, (ii) $y_0 = v_\tau$, (iii) $y_L \geq v_{\tau_n - T_i(\tau_n|\hat{\tau}(\tau))}$, (iv) $y_{L+1} = 0$, (v) $n_0 = 0$, and (vi) $n_l \leq n_{l+1}$. All properties follow directly from (55) and the fact that the optimal sequence v_τ has decreasing values. Consider then the rhs of (53) after inserting (54):

$$\sum_{\tau=0}^{t_n - \hat{T}(t_n)} \delta^{t(\tau)} m_{\hat{\tau}(\tau)}(\hat{\tau}(\tau)) - \sum_{k=1}^{n} \frac{\left(1 - \delta^{t(T_j(t_k))}\right) \delta^{t(t_k - T_j(t_k))}}{1-\delta} v_{t_k}.$$

It then follows that with every $m_{\hat{\tau}(\tau)}(\hat{\tau}(\tau))$ on the right hand side of (53) we can associate a sequence of positive but increasingly discounted differences of the form displayed in (56) where y_{L+1} is replaced by $v_{\hat{\tau}_k}$ with $\hat{\tau}_k = \min\{t_k\,|t_k > \hat{\tau}(\tau)\}$. This representation exhausts the rhs of (53). But for every v_τ and associated terms on the rhs we can find one and only one term $\delta^{t(\tau)}(v_\tau - v_{\hat{\tau}_k}) \geq 0$ on the lhs, which weakly dominates every corresponding term on the rhs and hence the inequality follows.

2. and **3.** The payoff characterization follows immediately from Theorem 2 in conjunction with Theorem 3. ■

The proof of Theorem 8 is facilitated by first proving the result for the deterministic model. The stochastic version is then shown by first creating a deterministic version for every sample path ω, which satisfies all the properties of the deterministic model, and then showing that all the important properties are preserved when taking expectations.

Consider the following deterministic matching problem. Each principal i has an employment opportunity for the agent. The productivity of the agent in the job offered by principal i is $v_i(T_i)$, where T_i is number of times the agent previously worked with the principal. The productivity $v_i(\cdot)$ is an arbitrary function of past history T_i with the principal i. In particular we don't assume monotonicity in T_i. This model is a deterministic version of multi-armed bandit model. The optimal policy is an index policy, where the index $\phi_i(T_i)$ is given by

$$\phi_i(T_i(t)) = \max_{\tau} \left[\left. \frac{\sum_{s=T_i}^{\tau} \delta^{(s-T_i)} v_i(s)}{\sum_{s=T_i}^{\tau} \delta^{(s-T_i)}} \right| T_i(t) \right]. \tag{57}$$

The alternative i which has the highest index at $T(t) = (T_1(t), ..., T_I(t))$:

$$\phi(t) \triangleq \max_{i \in S} \{\phi_i(T_i(t))\} \tag{58}$$

is selected. See Whittle (1982) or Karatzas (1984) for more details.

Theorem 9

1. *The deterministic job matching model has a unique truthful equilibrium.*

2. *The employing principal i receives: $m_i(\theta_t) = v_i(\theta_t) - r_i(i, \theta_t)$ and all other principals, $j \neq i$, receive $m_j(\theta_t) = 0$.*

3. *The agent receives only spot wages: $r(i, \theta_t) = r_i(i, \theta_t)$.*

As the proof is rather lengthy, a brief overview is given first. At the center of the proof are two infinite sequences, one generated by the optimal program, the other by the program with a single deviation. The inequality to be established is between the difference of these two sequences, $W(t) - W(t|a_t)$, and a sum of differences, generated by the same sequences, $\sum_{i \in S} (M_i(t) - M_i(t|a_t))$. Here each individual term, $M_i(t)$ or $M_i(t|a_t)$ is by itself a difference based on of the sequences, respectively, and an accelerated version of the same sequence. As $M_i(t) = W(t) - W_{-i}(t)$, the acceleration is due to the removal of the $i-th$

alternative. We then use (in this order) three properties of the optimal policy, and the attendant index characterization to establish the inequality: (i) by the index policy, the order, but not the time, in which the alternatives are used are identical in two sequences for all alternatives but the one which starts the modified sequence; (ii) the values of any uninterrupted sequence of alternative i can be replaced by a constant, which is (almost) its index; (iii) the indices along the optimal path, computed exclusively at the switching times, are decreasing.

