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The Optimality of a Simple Market Mechanism

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Abstract

Strategic behavior in a finite market can cause inefficiency in the allocation, and market mechanisms differ in how successfully they limit this inefficiency. A method for ranking algorithms in computer science is adapted here to rank market mechanisms according to how quickly inefficiency diminishes as the size of the market increases. It is shown that trade at a single market-clearing price in the k-double auction is ***worst-case asymptotic optimal*** among all plausible mechanisms: evaluating mechanisms in their least favorable trading environments for each possible size of the market, the k-double auction is shown to force the worst-case inefficiency to zero at the fastest possible rate.

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1 Introduction

The rules that determine how trading proceeds within a market can be regarded as an algorithm for solving the problem of the market, which is to allocate units from the traders who initially own them to those who value them most highly. Market mechanisms and computer algorithms are more than just analogous. Almost every financial exchange in the world is developing a computerized trading system to complement or even substitute for its floor trading system. In the field of experimental economics, trade is commonly studied using a computer network to regulate exchange among subjects. The market mechanism in each of these cases is explicitly a computer algorithm.

The relationship between market mechanisms and algorithms is used in this paper to show that a family of common market mechanisms is optimal in a sense motivated by the study of algorithms in computer science. This is part of an effort to develop a theory of market mechanisms, analogous to the theory of auctions. Currently, economic theory provides little guidance to financial exchanges in the selection of computer algorithms and floor procedures for trading. A theory of market mechanisms would both provide such guidance and also complement the rich literature on markets in experimental economics, which is currently the main source of guidance in the design of market mechanisms.

The family of *k-double auction* (or k-DA) mechanisms is proven to be optimal here, where each choice of the index $k \in [0, 1]$ determines a different mechanism in the family. The k-DA operates as follows. Bids and offers are simultaneously submitted by the traders and then aggregated to form supply and demand curves. Using the weight k, a market-clearing price $p = (1 - k)a + kb$ is selected from the interval $[a, b]$ of all possible market-clearing prices. Buyers whose bids were above p then trade with sellers whose offers were below p. The k-DA institutionalizes Marshall's model of supply and demand as a market mechanism. It is a practical method for organizing trade that is well-grounded in classical microeconomic thought.[1]

If the market is perfectly competitive, then the k-DA solves the problem of the market exactly in the sense that its allocation is efficient. If the market has only a finite number of traders, however, then the k-DA's allocation may be inefficient, i.e., there may be "error" in this algorithm's solution. This may occur because traders in a finite market who privately know their preferences need not act as price-takers and their strategic efforts to influence price in their favor can cause inefficiency in the allocation. Such inefficiency is common among market mechanisms, for every such mechanism must manage the strategic behavior of the traders as each attempts to manipulate the market's outcome in his favor. Interpreted as an algorithm, the error of a market mechanism in computing the gains from trade is the fraction of the expected potential gains from trade that are inefficiently not achieved by the traders because of their strategic behavior. Market mechanisms differ in how successfully they limit this error. Reflecting

[1] There are a multitude of other market mechanisms that are used in practice, studied theoretically, and tested in experiments. See Friedman (1993) for a survey of these mechanisms, including the k-DA.

the theory of perfect competition, however, the error in any reasonable mechanism should converge to zero as the number of traders on each side of the market increases to infinity.

It is common in computer science to evaluate an algorithm by bounding its error with a function of some measure m of a binding constraint on the operation of the algorithm. The number m could, for instance, be the number of numerical inputs into the algorithm, the number of iterations the algorithm is permitted, or some measure of the amount of time that the algorithm is allowed to approximate the exact solution to the problem. A bound of this kind expresses the rate at which error diminishes as the constraining mesure m is relaxed, with error converging to zero as m goes to infinity. An algorithm with a faster rate of convergence is deemed superior to an algorithm with a slower rate of convergence because, for sufficiently large m, it approximates the exact solution of the problem more accurately than the slower algorithm.

We adapt this methodology from computer science to rank the k-DA relative to other market mechanisms in a simple model of trading. The trading model is as follows. There are m buyers, each of whom wishes to purchase at most one unit of an indivisible, homogeneous good, and m sellers, each of whom has one unit of the good to sell. The number m is the *size* of the market. Each buyer i and each seller j privately knows the value v_i or cost c_j that he places on a unit. Buyer i receives a payoff of $v_i - x_i$ when he purchases a unit and pays x_i while seller j receives payoff $y_j - c_j$ when he sells his unit and receives a payment of y_j. A trader who does not trade receives zero as his payoff. A trader privately knows his own value/cost and regards the values of all buyers as independent draws from the distribution $G(\cdot)$ and the costs of all sellers as independent draws from the distribution $F(\cdot)$. All of the above is common knowledge among the traders. A pair (G, F) is an *environment*.[2] It is assumed in this paper that the distributions G and F are C^1 functions with support $[0, 1]$ and respective densities g and f. An independent private values model with quasilinear utility is thus assumed here and the Bayesian game approach of Harsanyi (1967-68) is used to analyze the strategic behavior of traders.

A "maximin" approach is used to evaluate mechanisms. For each m, the error of a mechanism in a particular environment (G, F) is the fraction of the expected potential gains from trade (calculated with respect to G and F) that are inefficiently not achieved by the mechanism in equilibrium. Error defined in this manner measures ex ante incentive efficiency in the sense of Hölmstrom and Myerson (1983). The worst-case error of the mechanism is computed by maximizing this error over a set of possible environments. The rate at which their worst-case errors converge to zero is then used to compare mechanisms.

[2] A model of Telser (1978) explains (G, F) as the demand and supply of the limiting continuum market: $1 - G(p)$ is the mass of buyers and $F(p)$ the mass of sellers in a continuum market who can profitably trade at the price p. The finite market that we consider is obtained by independently sampling m buyers from the demand curve $1 - G$ and m sellers from the supply curve F. Common knowledge and symmetry of beliefs in the finite market both follow from common knowlege of demand and supply in the continuum market together with common knowledge of the sampling process.

The meaning of this comparison is illustrated by the problem of selecting a mechanism for a market in which the environment and the number m of traders on each side of the market may vary: a mechanism whose worst-case error converges to zero at a faster rate than that of some alternative mechanism guarantees superior performance over the class of environments whenever the market is sufficiently large.

The k-DA is compared in this paper to mechanisms that satisfy for each m and each environment both *interim individual rationality* (i.e., each trader's conditional expected payoff as a function of his value/cost is nonnegative) and *ex ante budget balance* (the expected sum of the transfers among the traders is nonnegative, so that the mechanism on average does not require a subsidy to operate). These rather weak restrictions are satisfied by most common mechanisms for trading.[3] The main result of this paper is that the k-DA is *worst-case asymptotic optimal* among all mechanisms for organizing trade that satisfy these two constraints. "Asymptotic" refers here to the ranking of mechanisms using rates of convergence and "worst-case" refers to the evaluation of each mechanism in its least favorable environment for each value of m. Stated simply, this result means that the k-DA's worst-case error over a set of environments converges to zero at the fastest possible rate. Our result complements an earlier result of Wilson (1985) concerning the optimality of the k-DA. He showed that it is interim incentive efficient in the Hölmstrom-Myerson sense when m is sufficiently large.

It is noteworthy that our optimality result concerns a property of this simple, well-motivated mechanism across a range of possible environments and sizes of the market, rather than simply for a single, fixed environment and size of market. Our result thus responds to the "Wilson critique" (Wilson, 1987) of mechanism design. Wilson criticized this field for focusing upon the problem of designing a mechanism explicitly for each specific problem (e.g., as determined here by the specification of an environment and a market size). An economic consultant asked for advice on the selection of a mechanism may not know all the parameters that specify the problem, and the parameters may change over time; theoretical results that describe how the mechanism should be chosen assuming detailed knowledge of the problem may thus have little value to the consultant. The more meaningful task for mechanism design is thus to establish the sense in which a simple mechanism performs reasonably well across the variety of problems that might be encountered in practice, which is the nature of our result.

We discuss below in the next four sections (i) the model, (ii) a result on the k-DA, (iii) a formal statement of our main result, and (iv) the analogy to the asymptotic analysis of algorithms in computer science. Proofs then follow in

[3] Most common mechanisms (such as the k-DA) satisfy the stricter constraints of *ex post budget balance* (the transfers among the traders balance for every sample of values/costs) and *ex post individual rationality* (no trader is ever forced to accept an unprofitable trade). We use the weaker constraints of interim individual rationality and ex ante budget balance here because optimizing over this larger class of mechanisms strengthens the sense in the k-DA is deemed optimal.

sections 6 and 7. Besides guaranteeing superior performance in a sufficiently large market, it is reasonable to hypothesize that a market mechanism exhibiting a faster rate of convergence than another mechanism will also be more efficient in small markets. Such hypotheses concerning algorithms in computer science are commonly tested through numerical computations that compare the fast algorithm to slower algorithms in a variety of small problems. In the next-to-last section we report the results of a set of such computations that compare the k-DA across a variety of environments and small sizes of markets to the constrained efficient mechanism. The constrained efficient mechanism achieves, for each choice of the environment and each size of market, the maximum possible gains from trade obtainable by an interim individually rational and ex ante budget balanced mechanism. The gains from trade achieved in the k-DA are almost indistinguishable in these computations from those of the constrained efficient mechanism, even in markets with as few as eight traders on a side. These computations thus support the hypothesis that the k-DA performs almost as well as any market mechanism can, even in very small markets.

2 The model

Having defined the trading problem in the introduction, we next define market games and mechanisms. To begin, a generic set of environments is denoted as E. A *market game* ϕ_m *of size* m *over the set of environments* E consists of:

1. a strategy set A_i for each of the $2m$ traders;
2. an outcome mapping $\zeta_m : \left(\prod_{i=1}^{2m} A_i\right) \times E \longrightarrow ([0,1] \times \mathbb{R})^{2m}$ that specifies for each trader his probability of receiving an unit along with a monetary transfer as functions of the profile of strategies and the environment;
3. the selection of a Bayesian-Nash equilibrium in the game defined by (1) and (2) for each environment $(G, F) \in E$.[4]

A *market mechanism over* E is a sequence $\Phi \equiv (\phi_m)_{m \in \mathbb{N}}$ in which ϕ_m is a market game of size m over E.

Efficiency dictates that in each sample of $2m$ values/costs the m units must be allocated to the traders with the m highest values/costs. In the efficient allocation, buyers whose values are among the top m values/costs purchase units from sellers whose values are among the m smallest values/costs. Let $\Gamma_m(G, F)$ denote the expected potential gains from trade among the $2m$ traders, computed with respect to the joint distribution of their $2m$ values/costs. The value $\phi_m(G, F)$ denotes the expected gains from trade achieved by the $2m$ traders

[4] Our definition of a market game is unusual in that (i) the outcome mapping ζ_m can depend upon the environment and (ii) an equilibrium is specified for each environment. Property (i) allows the market game to be chosen optimally for each environment, which is a central theme in mechanism design. Property (ii) is part of the definition of the game purely because this simplifies the discussion.

in the selected equilibrium of the market game ϕ_m when (G, F) is the environment. Our measure of inefficiency of a market game is *relative inefficiency* $e(\phi_m, G, F)$,which is the fraction of the expected potential gains from trade in the environment (G, F) that are inefficiently not achieved in the selected equilibrium of ϕ_m:

$$e(\phi_m, G, F) \equiv \frac{\Gamma_m(G, F) - \phi_m(G, F)}{\Gamma_m(G, F)}. \tag{1}$$

Myerson and Satterthwaite (1983) showed in the case of bilateral trade ($m = 1$) that $e(\phi_m, G, F) > 0$ in any market game ϕ_m satisfying interim individual rationality and ex ante budget balance. This result was later extended to arbitrary values of m by Williams (1997, Thm. 4). These results imply that an interim individually rational and ex ante budget balanced mechanism Φ is necessarily inefficient, regardless of the size of the market m.

