Discussion Paper No. 1246

# Cycling of Simple Rules in the Spatial Model

by

David Austen-Smith
Northwestern University
Economics and Political Science Departments
dasm@nwu.edu

and

Jeffrey S. Banks
Division of Humanities and Social Science
California Institute of Technology

Math Center web site:
http://www.kellogg.nwu.edu/research/math

# Cycling of Simple Rules in the Spatial Model*

David Austen-Smith
Department of Political Science
Northwestern University
Evanston, IL 60208

Jeffrey S. Banks
Division of Humanities and Social Sciences
California Institute of Technology
Pasadena, CA 91125

June 1997; revised July 1998

**Abstract**

McKelvey [4] proved that for strong simple preference aggregation rules applied to multidimensional sets of alternatives, the typical situation is that either the core is nonempty or the top-cycle set includes all available alternatives. But the requirement that the rule be strong excludes, *inter alia*, all supermajority rules. In this note, we show that McKelvey's theorem further implies that the typical situation for any simple rule is that either the core is nonempty or weak top-cycle set (equivalently, the core of the transitive closure of the rule) includes all available alternatives. Moreover, it is often the case that both of these statements obtain.

## 1 Introduction

One of the more celebrated results in the theory of preference aggregation over continua is McKelvey's theorem (McKelvey [3][4]). McKelvey [3] considered the strict majority preference relation for a finite set of $n$ individuals, each with Euclidean preferences over a convex set of alternatives in $\Re^k$. In

*We are grateful to John Duggan, Michel Le Breton and an anonymous referee for useful comments on an earlier version of the paper.

this setting, he proved that *either* the majority preference core is non-empty *or* the majority preference top-cycle set includes all of the available alternatives (see also Cohen [2]). In his subsequent paper, McKelvey [4] proved a deeper result that essentially implies McKelvey [3] extends to virtually any preference domain, any set of alternatives representable as topological space, and any *strong* simple preference aggregation rule (see also Schofield [6][7]). Here, an aggregation rule is simple if it is completely characterized by its decisive sets, or winning coalitions, and it is strong if, for any coalition of individuals, either the coalition or its complement is decisive (but not both).

An important, but by no means exhaustive, class of simple rules are $q$-rules, whereby a set of individuals is decisive if and only if it includes at least $q$ members with $q$ strictly greater than half the population. However, the only strong $q$-rule is the strict majority rule (and then only when the number of individuals is odd); thus McKelvey's theorems do not apply to any supermajority rule.

In this note, we review McKelvey's theorem [4] and study its implications for the entire class of simple rules. The main result is that it is the weak top-cycle set, or the core of the transitive closure of any simple rule, rather than the top-cycle set that (typically) includes all alternatives if the core is empty. So although it may not be the case for non-strong simple rules that any two alternatives can be connected by a finite sequence of strict preference steps, it is the case that any two alternatives can be connected by a finite sequence of weak preference steps. Furthermore, it can be quite generally the case both that the core of a simple rule is nonempty *and* that the core of the transitive closure of the rule includes all available alternatives.

## 2 Model

The model we use is somewhat more restrictive than necessary for the results to follow. It is, however, the canonic form for the spatial model used in applied work. Let $N = \{1, 2, .., n\}$ be a finite set of individuals and let $X \subset \Re^k$ be a convex set of feasible alternatives (with $X$ assumed to be of full dimension).

### 2.1 Preferences

For all $i \in N$, assume that $i$'s preferences are a weak order on $X$, representable by a continuous and strictly quasi-concave utility function, $u_i : X \to \Re$. Let $U$ denote the set of all continuous and strictly quasi-concave utility functions on $X$ and let $U^n = \mathcal{U}$. A *preference profile* on $X$ is an $n$-tuple $u = (u_1, u_2, ..., u_n) \in \mathcal{U}$. For any $i \in N$ and $y \in X$, let $I_i(y) = \{x \in X : u_i(x) = u_i(y)\}$. Say that a profile $u \in \mathcal{U}$ satisfies *diversity* if and only if, for all $y \in X$, for all distinct $i, j \in N$, the interior of $I_i(y) \cap I_j(y)$ in the relative topology on $I_i(y)$ is empty. For example, Euclidean preferences with distinct ideal points satisfy diversity. Let $\mathcal{U}^* \subset \mathcal{U}$ denote the set of profiles satisfying diversity.

