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Strategy-proof Location on a Network

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Abstract

We consider rules that choose a location on a graph (*e.g.* a network of roads) based on the report of agents' symmetric, single-peaked preferences over points on that graph. We show that while a *strategy-proof, onto* rule is not necessarily *dictatorial*, the existence of a cycle on the graph grants one agent a certain amount of decisive power. This result surprisingly characterizes the class of *strategy-proof, onto* rules both in terms of a certain *subclass* of such rules for trees and in terms of a parameterized set of generalized median voter schemes.

1 Introduction

The work of Gibbard (1973) and Satterthwaite (1975) has inspired a cottage industry of papers devoted to classifying the economic environments in which *non-dictatorial, strategy-proof* rules do or do not exist (see Thomson, 1998). One much-studied environment involves the provision of a pure public good. Consider a society in which the level of provision of public goods is given by a point in Euclidean space. Zhou (1991) shows that whenever the set of quadratic preferences over the space is admissible, any *strategy-proof* rule with a range of at least two dimensions is *dictatorial*.

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In contrast is the case in which the range of a rule is one-dimensional. Here, (strictly) convex preferences are single-peaked. Moulin (1980) introduces the class of "generalized median voter schemes," which are the only *strategy-proof*, *onto* rules when agents have single-peaked preferences over a one-dimensional set of alternatives (see Ching, 1997).

This important result has led to the analysis of various related models in which the concept of a generalized median voter scheme can be extended. In particular, Border and Jordan (1983) show that when preferences are both quadratic and separable on higher dimensions, a *strategy-proof*, *onto* rule must be decomposable into such a scheme for each dimension.[1]

Danilov (1994) shows that when agents have single-peaked preferences over a tree, *strategy-proof* rules satisfying a condition known as "peak-only" can be recursively decomposed into the "median" of triples of constant or dictatorial rules. Though this result may appear negative, it actually generalizes the class of generalized median voter schemes on a line.

In this paper, we examine the consequences of enlarging the domain of trees to include graphs with cycles. We consider a simpler structure on preferences, and assume them to be quadratic (i.e. symmetric and single-peaked) with respect to (shortest) paths along the graph from some peak. That is, each agent has a most preferred location on the graph—a peak—and preference over points is inversely related to (minimal) distance from the peak.[2]An example of an application of such preferences exists when agents have preferences over locations on a road network represented by distance traveled on the network, and distance is measured identically by any two agents.

Our results show that the presence of cycles in the graph implies that *strategy-proof*, *onto* rules must give one agent a degree of power in determining the chosen location. In particular, (Theorem 1) when all agents' peaks are on the same cycle, that one agent's peak is the chosen location. Second, (Theorem 2) in any other situation, this agent must consider the chosen location to be as good as any point that is on or between any cycle(s) in the graph.

This second result can be used to characterize the class of *strategy-proof*,

[1]See Peremans et al. (1997) for a characterization without the *onto* requirement.

[2]Hansen and Thisse (1981) and Demange (1982) restrict attention to graphs that are trees, and derive existence results for that model concerning Condorcet winners and the core.

onto rules in terms of a subclass of such rules for trees. For example, Danilov's characterization can be applied, leading to one characterization of *strategy-proof, onto* rules for graphs. Corollaries 1 and 2 formalize this style of characterization, stressing the importance of finding a "closed-form" characterization of such rules for trees, which at this point is an open question. For the case in which the graph is not acyclic, Theorem 2 can be used to provide a closed-form characterization in terms of certain families of generalized median voter schemes. In Theorem 3, we show that a *strategy-proof, onto* rule essentially gives one agent the power to offer the remaining agents his choice from among a set of generalized median voter schemes acting on certain paths on the graph.

Section 2 contains a formalization of our model. Section 3 contains Theorem 1, which restricts attention to any one cycle on a graph. Section 4 contains Theorem 2, which provides the result necessary to characterize the class of *strategy-proof, onto* rules. Finally, Section 5 contains the characterization results.

2 The Model

There is a set of agents, $N = \{1, 2, \ldots, n\}$, with arbitrary agents denoted i, j, etc. There is a "road network" represented by a graph, G, formalized below. A point (location) is to be chosen on G, based on the agents' preferences over points on G.

A (finite) *graph*[3] is a subset of Euclidean space, $G \subset \mathbb{R}^k$, that satisfies the following conditions:

1. G is the union of a finite number of rectifiable[4] curves of finite length.

2. Any two of these curves intersect at most at their extremities.

3. G is connected.

Each rectifiable curve is called an *edge*. Each of the two extremities of an edge is called a *vertex*. A vertex that lies on a single edge is called a *leaf*.

A *path* between $x, y \in G$ is a minimal connected subset of G that contains x and y. The length of a path is simply the sum of the lengths of the edges (or

[3]For a more complete formalization of graphs and related concepts, see Berge (1963).

[4]Roughly speaking, a curve is *rectifiable* if it can be approximated with a sequence of line segments. See Berge (1963).

portions thereof) whose union make up the path. The *distance* between any two points $x, y \in G$, denoted $d(x, y)$, is the minimum path-length between the two points. Denote the set of minimal-length paths between x and y by $[x, y] \equiv \{z \in G : d(x, z) + d(y, z) = d(x, y)\}$. Typically, $[x, y]$ is a single path.

A *cycle* in G is the union of two paths that intersect only at their endpoints. There is a cycle through any two distinct points $x, y \in G$ if there are at least two paths between them that intersect only at x and y. As a normalization, we assume that the distance around each cycle is at least 1, *i.e.* for any cycle $C \subset G$, there exist $x, y \in C$ such that $d(x, y) \geq 1/2$.

Each agent has a quadratic preference relation over G: there exists a point, p_i, called the peak, such that the agent's preferences are represented by the utility function $u(x) = -d(p_i, x)$.

Note that preference relations are uniquely defined by their peaks. Arbitrary peaks, though also points on the graph, will be denoted p_i, p'_j, etc. In standard fashion, for a list of peaks $p \in G^n$, the list obtained by replacing agent i's peak p_i with p'_i is written (p'_i, p_{-i}).