3 Results on the k-DA

A *k-DA mechanism* $\Phi^{k\text{-DA}} \equiv (\phi_m^{k\text{-DA}})_{m \in \mathrm{N}}$ is the sequence of market games described at the beginning of this paper[5] together with the selection for each m of an equilibrium for the market of size m in the environment (G, F) that has the following three properties:

1. The equilibrium is symmetric in the sense that each buyer uses the same function $B_m(\cdot)$ and each seller uses the same function $S_m(\cdot)$ to select his bid/ask as functions of his value/cost.
2. At every $v_i, c_j \in [0, 1]$, $B_m(v_i) \leq v_i$ and $S_m(c_j) \geq c_j$, i.e., traders do not use dominated strategies.
3. The sets $\{v_i \,|B_m(v_i) > 0\}$ and $\{c_j \,|S_m(c_j) < 1\}$ have positive measure, which implies that trade occurs with positive probability.

An equilibrium satisfying 1-3 is denoted $\langle B_m, S_m \rangle$. With our definition of a mechanism, each rule for selecting an equilibrium defines a different k-DA. The precise rule for choosing an equilibrium, however, is immaterial for our purposes as long as the selected equilibrium satisfies these three properties. Notice that property 2 insures that each equilibrium $\langle B_m, S_m \rangle$ satisfies interim individual rationality, and the rule that all trades in the k-DA are consummated at a market-clearing price insures that every equilibrium satisfies ex ante budget balance. Any k-DA mechanism thus satisfies these two constraints.

[5] The rules of the k-DA are defined in detail in Rustichini et. al. (1994, p. 1045). We will not be analyzing the operation of this mechanism in this paper, relying instead on results drawn from this earlier paper. It is thus sufficient here to understand that the game form ϕ_m^{k-da}operates as described in the introduction.

The following theorem concerning the rate at which the relative inefficiency of the k-DA mechanism converges to zero is the main result on the k-DA that is needed in this paper.

Theorem 1 (Rustichini, Satterthwaite and Williams (1994)) *There exists a continuous function* $\kappa : \mathbb{R}^{+^2} \to \mathbb{R}^{+^2}$ *such that*

$$e(\phi_m^{k\text{-}DA}, G, F) \leq \frac{\kappa(\underline{q}, \overline{q})}{m^2} \tag{2}$$

in any environment (G, F) *satisfying* $0 < \underline{q} \leq g, f \leq \overline{q}$.

In other words, $e(\phi_m^{k\text{-DA}}, G, F) = O\left(\frac{1}{m^2}\right)$. This theorem follows from Theorem 3.2 of Rustichini, Satterthwaite, and Williams (1994), which states that $e(\phi_m^{k\text{-DA}}, G, F)$ is bounded above by $\frac{\xi}{m^2}$ for some function ξ of G, F, and k. In (2) we have replaced ξ with a bound that holds for all $k \in [0, 1]$ and that expresses the dependence of the bound on G and F explicitly in terms of the bounds $\underline{q}$ and $\overline{q}$ on the densities. A function $\kappa(\underline{q}, \overline{q})$ that satisfies (2) can be obtained by working through the proofs of Theorems 3.1 and 3.2 in Rustichini, Satterthwaite, and Williams (1994).

The existence of a k-DA mechanism for an environment (G, F) requires the existence for each m of an equilibrium of the form $\langle B_m, S_m \rangle$ in the market of size m. As with many games of incomplete information with continuous type spaces, one expects that equilibria exist in the k-DA for each size of market m, but existence is difficult to prove. The issue of existence of equilibria is sidestepped in this paper by assuming for each $0 < \underline{q} \leq 1 \leq \overline{q}$ the existence of a set of environments $E(\underline{q}, \overline{q})$ with the following properties:

1. Each $(G, F) \in E(\underline{q}, \overline{q})$ satisfies the bounds $0 < \underline{q} \leq g, f \leq \overline{q}$.

2. For each $(G, F) \in E(\underline{q}, \overline{q})$, an equilibrium of the form $\langle B_m, S_m \rangle$ exists in $\phi_m^{k\text{-DA}}$.

3. The *uniform environment* (G^u, F^u) in which both G and F are uniform on $[0, 1]$ is in $E(\underline{q}, \overline{q})$.

Restriction 1 is imposed with the bound (2) in mind: the densities g and f are bounded above and away from zero in order to define a single bound on relative inefficiency in the k-DA that holds over the entire set $E(\underline{q}, \overline{q})$. This will be important in the worst-case analysis that follows. The existence of a set of environments $E(\underline{q}, \overline{q})$ for which equilibria in the k-DA exist is supported by Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989), which prove the existence of a variety of equilibria in the case of $\phi_1^{k\text{-DA}}$ (i.e., the bilateral k-DA) over a broad class of environments, and by Williams (1991), who proved existence of a unique smooth equilibrium in $\phi_m^{1\text{-DA}}$ (the 1-DA) for a generic set of environments, including the uniform case (G^u, F^u). The existence of $E(\underline{q}, \overline{q})$ is also supported by our experience with computing equilibria in the k-DA, as illustrated below in section 8.

4 The main result

A market game ϕ_m is evaluated over a set of environments E according to its *worst-case relative inefficiency* $e^{wor}(\phi_m, E)$, which is defined as

$$e^{wor}(\phi_m, E) \equiv \sup_{(G,F)\in E} e(\phi_m, G, F). \tag{3}$$

Given a set E of environments, a mechanism Φ defines a sequence of worst-case relative inefficiency values. A mechanism Φ is *worst-case asymptotic optimal* over E among some set M of mechanisms defined on E if the sequence of worst-case relative inefficiency values for any other mechanism in M cannot converge to zero at a faster rate than the sequence defined by the mechanism Φ. This notion of optimality is captured by the following definition.

Definition 2 *Given a set E of environments and a set M of mechanisms defined on E, a mechanism Φ is* ***worst-case asymptotic optimal*** *over E among mechanisms in M if, for any other mechanism $\Phi^* \in M$, there exists a constant $\eta \in \mathbb{R}^+$ such that*

$$e^{wor}(\phi_m, E) \leq \eta \cdot e^{wor}(\phi_m^*, E) \tag{4}$$

for all $m \in \mathbb{N}$.

The main theorem of this paper can now be stated.

Theorem 3 *For each $\underline{q}, \overline{q}$ satisfying $0 < \underline{q} \leq 1 \leq \overline{q}$, a k-DA mechanism Φ^{k-DA} is worst-case asymptotic optimal over the set of environments $E(\underline{q}, \overline{q})$ among all interim individually rational and ex ante budget-balanced mechanisms defined on $E(\underline{q}, \overline{q})$.*

The strength of this result is emphasized by noting that the constraints that it imposes on a mechanism are weak enough to allow the possibility that the mechanism operates over time, consummates trades at a number of different prices, runs surpluses and deficits that cancel only in expectation, or compels traders on occasion to accept losses ex post. A great variety of market mechanisms are thus covered by Theorem 3. While at first glance the restriction to the set of environments $E(\underline{q}, \overline{q})$ in the theorem seems to weaken the sense in which the k-DA is proven optimal, there are two points concerning the freedom to choose $\underline{q}$ and $\overline{q}$ that show that this is not the case. First, $E(\underline{q}, \overline{q})$ can include a great range of possible environments if $\underline{q}$ chosen to be small and $\overline{q}$ large. Second, because the theorem fundamentally concerns worst-case relative inefficiency, there is a sense in which the optimality of the k-DA is strengthened by choosing $E(\underline{q}, \overline{q})$ to be an arbitrarily small set, for it is in this way made clear that optimality of the k-DA does not depend upon consideration of extreme environments. As we next discuss, the special case of the theorem in which $\underline{q} = \overline{q} = 1$ and $E(\underline{q}, \overline{q})$ contains only the uniform environment is in fact the basis of the proof of the theorem.

Let $\Phi^{ce} = (\phi_m^{ce})_{m \in \mathbb{N}}$ denote a mechanism with the property that, for each m and each environment (G, F) in E, ϕ_m^{ce} maximizes the achieved gains from trade in the environment (G, F) subject to the constraints of interim individual rationality and ex ante budget balance.[6] Alternatively, ϕ_m^{ce} can be described as a market game ϕ_m that solves the constrained optimization problem

$$\min_{\phi_m} e(\phi_m, G, F) \tag{5}$$

subject to the constraints on ϕ_m of interim individual rationality and ex ante budget balance. A market game ϕ_m^{ce} that solves (5) is *constrained efficient* in the environment (G, F); ϕ_m^{ce} is constrained efficient in E if it solves (5) for every $(G, F) \in E$. A mechanism Φ^{ce} is *constrained efficient* in (G, F) (or E) if each of its market games ϕ_m^{ce} is constrained efficient in (G, F) (or E). Solving for ϕ_m^{ce} is the central problem in Bayesian mechanism design. The existence and the properties of Φ^{ce} will be discussed in section 6. Notice that any interim individually rational and ex ante budget-balanced mechanism Φ defined on $E(\underline{q}, \overline{q})$ satisfies

$$e^{wor}(\phi_m, E(\underline{q}, \overline{q})) \geq e(\phi_m, G^u, F^u) \geq e(\phi_m^{ce}, G^u, F^u) \tag{6}$$

for each m, where the first inequality is true because $(G^u, F^u) \in E(\underline{q}, \overline{q})$ and the second holds because ϕ_m^{ce} solves (5). As demonstrated below, Theorem 3 is a consequence of the following theorem.

Theorem 4 *There exists a positive number γ such that*

$$e(\phi_m^{ce}, G^u, F^u) \geq \frac{\gamma}{m^2}. \tag{7}$$

Establishing this lower bound on the relative inefficiency of the constrained efficient mechanism in the uniform environment constitutes most of the formal analysis of the paper. This theorem is proven in sections 6 and 7. We now show how the proof of our main result follows directly from it.