## 2.2 Simple rules

Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on $X$. An *aggregation rule* is a map, $f : \mathcal{U} \to \mathcal{B}$. Given an aggregation rule $f$, a profile $u \in \mathcal{U}$, and any pair of alternatives $x$, $y \in X$, write $xR_{f(u)}y \equiv xf(u)y$, and let $P_{f(u)}$ denote the asymmetric part of $R_{f(u)}$.

For any $x, y \in X$ and $u \in \mathcal{U}$, let $P(x, y; u) = \{i \in N : u_i(x) > u_i(y)\}$. A coalition $L \in 2^N$ is *decisive under* $f$ if and only if,

$$\forall x, y \in X,\ \forall u \in \mathcal{U},\ [L \subseteq P(x, y; u) \Rightarrow xP_{f(u)}y].$$

Any aggregation rule $f$ induces a (possibly empty) family of decisive coalitions on $2^N$. Let $\mathcal{L}(f) \subseteq 2^N$ denote the set of all decisive coalitions under $f$. It is immediate from the definition of an aggregation rule and a decisive coalition that $\mathcal{L}(f)$ is *proper* ($L \in \mathcal{L}(f) \Rightarrow N\backslash L \notin \mathcal{L}(f)$) and *monotonic* ($[L \in \mathcal{L}(f)$ and $M \supset L] \Rightarrow M \in \mathcal{L}(f)$). An aggregation rule $f$ is *simple* if it is completely characterized by its decisive coalitions. More precisely, given any proper family of coalitions $\mathcal{L} \subset 2^N$, define the aggregation rule $f_{\mathcal{L}}$ by:

$$\forall x, y \in X,\ \forall u \in \mathcal{U},\ xP_{f_{\mathcal{L}}(u)}y \Leftrightarrow [\exists L \in \mathcal{L} : L \subseteq P(x, y; u)].$$

Then $f$ is simple if and only if $f \equiv f_{\mathcal{L}(f)}$. Examples of simple rules include all $q$-rules, $f_q : xP_{f_q(u)}y$ iff $|P(x, y; u)| \geq q$, $n/2 < q \leq n$. More generally, it can be shown that an aggregation rule $f$ is simple if and only if it is *monotonic* (if $xP_{f(u)}y$ and, under some new profile $u'$, $x$ does not fall relative to $y$ in any individual's ordering, then $xP_{f(u')}y$), *decisive* (if $xP_{f(u)}y$ and, under some new profile $u'$, the set of individuals strictly preferring $x$ to $y$ is unchanged, then $xP_{f(u')}y$) and *neutral* (the rule is symmetric with respect to alternatives): see, for example, Austen-Smith and Banks [1, Theorem 3.1].

A simple rule $f$ is *collegial* if and only if there is some individual who is a member of all decisive coalitions (i.e. $\cap_{L \in \mathcal{L}(f)} L \neq \emptyset$), and it is *noncollegial* otherwise. All $q$-rules are noncollegial except for the unanimity rule, $q = n$. The *Nakamura number*, $s(f)$, of a simple rule is infinity if the rule is collegial, and is equal to the smallest cardinality of any family of decisive coalitions in $\mathcal{L}(f)$ with empty intersection otherwise; i.e. $f$ noncollegial implies $s(f) = \min\{|\mathcal{L}| : \mathcal{L} \subseteq \mathcal{L}(f) \text{ and } \cap_{L\in\mathcal{L}(f)} L = \emptyset\}$. The Nakamura number of any noncollegial $q$-rule is known to be the smallest integer greater than or equal to $n/(n-q)$. So the Nakamura number for strict majority rule is 3 (unless $n = 4$ and $q = 3$, when it is 4), and the Nakamura number of the $q = n - 1$ rule is $n$. More generally, the Nakamura number of any noncollegial rule falls between 3 and $n$.

