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INFINITE VOTERS AND THE POSSIBILITY OF
A CHEATPROOF SOCIAL CHOICE FUNCTION

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Arrow's conditions for social welfare functions have been shown by Fishburn [2] to be consistent with the hypothesis that the set of individuals is infinite. Focusing, like Fishburn, on the mathematical rather than the interpretive aspects of the problem of social choice we investigate here the possibility of a nondictatorial and nonimposed cheatproof social choice function for the case of an infinite set of individuals. As the results obtained by Gibbard [3] and Satterthwaite [6] imply that any cheatproof social choice function is either imposed or dictatorial when the number of individuals is finite, it is of interest to check whether the apparent duality in the finite case between Arrow's impossibility theorem and the negative cheat proofness result carries over to the case of an infinite set of individuals. The Theorem presented in this note answers this duality question in the affirmative as it asserts the existence of a nonimposed and nondictatorial cheatproof social choice function when the set of individuals is infinite.

Let \mathcal{A} denote a finite set of alternatives and let Σ denote the class of strong orderings (complete, asymmetric and transitive binary relations) on \mathcal{A} . Let V denote an infinite set of individuals. A Social Choice

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Function (SCF) F is a function $F: \Sigma^V \rightarrow \mathcal{A}$. An element of the domain of F is called a preference profile and is denoted by (P_1, P_2, \dots) where $P_i \in \Sigma$ and $i \in V$. A SCF is manipulable at $(P_1, P_2, \dots) \in \Sigma^V$ if there exists $P'_{i_0} \in \Sigma$ such that $F(P'_1, P'_2, \dots) \neq F(P_1, P_2, \dots)$ where $P'_i = P_i$ for all $i \neq i_0$. A SCF F is cheatproof if there is no preference profile at which it is manipulable; F is dictatorial if there exists an i in V such that for all (P_1, P_2, \dots) and for all $x \neq F(P_1, P_2, \dots)$ in the range of F , $F(P_1, P_2, \dots) = P_i x$; F is imposed if its range is a singleton. Finally, F will be said to satisfy the unanimity criterion if whenever for all $i \in V$ and for all $a \neq x$ in \mathcal{A} , $F(P_1, P_2, \dots) = x$. As shown by Gibbard [3] and Satterthwaite [6] (a simple proof of which is provided in Schmeidler and Sonnenschein [7]), when the number of individuals is finite there exists no cheatproof and nondictatorial SCF if the range of F contains at least three alternatives. We will show, much as in Fishburn [2], that this impossibility result does not carry over to the case of an infinite set of individuals.

Before proceeding to the main theorem, we recall some definitions and results which will be used in the proof below. For proofs and as a general reference, see Bourbaki [1].

Definition 1: Let V denote an infinite set. A set \mathcal{F} of subsets of V is a filter on V if:

- (a) Any subset of V containing an element of \mathcal{F} belongs to \mathcal{F} .
- (b) Any finite intersection of elements of \mathcal{F} belongs to \mathcal{F} .
- (c) The empty set does not belong to \mathcal{F} .

Definition 2: A set \mathcal{U} of subsets of V is an ultrafilter on V if:

- (a) \mathcal{U} is a filter.
- (b) For every subset S of V , either $S \in \mathcal{U}$ or $S^{\text{comp}} \in \mathcal{U}$.
(where $S^{\text{comp}} = V \setminus S$).

Proposition 1: Given any filter \mathcal{F} , there exists an ultrafilter \mathcal{U} such that for each $S \in \mathcal{F}$ it is also true that $S \in \mathcal{U}$.

Definition 3: \mathcal{F} is a principal filter if there exists $i \in V$ such that $\{i\} \in \mathcal{U}$. If no such i exists, \mathcal{F} is called a nonprincipal filter.

Example: Let \mathcal{F} be the set of subsets S of V such that S^{comp} is finite; \mathcal{F} is a nonprincipal filter.

Proposition 2: If \mathcal{F} is a nonprincipal filter then there exists an ultrafilter \mathcal{U} which is a nonprincipal filter and for which $S \in \mathcal{F}$ implies $S \in \mathcal{U}$.

Proposition 3: For any finite partition V_1, V_2, \dots, V_n of a nonempty set V ($\bigcup_{i=1}^n V_i = V, V_i \cap V_j = \emptyset$ for $i \neq j$), if \mathcal{U} is an ultrafilter on V then there exists an i_0 ($i_0 = 1, \dots, n$) such that $V_{i_0} \in \mathcal{U}$.

THEOREM: If the number of alternatives is finite, there exists a cheatproof social choice function which is neither imposed nor dictatorial provided the number of individuals is not finite.

Proof: Let \mathcal{U} be a nonprincipal ultrafilter on V . For all $a \in \mathcal{A}$ and for all $(P_1, P_2, \dots) \in \Sigma^V$, let $D(a, P_1, P_2, \dots)$ be the set of all v in V such that a is a maximal element with respect to P_v , i.e. $D(a, P_1, P_2, \dots) = \{v \in V \mid a \text{ is maximal with respect to } P_v\}$. Consider the sets $D(a_1, P_1, P_2, \dots), D(a_2, P_1, P_2, \dots), \dots, D(a_m, P_1, P_2, \dots)$ where $m = |\mathcal{A}|$. Since $\bigcup_{i=1}^m D(a_i, P_1, P_2, \dots) = V$ and $\bigcap_{k=i,j} D(a_k, P_1, P_2, \dots) = \emptyset$ for any $i \neq j$ ($i, j = 1, 2, \dots, m$) there exists an i_0 such that $D(a_{i_0}, P_1, P_2, \dots)$ belongs to \mathcal{U} ($i_0 = 1, 2, \dots, m$). Without loss of generality, suppose that $D(a_m, P_1, P_2, \dots) \in \mathcal{U}$. Define then the social choice function F by $F(P_1, P_2, \dots) = a_m$. Since \mathcal{U} is nonprincipal, F is clearly nondictatorial; it is not imposed since it satisfies the unanimity criterion.

We prove now that F is cheatproof. Suppose to the contrary that F is manipulable; then there exists an i_0 in V and a $P'_{i_0} \neq P_{i_0}$ such that $F(P'_1, P'_2, \dots, P'_{i_0}, \dots) = a_\ell, \ell \neq m$ where $P'_i = P_i$ for all $i \neq i_0$, and $a_\ell P_{i_0} a_m$. By the definition of F , it follows then that $D(a_\ell, P'_1, P'_2, \dots) \in \mathcal{U}$. But $D(a_\ell, P'_1, P'_2, \dots) \cap D(a_m, P_1, P_2, \dots) = \{i_0\} \neq \emptyset$, contradicting the assumption that \mathcal{U} is a nonprincipal ultrafilter on V . This completes the proof.

We mention without proof that the results carry over to the case where Σ is the class of weak orderings ("indifference" allowed) on \mathcal{A} and also to the case where the set of alternatives is infinite. Regarding the interpretation of the Theorem, the "invisible dictator" argument presented in Kirman and Sonderman [4] may help to rationalize the results obtained here (with the notion of the invisible dictator suitably redefined for the present SCF case). On the other hand, it is shown in [5] that as the number of individuals approaches infinity the proportion of profiles at which the plurality rule SCF is cheatproof approaches one. This result regarding the approximate cheatproofness of the plurality SCF in large enough finite societies complements nicely the nonconstructive, but exact, cheatproofness existence theorem for the infinite case presented here.

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