Proof of Theorem 3. Letting Φ denote an alternative mechanism defined on $E(\underline{q}, \overline{q})$, we need to find a positive number η that satisfies, for all $m \in \mathbb{N}$,

$$e^{wor}(\phi_m^{k-\mathrm{DA}}, E(\underline{q}, \overline{q})) \leq \eta(e^{wor}(\phi_m, E(\underline{q}, \overline{q}))).$$

Inequalities (6) and (7) together imply

$$e^{wor}(\phi_m, E(\underline{q}, \overline{q})) \geq \frac{\gamma}{m^2} = \left(\frac{\gamma}{\kappa(\underline{q}, \overline{q})}\right)\left(\frac{\kappa(\underline{q}, \overline{q})}{m^2}\right).$$

Theorem 1 then implies

$$e^{wor}(\phi_m, E(\underline{q}, \overline{q})) \geq \left(\frac{\gamma}{\kappa(\underline{q}, \overline{q})}\right) e^{wor}(\phi_m^{k-\mathrm{DA}}, E(\underline{q}, \overline{q})).$$

The proof is then completed by setting $\eta = \frac{\kappa(\underline{q}, \overline{q})}{\gamma}$. ∎

[6] Recall that the constraint of incentive compatibility is implicit in our definition of a mechanism.

5 An analogy to a method for analyzing algorithms in computer science

Theorem 3 applies a method used in computer science for the asymptotic analysis of algorithms. We now discuss this method in order to clarify the meaning of our result. A general theory of the asymptotic analysis of algorithms is developed in Traub, Wasilkowski, and Woznikowski (1988, Ch. 10). For simplicity, we discuss this method here in the familiar problem of approximating an integral of a function, where the analogy to the market problem is transparent. This analogy is summarized in Table 1.

In the same way that a market problem is specified by an environment (G, F), an integration problem is specified by a continuous function $h : [0,1] \to \mathbb{R}$. The *solution* $\Gamma(h)$ of the integration problem is the exact value of $\int_0^1 h(t)dt$. The goal of the integration problem is to approximate $\Gamma(h)$ knowing only the values of h at the $m+1$ points $\{\frac{i}{m} \mid 0 \le i \le m\}$. An *algorithm* Φ for approximating $\Gamma(h)$ is a sequence $\Phi = (\phi_m)_{m \in \mathbb{N}}$ in which ϕ_m is a function of the $m+1$ values $h(0), h(\frac{1}{m}), \ldots, h(\frac{m-1}{m}), h(1)$. Given an algorithm Φ, the solution $\Gamma(h)$ for an arbitrary C^2 function h cannot be computed exactly by ϕ_m for any finite value of m. This is analogous to the problem an outsider would face in trying to achieve all gains from trade knowing only the reported values/costs of the traders: error is fundamentally part of each problem.

It is common to measure an algorithm in the integration problem using *absolute error*, which is defined for an algorithm Φ, a function h, and a value of m as

$$e(\phi_m, h) \equiv |\Gamma(h) - \phi_m(h)| .$$

It is sometimes more meaningful in the evaluation of algorithms to study *relative error*, i.e.,

$$\frac{|\Gamma(h) - \phi_m(h)|}{|\Gamma(h)|}.$$

Our notion of relative inefficiency is analogous to relative error. We use a relative measure of error in part because the solution (or expected potential gains from trade) $\Gamma_m(G, F)$ that is approximated in the market problem grows linearly with the size of the market m.

Any reasonable algorithm for the integration problem should approximate $\Gamma(h)$ more accurately as m increases. Formally, this means

$$\lim_{m \longrightarrow \infty} e(\phi_m, h) = 0$$

for all h. Algorithms differ, however, in how quickly their errors converge to zero. In order to pursue a worst-case analysis of algorithms for the integration problem, we now assume that h is C^2 and satisfies $\left|\frac{dh}{dx}\right|, \left|\frac{d^2h}{dx^2}\right| \le q$ for some

Table 1: The Analogy Between the Market Allocation Problem and the Integration Problem

	Market	Integration
Problem:	achieve $\Gamma_m(G,F)$, the expected gains from trade in the market of size m when (G,F) is the environment	approx. $\Gamma(h)=\int_0^1 h(x)dx$ using $(h(x_i))_{0\le i\le m}$, where $x_i=\frac{i}{m}$
Class of problems E:	$(G,\ F)\in E(\underline{q},\overline{q})\Rightarrow$ G,F are C^1 on $[0,1]$ $\ni$ $\underline{q}\le f,g\le\overline{q}$ for $0<\underline{q}\le 1\le\overline{q}$	$h\in E(q)\Leftrightarrow$ h is C^2 on $[0,1]$ $\ni \left\lvert\frac{dh}{dx}\right\rvert$, $\left\lvert\frac{d^2h}{dx^2}\right\rvert\le q$ for $q\in\mathbb{R}^+$
Algorithm $\Phi=(\phi_m)$:	ϕ_m is a game along with a rule for selecting an equilibrium for each (G,F)	$\phi_m(h(x_0),\dots,h(x_m))$ is any function
Error:	relative error: $e(\phi_m,G,F)$ $=\frac{\Gamma_m(G,F)-\phi_m(G,F)}{\Gamma_m(G,F)}$	absolute error: $e(\phi_m,h)=$ $\lvert\Gamma(h)-\phi_m((h(x_i)))\rvert$
Bound on $e^{wor}(\phi_m,E)$ for all ϕ_m:	$e^{wor}(\phi_m,E(\underline{q},\overline{q}))\ge\frac{\gamma}{m^2}$ where $\gamma\in\mathbb{R}^+$	$e^{wor}(\phi_m,E(q))\ge\frac{\alpha q}{m^2}$ where $\alpha\in\mathbb{R}^+$
Worst-case asymptotic optimal algorithm:	the k-double auction Φ^{k-DA}, $e(\phi_m^{k-da},G,F)\le\frac{\kappa(\underline{q},\overline{q})}{m^2}$ where $\kappa(\underline{q},\overline{q})\in\mathbb{R}^+$	the trapezoid rule Φ^{tr}, $e(\phi_m^{tr},h)\le\frac{q}{12m^2}$
An inferior algorithm:	the fixed-price mechanism Φ^{fp}, $e^{wor}(\phi_m^{fp},E(\underline{q},\overline{q}))\ge\frac{\beta}{\sqrt{m}}$ where $\beta\in\mathbb{R}^+$	the rectangle rule Φ^{rec}, $e^{wor}(\phi_m^{rec},E(q))=\frac{q}{2m}$

$q \in \mathbb{R}^+$. Let $E(q)$ denote the set of C^2 functions on $[0,1]$ that satisfy these bounds. It is well-known among those who study the integration problem that

$$e^{wor}(\phi_m, E(q)) \equiv \sup_{h \in E(q)} e(\phi_m, h) \geq \frac{\alpha q}{m^2} \tag{8}$$

for some constant α.[7] This is analogous to our Theorem 4. The trapezoid algorithm Φ^{tr} approximates $\int_0^1 h(x)\,dx$ by

$$\phi_m^{tr}\left(h(0), h\left(\frac{1}{m}\right), \ldots, h(1)\right) \equiv \frac{1}{m}\left(\frac{h(0)+h(1)}{2} + \sum_{i=1}^{m-1} h\left(\frac{i}{m}\right)\right).$$

Its error satisfies the bound

$$e(\phi_m^{tr}, h) \leq \frac{q}{12m^2} \tag{9}$$

for all $h \in E(q)$ (Davis and Rabinowitz, 1975, p.42). Analogous to the k-DA, it is clear from (8) and (9) that the trapezoid rule is a worst-case asymptotic optimal algorithm over $E(q)$.

While worst-case asymptotic optimality does not determine a unique algorithm in either of these problems, it does distinguish some algorithms as markedly inferior. In the integration problem, the midpoint algorithm (defined for even values of m) is also worst-case asymptotic optimal over $E(q)$ (Davis and Rabinowitz (1975, p.42)), as is Simpson's Algorithm. The rectangle algorithm Φ^{rec} is defined by the formula

$$\phi_m^{rec}\left(h(0), h\left(\frac{1}{m}\right), \ldots, h(1)\right) \equiv \frac{1}{m}\sum_{i=0}^{m-1} h\left(\frac{i}{m}\right).$$

Its worst-case error satisfies

$$e^{wor}(\phi_m^{rec}, E(q)) = \frac{q}{2m},$$

and so it is an inferior algorithm.[8]

[7] This is discussed in Traub et. al. (1988, p. 375-76). Early results of this kind can be found in Bakhvalov (1959). A proof in the special case of q = 1 can be found in Novak (1988, p. 37). It is straightforward to generalize this proof to arbitrary values of q.

[8] There is a close relationship in the integration problem between worst-case error and the degree of differentiability of the class of functions considered: if h is restricted to the class

$$E^n(q) \equiv \left\{h \in C^n([0,1], \mathbb{R}) \mid \left|\frac{d^i h}{dx^i}\right| \leq q, 1 \leq i \leq n\right\},$$

then $e^{wor}(\phi_m, E^n(q)) \geq \frac{\beta q}{m^n}$ for some $\beta \in \mathbb{R}^+$ in any algorithm Φ. If the typical integration problem consists of a C^4 function $h \in E^4(q)$, then the widespread use of Simpson's Algorithm is consistent with the fact that it is worst-case asymptotic optimal over $E^4(q)$, while the trapezoid algorithm is not. The analogy between the integration problem and the market breaks down at this point, for there is no economic reason to believe that the selection of a market mechanism should be closely tied to the degree of differentiability of the distributions G and F.

In the market problem, the generalization of the fixed-price mechanism of Hagerty and Rogerson (1985) to markets of arbitrary size has a relative inefficiency that is at least $\frac{\beta}{\sqrt{m}}$ for some $\beta \in \mathbb{R}^+$ (Gresik and Satterthwaite (1989, p. 319)). The asymptotic approach ranks this mechanism as inferior, which supports common economic intuition. There are two other mechanisms besides the k-DA that are known to be worst-case asymptotic optimal. First, the constrained efficient mechanism Φ^{ce} is itself worst-case asymptotic optimal over any set E of environments on which it is defined. Second, McAfee (1992) designed an interim individually rational mechanism that generates a monetary surplus. Ex ante payments can be devised to return the expected surplus to the traders and thereby insure that the ex ante budget constraint is satisfied with equality. If such payments are included as part of the mechanism, then it too is worst-case asymptotic optimal over $E(\underline{q}, \overline{q})$.

Such ex ante payments, however, must vary with the environment. Altered in this way, McAfee's mechanism shares the flaw of the constrained efficient mechanism, which is that its outcome function depends upon the environment. This flaw renders a market mechanism implausible for actual use. If such payments are disallowed in McAfee's mechanism and the surplus is instead regarded as a cost of arranging trade, then its worst-case error is at least $\frac{\delta}{m}$ for some $\delta \in \mathbb{R}^+$ (Rustichini, Satterthwaite, and Williams, 1992). McAfee's mechanism without payments is thus inferior to the k-DA in a worst-case asymptotic analysis. Though we suspect that other mechanisms besides the k-DA can be both worst-case asymptotic optimal and robust in the sense that their outcome functions do not vary with the environment, examples of such mechanisms have not yet been found.