For any aggregation rule $f$ and profile $u \in \mathcal{U}$, the *core* for $(f, u)$ in $X$ is the set of best alternatives with respect to the preference relation $f(u)$ :

$$C_f(u) = \{x \in X : \forall y \in X,\ xR_{f(u)}y\}.$$

Given a simple rule $f$, a preference profile $u \in \mathcal{U}$, and $X \subset \Re^k$ convex and compact, the core $C_f(u)$ is nonempty for all $u \in \mathcal{U}$ if and only if $k < s(f) - 1$ (see, for example, Schofield [8]). For example, majority core points are only guaranteed to exist when $X$ is one-dimensional and, if $X$ is at least $(n-1)$-dimensional, then cores fail to exist for all noncollegial simple rules at some profiles. Moreover, for sufficiently high dimensional spaces ($k = 2$ in the case of majority rule with $n$ odd), the set of smooth profiles for which cores do exist are non-generic (Saari [5]).

A simple rule $f$ is *strong* if and only if, for all coalitions $L \in 2^N$, if the complement of $L$ in $N$ is not a decisive coalition, then $L$ itself is a decisive coalition (i.e. $N\backslash L \notin \mathcal{L}(f) \Rightarrow L \in \mathcal{L}(f)$). Thus the unique strong $q$-rule is strict majority rule ($q = q_m$) if $n$ is odd, and there exist no strong $q$-rules if $n$ is even. Finally, it is worth noting two easily checked properties of strong simple rules, useful for interpreting the results to follow. First, the Nakamura number of any noncollegial strong simple rule is 3 and, second, under the assumption of strictly convex preferences the core of any strong rule is either singleton or empty.

## 3 McKelvey's theorem

The statement of McKelvey's theorem uses the following concepts.

**Definition 1** *Let $f$ be a strong simple rule, $x, y \in X$, $u \in \mathcal{U}$, and $i, j \in N$. Say that, with respect to $\{x, y\}$ :*

*$i$ is a dummy voter at $u$ if*, $\forall L \subseteq N\backslash\{i\}$,

$$[P(x,y;u)\backslash\{i\} \subseteq L \subseteq N\backslash P(y,x;u)] \Rightarrow [L \cup \{i\} \in \mathcal{L}(f) \Rightarrow L \in \mathcal{L}(f)];$$

*$i$ is as strong as $j$ at $u$ if*, $\forall L \subseteq N\backslash\{i,j\}$,

$$[P(x,y;u)\backslash\{i,j\} \subseteq L \subseteq N\backslash P(y,x;u)] \Rightarrow [L \cup \{j\} \in \mathcal{L}(f) \Rightarrow L \cup \{i\} \in \mathcal{L}(f)].$$

Thus, if $i$ is a dummy voter then $i$ cannot influence the collective preference irrespective of how indifferent individuals are treated; and if $i$ is as strong as $j$ then a coalition is decisive with $j$ only if that coalition is decisive with $i$.

Given a simple rule $f$, a profile $u$, and a point $x \in X$, the set of feasible alternatives reachable from $x$ via the strict preference relation $P_{f(u)}$ is given by:

$$\begin{aligned} Q_{f(u)}(x) = \quad & \{y \in X : \exists\{a_0, a_1, \ldots, a_r\} \subset X \text{ such that} \\ & a_0 = x, a_r = y, r < \infty \text{ and}, \forall t \leq r-1,\ a_{t+1} P_{f(u)} a_t\}. \end{aligned}$$

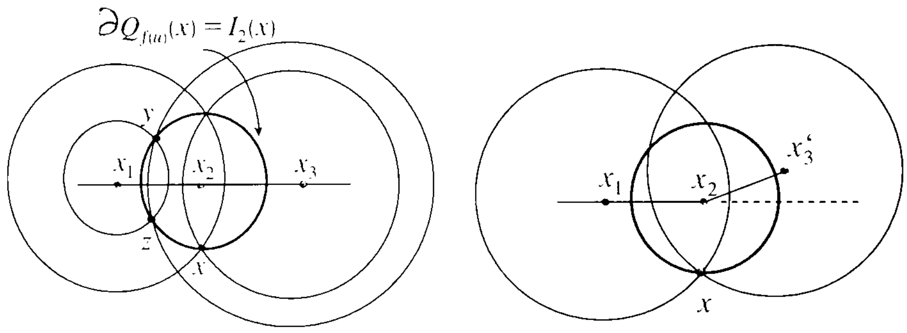
Note that $Q_{f(u)}(x) = \emptyset$ if and only if $x \in C_f(u)$. For any set $Y \subseteq X$, let $\partial Y$ be the boundary of $Y$.