Proof of Theorem 9.1 By Theorem 5, it is sufficient to show that the inequality

$$\sum_{i\in S}\left(M_i(t)-M_i(t\,|a_t)\right)\leq W(t)-W(t\,|a_t)\,,\ \forall t,\forall S,\ \forall a_t, \tag{59}$$

is satisfied. Without loss of generality, assume that $t=0$. Consider the optimal policy a and the modified policy $\hat{a}$. The modified policy starts with an assignment $\hat{a}(0)=k$ with $\phi_k(0)<\phi(0)$, but thereafter continues optimally.

Define the counters $T_i(t)=\{\#s\,|a(s)=i,s\leq t\}$ and $\hat{T}_i(t)=\{\#s\,|\hat{a}(s)=i,\ s\leq t\}$. For each alternative i, we define switching times along program a and $\hat{a}$ by starting times:

$$r_{i_n}=\min\left\{t\,\middle|a(t)=i,\ t>r^{i_{n-1}}\right\},\ \text{for } n=0,1,\ldots \text{ and } r^{i_{-1}}=-1.$$

and stopping times:

$$r^{i_n}=\min\left\{t\,|a(t+1)\neq i,\ t\geq r_{i_n}\right\},\ \text{for } n=0,1,\ldots$$

Similar times are defined for the program $\hat{a}$, with the additional restriction that $r_{j_0}>0$ for all j, as we start to count switching times only along the optimal continuation paths. Consider next the alternative k and its first use between r_{k_0} and r^{k_0} in the optimal program. Without loss of generality, suppose that the index at r_{k_0} is given by:

$$\phi\left(r_{k_0}\right)=\phi_k\left(r_{k_0}\right)=\frac{\sum_{s=r_{k_0}}^{\tau}\delta^{\left(s-r_{k_0}\right)}v_i(s)}{\sum_{s=r_{k_0}}^{\tau}\delta^{\left(s-r_{k_0}\right)}}$$

with $\tau=r^{k_0}$. If the maximal index is obtained with $\tau<r^{k_0}$, then we simply create a new switching time at τ so that instead of a single interval $\left\{r_{k_0},\ldots,r^{k_0}\right\}$,

we obtain two intervals over which we employ k without interruption, namely $\{\bar{r}_{k_0}, ..., \bar{r}^{k_0}\}$ and $\{\bar{r}_{k_1}, ..., \bar{r}^{k_1}\}$ with $r_{k_0} = \bar{r}_{k_0}$, $\bar{r}^{k_0} = \tau$, $\bar{r}_{k_1} = \tau + 1$, $\bar{r}^{k_1} = r^{k_0}$. In the later case, we simply need to replace all arguments involving r_{k_0} and r^{k_0} by $\bar{r}_{k_0}$ and $\bar{r}^{k_0}$.

By the optimality of the index policy,

$$v_t = \hat{v}_t \text{ for all } t \geq r^{k_0}. \tag{60}$$

We can therefore restrict our attention to $t \leq r^{k_0}$. Notice that r^{k_0} may be infinite. Moreover, the optimal order of employment among all alternatives but k remains unchanged when comparing the optimal sequence to $\hat{a}$. Formally, for all t with $\hat{a}(t) \neq k$, we have

$$\hat{v}_t = v_{t - \hat{T}_k(t)}, \tag{61}$$

and thus all terms involving $i \neq k$ appear also the modified program, only later.

Before we prove the inequality (59) we rewrite $W(0)$ and $W(0|k)$ to facilitate the proof. Consider the modified program $\hat{a}$. Without loss of generality we can redefine $\hat{v}_t$ to be $\hat{v}_t \triangleq \hat{v}_{i_n}$ for all $\hat{t} \in \{\hat{r}_{i_n}, ..., \hat{r}^{i_n}\}$ where

$$\phi\left(\hat{r}^{i_n} + 1\right) \leq \hat{v}_{i_n} \leq \phi\left(\hat{r}_{i_n}\right),$$

and

$$\sum_{t=\hat{r}_{i_n}}^{\hat{r}^{i_n}} \delta^{t-\hat{r}_{i_n}} \hat{v}_t = \hat{v}_{i_n} \sum_{t=\hat{r}_{i_n}}^{\hat{r}^{i_n}} \delta^{t-\hat{r}_{i_n}}.$$