6 The constrained efficient mechanism

All that remains to be proven is Theorem 4, which bounds below the error $e(\phi_m^{ce}, G^u, F^u)$ of the constrained efficient market game ϕ_m^{ce} in the uniform environment (G^u, F^u) by $\frac{\gamma}{m^2}$. The purpose of this section is to establish several results concerning Φ^{ce} that are needed in section 7 for the proof of Theorem 4. Let $\alpha \in [0,1]$. A key to characterizing this mechanism is a trader's *α-virtual utility*, which is defined for a buyer/seller as a function of his privately-known value/cost by the following formulas: a buyer's α-virtual utility function is

$$\Psi_\alpha^b(v_i) \equiv v_i + \alpha \frac{G(v_i) - 1}{g(v_i)},$$

and a seller's α-virtual utility function is

$$\Psi_\alpha^s(c_j) \equiv c_j + \alpha \frac{F(c_j)}{f(c_j)}.$$

The environment (G, F) is *regular* if, for each α, the functions $\Psi_\alpha^b(\cdot), \Psi_\alpha^s(\cdot)$ are increasing on $[0,1]$. The uniform environment is regular and the discussion in this section is restricted to regular environments.

For a sample of $2m$ values/costs of the traders, let $t_{(j)}$ denote the jth smallest α-virtual utility among the corresponding $2m$ α-virtual utilities:

$$t_{(1)} \leq t_{(2)} \leq \ldots \leq t_{(2m)}.$$

Recall that the definition of a market mechanism Φ specifies an equilibrium in each market game ϕ_m for each environment (G, F). For a given $\alpha \in [0, 1]$, a market game ϕ_m is an *α-market game in the environment* (G, F) if ϕ_m allocates the m units to buyers and sellers whose α-virtual utilities are among the top m values $t_{(m+1)} \leq t_{(m+2)} \leq \ldots \leq t_{(2m)}$. Trades thus occur in an α-market game between buyers whose α-virtual utilities are at least $t_{(m+1)}$ and sellers whose α-virtual utilities are no more than $t_{(m)}$. For a sequence $A = (\alpha_m)_{m \in \mathrm{N}}$, a market mechanism Φ is an *A-mechanism in the environment* (G, F) if for each m the market game ϕ_m is an α_m-market game in this environment. As summarized in the theorem below, the key insight of the Gresik-Satterthwaite derivation of the constrained efficient mechanism is that an interim individually rational, ex ante budget balanced mechanism is constrained efficient in the environment (G, F) if and only if it is an A^*-mechanism for a particular sequence $A^* \equiv (\alpha^*_m(G, F))_{m \in \mathrm{N}}$ defined below. This insight is a general principle of Bayesian mechanism design.[9]

Because allocation according to α-virtual utilities is central to the discussion of constrained efficiency, it is helpful to develop some intuition before proceeding to the theorem. Notice that

$$\Psi^b_0(v_i) \equiv v_i \text{ and } \Psi^s_0(c_j) \equiv c_j,$$

and so the m items are allocated in an $\alpha = 0$ market game according to the true values/costs of the traders. An $\alpha = 0$ market game is thus efficient; as noted earlier, however, it cannot also be both ex ante budget balanced and interim individually rational. Notice also that $\Psi^b_\alpha(0) < 0$, $\Psi^s_\alpha(1) > 1$,

$$v - \Psi^b_\alpha(v) = \alpha \frac{1 - G(v)}{g(v)} \geq 0,$$

and

$$\Psi^s_\alpha(c) - c = \alpha \frac{F(c)}{f(c)} \geq 0.$$

Except in the case of $v = 1$, $c = 0$ or $\alpha = 0$, a buyer's α-virtual utility is thus less than his value and a seller's α-virtual utility exceeds his cost, with the differences increasing in α. Increasing the value of α distorts the allocation

[9] In a general mechanism design problem, efficiency is a mapping that specifies an outcome as a function of the true types of the agents. The general principle is that the outcome in a constrained efficient revelation mechanism is determined by applying the efficiency mapping to the α-virtual utilities of the agents, where α is chosen so that the mechanism is ex ante budget balanced. This principle originated in Myerson (1981), which concerned the constrained efficient mechanism for auctioning an item. It is derived as a general principle in Wilson (1993) and it appears in almost all derivations of constrained efficient mechanisms.

of a α-market game further and further away from efficiency. Allocating the m units to those traders with the largest α-virtual utilities therefore becomes increasingly inefficient as α increases. It should not be surprising that the key issues in the remainder of the paper are (i) the rate at which the sequence $(\alpha_m^*(G^u, F^u))_{m\in\mathrm{N}}$ that determines the constrained efficient mechanism in the uniform environment decreases to zero as m increases to infinity, and (ii) the relationship between this rate and the rate at which $e(\phi_m^{ce}, G^u, F^u)$ converges to zero.

Some notation is needed to state the theorem. Let $\sigma_m \equiv (v_1, ..., v_m, c_1, ...c_m)$ denote a sample of $2m$ values/costs. Given σ_m and $\alpha \in [0,1]$, for buyer i define $p_\alpha^i(\sigma_m)$ as

$$p_\alpha^i(\sigma_m) \equiv \begin{cases} 1 \text{ if } \Psi_\alpha^b(v_i) \geq t_{(m+1)} \\ 0 \text{ if } \Psi_\alpha^b(v_i) < t_{(m+1)} \end{cases},$$

and for the jth seller define $q_\alpha^j(\sigma_m)$ as

$$q_\alpha^j(\sigma_m) \equiv \begin{cases} 1 \text{ if } \Psi_\alpha^s(c_j) \leq t_{(m)} \\ 0 \text{ if } \Psi_\alpha^s(c_j) > t_{(m)} \end{cases}.$$

These are indicator functions that equal one if and only if the trader trades in the given sample σ when items are allocated by an α-market game. Define the function $\mathrm{Sur}(\alpha, m, G, F)$ by the formula

$$\mathrm{Sur}(\alpha, m, G, F) \equiv \mathcal{E}\left[\left(\sum_{i=1}^{m} \Psi_1^b(v_i) p_\alpha^i(\sigma_m)\right) - \left(\sum_{j=1}^{m} \Psi_1^s(c_j) q_\alpha^j(\sigma_m)\right)\right]. \quad (10)$$

This function is crucial because the equation $\mathrm{Sur}(\alpha, m, G, F) = 0$ determines the value of $\alpha_m^*(G, F)$ that characterizes the constrained efficient market game in this environment. The rather complicated formula (10) fits the standard format of such formulas in Bayesian mechanism design problems. Fortunately, we will develop below an alternative to (10) that avoids much of its complexity.

Theorem 5 (Gresik and Satterthwaite, 1983) *The following statements are true in the case of a regular environment (G, F).*

1. *For each $m \geq 1$, there exists a unique $\alpha_m^*(G, F) \in (0,1)$ that satisfies $\mathrm{Sur}(\alpha_m^*(G, F), m, G, F) = 0$.*

2. *Let $A^* \equiv (\alpha_m^*(G, F))_{m\in\mathrm{N}}$. A constrained efficient mechanism exists in the environment (G, F) and is an A^*-mechanism. Conversely, any A^*-mechanism that satisfies interim individual rationality and ex ante budget balance is constrained efficient in this environment.*

Theorem 5 combines a number of results in Gresik and Satterthwaite (1983) to characterize the constrained efficient market game.[10,11] Of greatest interest for our purposes is an interpretation of $\text{Sur}(\alpha, m, G, F)$ that follows directly from their derivation of this game. Consider any α-market game ϕ_m in the market of size m with the property that any buyer i whose value v_i equals zero and any seller j whose cost c_j equals one has, conditional on his value/cost, an expected payoff equal to zero from participating in ϕ_m. Assume also that ϕ_m is interim individually rational but not necessarily ex ante budget balanced, i.e., the expected gains from trade that ϕ_m generates may differ from the sum of the expected payoffs that it distributes to the traders. The quantity $\text{Sur}\,(\alpha, m, G, F)$ equals this market game's *expected surplus*, i.e.,

$$\begin{aligned} \text{Sur}\,(\alpha, m, G, F) \;=\; & \textit{the expected gains from trade generated} \\ & \quad \textit{by the } \alpha\textit{-market game } \phi_m \qquad\qquad (11) \\ & - \textit{the sum of the ex ante expected payoffs received} \\ & \quad \textit{by the 2m traders in this market game.} \end{aligned}$$

This interpretation of $\text{Sur}\,(\alpha, m, G, F)$ is important because it will be used to establish a formula for this function that is substantially simpler than (10).[12] We consider next the family of *two-price A-mechanisms*, which is a particular family of A-mechanisms in which a buyer with value zero and a seller with cost one each have an interim expected payoff of zero. A formula for the expected surplus $\text{Sur}\,(\alpha, m, G, F)$ is derived below from the features of this family of mechanisms. The interpretation of $\text{Sur}\,(\alpha, m, G, F)$ as the expected surplus establishes that this alternative formula is equivalent to (10), which would be

[10] This theorem follows from Theorems 2 and 3 of their paper together with the following three observations. First, while Gresik and Satterthwaite assume the stronger constraint of ex post budget balance, only the weaker constraint of ex ante budget balance is needed to derive Theorem 5 above. They showed that transfers in a constrained efficient mechanism can always be altered to satisfy ex post budget balance without disturbing the constrained efficiency of the mechanism. Second, Williams (1997, Thm. 4) proved that $\text{Sur}(0, m, G, F) < 0$. This inequality implies both that ϕ_m^{ce} cannot be an $\alpha = 0$ market game (which is one of the possible conclusions of their theorems) and also (together with their Theorem 3) the existence of a solution to $\text{Sur}(\alpha_m^*(G, F), m, G, F) = 0$. Third, any $\alpha_m^*(G, F)$ that solves this equation is shown by Gresik and Satterthwaite to define a constrained efficient allocation. Because inefficiency is increasing in α, the solution $\alpha_m^*(G, F)$ must therefore be unique.

[11] Results of this kind are now standard in the derivation of constrained efficient mechanisms. Because the relevant material of Gresik and Satterthwaite (1983) is unpublished, the reader may wish to consult the derivation in Myerson and Satterthwaite (1983), which presents the main ideas of the analysis in the simplified setting of bilateral trade ($m = 1$), or the general discussion in Wilson (1993).

[12] In the Gresik-Satterthwaite derivation, the constraints of incentive compatibility and interim individual rationality are first applied to show that the search for a constrained efficient market game can be restricted to the family of α-market games. The expected surplus is then computed in an arbitrary α-market game by subtracting the expected payoffs of the $2m$ traders from the total expected gains from trade. It is then clear in their analysis that the interim expected payoff of a buyer with value zero or a seller with cost one both equal zero in the constrained efficient market game, in which case the expected surplus reduces to $\text{Sur}\,(\alpha, m, G, F)$. The equation $\text{Sur}\,(\alpha_m^*(G, F), m, G, F) = 0$ that characterizes the constrained efficient mechanism is thus simply the ex ante budget constraint applied to this reduced form of the optimization problem, which completes their derivation of the constrained efficient mechanism.

difficult to prove by directly reducing one of these formulas into the other. This alternative formula for $\mathrm{Sur}(\alpha, m, G, F)$ makes it much easier to examine how the value $\alpha_m^*(G, F)$ that characterizes the constrained efficient market game varies as m increases.

