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Efficient Design with Interdependent Valuations

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Abstract

We study efficient, Bayes-Nash incentive compatible mechanisms in a general social choice setting that allows for informationally interdependent valuations and for allocative externalities. We show that such mechanisms exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multi-dimensional, the congruence condition is determined by a complex integrability constraint, and it can hold only in non-generic cases such as the private value case or the symmetric case. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied.

We apply the results to the study of multi-object auctions, and we discuss why such auctions cannot be reduced to one-dimensional models without loss of generality.

1. Introduction

During the last few years several national agencies (including, most prominently, the U.S. Federal Communication Commission) have conducted auctions of spectrum licenses. Spectrum auctions share with other large auction experiments (such as the recent sale of the 12 parts of Telebras) several main features:

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- There are several, heterogenous objects to be sold.
- Valuations for the objects are interdependent.
- There are potential asymmetries among bidders (e.g., incumbents versus new entrants, regional versus national interests)
- The ensuing allocation affects future market structure (and hence future payoffs.)
- The stated main goal of the auction's organizer (e.g., government agencies) is allocative efficiency.

A model for spectrum allocation problems (and for many other similar situations) must therefore include both informational and allocative externalities in a context where agents may be asymmetric, and where several objects are being sold.

Even if values are purely private and no externalities of any kind are present, a general treatment of an m-object auction requires signals having at least $2^m - 1$ dimensions (i.e., at least a real-valued signal about each possible bundle that may be acquired at the auction). The critical aspect of multidimensionality is the fact that the payoff relevant part of the signal varies with the chosen alternative. As we show below, such a framework cannot be reduced to a one-dimensional model without a serious restriction of generality[1].

There exists an extensive literature on efficient auctions and mechanism design. A situation that has received a lot of attention is the case where each agent i has a quasi-linear utility function that depends on information (or signal) privately known to i, and on a monetary transfer, but does not depend on information obtained by other agents. In this framework, a prominent role is played by the so called Clarke-Groves-Vickrey (CGV) mechanisms (see Clarke, 1971, Groves, 1973, Vickrey, 1961). These are mechanisms that ensure both that an efficient decision is taken and that truthful revelation of privately held information is a dominant strategy for each agent. This result holds for arbitrary dimensions of signal spaces and for arbitrary signals' distributions. [2].

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[1] An exception is the case where all m goods are perfect substitutes.

[2] It is well known that, generally, CGV mechanisms do not satisfy budget-balancedness, i.e., these mechanisms generate either a monetary surplus or a monetary deficit. d'Aspremont, Gerard-Varet, 1979, d'Aspremont, Cremer, and Gerard-Varet, 1990, Matsushima, 1990, Fudenberg, Levine, and Maskin, 1994, and Aoyagi, 1998 have given conditions under which Bayesian

It is worth noting that in the private values case with quasi-linear utilities and independent signals we can find for *any* Bayesian incentive compatible and efficient mechanism a CGV mechanism that yields the same allocation and the same *expected* transfers. In fact results such as the Revenue Equivalence Theorem for private values auctions and Myerson and Satterthwaithe's Impossibility Theorem are straightforward consequences of this result (which holds no matter what the dimension of agents' signal spaces is - see Williams, 1994[3]).

Although there are many auction papers that go beyond the private values case (e.g., the literature following Milgrom and Weber, 1982), almost all of them restrict attention to situations where signals are one-dimensional, agents are ex-ante symmetric and do not care about other agents receive at the auction.

In this paper we look at the case where each agent has a quasi-linear utility function having as arguments the signals received by *all* agents and the chosen social alternative. Hence, we allow for both informational and allocative externalities. Signals, which are independently drawn from infinite spaces, may be multi-dimensional, allowing, among other things, for a consistent treatment of general multi-object auctions. Signal independence is the most seriously restrictive assumption in an otherwise rather general model (but note that distributional assumptions do not play any role in the several positive results obtained in the paper).

Our general aim is to embed the mechanism design problem in a broader economic context: the idea is that the agents are engaged in some future interaction, and that the outcome of that interaction depends both on the allocation chosen at the prior stage, and on the agents' signals. Hence, agents' valuations, which depend on the chosen alternative and on all signals, stand as reduced forms for payoffs in future interaction[4], and future market structure considerations are introduced in the analysis.

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incentive compatible mechanisms can achieve both efficiency and budget-balancedness. Myerson and Satterthwaithe, 1979 have shown that the above conditions are, together, not compatible with the requirement of individual rationality.

[3]The simple idea behind this result has been observed in various settings by many authors - see also the survey of Laffont and Maskin (1982). Williams explicitly deals with multidimensional signals and points out the applications.

[4]If signals do not become common-knowledge beetween the prior stage and the interaction stage, signalling effects may also play a role. We abstract here from these effects: while they can be formally thought of as part of the reduced forms, some finer phenomena cannot be studied through such modeling.