A *(social choice) rule* is a function $f: G^n \to G$ mapping lists of agents' peaks into points on the graph. We are interested in *onto* rules that are non-manipulable in the sense of being *strategy-proof*:

$$\forall p \in G^n, i \in N, p'_i \in G,\ d(p_i, f(p)) \leq d(p_i, f(p'_i, p_{-i})) \tag{1}$$

A standard result in the literature states that any *strategy-proof* rule that is also *onto* satisfies what is known as *unanimity*: for all $p \in G^n$ and $i \in N$, if for all $j \in N$, $p_j = p_i$, then $f(p) = p_i$.

3 The Behavior of Rules along Cycles

In this section we prove that the restriction of a *strategy-proof*, *onto* rule to a cycle must be dictatorial on the cycle.

Consider a small interval (a "short" path) on the graph G. Furthermore, consider only those profiles of preferences for which each agent's peak is located within that interval. As we show in Lemma 3 below, for such profiles, a *strategy-proof*, *onto* rule must choose a location within this interval. Furthermore, the restriction of such preferences to this interval induce what is analogous to single-peaked preferences on a line segment. Therefore, the restriction of the rule to such a set of profiles must be analogous to some *strategy-proof*, *onto* rule mapping single-peaked preferences over a line segment into points on that segment.

In other words, the choice problem on this subdomain of preference profiles is, essentially, one that is mathematically equivalent to the problem of Border and Jordan (1983) for single-peaked preferences on a line. Thus, it follows from the results of Border and Jordan that on small intervals, f must behave like a generalized median voter rule. This fact is presented as Proposition 1 below.

Now consider a cycle, $C \subset G$. Proposition 1 requires a *strategy-proof*, *onto* rule to behave like a generalized median voter scheme on each small interval around C. In order for all of the schemes for these small intervals to be consistent with each other around the cycle C, each one must be *dictatorial.* This notion is used to prove the stronger statement of Theorem 1: there must be a dictator on the entire cycle (and not only on small intervals along the cycle).

For the remainder of the section, we take as given a *strategy-proof*, *onto* rule, f, on the graph G. The results hold for any graph, but for Lemmas 4 and 5 to be nontrivial, G should contain at least one cycle.

The following lemma is similar to Border and Jordan's result saying that *strategy-proofness* implies what they call *uncompromisingness*—moving an agent's peak closer to the chosen location should not change the choice of location. On graphs, this conclusion is true in a neighborhood of the originally chosen location.

Lemma 1 (limited uncompromisingness) *For all $i \in N$ and all $x, p_i \in G$, there exists $\epsilon > 0$ such that for all $p_{-i} \in G^{n-1}$ and all $p_i' \in [p_i, f(p)]$, if $f(p) = x$ and $d(p_i', f(p)) < \epsilon$, then $f(p_i', p_{-i}) = f(p)$.*

Proof: Let $p_i' \in [p_i, f(p)]$, $x = f(p)$, and $y = f(p_i', p_{-i})$. By eqn. (1) above, we have both $d(p_i, x) \leq d(p_i, y)$ and $d(p_i', y) \leq d(p_i', x)$. If p_i' is sufficiently close to $f(p)$, it follows from *strategy-proofness* that $x = f(p)$, regardless of the values of p_{-i}. □

Since a *strategy-proof*, *onto* rule must satisfy *unanimity*, this result can be used to show that the point chosen by the rule must lie "between" the agents' peaks, in the sense that the shortest paths from the agents' peaks to the chosen point must jointly intersect only at the chosen point.

Lemma 2 (no intersecting shortest paths) *For all $p \in G^n$ and all $i \in N$, there exists $j \in N$ such that $[p_i, f(p)] \cap [p_j, f(p)] = \{f(p)\}$.*

Proof: Suppose for a contradiction that

$$P \equiv \bigcap_{k \in N} [p_k, f(p)] \supsetneq \{f(p)\}$$

Hence there exists $x \in P$ with $x \neq f(p)$, such that $[x, f(p)] \subset P$. By Lemma 1, for all $i \in N$, there exists $p'_i \in [x, f(p)]$ with $p'_i \neq f(p)$ such that, $f(p'_i, p_{-i}) = f(p)$.

Let $j = \arg\min_{k \in N} d(p'_k, f(p))$. For all $i \in N$, let $p''_i = p'_j$. Then by repeated application of Lemma 1, $f(p') = f(p)$, contradicting *unanimity*. □

The next lemma states that when agents' peaks lie in a sufficiently small neighborhood, a *strategy-proof*, *onto* rule must choose an efficient point—a point lying in the union of the shortest paths between peaks. Recall that the length of each cycle has been assumed to be at least 1.

Lemma 3 (limited efficiency) *Let $p \in G^n$ be such that for all $i, j \in N$, $d(p_i, p_j) \leq 1/8$. Then $f(p) \in \bigcup_{i,j \in N} [p_i, p_j]$.*

Proof: Suppose in contradiction to the Lemma that $f(p) \notin \bigcup_{i,j \in N} [p_i, p_j]$.

Claim: For all $i, j \in N$, if $[p_i, f(p)] \cap [p_j, f(p)] = \{f(p)\}$, then, where $p'_j = p_i$, $f(p'_j, p_{-j}) \notin \bigcup_{\ell,m \in N} [p_\ell, p_m]$.

Suppose $i, j \in N$ are such that $[p_i, f(p)] \cap [p_j, f(p)] = \{f(p)\}$. Since $f(p) \notin [p_i, p_j]$ by assumption, $[p_i, f(p)] \cup [p_j, f(p)] \cup [p_i, p_j]$ contains a cycle. Let $d^i = d(p_i, f(p))$, $d^j = d(p_j, f(p))$, and $d^{ij} = d(p_i, p_j)$. As the minimum length of a cycle is 1, we have $d^i + d^j + d^{ij} \geq 1$. Since $d^{ij} \leq 1/8$, we have either $d^i \geq 7/16$ or $d^j \geq 7/16$ (or both). In the former case, the triangular inequality implies $d^j + d^{ij} \geq d^i$, hence $d^j \geq 5/16$ in all cases.