**Theorem 1 (McKelvey)** *Let $f$ be a strong simple rule and suppose $u \in \mathcal{U}^*$. Then for any $x \in X$, either $\partial Q_{f(u)}(x) = \emptyset$ or both (1) and (2) obtain:*

*(1) There exists some $j \in N$ such that, $\forall y \in \partial Q_{f(u)}(x)$, $\partial Q_{f(u)}(x) \subseteq I_j(y)$;*

*(2) If $y, z \in \partial Q_{f(u)}(x)$ with $z \in I_i(y)$ for some $i \in N\backslash\{j\}$ and either $i$ is not a dummy voter or $i$ is as strong as $j$ with respect to $\{y,z\}$ at $u$, then there exists $\ell \in N\backslash\{i,j\}$ such that $z \in I_\ell(y)$.*

As McKelvey [4, p.1097] argues, the symmetry properties (1) and (2) are knife-edge in the extreme and so unlikely to occur. Figure 1 illustrates these observations for a society $N = \{1,2,3\}$, $X = \Re^2$ and strict majority rule, $f \equiv f_m$.




(a) (1) and (2) satisfied at $x$ (b) (1) and (2) fail at $x$

Figure 1: Conditions in McKelvey's Theorem

In Figure 1(a), individual preferences are Euclidean, i.e. the profile $u = (u_1, u_2, u_3)$ is such that, for each $i \in N$ and $y \in X$, $u_i(y) = -\|y - x_i\|$, with $\|x_i\| < \infty$. Furthermore, the ideal points $\{x_1, x_2, x_3\}$ are colinear. Then the boundary $\partial Q_{f(u)}(x) = I_2(x)$ and condition (2) holds for every pair $\{y, z\} \subset \partial Q_{f(u)}(x)$. Figure 1(b) describes the same situation except that individual 3's ideal point is perturbed off the line through $x_1$ and $x_2$; so $u' = (u_1, u_2, u_3')$ In this case, both conditions (1) and (2) fail and it can be checked (using, for example, the construction in McKelvey [3]) that $\partial Q_{f(u')}(x) = \emptyset$. In general, therefore, for any $x \in X$, the boundary $\partial Q_{f(\cdot)}(x)$ is empty. The consequences of this fact are fairly dramatic.

Because $X$ is a convex set in $\Re^k$, if the boundary of any subset $Y \subseteq X$ is empty then either $Y = \emptyset$ or $Y = X$.[1] Consequently, if $x \in X$ and $\partial Q_{f(u)}(x) = \emptyset$ then, by definition of $Q_{f(u)}(\cdot)$ and the core, either $x \in C_f(u)$ or $Q_{f(u)}(x) = X$. In other words, if the boundary of the set of points reachable from $x$ via $P_{f(u)}$ is empty, either $x$ is a core point for $(f, u)$ or *every* point in $X$ is reachable from $x$.

[1]To see this, suppose $Y \neq \emptyset$ and $Y \neq X$; then there exist distinct points $x, y \in X$ such that $y \in Y$ and $x \in X \backslash Y$. By convexity, for all $\lambda \in (0, 1)$, $\lambda y + (1 - \lambda)x \in X$. Hence, there exists $z = \lambda' y + (1 - \lambda')x$, $\lambda' \in (0, 1)$, such that, for all $\epsilon > 0$, both $B(x, \epsilon) \cap Y \neq \emptyset$ and $B(x, \epsilon) \cap [X \backslash Y] \neq \emptyset$, where $B(x, \epsilon)$ is the epsilon open ball centred at $x$. But this contradicts $\partial Y = \emptyset$ (without $X$ convex, essentially the same result holds if we replace "$Y = X$" with "$closure(Y) = X$": see [4, Lemma 5]).