This operation achieves a constant payoff between $\hat{r}_{i_n}$ and $\hat{r}^{i_n}$while preserving the value collected between starting time $\hat{r}_{i_n}$ and stopping time $\hat{r}^{i_n}$. Observe also that the modified sequence displays payoffs which are weakly decreasing in time, since the sequence of optimal allocation indices $\phi(\cdot)$ is weakly decreasing in the starting times $r_{k_0} < ... < r_{i_n} < ...$, but not necessarily in time t. Assign the same values in the optimal program a by using the identity (61) and to the values v_t with $a(t) = k$ by the obvious relationship of $v_t = \hat{v}_{k_n}$ if $a(t) = k$, and $\hat{T}_k(r_{k_n}) \leq T_k(t) \leq \hat{T}_k(r^{k_n})$. Notice that the reassignment doesn't affect the marginal contributions $M_i(0)$ or $M_i(0|k)$ either as they are just obtained by differencing the respective social value functions. Finally, we normalize all flow values by subtracting $\phi(r_{k_0})$ from each flow value. We work in the following directly with these normalized values $\hat{v}_{i_n}$ which are weakly positive for $v_t, \hat{v}_t$ with $t < r^{k_0}$.

Consider next the inequality (59):

$$W(0) - \sum_{i \in \mathcal{S}} M_i(0) \geq W(0 \,|k) - \sum_{i \in \mathcal{S}} M_i(0 \,|k), \tag{62}$$

The flow marginal contributions are

$$m_i(t) = 0 \text{ if } a(t) \neq i, \tag{63}$$

and

$$m_i(t) = \sum_{s=t}^{\infty} \delta^{s-t} \left(v_{\sigma_i(s|t+1)} - v_{\sigma_i(s|t)} \right), \text{ if } a(t) = i, \tag{64}$$

with the convention that $v_{\sigma_i(t|t+1)} = v_t$, and similarly for the program $\hat{a}$ for all $t > 0$. The definition of $\sigma_i(s\,|t)$ is as in (23). Using (63) and (64), we can write the inequality (62) as

$$\sum_{t=0}^{\infty} \delta^t \left(v_t - m_{a(t)} \mathbb{I}_{\mathcal{S}}(a(t)) \right) \geq \sum_{t=0}^{\infty} \delta^t \left(\hat{v}_t - m_{\hat{a}(t)} \mathbb{I}_{\mathcal{S}}(\hat{a}(t)) \right) \tag{65}$$

with the indicator function

$$\mathbb{I}_{\mathcal{S}}(a(t)) = \begin{cases} 0 & \text{if } a(t) \notin \mathcal{S}, \\ 1 & \text{if } a(t) \in \mathcal{S}. \end{cases}$$

By (60), it follows that the maximal set necessary to consider is given by $\langle i \in \mathcal{S} \rangle \Leftrightarrow \langle \exists t, t < r_{k_0}, a(t) = i \rangle$, and hence the inequality (65) is equivalent to

$$\sum_{t=0}^{r^{k_0}} \delta^t \left(v_t - m_{a(t)} \mathbb{I}_{\mathcal{S}}(a(t)) \right) \geq \sum_{t=0}^{r^{k_0}} \delta^t \left(\hat{v}_t - \hat{m}_{\hat{a}(t)} \mathbb{I}_{\mathcal{S}}(\hat{a}(t)) \right). \tag{66}$$

Consider then a particular entry

$$\hat{v}_t - \hat{m}_{\hat{a}(t)}(t) \tag{67}$$

with $\hat{a}(t) \neq k$. By (61), we know that there is a corresponding element

$$v_{t - \hat{T}_k(t)} - m_{a\left(t - \hat{T}_k(t)\right)} \left(t - \hat{T}_k(t) \right)$$

occurring some $\hat{T}_k(t)$ periods earlier. Using (64), (67) can be written as

$$\hat{v}_t - \hat{m}_{\hat{a}(t)} = (1-\delta)\sum_{s=t}^{\infty}\delta^{s-t}v_{\sigma_{\hat{a}(t)}(s|t)},$$

and similarly for a. However due to (60), we can again truncate the infinite series as follows. For any t and $t-\hat{T}_k(t)$ in the modified and original problem, respectively, there exists an $\hat{s}_t$ and s_t defined by:

$$\hat{s}_t \triangleq \min\left\{s \left|\sigma_{\hat{a}(t)}(s|t) > r^{k_0}\right.\right\},$$

and

$$s_t \triangleq \min\left\{s \left|\sigma_{a\left(t-\hat{T}_k(t)\right)}\left(s \left|t-\hat{T}_k(t)\right.\right) > r^{k_0}\right.\right\}.$$