Letting $p'_j = p_i$ and $x = f(p'_j, p_{-j})$, *strategy-proofness* implies $d(p_j, x) \geq 5/16$. For any other agent $k \in N$, $d(p_j, p_k) + d(p_k, x) \geq d(p_j, x)$, so $d(p_k, x) \geq 3/16$. Therefore $x \notin \bigcup_{\ell,m \in N} [p_\ell, p_m]$, and the Claim is proven.

By Lemma 2, let $i, j \in N$ be such that $[p_i, f(p)] \cap [p_j, f(p)] = \{f(p)\}$. For all $k \in N$, let $p'_k = p_i$. By the Claim, we have $f(p'_j, p_{-j}) \notin \bigcup_{\ell,m \in N} [p_\ell, p_m]$.

Repeating the argument, by Lemma 2, there must exist $k \in N$ such that $[p_i, f(p'_j, p_{-j})] \cap [p_k, f(p'_j, p_{-j})] = \{f(p'_j, p_{-j})\}$. By the Claim, we have $f(p'_k, p'_j, p_{-jk}) \notin \bigcup_{\ell,m \in N} [p_\ell, p_m]$.

This argument can be repeated until we have $f(p') \notin \bigcup_{\ell,m \in N} [p_\ell, p_m]$, which contradicts *unanimity*. □

Fix a cycle $C \subset G$, and consider the restriction of peaks (i.e., a subdomain) to an interval on C of length no greater than $1/8$. The induced preference relations over this interval are single-peaked. Furthermore, Lemma 3 states that on this subdomain, a *strategy-proof*, *onto* rule must make its selection from this interval. Therefore, the choice problem on this subdomain is mathematically equivalent to the problem of Border and Jordan for single-peaked preferences on a line. Thus, it follows from Border and Jordan (1983) that on $[x, y]$, f must behave like a generalized median voter rule.

To present this result for points on a cycle, however, it is necessary to introduce a partial order over points on the cycle. To do so, imagine C as a circle. For any two points $x, y \in C$ such that $d(x, y) \leq 1/8$, we write $y \preceq x$ if and only if the (unique) shortest path from y to x moves in a clockwise direction on the circle. For example, looking at a clock, if x is at 11:00 and y is at 10:00, then $x \succeq y$.

Proposition 1 (generalized median voter schemes) *For all $x, y \in G$ such that $d(x, y) \leq 1/8$, there exist $2^{|N|}$ points in $[x, y]$, $(\alpha_S^{xy})_{S \subseteq N}$, such that (i) $S \subset T \subset N$ implies $\alpha_S^{xy} \succeq \alpha_T^{xy}$, and (ii) for all $p \in [x, y]^n$,*

$$f(p) = \max_{S \subset N} \min\{(p_i)_{i \in S}, \alpha_S^{xy}\}$$

where the max *and* min *operations are taken with respect to the partial order* $\succeq$.

Proof: (Sketch.) The restriction of f to $[x, y]$, say g, is itself a *strategy-proof*, *onto* rule for the interval. Furthermore, the restriction of peaks to $[x, y]$ defines a subdomain which is identical to the domain of Border and Jordan (1983).[5] Therefore, the result follows from Propositions 2 and 4 of Border and Jordan. □

To present the next set of results, we refer to the parameters described in Proposition 1. For all $x, y \in G$ such that $d(x, y) \leq 1/8$, let $(\alpha_S^{xy})_{S \subset N}$ be those parameters.

The following lemma states that these parameters lie at the extreme points of their respective intervals. Furthermore, for each coalition, the direction in which its parameter lies (i.e., the "right" or "left" of the interval)

[5]There is a slight technical difference in that Border and Jordan's range is the real line instead of a finite interval. Their characterization can easily be extended to intervals. A proof is available upon request.

is consistent across intervals. In essence, this implies that on intervals of length less than 1/8, a *strategy-proof, onto* rule can be described in terms of right- and left-coalitions.[6]

Lemma 4 (right-/left-coalitions) *There exists a family of (right-) coalitions, $\mathcal{S} \subset 2^N$, such that for all $S \subset N$,*

(right) if $S \in \mathcal{S}$, then for all $x, y \in C$ such that $d(x,y) \leq 1/8$ and $x \succeq y$, we have $\alpha_S^{xy} = x$

(left) if $S \notin \mathcal{S}$, then for all $x, y \in C$ such that $d(x,y) \leq 1/8$ and $x \succeq y$, we have $\alpha_S^{xy} = y$

Proof: Let $S \subset N$ and let $x, y \in C$ be such that $x \succeq y$ and $d(x,y) \leq 1/16$. Let $\alpha = \alpha_S^{xy}$. We will show that if $\alpha \neq y$, then for all $v, w \in C$ such that $v \succeq w$ and $d(v,w) \leq 1/8$, we have $\alpha_S^{vw} = v$, i.e., we will show that if $\alpha \neq y$, then S is a "right-coalition."

Let $z \in C$ be such that $y \succeq z$ and $d(x,z) \leq 1/8$. For all $i \in N$, let $p_i = x$ if $i \in S$ and $p_i = y$ otherwise. By definition of the α-parameters, we have $f(p) = \alpha$. Since $f(p) \neq y$, we also have

$$\alpha_S^{xz} = \alpha \tag{2}$$

For all $i \in N$, let $p'_i = y$ if $i \in S$ and $p'_i = z$ otherwise. By Proposition 1, $f(p) \succeq \min\{(p_i)_{i \in S}, \alpha_S^{xz}\} = y$. Therefore Lemma 3 implies $f(p) = y$. Hence,

$$\alpha_S^{yz} = y \tag{3}$$

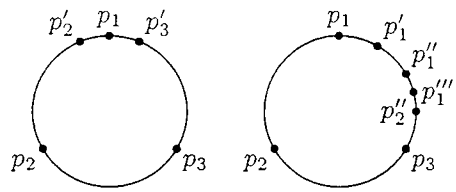
Repeating the arguments that lead to eqns. (2) and (3), we can show that for all $w \in C$ such that $w \succeq z$ and $d(w,y) \leq 1/8$, we have $\alpha_S^{yw} = y$ and $\alpha_S^{zw} = z$.