**Definition 2** *For any aggregation rule $f$ and profile $u \in \mathcal{U}$, the top-cycle set for $(f, u)$ in $X$ is given by*

$$T_f(u) = \begin{array}{l} \{x \in X : \forall y \in X \backslash \{x\}, \exists \{a_0, a_1, \ldots, a_r\} \subset X \textit{ such that} \\ a_0 = y, a_r = x, r < \infty \textit{ and}, \forall t \leq r-1,\ a_{t+1} P_{f(u)} a_t\}. \end{array}$$

Thus, $x$ is in the top cycle set if and only if we can get to $x$ from *any* $y$ via the asymmetric part of $f$ in a finite number of steps. And clearly, if $x, y \in T_f(u)$, $x \neq y$, then $x \in Q_{f(u)}(y)$ *and* $y \in Q_{f(u)}(x)$. Then the preceding remarks imply that if, for all $x \in X$, $\partial Q_{f(u)}(x) = \emptyset$ then either $C_f(u) \neq \emptyset$ or $T_f(u) = X$.

McKelvey's theorem essentially asserts that for strong simple rules and *any* $x \in X$, the boundary of the set of points reachable from $x$ is typically empty; that is, $\partial Q_{f(u)}(x) = \emptyset$ for all $x \in X$ describes the general case for strong simple rules and profiles $u \in \mathcal{U}^*$. Hence, with McKelvey's theorem, we have that in general, for any strong simple rule $f$ and most profiles $u \in \mathcal{U}^*$, *either* the core for $(f, u)$ is nonempty *or* the top-cycle set for $(f, u)$ includes all points in $X$. And in view of the genericity of the set of profiles with an empty core, the second alternative is the usual case in sufficiently high dimensional spaces.

It is worth emphasising what McKelvey's theorem does and does not imply. The theorem does not imply anything about core existence. Moreover, the theorem does *not* imply that observed *choices* under any strong simple rule $f$ are "chaotic", only that if the core is empty then (typically) *there exists a preference path* linking any two alternatives. It is a theorem on the analytical structure of a class of aggregation rules and not on the empirical behaviour of polities using any rule within the class. Having said this, it is also important to observe that the class of rules covered by the theorem is relatively small; in particular, as remarked above, the result does not include any of the $q$-rules beyond strict majority rule, and includes the latter only when $n$ is odd. It is therefore of some interest to analyse what if anything the result implies for simple rules as a whole.

## 4 The transitive closure of simple rules

The extension of the results reported in section 3 to arbitrary simple rules is of the following form: if the core of $f$ at a profile $u$ is empty then, for all $x \in X$, every alternative $y \in X$ is reachable from $x$ via the *weak* collective preference relation $R_{f(u)}$.

**Definition 3** *For any aggregation rule $f$ and profile $u \in \mathcal{U}$, the weak top-cycle set for $(f, u)$ in $X$ is given by*

$$\begin{aligned} T_f^w(u) = \quad & \{x \in X : \forall y \in X\backslash\{x\}, \exists\{a_0, a_1, \ldots, a_r\} \subset X \text{ such that} \\ & a_0 = y, a_r = x, r < \infty \text{ and}, \forall t \leq r-1, \ a_{t+1} R_{f(u)} a_t\}. \end{aligned}$$

Alternatively, we can define this set via the transitive closure of the induced preference relation, $R_{f(u)}$. Let $f$ be any aggregation rule, $u \in \mathcal{U}$, $x \in X$ and define:

$$\begin{aligned} Q_{f(u)}^w(x) = \quad & \{y \in X : \exists\{a_0, a_1, \ldots, a_r\} \subset X \text{ such that} \\ & a_0 = x, a_r = y, r < \infty \text{ and}, \forall t \leq r-1, \ a_{t+1} R_{f(u)} a_t\}. \end{aligned}$$