By (60), it follows that

$$\sum_{s=s_t}^{\infty}\delta^{s-t}v_{\sigma_{a(t)}(s|t)} = \sum_{s=s_{\hat{t}}}^{\infty}\delta^{s-t}v_{\sigma_{\hat{a}(t)}(s|t)}. \tag{68}$$

Hence when considering the inequality (66), we can not only truncate the comparison at r^{k_0}, but also the differences in the flow values of the marginal contribution can be exclusively attributed to payoff differences before r^{k_0}. Summarizing, we can write (66) as

$$\begin{aligned} &(1-\delta)\sum_{t=0}^{r^{k_0}}\mathbb{I}_S(a(t))\sum_{s=t}^{s_t-1}\delta^{s-t}v_{\sigma_{a(t)}(s|t)} + \sum_{t=0}^{r^{k_0}}\delta^t\left(v_t\left(1-\mathbb{I}_S(a(t))\right)\right) \\ \geq \; & \\ &(1-\delta)\sum_{t=0}^{r^{k_0}}\mathbb{I}_S(\hat{a}(t))\sum_{s=t}^{s_{\hat{t}}-1}\delta^{s-t}v_{\sigma_{\hat{a}(t)}(s|t)} + \sum_{t=0}^{r^{k_0}}\delta^t\left(\hat{v}_{\hat{t}}\left(1-\mathbb{I}_S(a(t))\right)\right) \end{aligned} \tag{69}$$

Observe next that when comparing corresponding elements

$$\delta^{t-\hat{T}_k(t)}\left(v_{t-\hat{T}_k(t)} - m_{a\left(t-\hat{T}_k(t)\right)}\right) = (1-\delta)\sum_{s=t-\hat{T}_k(t)}^{s_t}\delta^s v_{\sigma_{a\left(t-\hat{T}_k(t)\right)}(s|t)}, \tag{70}$$

and

$$\delta^t\left(\hat{v}_t - \hat{m}_{\hat{a}(t)}\right) = (1-\delta)\sum_{s=t}^{\hat{s}_t}\delta^s v_{\sigma_{\hat{a}(t)}(s|t)}, \tag{71}$$

the precedence order (61) still holds for $\hat{a}\left(\sigma_{\hat{a}(t)}\left(s\,|t\,\right)\right) \neq k$. For every $\delta^s v_{\sigma_{\hat{a}(t)}(s|t)}$ in (71), there is a $\delta^{s-\hat{T}_k(s)} v_{\sigma_{a(s-\hat{T}_k(s))}\left(s-\hat{T}_k(s)|t\right)}$ in (70) such that

$$v_{\sigma_{\hat{a}(t)}(s|t)} = v_{\sigma_{a(s-\hat{T}_k(s))}\left(s-\hat{T}_k(s)|t\right)}.$$

Hence it follows that if we set $\hat{v}_t = 0$ for all $t \leq r^{k_0}$ and $\hat{a}(t) \neq k$, and the inequality (69) still holds, then it holds a fortiori with the original $\hat{v}_t \geq 0$. Thus suppose we do set them zero. Then the lhs of (69) is equal to zero, where we recall the earlier normalization. The rhs can be rewritten after dividing by $(1-\delta)$ as

$$\sum_{t=0}^{r^{k_0}} \mathbb{I}_S\left(\hat{a}\left(t\right)\right) \sum_{s=t}^{s_{\hat{t}}-1} \delta^{s-t} v_{\sigma_{\hat{a}(t)}(s|t)} \left(1 - \mathbb{I}_S\left(\hat{a}\left(\sigma_{\hat{a}(t)}\left(s\,|t\,\right)\right)\right)\right) + \sum_{t=0}^{r^{k_0}} \frac{\delta^t}{1-\delta}\left(\hat{v}_{\hat{t}}\left(1 - \mathbb{I}_S\left(a\left(t\right)\right)\right)\right), \tag{72}$$

where all but the trivial terms are realizations of the $k-th$ alternative in form of different geometric series. Let $\hat{t}_0, \hat{t}_1, ..., \hat{t}_K$ be the times at which alternative k is realized between $0 \leq t \leq r^{k_0}$, and denote by $\hat{v}_{\hat{t}_0}, \hat{v}_{\hat{t}_1}, ..., \hat{v}_{\hat{t}_K}$ the values at these times. Then (72) can be rewritten as