Therefore, by choosing an appropriate sequence of points around C, the same arguments show that for all $v, w \in C$ such that $v \succeq w$ and $d(v,w) \leq 1/8$, we have $\alpha_S^{vw} = v$.

If instead we had $\alpha = y$, then we would have shown that S is a "left-coalition." $\square$

Lemma 5 (peak selection) *For all $p \in C^n$, $f(p) \in \{p_1, p_2, \ldots, p_n\}$.*

[6] This terminology should not be confused with the literature's standard description of generalized median voter schemes in terms of "right/left coalition systems."



(a) (b)

Figure 1: Proof of Proposition 2.

Proof: Let $p \in C^n$. Suppose in contradiction that $f(p) \notin \{p_1, p_2, \ldots, p_n\}$. By Lemma 1, there exists $p'_1 \in [p_1, f(p)]$ such that $0 < d(p'_1, f(p)) \leq 1/16$ and $f(p'_1, p_{-1} = f(p)$.

Similarly, there exists $p'_2 \in [p_2, f(p)]$ such that $0 < d(p'_2, f(p)) \leq 1/16$ and $f(p'_1, p'_2, p_{-1,2}) = f(p)$.

Repeating the construction for the other agents, we have $f(p') = f(p)$, $d(p'_i, p'_j) \leq 1/8$ for all $i, j \in N$, and $p'_i \neq f(p')$ for all $i \in N$, contradicting Lemma 4. $\square$

Proposition 2 *If* $|N| = 3$, *then there exists* $i \in N$ *such that for all* $p \in C^3$, $f(p) = p_i$.

Proof: Let $p \in C^3$ be such that $d(p_i, p_j) = 1/3$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$. Assume without loss of generality (and by Lemma 5) that $f(p) = p_1$. We will show that for all $p \in C^3$, $f(p) = p_1$.

Let $\mathcal{S}$ be the set of coalitions described in Lemma 4. Note that by Proposition 1, it is sufficient to show that both (i) $\{1\} \in \mathcal{S}$ and (ii) if $S \in \mathcal{S}$ and $|S| = 1$, then $S = \{1\}$.[7] Notice also that if $S \in \mathcal{S}$, then $S \subset S' \subset N$ implies $S' \in \mathcal{S}$.

For $i \in \{2, 3\}$, let $p'_i \in [p_i, p_1]$ be such that $0 < d(p'_i, p_1) \leq 1/16$ and $f(p_1, p'_2, p'_3) = f(p)$ (see Figure 1a). By Proposition 1, we have $\{3\} \notin \mathcal{S}$.

Let $p'_1, p''_1, p'''_1 \in [p_1, p_3]$ satisfy $d(p'_1, p_1) = 1/12$, $d(p''_1, p_1) = 1/6$, and $d(p'''_1, p_3) = 1/8$ (see Figure 1b). Since f satisfies *peak selection*, *strategy-proofness* implies $f(p'_1, p_2, p_3) = p'_1$. Similarly, we have $f(p''_1, p_2, p_3) = p''_1$ and $f(p'''_1, p_2, p_3)$. *Strategy-proofness* and *peak selection* also imply that for

[7]These two conditions are what define a dictator for our class of median voter schemes.

any $p_2'' \in [p_1''', p_3]$, $f(p_1''', p_2'', p_3) = p_1'''$. Therefore, $\{2,3\} \notin \mathcal{S}$, which implies $\{2,\} \notin \mathcal{S}$.

The symmetric argument, with $\tilde{p}_1 \in [p_1, p_2]$ satisfying $d(\tilde{p}_1, p_2) = 1/8$ and $p_3'' \in [p_1''', p_2]$, demonstrates that $f(\tilde{p}_1, p_2, p_3'') = \tilde{p}_1$. Therefore, $\{1\} \in \mathcal{S}$. $\square$

Now we prove the general result. The proof works by showing that if a *strategy-proof*, *onto* rule is *non-dictatorial* for the general case, then there must be such a rule for the 3-agent case, contradicting Proposition 2. The method of proof is similar to that found in Kalai and Muller (1973), Aswal and Sen (1996), and Schummer (1998).

Theorem 1 (cycle dictator) *Let $C \subseteq G$ be a cycle on the graph. There exists an agent $i \in N$ such that for all $p \in C^n$, $f(p) = p_i$.*

Proof: The proof is by induction on $|N|$. Proposition 2 proves the result when $|N| = 3$.

Suppose that for all $|N| \leq n$, the result is true. Let $f\colon C^{n+1} \to C$ be a *strategy-proof*, *onto* rule. Define two n agent rules, g and g', as follows:

$$\forall p \in C^n,\ g(p_1, p_2, \ldots, p_n) = f(p_1, p_2, \ldots, p_n, p_n)$$

$$\forall p \in C^n,\ g'(p_1, p_2, \ldots, p_n) = f(p_1, p_1, p_2, \ldots, p_n)$$

That is, g is defined by creating a "copy" of agent n and applying the rule f. Similarly, g' is defined by duplicating agent 1.

Step 1: g and g' are both *strategy-proof* and *onto.*

Since a *strategy-proof*, *onto* rule must satisfy *unanimity*, it follows that g is *onto*. It is also clear that agents 1 through $n-1$ cannot manipulate the rule g. Thus, to demonstrate the *strategy-proofness* of g it suffices to prove that for all $p \in C^n$ and all $p_n' \in C$, $d(g(p), p_n) \leq d(g(p_n', p_{-n}), p_n)$.

By the *strategy-proofness* of f, for all $p \in C^n$ and all $p_n' \in C$,

$$d(f(p_1, \ldots, p_n, p_n), p_n) \leq d(f(p_1, \ldots, p_n, p_n'), p_n) \leq d(f(p_1, \ldots, p_n', p_n'), p_n)$$

Hence g is *strategy-proof.*

Similarly, g' satisfies both properties.