So $y \in Q_{f(u)}^w(x)$ if and only if $y$ can be reached from $x$ via a finite sequence under the weak social preference relation, $R_{f(u)}$ (and clearly, $Q_{f(u)}(x) \subseteq Q_{f(u)}^w(x)$). Now, define the *transitive closure* of $R_{f(u)}$, $R_{f(u)}^T$, by:

$$\forall x, y \in X, \ x R_{f(u)}^T y \Leftrightarrow x \in Q_{f(u)}^w(y).$$

Thus, if $x R_{f(u)}^T y$ then $x$ is ranked indirectly to be at least as good as $y$, since $x$ can be reached from $y$ via $R_{f(u)}$ in a finite number of steps. Then the weak top-cycle set for $(f, u)$ in $X$ is the set of maximal elements in $X$ under the transitive closure relation, $R_{f(u)}^T$:

$$T_f^w(u) = \{x \in X : \forall y \in X, \ x R_{f(u)}^T y\}.$$

Let $f$ and $\hat{f}$ be two distinct simple rules. $\hat{f}$ is said to be *more resolute* than $f$ if, for all $x, y \in X$ and all $u \in \mathcal{U}$, $x P_{f(u)} y$ implies $x P_{\hat{f}(u)} y$. In particular, if $\hat{f}$ is more resolute than $f$ then $x P_{\hat{f}(u)} y$ implies $x R_{f(u)} y$.

**Theorem 2** *Let $u \in \mathcal{U}$, $f$ be any simple rule and $\hat{f}$ be a strong simple rule, more resolute than $f$. If, for all $x \in X$, $\partial Q_{\hat{f}(u)}(x) = \emptyset$ then $C_f(u) = \emptyset$ implies $T_f^w(u) = X$.*

**Proof.** If $f$ is strong, then the result is proved above. Assume $f$ is not strong and let $\hat{f}$ be the relevant strong rule, more resolute than $f$. By assumption, $\forall x \in X$, $\partial Q_{\hat{f}(u)}(x) = \emptyset$. So, by an earlier argument, either $C_{\hat{f}}(u)$ is empty or $T_{\hat{f}}(u) = X$. But since $\hat{f}$ is more resolute than $f$,

$$C_{\hat{f}}(u) \subseteq C_f(u) \text{ and } T_f(u) \subseteq T_{\hat{f}}(u) \subseteq T_f^w(u)$$

Therefore, $C_f(u) = \emptyset$ implies $T_f^w(u) = X$, as required.□

McKelvey's theorem, Theorem 1, shows that (typically) if a simple rule is strong and the core is empty, then the top-cycle set includes all of the alternatives in $X$; that is, we can construct a strict social preference cycle that includes all of $X$. When the rule is simple but not strong, this may not be possible. However, Theorem 2 shows that the price paid for having less than all-inclusive strict preference cycles is that instead we have all-inclusive weak preference "cycles". Equivalently, Theorem 2 says that if the core of a simple rule $f(u)$ is empty, then the transitive closure of the underlying preference relation $R_{f(u)}$ declares all alternatives socially indifferent: for all $x, y \in X$, $xR^T_{f(u)}y$ *and* $yR^T_{f(u)}x$. The following example illustrates Theorem 2.

**Example 1** *Suppose $n$ is odd and let $f$ be any $q$-rule with $q_m < q < n$. Then majority rule, $f_m$ (where $q = q_m$) is a strong rule and $f_m$ is more resolute than $f$. If preferences are Euclidean and, for all $i \neq j$, $x_i \neq x_j$, then by McKelvey [3] either $C_{f_m}(u) \neq \emptyset$ or $T_{f_m}(u) = X$. By Theorem 2, therefore, if $C_f(u) = \emptyset$ then $T_f^w(u) = X$.*□

Of course, Theorem 2 is predicated on the existence of a suitable more resolute strong rule $\hat{f}$ for any simple rule $f$. Example 1 shows that such existence is immediate for $q$-rules when $n$ is odd; our next result insures such existence quite generally.