For a given sequence $A \equiv (\alpha_m)_{m\in\mathbb{N}}$, define the two-price A-mechanism $\Phi^{2,A} = \left(\phi_m^{2,A}\right)_{m\in\mathbb{N}}$ as follows. For given m, starting with values/costs as reported by the traders, compute the α_m-virtual utilities as functions of these reports and rank the results. Allocate the m units to the traders whose α_m-virtual utilities are the m largest (i.e., those at or above $t_{(m+1)}$); items are allocated in the case of a tie of $t_{(m)} = t_{(m+1)}$ first by assigning items to those traders whose α_m-virtual utilities are strictly above $t_{(m+1)}$, second to buyers whose α_m-virtual utilities equal $t_{(m+1)}$, and last to sellers whose α_m-virtual utilities equal $t_{(m+1)}$, using a fair lottery whenever necessary. Each buyer who purchases a unit pays $(\Psi^b_{\alpha_m})^{-1}(t_{(m)})$ as his price and each seller who sells his unit receives $(\Psi^s_{\alpha_m})^{-1}(t_{(m+1)})$. Traders who fail to trade neither receive nor pay a monetary transfer. As discussed below, reporting one's true value/cost is the unique dominant strategy for each trader in this market game. In this dominant strategy equilibrium, the m items are allocated to traders whose α_m-virtual utilities are among the m largest, which confirms that this is an α_m-market game. The selection of this equilibrium for each m and each (G, F) completes the definition of the two-price A-mechanism.

In order to verify that honest reporting defines the unique dominant strategy equilibrium, consider a buyer with value v who considers reporting v^*. Let $u_{(m)}$ denote the mth smallest α_m-virtual utility among the $2m-1$ values computed using the reports of the other traders. The selected buyer's ex post payoff is:

$$\begin{aligned} v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)}) \; if \; \Psi^b_{\alpha_m}(v^*) &> u_{(m)}; \\ \pi(v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)})) \; if \; \Psi^b_{\alpha_m}(v^*) &= u_{(m)}; \\ 0 \; if \; \Psi^b_{\alpha_m}(v^*) &< u_{(m)}. \end{aligned} \tag{12}$$

In (12) π represents the probability that the selected buyer receives an item if randomization is needed to complete the allocation. The value of π depends only upon the values and costs reported by the $2m-1$ other traders. It is clear from (12) that the selected buyer maximizes his ex post payoff through his choice of v^* if he receives $v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)})$ when it is positive and zero when it is not. Regularity implies that

$$v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)}) > 0 \Leftrightarrow \Psi^b_{\alpha_m}(v) > u_{(m)}.$$

This equivalence implies that $v^* = v$ is the unique report that guarantees the selected buyer receives $v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)})$ exactly when it is positive. Similar to the second-price Vickrey auction, it follows that $v^* = v$ is the selected buyer's unique dominant strategy. A similar argument establishes that honestly reporting his cost is the unique dominant strategy of every seller. Notice that ties among the α_m-virtual utilities occur with probability zero in this dominant strategy equilibrium. Consequently we ignore this event in the remainder of this paper.

Two properties of this mechanism are noteworthy. First, the dominant strategy equilibrium is interim individually rational. A buyer with value v_i trades only if $\Psi^b_{\alpha_m}(v_i) \geq t_{(m)}$, which, by the assumption of regularity, implies that $v_i \geq (\Psi^b_{\alpha_m})^{-1}(t_{(m)})$. The price $(\Psi^b_{\alpha_m})^{-1}(t_{(m)})$ he pays when he buys is thus no more than his value v_i. A similar argument shows that the price received by a seller is at least as large as his cost. Second, a buyer with value $v_i = 0$ or a seller with cost $c_j = 1$ has an expected payoff equal to zero because a trader with this value or cost never trades. If buyer i, for instance, has value $v_i = 0$, then his virtual utility is $\Psi^b_{\alpha_m}(0) < 0$. The α_m-virtual utility of a seller with cost $c_j = 0$ is $\Psi^b_{\alpha_m}(0) = 0$. Regularity thus implies that the α_m-virtual utilities of the m sellers surely exceed the α_m-virtual utility of the ith buyer. The ith buyer thus never trades, for his α_m-virtual utility cannot be among the m largest.

Recall from the discussion of (11) that because a two-price A-mechanism has these two properties, the expected surplus in $\phi^{2,A}_m$ is $\mathrm{Sur}(\alpha_m, m, G, F)$. We now derive a formula for $\mathrm{Sur}(\alpha_m, m, G, F)$ from this mechanism. Let $H\left(t_{(m)}, t_{(m+1)}\right)$ denote the expected number of trades conditional on the values of the mth and the $(m+1)$st α_m-virtual utilities $t_{(m)}$ and $t_{(m+1)}$. The expected surplus conditional on $t_{(m)}$ and $t_{(m+1)}$ is

$$H\left(t_{(m)}, t_{(m+1)}\right)\left(\left(\Psi^b_{\alpha_m}\right)^{-1}\left(t_{(m)}\right) - \left(\Psi^s_{\alpha_m}\right)^{-1}\left(t_{(m+1)}\right)\right), \tag{13}$$

because $\left(\Psi^b_{\alpha_m}\right)^{-1}\left(t_{(m)}\right)$ is the price that buyers pay and $\left(\Psi^s_{\alpha}\right)^{-1}\left(t_{(m+1)}\right)$ is the price that sellers receive. Notice that

$$(\Psi^b_{\alpha_m})^{-1}(t_{(m)}) \geq t_{(m)} \text{ and } (\Psi^s_{\alpha_m})^{-1}(t_{(m+1)}) \leq t_{(m+1)}.$$

Consequently, even though $t_{(m+1)} \geq t_{(m)}$, the price $(\Psi^b_{\alpha_m})^{-1}(t_{(m)})$ paid by buyers can be either above, below or equal to the price $(\Psi^s_{\alpha_m})^{-1}(t_{(m+1)})$ received by sellers, depending on the values of α_m, $t_{(m)}$ and $t_{(m+1)}$. The conditional expected surplus (13) can thus be either positive, negative, or zero. Taking expectations with respect to the joint distribution of $t_{(m)}$ and $t_{(m+1)}$ and replacing α_m with the generic parameter α produces the desired formula for $\mathrm{Sur}(\alpha, m, G, F)$:

$$\begin{aligned}&\mathrm{Sur}(\alpha, m, G, F) = \\ &\mathcal{E}\left[H\left(t_{(m)}, t_{(m+1)}\right)\left(\left(\Psi^b_{\alpha}\right)^{-1}\left(t_{(m)}\right) - \left(\Psi^s_{\alpha}\right)^{-1}\left(t_{(m+1)}\right)\right)\right].\end{aligned} \tag{14}$$

Eq. (14) is clearly different from the standard representation of $\mathrm{Sur}(\alpha, m, G, F)$ in (10), for (14) depends upon both the transfers and the allocation rule in the two-price A-mechanism while (10) depends only upon the allocation rule.

7 A lower bound on the inefficiency of the constrained efficient mechanism in the uniform environment

The alternative formula (14) for the expected surplus is valuable because it makes the equation $\mathrm{Sur}(\alpha_m^*(G^u, F^u), m, G^u, F^u) = 0$ solvable for a lower bound on $\alpha_m^*(G^u, F^u)$. This bound is derived below in Lemma 6. The bound is then used in Theorem 4 to establish the desired lower bound on $e(\phi_m^{ce}, G^u, F^u)$. Because these two results concern only the uniform environment (G^u, F^u) and a fixed market size m, "$\alpha_m^*(G^u, F^u)$" is replaced in this section by the generic parameter "α", except in the statement of the lemma.

We begin by reducing the equation $\mathrm{Sur}(\alpha, m, G^u, F^u) = 0$. Uniformity implies that:

$$\begin{aligned}
\Psi_\alpha^b(v_i) &= (1+\alpha)v_i - \alpha \Leftrightarrow \left(\Psi_\alpha^b\right)^{-1}\left(t_{(m)}\right) = \frac{t_{(m)} + \alpha}{1+\alpha}, \\
\Psi_\alpha^s(c_j) &= (1+\alpha)c_j \Leftrightarrow \left(\Psi_\alpha^s\right)^{-1}\left(t_{(m+1)}\right) = \frac{t_{(m+1)}}{1+\alpha}.
\end{aligned}$$

Substitution of the above formulas for $\left(\Psi_\alpha^b\right)^{-1}\left(t_{(m)}\right)$ and $\left(\Psi_a^s\right)^{-1}\left(t_{(m+1)}\right)$ into (14) implies

$$\mathrm{Sur}(\alpha, m, G^u, F^u) = \mathcal{E}\left[H\left(t_{(m)}, t_{(m+1)}\right)\left(\frac{t_{(m)} + \alpha}{1+\alpha} - \frac{t_{(m+1)}}{1+\alpha}\right)\right].$$

The equation $\mathrm{Sur}(\alpha, m, G^u, F^u) = 0$ can be then solved for α:

$$\alpha = \frac{\mathcal{E}\left[H\left(t_{(m)}, t_{(m+1)}\right)\left(t_{(m+1)} - t_{(m)}\right)\right]}{\mathcal{E}\left[H\left(t_{(m)}, t_{(m+1)}\right)\right]}.$$

The expected number of trades $H\left(t_{(m)}, t_{(m+1)}\right)$ given $t_{(m)}$ and $t_{(m+1)}$ is clearly no more than m, which implies

$$\alpha \geq \frac{\mathcal{E}\left[H\left(t_{(m)}, t_{(m+1)}\right)\left(t_{(m+1)} - t_{(m)}\right)\right]}{m}. \tag{15}$$

Notice that the right side of (15) still depends upon α because its value affects the distributions of $t_{(m)}$ and $t_{(m+1)}$.

Lemma 6 *There exists a constant $\tau \in \mathbb{R}^+$ such that the value $\alpha_m^*(G^u, F^u)$, which characterizes the constrained efficient mechanism in the uniform environment, is at least $\frac{\tau}{m}$ for all m.*

Proof. Starting from (15) it is sufficient to show that there exists a constant τ such that

$$\mathcal{E}[H(t_{(m+1)}, t_{(m)})(t_{(m+1)} - t_{(m)})] \geq \tau. \tag{16}$$

Because this proof concerns the distributions of the traders' α-virtual utilities, it is helpful to note that buyers' and sellers' α-virtual utilities are independently and uniformly distributed on $[-\alpha, 1]$ and $[0, 1+\alpha]$, respectively. Trade occurs only among buyers and sellers whose α-virtual utilities are in $[0,1]$, for an α-virtual utility of a buyer that is in $[-\alpha, 0)$ is surely below those of all sellers and the α-virtual utility of a seller in $(1, 1+\alpha]$ is surely above those of all buyers.

The left side of (16) is calculated by summing over the m^2 events distinguished by the number of α-virtual utilities from each of the two sides of the market that lie within $[0,1]$. For $1 \leq i,j \leq m$, define $A_{i,j}$ as the event in which exactly i buyers' α-virtual utilities and j sellers' α-virtual utilities lie in $[0,1]$. We have

$$\mathcal{E}\left[\left(t_{(m+1)} - t_{(m)}\right) H(t_{(m+1)}, t_{(m)}) \right] =$$

$$\sum_{1 \leq i,j \leq m} \mathcal{E}[\,(t_{(m+1)} - t_{(m)}) H(t_{(m+1)}, t_{(m)}) \mid A_{i,j}\,] \cdot \Pr(A_{i,j}), \tag{17}$$

where the events in which either $i = 0$ or $j = 0$ are omitted because no trades occur in these cases (i.e., $H(t_{(m+1)}, t_{(m)}) = 0$).