Step 2: if $i < n$ is a g-dictator, then i is an f-dictator.

By the induction hypothesis above, there exists $i \in \{1, 2, \ldots, n\}$ such that for all $p \in C^n$, $g(p) = p_i$. Suppose $i \neq n$. We show that for all $p \in C^{n+1}$, $f(p) = p_i$.

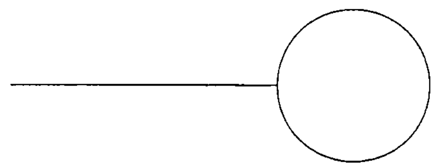


Figure 2: A simple graph with a cycle, admitting a *non-dictatorial*, *strategy-proof*, *onto* rule.

Let $p \in C^{n+1}$. For all $j \in \{1, 2, \ldots, n\}$ with $j \neq i$, let $p'_j = f(p)$, and let $p'_i = p_i$. Then by repeated application of Lemma 1, $f(p') = f(p)$. By the definitions of g and i, we also have $f(p') = g(p'_1, \ldots, p'_n) = p'_i = p_i$.

Step 3: if $i > 1$ is a g'-dictator, then $i + 1$ is an f-dictator.

This follows as in Step 2.

Step 4: either $i < n$ is a g-dictator or $i > 1$ is a g'-dictator.

Let $p \in C^{n+1}$ satisfy $p_1 = p_2 \neq p_n = p_{n+1}$. Then $g(p_1, \ldots, p_n) = f(p) = g'(p_1, p_3, \ldots, p_{n+1})$. Therefore it cannot be that both $g(p_1, \ldots, p_n) = p_n$ and $g'(p_1, p_3, \ldots, p_{n+1}) = p_1$.

Therefore f is dictatorial on C. □

4 The Behavior of Rules on Graphs

Consider the case in which G consists of a cycle and a line segment intersecting the cycle at one of its endpoints (as in Figure 2). Clearly a dictatorial rule on G is both *strategy-proof* and *onto*. A *non-dictatorial*, *strategy-proof*, *onto* rule also exists for this graph. One such rule can be constructed as follows: for each profile of preferences, if at least one agent's peak lies on the cycle, choose the point on the cycle closest to agent 1's peak; otherwise, choose the peak of the agent closest to the cycle.

For this rule, agent 1 plays the role of "cycle dictator" from Theorem 1. On the line segment, the rule behaves just like a (*non-dictatorial*) generalized median voter scheme. However, this generalized median voter scheme has the additional feature that from the perspective of agent 1, the chosen location (on the line segment) is at least as good as any location on the cycle. In fact, this notion—that the cycle dictator likes the chosen location as much as any location on a cycle—is what helps to characterize the *strategy-proof*, *onto* rules for graphs.

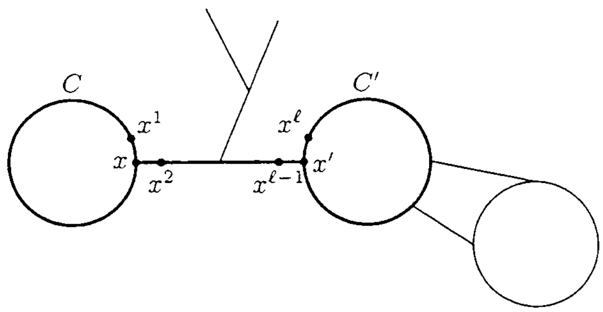


Figure 3: Drawn with thick lines are two cycles, and a path between them that intersects no other cycle on its interior.

Before we complete this characterization, we generalize Theorem 1 to more general cyclic subsets of graphs. We have shown that on any given cycle, a *strategy-proof, onto* rule must be dictatorial. This result extends to certain connected sets of cycles.

The following lemma says that each of the "cycle dictators" described by Theorem 1 (for each of the different cycles on the graph) are the same agent, i.e. there is one agent such that whenever all peaks are on the same cycle, that agent's peak is chosen.

Lemma 6 (unique cycle dictator) *Suppose C and C' are two cycles in G. There exists an agent $i \in N$ such that for all $p \in C^n \cup (C')^n$, $f(p) = p_i$.*

Proof: Without loss of generality, we will assume that C and C' can be connected by a path whose interior intersects no cycles. If the conclusion of the Lemma holds for this case, then the conclusion holds in the general case by repeating the argument.

Therefore, describe a path connecting C and C' by letting $x \in C$, $x' \in C'$ be such that for any cycle C'', $C'' \cap [x, x'] \subset \{x, x'\}$. (If C and C' intersect, let $x = x'$.) See Figure 3.

By Theorem 1, there exist $i, j \in N$ such that for all $p \in C^n$ and all $p' \in (C')^n$, we have $f(p) = p_i$ and $f(p') = p_j$. Let $x^1, x^2, \ldots, x^\ell \in G$ be such that $x^1 \in C \setminus \{x\}$, $x^k \in C' \setminus \{x'\}$, for $k \in \{1, 2, \ldots, \ell - 1\}$, $d(x^k, x^{k+1}) \leq 1/8$, and if $\ell \geq 2$, $x^2, \ldots, x^{\ell-1} \in [x, x']$.

By Proposition 1, the restriction of f to $[x^k, x^{k+1}]^n$ must be a generalized median voter scheme for each k. For each k, let $(\alpha_S^{x^k x^{k+1}})_{S \subseteq N}$ be the parameters for the rule in which we (arbitrarily) set the partial order $\preceq$ to satisfy $x^k \preceq x^{k+1}$.

Let $p_i = x^1$ and for all $k \neq i$, $p_k = x$. Since $p \in C^n$, we have $f(p) = p_1$, which implies that for all $S \subset N$ such that $i \notin S$, $\alpha_S^{x^1,x^2} = x^1$. As in the proof of Lemma 4, this implies that for all $m \in \{1, 2, \ldots, \ell - 1\}$, $\alpha_S^{x^m,x^{m-1}} = x^m$.