**Theorem 3** *For any noncollegial simple rule $f$, there exists a noncollegial strong simple rule $\hat{f}$ that is more resolute than $f$.*

**Proof.** If $f$ is strong then the result is trivial. So assume $f$ is not strong. Let

$$\mathcal{M} = \{L \subset N : L \notin \mathcal{L}(f) \ \& \ N\backslash L \notin \mathcal{L}(f)\}.$$

Since $f$ is not strong, $\mathcal{M} \neq \emptyset$ and $|\mathcal{M}| = 2t$ for some integer $t \geq 1$. Partition $\mathcal{M}$ into two subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ such that:

$$\begin{array}{lcl} |\mathcal{M}_i| = t, \quad i = 1,2; & & \\ L \in \mathcal{M}_1 & \Leftrightarrow & N\backslash L \in \mathcal{M}_2 \\ L \in \mathcal{M}_1 & \Rightarrow & |L| \geq |N\backslash L|. \end{array}$$

By definition of $\mathcal{M}$, $\mathcal{L}(f) \cap \mathcal{M}_i = \emptyset$, $i = 1, 2$. Define the preference aggregation rule $\hat{f}$ by:

$$\begin{aligned} \forall x, y &\in X, \forall u \in \mathcal{U}, \\ xP_{\hat{f}(u)}y &\Leftrightarrow [\exists L \in \mathcal{L}(f) \cup \mathcal{M}_1 : L \subseteq P(x, y; \rho)]. \end{aligned}$$

By definition, $\mathcal{L}(f) \cup \mathcal{M}_1$ is monotonic and both $\mathcal{L}(f)$ and $\mathcal{M}_1$ are proper. Suppose $\mathcal{L}(f) \cup \mathcal{M}_1$ is not proper. Then there exist $L \in \mathcal{L}(f)$ and $M \in \mathcal{M}_1$ such that $L \cap M = \emptyset$; hence, $M \subseteq N \backslash L$. By $M \in \mathcal{L}(f) \cup \mathcal{M}_1$ and $\mathcal{L}(f) \cup \mathcal{M}_1$ monotonic, $M \subseteq N \backslash L$ implies $N \backslash L \in \mathcal{L}(f) \cup \mathcal{M}_1$. By definition of $\mathcal{M}_1$, $N \backslash L \notin \mathcal{M}_1$; so $N \backslash L \in \mathcal{L}(f)$. But since $L \in \mathcal{L}(f)$, this contradicts $\mathcal{L}(f)$ proper. Therefore, $\mathcal{L}(f) \cup \mathcal{M}_1$ is proper. And since, by definition, $\hat{f}$ is neutral and decisive, $\hat{f}$ is a simple rule. Moreover, by construction, $\hat{f}$ is strong, noncollegial and more resolute than $f$ as required.□

Finally, unlike with strong simple rules, it is possible here to have $T_f^w(u) = X$ *and* $C_f(u) \neq \emptyset$; that is, the core is nonempty yet every alternative is reachable from every other alternative via $R_{f(u)}$ in a finite number of steps.

**Theorem 4** *Let $f$ be a noncollegial simple rule and $X$ be compact. If $2 \leq k \leq s(f) - 2$ then there exist profiles $u \in \mathcal{U}$ such that $C_f(u) \neq \emptyset$ and $T_f^w(u) = X$.*

**Proof.** Let $f$ be a noncollegial simple rule with Nakamura number $s(f) \geq k + 2$. By Theorem 3, there exists a noncollegial strong simple rule $\hat{f}$ more resolute than $f$. By an earlier remark, $s(\hat{f}) = 3$. So $k \geq 2$ implies there exist $u \in \mathcal{U}$ such that $C_{\hat{f}}(u) = \emptyset$; in particular, we can always choose such a profile $u$ to be Euclidean and to satisfy diversity (see, for example, [9, pp.146/147]). By Cohen's theorem [2], therefore, $T_{\hat{f}}(u) = X$ for such a profile $u$. By definition, $T_{\hat{f}}(u) \subseteq T_f^w(u)$; so $T_f^w(u) = X$. But $k \leq s(f) - 2$ implies $C_f(u) \neq \emptyset$ ([8]) and the result is proved.□

In some circumstances, a stronger statement than Theorem 4 holds. In particular, recall that on the class of smooth utility profiles, the majority rule core when $n$ is odd is generically empty when $k \geq 2$ ([5]). Moreover, when $n$ is odd and $f$ is a noncollegial $q$-rule other than simple majority rule $f_m$, $f_m$ is a noncollegial strong simple rule more resolute than $f$. Hence, on the class of smooth utility profiles, if $f$ is a noncollegial $q$-rule other than $f_m$ and $2 \leq k \leq s(f) - 2$, then generically both $C_f(u) \neq \emptyset$ and $T_f^w(u) = X$.