We next simplify three terms in (17) in the event $A_{i,j}$. First, observe that

$$\Pr(A_{i,j}) = \binom{m}{i}\binom{m}{j}\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)}. \tag{18}$$

This follows from the distributions of buyers' and sellers α-virtual utilities: for either a buyer or a seller, $\frac{1}{1+\alpha}$ is the probability that his α-virtual utility is in $[0,1]$ and $\frac{\alpha}{1+\alpha}$ is the probability that it is outside this interval. Second, consider $H(t_{(m+1)}, t_{(m)})$. In event $A_{i,j}$, the α-virtual utilities of exactly i buyers' and j sellers' are independently and uniformly distributed on $[0,1]$. Consequently, there are exactly $m-i$ α-virtual utilities of buyers below 0. The values $t_{(m)}$ and $t_{(m+1)}$ in the entire sample of $2m$ α-virtual utilities are thus respectively the ith and the $(i+1)$st among those within $[0,1]$. The expected number of trades $H(t_{(m+1)}, t_{(m)})$ in event $A_{i,j}$ given the values of $t_{(m)}$ and $t_{(m+1)}$ therefore equals the expected number of the i buyers' α-virtual utilities that are among the j largest in this sample of $i+j$ α-virtual utilities from the uniform distribution on $[0,1]$. In such a sample, $\frac{j}{i+j}$ is the probability that the α-virtual utility of any one of these $i+j$ traders is among the j largest. It follows that $\frac{ij}{i+j}$is the expected number of buyers whose α-virtual utilities are among the j largest, and so

$$H(t_{(m+1)}, t_{(m)}) = \frac{ij}{i+j} \text{ in the event} A_{i,j}. \tag{19}$$

Third, $t_{(m+1)} - t_{(m)}$ is the difference between the ith and $(i+1)$st values in this sample of $i+j$ α-virtual utilities that are independently and uniformly distributed on $[0,1]$. It follows from David ((1981), ex. 3.1.1, p. 35) that

$$\mathcal{E}[t_{(m+1)} - t_{(m)} \mid A_{i,j}] = \frac{1}{i+j+1}. \tag{20}$$

Substituting (18), (19), and (20) into (17) produces

$$\mathcal{E}[(t_{(m+1)} - t_{(m)})H(t_{(m+1)}, t_{(m)})] =$$

$$\sum_{1\leq i,j\leq m} \binom{m}{i}\binom{m}{j}\left(\frac{ij}{i+j}\right)\left(\frac{1}{i+j+1}\right)\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)}. \qquad (21)$$

The remainder of this proof is a calculation that bounds (21). It follows from the definition of a binomial coefficient that

$$\binom{m}{i}\binom{m}{j}\left(\frac{ij}{i+j}\right)\left(\frac{1}{i+j+1}\right) = \binom{m-1}{i-1}\binom{m-1}{j-1}(m^2)\left(\frac{1}{i+j}\right)\left(\frac{1}{i+j+1}\right).$$

Recall that $1 \leq i, j \leq m$, which implies that

$$m^2\left(\frac{1}{i+j}\right)\left(\frac{1}{i+j+1}\right) \geq \frac{m^2}{2m(2m+1)} = \frac{1}{4+\frac{2}{m}} \geq \frac{1}{6}.$$

The expression in (21) is thus at least

$$\frac{1}{6}\sum_{1\leq i,j\leq m}\binom{m-1}{i-1}\binom{m-1}{j-1}\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)} =$$

$$\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2\sum_{1\leq i,j\leq m}\binom{m-1}{i-1}\binom{m-1}{j-1}\left(\frac{1}{1+\alpha}\right)^{i+j-2}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)},$$

which after replacing i with $i+1$ and j with $j+1$ becomes

$$\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2 \times \sum_{1\leq i,j\leq m-1}\binom{m-1}{i}\binom{m-1}{j}\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2(m-1)-(i+j)}.$$

This expression factors as

$$\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2\left(\sum_{i=0}^{m-1}\binom{m-1}{i}\left(\frac{1}{1+\alpha}\right)^{i}\left(\frac{\alpha}{1+\alpha}\right)^{m-1-i}\right)^2.$$

Applying the binomial expansion, this equals

$$\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2\left(\left(\frac{1}{1+\alpha}+\frac{\alpha}{1+\alpha}\right)^{m-1}\right)^2 = \frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2.$$

We thus have

$$\mathcal{E}[(t_{(m+1)} - t_{(m)})H(t_{(m+1)}, t_{(m)})] \geq \frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2 \geq \frac{1}{24},$$

where the last inequality holds because $\alpha \in [0,1]$. ■

Using Lemma 6 we can now prove Theorem 4.

Proof of Theorem 4. Let $s_{(j)}$ denote the jth smallest value/cost in a sample of $2m$ buyers' values and sellers' costs in the uniform environment. A lower bound on the expected value of the unrealized gains from trade will be computed by bounding a portion of the losses in the event D that is defined by the following two conditions:

1. $s_{(m)}$ is a seller's cost and $s_{(m+1)}$ is a buyer's value;
2. $\psi^b_\alpha(s_{(m+1)}) < \psi^s_\alpha(s_{(m)}) \Leftrightarrow s_{(m+1)} - s_{(m)} < \frac{\alpha}{\alpha+1}$.

Recall that efficiency requires that the m items be assigned to the traders with the m highest values/costs while a constrained efficient market game ϕ^{ce}_m assigns the items to the traders with m highest α-virtual utilities. Condition 1 implies that both the buyer with value $s_{(m+1)}$ and the seller with cost $s_{(m)}$ should trade for the sake of efficiency, either with each other or with others. Because $\psi^b_\alpha(\cdot)$ is increasing and $c_j \leq \psi^s_\alpha(c_j)$, the $m-1$ α-virtual utilities of traders whose values/costs exceed $s_{(m+1)}$ are above $\psi^b_\alpha(s_{(m+1)})$. Condition 2 implies that there is an additional α-virtual utility above $\psi^b_\alpha(s_{(m+1)})$, for a total of at least m. The buyer whose value equals $s_{(m+1)}$ thus does not trade in ϕ^{ce}_m. A similar argument shows that the seller with cost $s_{(m)}$ also does not trade. The unrealized gains from trade are therefore at least $s_{(m+1)} - s_{(m)}$ in event D.

A lower bound on $\Gamma_m(G^u, F^u) - \phi^{ce}_m(G^u, F^u)$ will now be computed by integrating $s_{(m+1)} - s_{(m)}$ over event D. Define $w \equiv s_{(m+1)} - s_{(m)}$ and let $\rho(w;m)$ denote its density function. Notice first that, for any given value of w, the probability that $s_{(m+1)}$ is a buyer's value and $s_{(m)}$ is a seller's cost equals $\frac{1}{4}$. This is true because each trader's value/cost is independently drawn from the same distribution. A lower bound on $\Gamma_m(G^u, F^u) - \phi^{ce}_m(G^u, F^u)$ is thus given by (22):

$$\begin{aligned} \Gamma_m(G^u, F^u) - \phi^{ce}_m(G^u, F^u) &> \frac{1}{4}\int_0^{\frac{\alpha}{\alpha+1}} w\,\rho(w;m)\,dw \qquad (22) \\ &> \frac{1}{4}\int_0^{\frac{\tau}{2m}} w\rho(w;m)\,dw. \qquad (23) \end{aligned}$$

Because $\frac{\tau}{m} < \alpha \in [0,1]$, it follows that $\frac{\alpha}{\alpha+1} \geq \frac{\alpha}{2} > \frac{\tau}{2m}$. This implies (23).

The integral in (23) is straightforward to evaluate given that buyers' values and sellers' costs are distributed uniformly on $[0,1]$. Eq. (2.3.1) in David (1981,

p. 11) implies that

$$\rho\,(w;m) = \frac{2m!}{((m-1)!)^2}\int_0^{1-w} x^{m-1}(1-x-w)^{m-1}dx. \tag{24}$$

Integration by parts implies that for $j, k \leq 1$,

$$\int_0^{1-w} x^j(1-x-w)^k dx = \int_0^{1-w} \frac{k}{j+1}x^{j+1}(1-x-w)^{k-1}dx.$$

Applying this formula to (24) a total of $m-1$ times and then simplifying produces $\rho(w;m) = 2m\,(1-w)^{2m-1}$. This formula allows us to evaluate the integral in (23):

$$\int_0^{\frac{\tau}{2m}} w\,\rho(w;m)\,dw = \frac{1-(1+\tau)(1-\frac{\tau}{2m})^{2m}}{2m+1}. \tag{25}$$

The term $(1-\frac{\tau}{2m})^{2m}$ in (25) is positive and decreasing in m to $\lim_{m\to\infty}(1-\frac{\tau}{2m})^{2m} = e^{-\tau}$. Substitution into (22)-(23) thus implies

$$\Gamma_m(G^u, F^u) - \phi_m^{ce}(G^u, F^u) > \frac{1}{4} \times \frac{1-(1+\tau)e^{-\tau}}{2m+1} > \frac{\gamma}{m},$$

where

$$\gamma \equiv \frac{1}{4} \times \frac{1-(1+\tau)e^{-\tau}}{3}.$$

To show that $\gamma > 0$, regard τ as a variable and note that: (i) $1-(1+\tau)e^{-\tau} = 0$ at $\tau = 0$; (ii) $\frac{d}{d\tau}\left[1-(1+\tau)e^{-\tau}\right] = \tau e^{-\tau} > 0$ for $\tau > 0$. It follows that $\gamma > 0$ for the positive value of τ given by Lemma 6.

Turning finally to $e(\phi_m^{ce}, G^u, F^u)$, we have

$$e(\phi_m^{ce}, G^u, F^u) \equiv \frac{\Gamma_m(G^u, F^u) - \phi_m^{ce}(G^u, F^u)}{\Gamma_m(G^u, F^u)} > \frac{\gamma}{m\Gamma_m(G^u, F^u)}.$$

The expected potential gains from trade $\Gamma_m(G^u, F^u)$ are at most m because at most m trades can be made, each of value one or less. Therefore, $e(\phi_m^{ce}, G^u, F^u) > \frac{\gamma}{m^2}$. ■

8 A numerical comparison of the k-DA with the constrained efficient mechanism

Our main result is that the k-DA is worst case asymptotic optimal over $E(\underline{q}, \overline{q})$ among the class of interim individually rational and ex ante budget balanced mechanisms. There are two shortcomings of this statement of optimality. First, that it is worst case means that a set of environments may exist in $E(\underline{q}, \overline{q})$ for which some mechanism has a faster rate of convergence to efficiency than does the k-DA. Second, that it is asymptotic means that a mechanism may exist that is more efficient in some environments if the market is sufficiently small. The worst case asymptotic approach is in part a mathematical expedient that reduces the evaluation of algorithms over a range of problems to a single case (the worst one) and a single statistic (the rate of convergence of error to zero). This approach is largely justified by pragmatism, for it allows progress to be made in the difficult task of comparing algorithms.