A symmetric argument shows that for all $m \in \{1, 2, \ldots, \ell-1\}$, $\alpha_{\{j\}}^{x^m,x^{m-1}} = x^{m+1}$. Therefore $i = j$. In fact this also shows that when peaks lie within the same interval of length less than 1/8, agent i's peak is chosen. $\square$

The following lemma states that the unique cycle dictator described in Lemma 6 is a dictator over the minimal connected subgraph containing all cycles in the graph. We will refer to this (unique) minimal subgraph as the **cycles neighborhood**. In Figure 3, the cycles neighborhood consists of the part of the graph drawn with thick lines plus everything lying to the right of C'.

Lemma 7 (cycles neighborhood dictator) *Let $\mathcal{C} \subseteq G$ be the minimal connected subgraph of G containing all of the cycles in G (i.e., the cycles neighborhood of G). There exists an agent $i \in N$ such that for all $p \in \mathcal{C}^n$, $f(p) = p_i$.*

Proof: Let $p \in \mathcal{C}$. Let $i \in N$ be the cycles dictator described in Lemma 6. Note that Lemma 2 implies that $f(p) \in \mathcal{C}$. If $f(p) = p_i$, we are done. Otherwise, for all $j \neq i$, let $p'_j = f(p)$. By repeated application of *strategy-proofness*, $f(p_i, p'_{-j}) = f(p)$.

By Lemma 1, for all p'_i sufficiently close to $f(p)$, we have $f(p') = f(p)$. First suppose there exists such a p'_i not equal to $f(p)$ which lies on a cycle also containing $f(p)$. Then by Lemma 6, we have $f(p') = p'_i$, contradicting Lemma 1.

Otherwise, since $f(p) \in \mathcal{C}$, there exists such a p'_i not equal to $f(p)$ such that p'_i and $f(p)$ lie on the same path between two cycles, and $d(p'_i, f(p)) \leq 1/8$. As shown at the end of the proof of Lemma 6, in this situation we must have $f(p') = p'_i$, contradicting Lemma 1. $\square$

Our main result is that a *strategy-proof*, *onto* rule must choose a location along the unique path between the cycle dictator's peak and the cycles neighborhood. Therefore, whenever the cycle-dictator's peak lies in the cycle-neighborhood, (and, hence, when this path is a point,) that agent's peak is chosen.

Theorem 2 (cycle dictator's rationality) *Let $C \subseteq G$ be the cycles neighborhood of G. There exists an agent $i \in N$ such that for all $p \in G^n$, $f(p) \in \bigcap_{x \in C}[p_i, x]$.*

Proof: Let $i \in N$ be the cycle dictator described in Lemma 7. Without loss of generality, assume that for all $j \neq i$, $p_j = f(p)$ (as in the proof of Lemma 7). By Lemma 1, for $p'_i \in [p_i, f(p)]$ sufficiently close to $f(p)$, we have $f(p'_i, p_{-i}) = f(p)$. Therefore if $f(p) \in C$, then with Lemma 7, we must have $f(p) \in \bigcap_{x \in C}[p_i, x]$ (otherwise, there exists such a $p'_i \in C$ such that $f(p'_i, p_{-i}) \neq p'_i$).

If $f(p) \notin C$, then an argument similar to the one in the proof of Lemma 6 can be used, along a path from $f(p)$ to C, to show that if $d(p'_i, f(p)) \leq 1/8$, we must have $f(p) = f(p'_i, p_{-i}) \in [p'_i, x]$ for all $x \in C$. □

5 A Description of Strategy-proof Rules

Reconsider the example of a graph given in Figure 2. According to Theorem 2, under any *strategy-proof, onto* rule, there exists an agent, say agent 1, such that for any profile of preferences, (i) if agent 1's peak is on the cycle, his peak is chosen, and (ii) otherwise, the chosen location must lie on the interval between his peak and the cycle.

Conversely, the following method will always produce a *strategy-proof, onto* rule for this graph: (i) if agent 1's peak is on the cycle, choose his peak, and (ii) otherwise, on the line segment, use an *onto* generalized median voter scheme that always chooses a point between agent 1's peak and the point on the line segment intersecting the cycle.[8] Given Border and Jordan's (1983) characterization of generalized median voter schemes as the only *strategy-proof* rules for symmetric, single peaked preferences on a line segment, this method can be shown to provide the *only* way to construct *strategy-proof, onto* rules for this particular graph.

For more general graphs, a similar characterization holds, as we formalize below. That is, for any graph, each *strategy-proof, onto* rule can be constructed by choosing an agent, say agent 1, such that (i) if agent 1's peak is on the cycles neighborhood (if one exists), choose his peak, and (ii) otherwise, if agent 1's peak lies on some "subtree," use any *strategy-proof, onto*

[8] The arguments of the generalized median voter scheme are the points on the line segments closest to the agents' peaks on the graph—their "peaks" on the line segment.

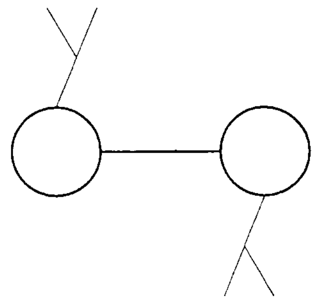


Figure 4: A graph with two (non-degenerate) maximal trees. The cycles neighborhood is drawn with thick lines.

rule for trees that always chooses a point between agent 1's peak and the unique intersection of the subtree with the cycles neighborhood.

To break up a graph into its relevant parts, we need to define the standard concept of a tree. A subgraph $T \subset G$ is a *tree* if it is connected and contains no cycles. Our description of *strategy-proof* rules depends on labeling the trees which, together with the cycles neighborhood, make up G. Let $C \subset G$ be the cycles neighborhood of G. A tree $T \subset G$ is a **maximal tree of** G if (i) T contains more than one point, (ii) $T \cap C$ contains at most one point, and (iii) there exists no tree $T' \subset G$ such that $T \subsetneq T'$ and $T' \cap C$ contains at most one point. See Figure 4.