In other words, on this set of preferences, noncollegial $q$-rules insure core existence at the cost of declaring all alternatives indirectly as good each other via the transitive closure, $R^T_{f(\cdot)}$. We close with an example illustrating these remarks.

**Example 2** *Let $n = 5$ and suppose $f$ is a $q$-rule with $q = 4$; then $s(f) = 5$. Let $X \subset \Re^2$ be a closed sphere centered at $(0,0)$ with arbitrarily large finite radius, and assume all individuals have Euclidean preferences over $X$ with ideal points: $x_1 = x_2 = (0,1)$; $x_3 = x_4 = (1,0)$; and $x_5 = (0,0)$. Then*

$$C_f(u) = \{x \in X : x = \lambda x_1 + (1-\lambda)x_3, \lambda \in [0,1]\}.$$

*In this case, $C_{f_m}(u) = \emptyset$ and $T_{f_m}(u) = X$ ([3]); hence, by Theorem 4, $T^w_f(u) = X$. To see this, consider Figure 2 and the points $w \in C_f(u)$ and $z \notin C_f(u)$. Clearly $wP_{f(u)}z$ and so $wR^T_{f(u)}z$. But it is also the case that $zR^T_{f(u)}w$: by Euclidean preferences and definition of $f$, it is easy to check that $a_1R_{f(u)}w$, $a_2P_{f(u)}a_1$, $a_3R_{f(u)}a_2$ and $zP_{f(u)}a_3$; hence $z \in Q^w_{f(u)}(w)$ and, therefore, $zR^T_{f(u)}w$. And it is apparent that this construction can be used for any pair $a, b \in X$ to show both $aR^T_{f(u)}b$ and $bR^T_{f(u)}a$, giving $T^w_f(u) = X$.*□

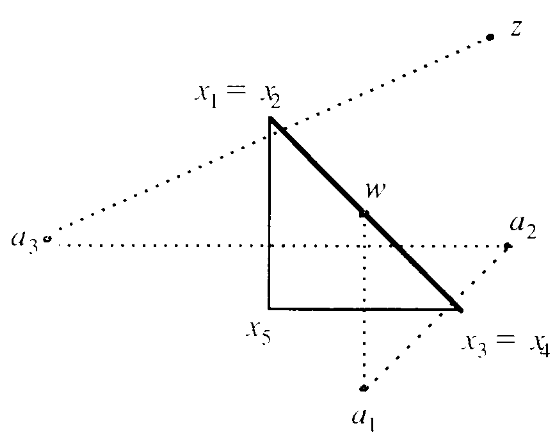



Figure 2: $C_f(u) \neq \emptyset$ and $T^w_f(u) = X$

## References


[1] Austen-Smith, D. and J.S. Banks 1998. *Positive Political Theory I: Collective Preference.* Ann Arbor: University of Michigan Press.

[2] Cohen, L. 1979. Cyclic sets in multidimensional voting models. *Journal of Economic Theory, 20:1-12.*

[3] McKelvey, R.D. 1976. Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory, 12:472-82.*

[4] McKelvey, R.D. 1979. General conditions for global intransitivities in formal voting models. *Econometrica, 47:1086-1112.*

[5] Saari, D.G. 1997. The generic existence of a core for $q$-rules. *Economic Theory, 9:219-62.*

[6] Schofield, N.J. 1980. Generic properties of simple Bergson-Samuelson welfare functions. *Journal of Mathematical Economics, 7:175-92.*

[7] Schofield, N.J. 1983. Generic instability of majority rule. *Review of Economic Studies, 50:695-705.*

[8] Schofield, N.J. 1984. Social equilibrium and cycles on compact sets. *Journal of Economic Theory, 33:59-71.*

[9] Schofield, N.J. 1985. *Social Choice and Welfare.* Berlin: Springer.