$$\sum_{t=0}^{r_{k_0-1}} \sum_{k=0}^{K} \delta^{\tau(t,k)} \hat{v}_{\hat{t}_k} + \sum_{k=0}^{K} \frac{\delta^{r_{k_0}+k}}{1-\delta} \hat{v}_{\hat{t}_k}. \tag{73}$$

with

$$\tau\left(t,k\right) = \begin{cases} \sigma_k\left(t\right) - \left(\hat{T}_k\left(\sigma_k\left(t\right)\right) - k\right), & \text{if} \quad \hat{t}_k < \sigma_k\left(t\right), \\ \hat{t}_s - T_{\hat{a}(\sigma_k(t))}\left(\hat{t}_s\left|\sigma_k\left(t\right)\right.\right), & \text{if} \quad \hat{t}_k > \sigma_k\left(t\right). \end{cases} \tag{74}$$

Notice that the second term is equal to zero as it is just the normalized sequence of realizations of k between r_{k_0} and r^{k_0} in the original program multiplied $1/\left(1-\delta\right)$. The double sum on the other hand is the same sequence of realization, but "punctured" as realizations of k are frequently interrupted by now realizations. But any such punctured series necessarily satisfies

$$\sum_{k=0}^{K} \delta^{\tau(t,k)} \hat{v}_{\hat{t}_k} \leq 0, \tag{75}$$

as the highest average discounted payoff is achieved by a sequence starting with the sequence from v_{t_0} to v_{t_K}, including in particular v_{t_K}. Thus any delay, which in particular postpones the realization of $\hat{v}_{\hat{t}_K}$ has a smaller average discounted payoff than the original sequence, and as the original sequence was normalized to be zero, the inequality (75) follows immediately. But then we conclude that the rhs is nonpositive, and hence the inequality (69) follows, which concludes the proof.

2. and **3.** The payoff characterization follows immediately from Theorem 2 in conjunction with Theorem 3. ■

We are now prepared to prove Theorem 8. Unless noted otherwise, the notation carries over from Theorem 9.

Proof of Theorem 8.1. Fix a sample path $\omega \in \Omega$. For every ω, the optimal policy induces a sequence of jobs $a(t) \triangleq a(t(\omega))$ and value realizations $v(t) \triangleq v_{a(t(\omega))}(t(\omega))$ and similar for any modified program $\hat{a}$. For every ω, we create an associated sequence of deterministic values v_t and $\hat{v}_t$. Neither the flow values of these sequences nor their aggregate discounted values are necessarily identical to their sample path realizations under ω. Denote the aggregate value of such a deterministic sequence associated with ω by $W(\omega)$. These new sequences have two important properties: (i) each ω is associated with a deterministic multi-arm bandit problem as defined in (57) and (58), (ii) the social value $W(0)$ of the original matching model satisfies:

$$W(0) = \int_{\Omega} W(\omega)\, dp_0(\omega), \tag{76}$$

where $p_0(\omega)$ is the prior over the sample space Ω. It is sufficient to show that we can construct such sequences satisfying (i) and (ii) to prove (1). Theorem 9 shows that any model satisfying (i) has a unique truthful equilibrium, and if the model satisfies (59) for every sample point ω, then it is also satisfied in expectation if condition (ii) is satisfied.

We begin the construction with the modified program $\hat{a}$. The construction relies on the characterization of the optimal policy by the dynamic allocation index. The optimal policy is to select the alternative with the highest index, where the index $\phi_i\left(\theta_t^i\right)$ of alternative i, is given by

$$\phi_i(t) = \max_{\tau} \frac{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} v_i(s) \left| \mathcal{F}_t^i \right.\right]}{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} \left| \mathcal{F}_t^i \right.\right]}, \tag{77}$$

and $\mathcal{F}_t^i$ is the information (filtration) in period t on the alternative i.