Corollary 1 *Let $C \subset G$ be the cycles neighborhood of G and let $T_1, T_2, \ldots, T_k$ be the maximal trees of G. A rule $f: G^n \longrightarrow G$ is* strategy-proof *and* onto *if and only if there exists $i \in N$ such that*

1. *for all $p \in G^n$, $p_i \in C$ implies $f(p) = p_i$,*
2. *for each T_j, $1 \leq j \leq k$, there exists a* strategy-proof, onto *rule, $f_j: T_j^n \longrightarrow T_j$, such that for all $p \in G^n$, $p_i \in T_j$ implies*
   (a) *$f(p) = f_j(\tilde{p})$, where for all $k \in N$, $\tilde{p}_k$ is the point in T_j closest to p_k, and*
   (b) *where $\{x\} = C \cap T_j$, we have $f(p) \in [p_i, x]$.*

The formal proof of this corollary is left to the reader.

There is a second way to describe *strategy-proof, onto* rules for graphs that more clearly demonstrates the following fact: These rules for graphs

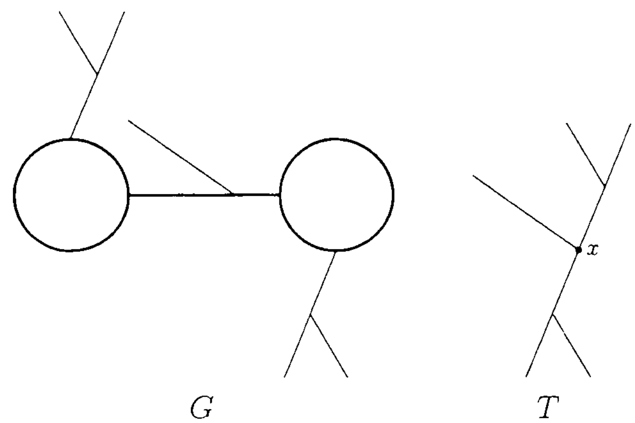


Figure 5: The tree T is obtained from the graph G by cycle reduction. Here, G has three maximal trees, and point x on T is associated with the cycle neighborhood of G.

should be thought of as a *subclass* of such rules for trees. We do this by transforming the graph, G, into a tree, T, which has the same structure as G except that the cycles neighborhood is replaced by a single point.

We will say that the tree T is *obtained from G by cycle-reduction* if T can be decomposed into subtrees, $T_1, T_2, \ldots, T_k$, such that (i) they have a one-to-one correspondence with the maximal trees of G, and (ii) they all intersect at the point corresponding to their intersection with the cycles neighborhood in G. See Figure 5. Hence, one should think of the intersection point of these subtrees as a node replacing the cycles neighborhood of G.

As a corollary to Theorem 2, it is easy to see that for the tree T obtained from G by cycle-reduction, there is a one-to-one correspondence between the class of *strategy-proof, onto* rules for G and the class of *strategy-proof, onto* rules for T that satisfy the following individual rationality condition on T. For an agent $i \in N$ and a point $x \in G$, the rule $f\colon G^n \to G$ satisfies *individual rationality for i with respect to x* if for all $p \in G^n$, $f(p) \in [p_i, x]$.[9]

Corollary 2 *Let T be the tree obtained by the cycle-reduction of G, and let $x \in T$ be the point associated with the cycles neighborhood of G. Then each*

[9]This condition is slightly stronger than its name suggests: not only must agent i weakly prefer the chosen location to point x, but the chosen point must lie on the path between his peak and x.

strategy-proof, onto *rule* $f: G^n \to G$ *for* G *is equivalent to some rule for* T, $f': T^n \to T$, *that is* strategy-proof, *is* onto, *and, for some* $i \in N$, *is* individually rational for i with respect to x.

The proof of this result is trivial. We state it only to emphasize the point that *strategy-proof, onto* rules for graphs containing at least one cycle should be thought of as a particular subclass of the set of *strategy-proof, onto* rules for trees.

5.1 A Characterization

Part 2b of Corollary 1 suggests that a characterization of all *strategy-proof, onto* rules is possible if G contains at least one cycle. Consider a situation in which the cycle dictator's peak lies on a maximal tree of G. The corollary states that in this situation, a *strategy-proof, onto* rule f operates in agreement with some *strategy-proof, onto* rule for the maximal tree. Furthermore, holding the cycle dictator's peak fixed, the range of the rule is restricted to an interval—the path between his peak and the cycle neighborhood—along which agents have single peaked preferences.

Therefore, fixing the cycle dictator's preferences, p_i, we can essentially view the rule $f(p_i, \cdot)$ as an $(n-1)$-agent rule mapping single-peaked preferences over that interval into points on that interval.[10] Hence, the Border and Jordan characterization can be applied, stating that with p_i fixed, f behaves like a generalized median voter scheme (g.m.v.s.) with respect to the remaining $(n-1)$ agents.

This leads to a characterization in terms of a family of generalized median voter schemes—one for each peak $p_i \notin C$—that satisfy a certain consistency condition, which we now define. We now assume G to contain at least one cycle.

Let $T_1, T_2, \ldots, T_k$ be the maximal trees of G. For each T_j, let $\{x_j\} = T_j \cap C$. For each T_j, and each $x \in T_j \setminus C$ (to be interpreted as the cycle dictator's peak), let $g^x: [x, x_j]^{n-1} \to [x, x_j]$ be an $(n-1)$-agent g.m.v.s. on $[x, x_j]$ for $N \setminus i$, with parameters $\{\alpha_S^x\}_{S \subseteq N \setminus i}$, where the order is chosen so that $x \preceq x_j$. The interpretation to follow is that when the cycle dictator's peak is p_i, the remaining agents face the g.m.v.s. g^{p_i}.

[10]There is a technical issue regarding the cases in which a change in an agent's preferences over G does not change his preferences over the interval. We will deal with this issue later.

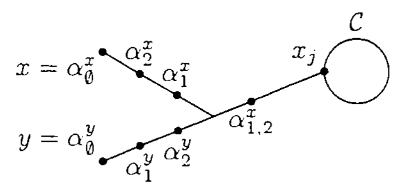


Figure 6: Consider two g.m.v.s.'s corresponding to points x and y, defined for the subset of agents $\{1, 2\}$ (and imagine agent 3 as the cycle dictator). For them to satisfy eqn. (5), it must be the case that $\alpha^y_{\{1,2\}} = \alpha^x_{\{1,2\}}$.