The rate at which worst-case error converges to zero is a meaningful statistic for ranking algorithms if it reflects practical experience in applying the algorithms to solve problems. The rapid rate of convergence of Simpson's rule relative to the rectangle rule in the integration problem, for instance, is meaningful because it reflects the superiority of Simpson's rule as a practical method in many problems. A proof of worst case asymptotic optimality of the k-DA should therefore be supported with a panel of numerical experiments that, for small markets and a variety of different environments, demonstrates that this mechanism's worst case asymptotic optimality accurately reflects its performance both absolutely and relative to other mechanisms. This numerical testing is important not only for checking the robustness of our worst-case asymptotic optimality result, but also for formulating precise predictions of equilibrium behavior that can be tested in the laboratory using human subjects.[13]

This section initiates such a numerical test. We compare the 0.5-DA to the constrained efficient mechanism across four environments and for the market sizes $m = 2$, 4, and 8. These numerical experiments suggest, irrespective of the tested environment, that the 0.5-DA achieves approximately quadratic convergence even in small markets and nearly matches the constrained efficient mechanism's performance in markets as small as $m = 8$. While examining four environments and three sizes of markets is limited as a test, the results are notably consistent for this cross section of economically plausible environments.[14]

Table 2 summarizes the results of this test. To understand the table fully, we must first explain Figures 1-4, each of which depicts equilibria in the 0.5-DA for m=2, 4 and 8 in one of the four environments. In each figure the upper left square depicts the density functions g and f for the environment (G, F). The next three squares depict for market sizes 2, 4, and 8 a set of equilibria $< B_m, S_m >$ that are symmetric, undominated, result in trade with positive probability, and in which each of the strategies B_m and S_m is a smooth function over the closure of the interval of those values/costs for which the trader's conditional probability of trading is positive. The diagonals in the squares represent honest reporting of a trader's value/cost through his choice

[13] Experimental evaluation of the k-DA has been initiated by Kagel and Vogt (1993) and by Cason and Friedman (1997). Both rely upon computational work in Satterthwaite and Williams (1989b) and Rustichini, Satterthwaite and Williams (1992).

[14] To our knowledge these are the first numerical experiments with the k-DA and the constrained efficient mechanism beyond the uniform environment in the case of $m > 1$. Leininger, Linhart, and Radner (1989) computed equilibria in the k-DA in the bilateral case in non-uniform enviroments.

of a bid/offer. The gap between a strategy and the diagonal thus reflects deviation from price-taking behavior. Each strategy B_m lies below the diagonal, representing underbidding by each buyer, while each strategy S_m lies above the diagonal, reflecting the effort of each seller to push the price upwards. Every strategy B_m pairs with a particular strategy S_m in the figure to define an equilibrium. As detailed in Satterthwaite and Williams (1993), these equilibria are computed by numerically solving the system of differential equations defined by the first order conditions for equilibrium. The initial points for the solutions shown are a sequence of points on an appropriately chosen line. Consequently, these graphs approximate a slice through the set of smooth equilibria for the given environment and market size.

The uniform environment (G^u, F^u) is examined in Figure 1. It is included here because it is both the cornerstone of the worst case analysis of this paper and because it is commonly used in computing examples. We consider bell-shaped density functions in the remaining three figures, consistent with common intuition about economically relevant distributions. Figure 2 depicts the case in which both G and F equal the Beta distribution with parameters 5 and 5 ($B(5,5)$ symbolically). The remaining two figures examine the effect of shifting the mass of this distribution leftward and rightward. Figure 3 depicts the case of $G = B(5,3)$ and $F = B(3,5)$, while Figure 4 depicts the case of $G = B(3,5)$ and $F = B(5,3)$. The two parameters of the Beta distribution are chosen in each case both to produce the desired shape of the density function and also to insure that the distribution satisfies regularity. This property is needed to insure the sufficiency of the first order conditions for equilibria in the 0.5-DA and because the constrained efficient mechanism is well-understood only when the distributions are regular.

Each figure illustrates convergence of equilibria to truthful revelation as the size of the market increases from 2 to 4 and then to 8. As can be confirmed with a ruler, the rate of this convergence is consistent with the $O\left(\frac{1}{m}\right)$ rate established in Rustichini, Satterthwaite, and Williams (1994, Thm. 3.1). In addition to this convergence, each figure shows that the bundles of buyers' strategies and of sellers' strategies become smaller as m increases, even more rapidly than required by the convergence to truthful revelation. For $m = 8$ the equilibrium is nearly unique. While this does not formally resolve the issue of equilibrium selection (it appears that a continuum of equilibria continues to exist), this issue clearly becomes less and less interesting as the size of the market increases. Except in the the uniform environment, convergence to honest reporting seems to lag a bit for buyers with values near $v = 1$ and for sellers with costs near $c = 0$. As can be inferred from the figures, however, the bid of a buyer whose value is near 1 is almost certain to be among the largest of the $2m$ bids/offers; as a consequence, such a buyer almost certainly trades without affecting the price. Similar remarks apply to a seller whose cost c is near 0. Therefore, as Table 2 verifies, the misreporting at extremes of value and cost is unlikely to cause inefficiency and therefore has almost no effect on the relative inefficiency

Table 2. A Comparison of the 0.5-DA Mechanism $\Phi^{0.5-\mathrm{DA}}$ with the Constrained Efficient (CE) Mechanism Φ^{ce} across Four Environments (G, F) and Three Market Sizes m

			0.5-DA $e\left(\phi_m^{0.5\text{-DA}}, \cdot\right)$		CE Mechanism	
(G, F)	m	$\Gamma(\cdot)$	$\min\limits_{\langle B_m, S_m\rangle}$	$\max\limits_{\langle B_m, S_m\rangle}$	$e(\phi^{ce}, \cdot)$	$\alpha^*(\cdot)$
$G = G^u =$	2	0.400	0.058	0.065	0.058	0.226
$F = F^u =$	4	0.889	0.015	0.016	0.015	0.123
$B(1,1)$	8	1.882	0.0037	0.0037	0.0037	0.062
$G = B(5,5)$	2	0.203	0.054	0.059	0.054	0.233
$F = B(5,5)$	4	0.445	0.011	0.013	0.011	0.123
	8	0.936	0.0032	0.0032	0.0032	0.062
$G = B(5,3)$	2	0.553	0.029	0.036	0.029	0.141
$F = B(3,5)$	4	1.137	0.0070	0.0079	0.0070	0.074
	8	2.317	0.0017	0.0022	0.0017	0.039
$G = B(3,5)$	2	0.0529	0.12	0.16	0.12	0.354
$F = B(5,3)$	4	0.137	0.052	0.057	0.051	0.253
	8	0.317	0.014	0.014	0.014	0.140

of the equilibria.[15]

Table 2 reports the error in both the 0.5-DA and the constrained efficient mechanism over these four environments and three sizes of market. The first three columns specify the environment (G, F), the size of the market m, and the expected potential gains from trade $\Gamma(\cdot)$. The next two columns list relative inefficiencies in the 0.5-DA. Figures 1-4 display a multiplicity of equilibria in each environment and for each size of market, and each of these equilibria has a different relative inefficiency. In the fourth and fifth columns we report the minimum and the maximum values of $e(\phi_m^{0.5-da}, G, F)$ over the equilibria that are graphed. While not extrema in a formal sense, they do approximate the inefficiency of the most efficient and the least efficient smooth equilibria. The last two columns concern the constrained efficient mechanism. The column labeled $e(\phi_m^{ce}, G, F)$ is the minimum possible error among all interim individually rational and ex ante budget balanced mechanisms in the market determined by

[15]Formally, this lag in convergence at $c = 0$ and $v = 1$ is attributable to the density of the Beta distribution in each of these cases equaling zero at these points. The $O(\frac{1}{m})$ convergence result of Rustichini et. al. (1994) assumes that the densities are bounded both above and below, away from zero. It is easy to infer from the proof of this result that the $O(\frac{1}{m})$ holds in the cases considered in Figures 2-4 on any proper subinterval $[\varepsilon, 1-\varepsilon]$ of $[0,1]$.

(G, F) and by m. It is the benchmark against which we evaluate the 0.5-DA. Finally, the column labeled $\alpha_m^*(G, F)$ lists the value of α that characterizes the constrained efficient market game in each environment and size of market.

The first three rows of Table 2 concern the uniform environment (G^u, F^u). Notice that $\alpha_m^*(G^u, F^u)$ decreases approximately by a factor of 2 each time that m doubles, consistent with Lemma 6. Similarly, $e(\phi_m^{ce}, G^u, F^u)$ decreases by a factor of $2^2 = 4$ as m doubles, consistent with Theorem 4. With an eye towards the worst case and the asymptotic nature of our optimality result, note the following two points:

1. As m increases from 2 to 4 and then to 8 in each of the four environments, the relative inefficiency of the least efficient (and hence of each) smooth equilibrium of the 0.5-DA rapidly approaches that of the constrained efficient mechanism. In each of the four environments for market size $m = 8$ the relative inefficiency of each equilibrium is almost indistinguishable from that of the constrained efficient mechanism.

2. In each of the three non-uniform environments, both $\alpha_m^*(G, F)$ and $e(\phi_m^{ce}, G, F)$ replicate the convergence rates observed in the uniform environment. The rates of convergence established for the constrained efficient mechanism in Lemma 6 and Theorem 4 thus appear to hold even in these small markets and non-uniform environments.

Point 1 supports our use of rate of convergence as a summary statistic for measuring the performance of the $k-$DA, for its worst-case asymptotic optimality mirrors its nearly optimal performance in these small markets. Point 2 suggests that the uniform environment is typical of environments and not an oddity, which supports our use of it as a convenient worst case. This also is supported by the theory, for uniformity is invoked in our proofs only to simplify several difficult formulas involving order statistics: nothing in the proofs suggests that relative inefficiency in the constrained efficient mechanism depends crucially on the assumption of uniformity. We thus conclude by conjecturing that the k-DA is in fact *asymptotically optimal* over $E(\underline{q}, \overline{q})$, i.e., it exhibits the fastest possible rate of convergence of relative efficiency to zero in the case of each environment (G, F) in $E(\underline{q}, \overline{q})$.