Fix ω. For every ω the decision maker is choosing an optimal action given $\mathcal{F}_t^i(\omega)$. Consider next an arbitrary switching time t where $\hat{a}(t) = i$, but $\hat{a}(t-1) \neq i$. Since the first choice under $\hat{a}$ in period 0 is not optimal, we consider $t = 1$ as a switching time as well even if $\hat{a}(0) = \hat{a}(1)$, whereas $t = 0$ is not considered as a switching time. By the optimality of the index criterion, it follows that

$$\max_{\tau} \frac{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} v_i(s) \left| \mathcal{F}_t^i(\omega) \right.\right]}{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} \left| \mathcal{F}_t^i(\omega) \right.\right]} \geq \max_{\tau} \frac{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} v_j(s) \left| \mathcal{F}_t^j(\omega) \right.\right]}{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} \left| \mathcal{F}_t^j(\omega) \right.\right]},$$

for all $j \neq i$. Consider now all ω' with $\mathcal{F}_t(\omega') = \mathcal{F}_t(\omega)$ and denote the set by $\Omega(\omega) = \{\omega' \in \Omega \,|\, \mathcal{F}_t(\omega') = \mathcal{F}_t(\omega)\}$. Next we contrast the stopping time τ which maximizes (77) with the random time $\hat{\tau}$ which is the defined to be the last time the optimal policy employs i before it switches to another alternative j in period $\hat{\tau} + 1$. By the definition of the index policy it follows that $\tau \leq \hat{\tau}$. Define the expected average value over all $\omega' \in \Omega(\omega)$ until $\hat{\tau}$ as

$$\hat{v}_t \triangleq \frac{\mathbb{E}\left[\sum_{s=t}^{\hat{\tau}} \delta^{s-t} v_i(s) \,|\mathcal{F}_t(\omega)\right]}{\mathbb{E}\left[\sum_{s=t}^{\hat{\tau}} \delta^{s-t} \,|\mathcal{F}_t(\omega)\right]}. \tag{78}$$

By construction, $\hat{v}_t$ is then bracketed by

$$\max_{\tau} \frac{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} v_i(s) \left| \mathcal{F}_t^i(\omega) \right.\right]}{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} \left| \mathcal{F}_t^i(\omega) \right.\right]} \geq \hat{v}_t \geq \max_{\tau} \frac{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} v_j(s) \left| \mathcal{F}_t^j(\omega) \right.\right]}{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t} \left| \mathcal{F}_t^j(\omega) \right.\right]} \tag{79}$$

Now we replace the realized payoff for the sample path ω' between $t(\omega')$ and $\hat{\tau}(\omega')$ by the constant $\hat{v}_t$, for all $\omega' \in \Omega(\omega)$. This operation changes the (expected) payoff for every sample path, but the (expected) payoff across all $\omega' \in \Omega(\omega)$ remains unchanged, or

$$\int_{\Omega(\omega)} \left(\sum_{s=t}^{\hat{\tau}(\omega')} \delta^{s-t} v_i(s(\omega')) \right) dp_0(\omega') = \hat{v}_t \int_{\Omega(\omega)} \left(\sum_{s=t}^{\hat{\tau}(\omega')} \delta^{s-t} \right) dp_0(\omega'). \tag{80}$$

By performing this substitution for all switching times, this process assigns values for the modified program $\hat{a}$, which are weakly decreasing over time. Consider then the optimal program a and its sequence of value realizations. The sequence v_t is derived from the sequence $\hat{v}_t$ by setting

$$\left\langle v_s \triangleq \hat{v}_t \right\rangle \Leftrightarrow \left\langle \hat{T}_k(t) = T_k(s),\ \hat{T}_k(t) \neq \hat{T}_k(t-1),\ T_k(s) \neq T_k(s-1) \right\rangle,$$

for all t with $T_k(t) \leq \hat{T}_k(t)$, and otherwise $v_t \triangleq \hat{v}_t$. Finally, we need to assign a value to the first realization of k in the modified program. We define it to be

$$\hat{v}_0 \triangleq \phi(r_{k_0}) \sum_{s=r_{k_0}}^{r^{k_0}} \delta^{s-t} - \sum_{s=r_{k_0}+1}^{r^{k_0}} \delta^{s-t} v_s \tag{81}$$

where r_{k_0} and r^{k_0} are the first switching times for k along the original program, and to preserve the value of the program we set $v_{r_{k_0}} \triangleq \hat{v}_0$. By construction, the sequences v_t and $\hat{v}_t$ now satisfy the requirements of a deterministic multi-armed bandit problem, and hence for every ω, the inequality (59) holds. By (80) and (81), the expectation over all $\omega \in \Omega$ satisfies (76), both for a and $\hat{a}$. Finally, since $M_i(0)$ and $M_i(0\,|k)$ are obtained as differences between two different social values, their values are also preserved under expectations, and hence a marginal contribution equilibrium exists for the stochastic job matching model as well.

2. and **3.** The payoff characterization follows immediately from Theorem 2 in conjunction with Theorem 3. ■

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