It is easy to see that in order for f to satisfy *unanimity*, it should be the case that when each agents peak is equal to p_i, g^x should chose that location. In other words, it should be the case that for each $x \in \bigcup T_j \setminus C$,

$$\alpha^x_\emptyset = x \tag{4}$$

Obviously, each g.m.v.s. is *strategy-proof* when viewed as an $(n - 1)$-agent rule. For f to be *strategy-proof*, however, the cycle dictator also must be unable to gain by misrepresenting his preferences. This means that he should not want to offer the other agents a g.m.v.s. different from the one they are currently using. This implies a form of consistency between the g.m.v.s.'s within each maximal tree of G. Particularly, when the range of two such g.m.v.s.'s intersect, the α-parameters should coincide over that intersection. In other words, for all T_j, all $x, y \in T_j$, and all $S \subseteq N \setminus i$,

$$\alpha^x_S \in [x, x_j] \cup [y, x_j] \setminus [x, y] \text{ implies } \alpha^x_S = \alpha^y_S, \text{ and} \tag{5}$$
$$\alpha^y_S \in [x, x_j] \cup [y, x_j] \setminus [x, y] \text{ implies } \alpha^x_S = \alpha^y_S \tag{6}$$

See Figure 6.

Any set of g.m.v.s.'s $\{g^x\}_{x \in G \setminus C}$ satisfying eqns. (4), (5), and (6) is said to be a **consistent family of g.m.v.s.'s for G with respect to $i \in N$**.

Theorem 3 *Suppose G contains at least one cycle. The rule f is* strategy-proof *and* onto *for G if and only if there exists an agent $i \in N$ and a consistent family of g.m.v.s.'s for G with respect to i, $\{g^x\}_{x \in G \setminus C}$, such that for all $p \in G^n$,*

$$f(p) = \begin{cases} p_i & \text{if } p_i \in C \\ g^{p_i}(\hat{p}_{-i}) & \text{if } p_i \in T_j, \text{ where } \hat{p}_j \text{ is the point in } [p_i, x_j] \text{ closest to } p_j \end{cases}$$

Proof: It is left to the reader to formally verify that such rules are *strategy-proof* and *onto*: Clearly for any fixed $p_i \in G$, the agents in $N \setminus i$ either face a (*strategy-proof*) $(n-1)$-agent g.m.v.s., or cannot alter the outcome of such a rule. Eqns. (5) and (6) guarantee that agent i cannot manipulate f. Eqn. (4) implies that f is *onto*.

Conversely, suppose that f is a *strategy-proof, onto* rule for G. Consider any maximal tree of G, T_j, and any $x \in T_j$. Let $\{x_j\} = T_j \cap C$, and let $I = [x, x_j]$. Let $g: I^n \to I$ be the restriction of f to I: for all $p \in I^n$, $g(p) = f(p)$. (By Theorem 2, $g(p) \in I$, so g is well-defined.) Note that g is a *strategy-proof, onto* rule on I. Therefore, it follows from Border and Jordan (1983) that g can be written as a g.m.v.s.

Let $i \in N$ be the cycle dictator described in Theorem 2. For any $p_i \in I$, let $g^{p_i}: I^{n-1} \to I$ satisfy for all $p_{-i} \in I^{n-1}$, $g^{p_i}(p_{-i}) = f(p_i, p_{-i})$. This restriction of an n-agent g.m.v.s. to $n-1$ agents is itself a g.m.v.s. (This follows simply from the definition of a g.m.v.s.) Furthermore, Theorem 2 implies that for all $p_{-i} \in I^{n-1}$, $g^{p_i}(p_{-i}) \in [p_i, x_j]$. Since f satisfies *unanimity*, g^{p_i} must satisfy eqn. (4). By a simple but tedious argument, the following is also true: for any $p_{-i} \in G^n$, where for each $j \in N \setminus i$, $\hat{p}_j$ is the point in I closest to p_j, we must have $f(p) = g^{p_i}(\hat{p}_{-i})$. That is, f should be insensitive to changes in the peaks of agents in $N \setminus i$ when the restriction of those agents' preferences to I does not change.

Finally, suppose that eqn. (5) does not hold. Suppose that for some $S \subseteq N \setminus i$, and for some $p_i, p_i' \in T_j$, $\alpha_S^{p_i} \in [p_i, x_j] \cup [p_i', x_j] \setminus [p_i, y]$ and $\alpha_S^{p_i} \prec \alpha_S^{p_i'}$. For all $j \in S$, let $p_j = \alpha_S^{p_i'}$ and for all $j \in N \setminus (S \cup i)$, let $p_j = \alpha_S^{p_i}$. Then, $f(p_i, p_{-i}) = \alpha_S^{p_i}$, and $f(p_i', p_{-i}) = \alpha_S^{p_i'}$, which contradicts *strategy-proofness*. Similar arguments apply to the cases in which $\alpha_S^{p_i} \succ \alpha_S^{p_i'}$, and in which eqn. (6) does not hold. $\square$

6 Conclusion

We have derived a characterization of the class of *strategy-proof, onto* rules that choose locations on networks (graphs) containing at least one cycle. Our results are partially negative and partially positive; one agent acts as a dictator on or between all cycles on the network, but exercises more limited power on other parts of the network. If the network is thought of as representing a highway network, with cycles around an urban center, and subtrees branching out into the suburbs, the rules can be roughly described as follows: one agent chooses the suburb in which the location should lie (or chooses an exact location within the urban area), and the remaining agents choose the precise location within the suburb (according to a generalized median voter scheme particular to that suburb).

One question that remains is whether a similar characterization exists for networks without cycles (*i.e.*, trees). We describe a way in which our rules can be described as a subclass of *strategy-proof*, *onto*rules for trees. At this time we leave open the generalization of our class of rules.

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