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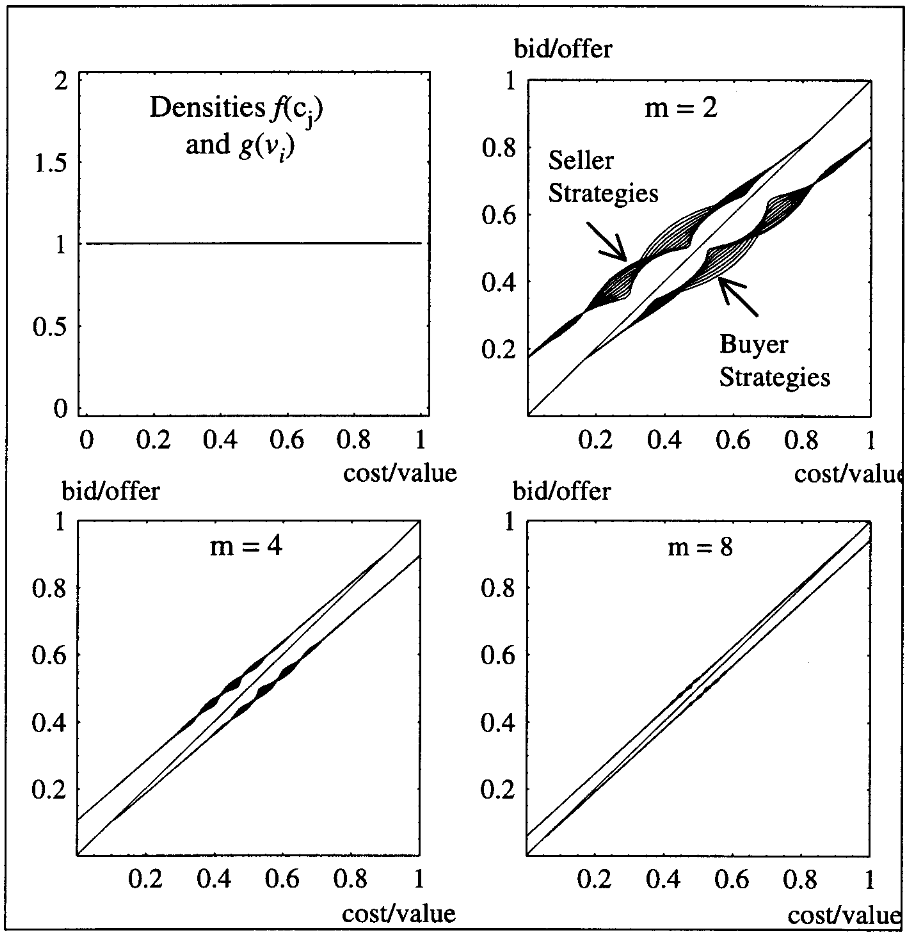

Figure 1: Bundles of equilibrium strategies if $v_i \sim G = \mathrm{B}(1,1)$ and $c_j \sim F = \mathrm{B}(1,1)$.

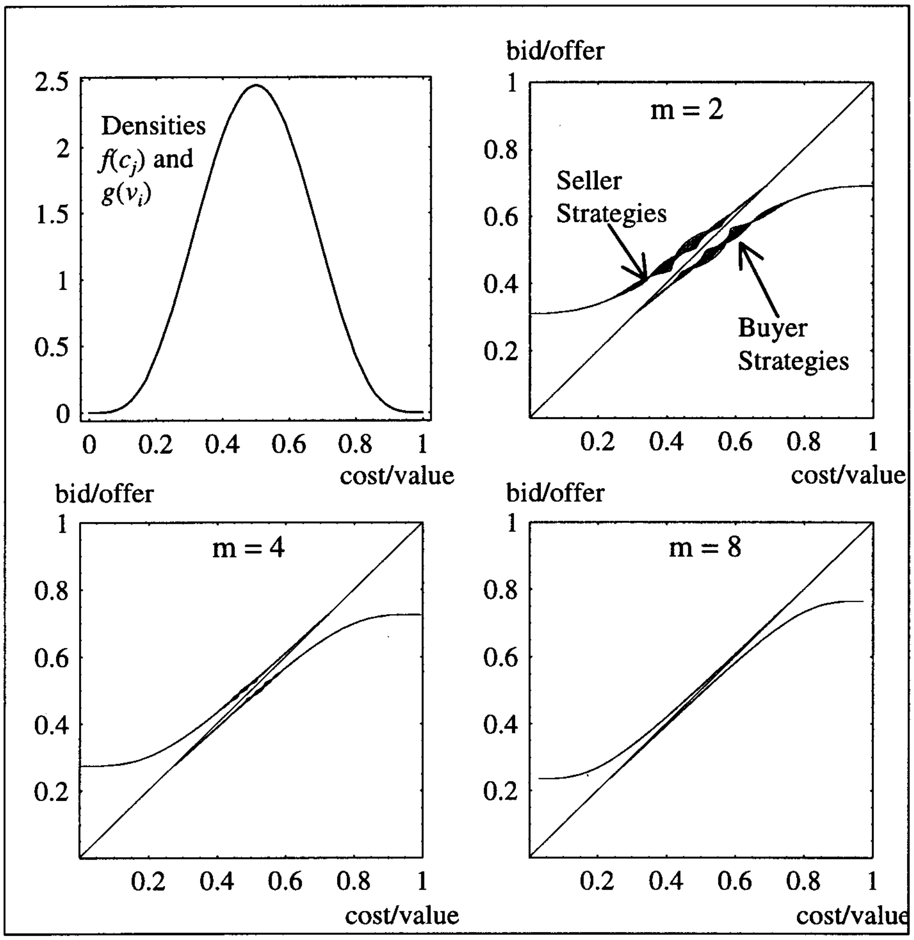

Figure 2: Bundles of equilibrium strategies if $v_i \sim G = \mathrm{B}(5,5)$ and $c_j \sim F = \mathrm{B}(5,5)$.

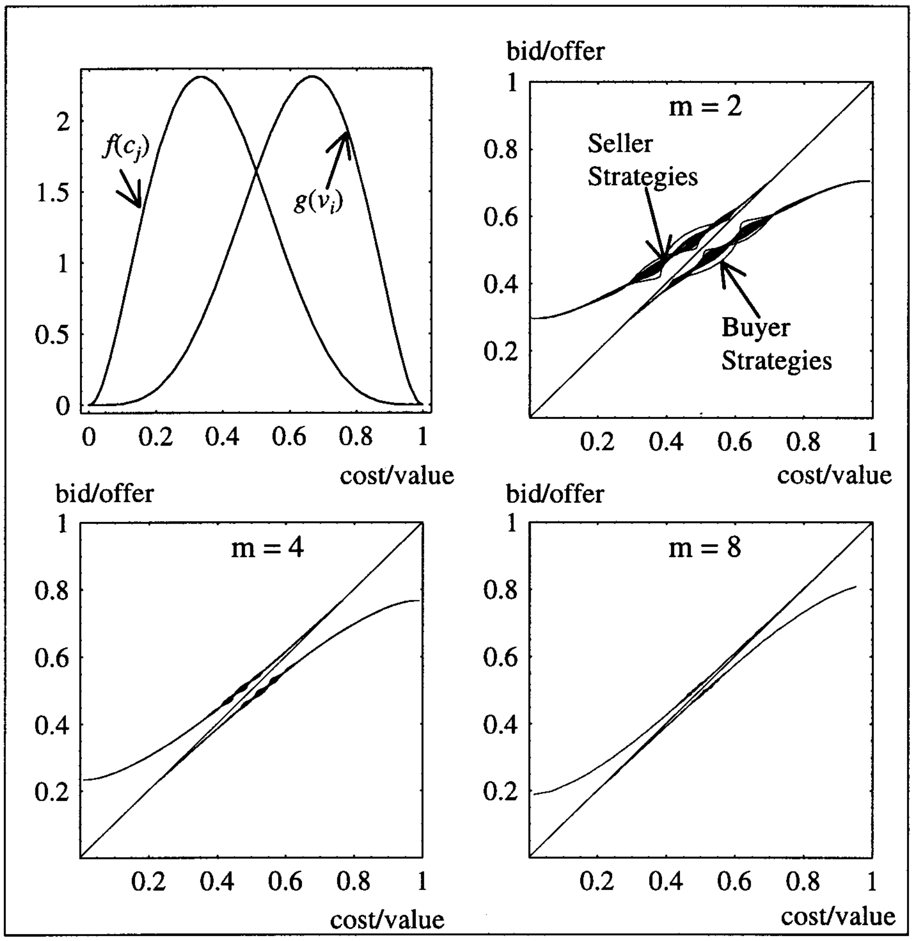

Figure 3: Bundles of equilibrium strategies if $v_i \sim G = \mathrm{B}(5,3)$ and $c_j \sim F = \mathrm{B}(3,5)$.

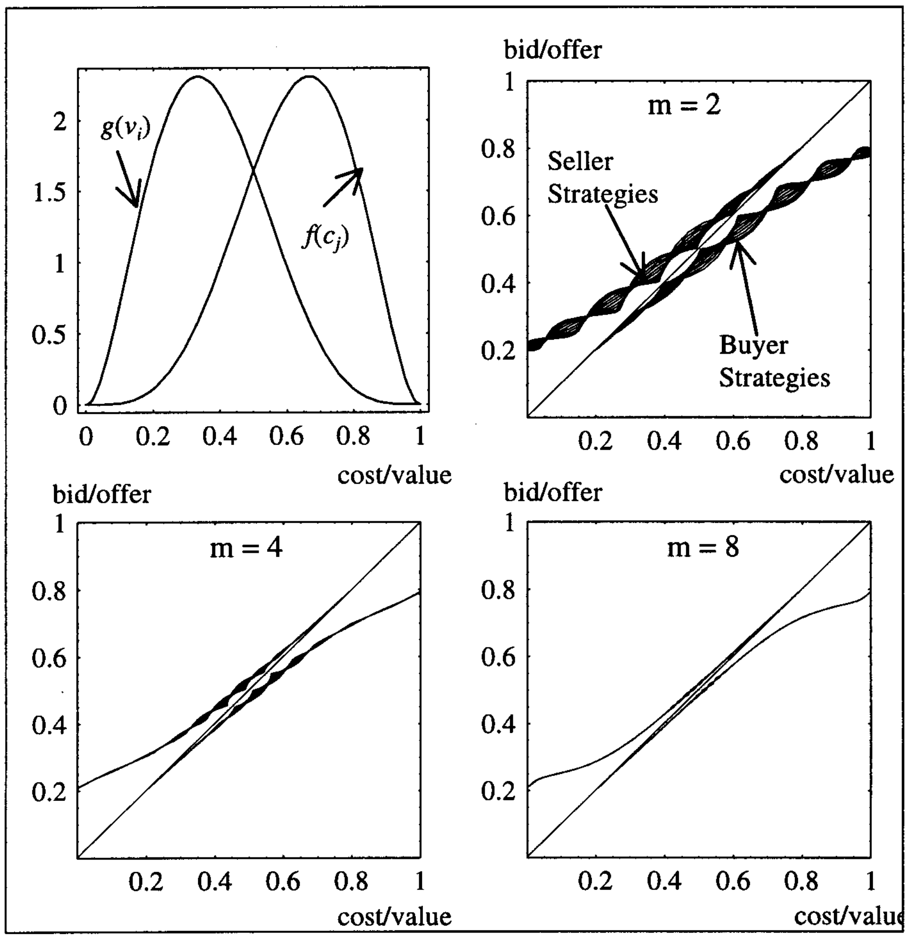

Figure 4: Bundles of equilibrium strategies if $v_i \sim G = \mathrm{B}(3,5)$ and $c_j \sim F = \mathrm{B}(5,3)$.