The importance of "market structure" considerations has been articulated in the context of single-object auctions in Jehiel, Moldovanu, and Stacchetti (1996, 1999), and Jehiel, Moldovanu (1996,1997).

In the social choice framework considered here, Williams and Radner (1988) have shown that, in general, no efficient, dominant-strategy incentive compatible mechanisms exist[5]. Important insights about auctions with interdependent valuations (but without allocative externalities) and can be found in Maskin (1992) and Dasgupta and Maskin (1998) . These authors have shown that, under a set of assumptions concerning marginal valuations, an English auction for a single good (and the corresponding direct mechanism) is efficient if signals are one dimensional[6]. Maskin (1992) observes that, in general, no efficient, incentive-compatible auctions exist if the signal affecting a buyer's valuation for the good is multi-dimensional. Finally, Dasgupta and Maskin (1998) construct a modification of the CGV mechanism that achieves efficient allocations (under appropriate conditions on marginal valuations) if signals are one-dimensional. Perry and Reny (1998) present a bidding procedure that achieves efficient allocations for a one-dimensional model where several identical goods are allocated to buyers with decreasing marginal valuations.

This paper is organized as follows: In Section 2 we present the social choice model. In Section 3 we offer some illustrations for auction theory. In Section 4 we obtain a characterization Theorem for Bayesian incentive compatible mechanisms. In Section 5 we briefly look at Groves mechanisms in the private values case. In Section 6 we exhibit several impossibility results about efficient, Bayesian incentive compatible mechanisms. We only require value maximization regarding the chosen social alternative, and we completely ignore budget-balancedness and any other properties. Hence, we show that correct informational incentives are not compatible even with another very weak efficiency requirement.

Crucial roles are in the analysis played by the dimensions of the agents' signal spaces (e.g., by the fact that the payoff-relevant part of an agent's signal varies

[5] Cremer and McLean (1985,1988) and McAfee and Reny (1992) have given conditions under which a principal can extract the full surplus available when types are correlated. Full extraction mechanisms are, in particular, efficient. Neeman (1998) shows that these results do not hold in a model that can be interpreted as one where agents have multidimensional signals, and signals have some private and some common components.

[6] The strength of this remarkable result comes from the fact that no symmetry assumption is made. Note that in the private values case, a first-price auction, say, is efficient only if agents are symmetric. Moreover, with informational externalities, also the second-price auction need not be efficient if there are more than 2 (asymmetric) bidders.

with the implemented social alternative), and by a condition that compares private and social rates of informational substitution.

Theorem 6.2 shows the impossibility of efficient, incentive compatible mechanisms in situations where there is at least one agent possessing essential information that affects other agents, but does not directly affect the owner of that information. The literature on *credence goods* deals extensively with such frameworks. A corollary of this result is exhibited in Example 6.4: generically, there are no efficient, incentive compatible mechanisms if there exist an alternative k and an agent i such that agent i's signal affecting her valuation for alternative k is multidimensional.

The above results imply that, generically, an incentive compatible mechanism for gent i cannot condition on more than a piece of scalar information per social alternative[7]. Our main impossibility result is Theorem 6.5. We consider the critical framework where each agent i has a $K-$dimensional signal s^i (where K is the number of alternatives). The coordinate s^i_k is a **one-dimensional** signal affecting the valuations of **all** agents for alternative k. In this framework none of the above inefficiency results apply, since no change of variables yields the needed features. The Theorem shows that efficient, incentive compatible mechanisms can exist only if a congruence condition pertaining to private and social rates of informational substitution is satisfied: As a consequence of the integrability constraint associated with multidimensional design problems, incentive compatible mechanisms necessarily "pool" together different types along certain lines. Efficient mechanisms will pool types along other lines. Efficient, incentive compatible mechanisms are possible only if the slopes of the two varieties of pooling lines are equal. Unfortunately, this last condition cannot hold generically[8].

Since modeling general multi-object auctions always requires that an agent's payoff-relevant signal depends on the bundle that this agent may acquire, our Theorem implies that the quest for full efficiency in multi-object auctions is elusive. Also, it is very difficult to exhibit a second-best procedure[9] because further necessary dimensionality reductions become endogenous.

[7]Hence, any model can be first reduced to a model where signals are $K-$dimensional without loss of efficiency.

[8]We show that the congruence condition is satisfied in non-generic situations where either symmetry, or the private values assumption hold.

[9]Jehiel, Moldovanu and Stacchetti (1998) discuss the methodologically related question of revenue maximization in a multidimensional private values model. The integrability constraint boils down to a certain partial differential equation. For some special cases, the equation is an ordinary one, and examples can be analytically computed.

In Section 7 we focus on one-dimensional signal spaces, and we define *generalized Groves mechanisms.* Our definition is based on the old idea (which can be traced back to Pigou) that transfers should stand for the cumulative effect of one's action (here a signal report) on all other agents. We show that such mechanisms are efficient and Bayesian incentive compatible if each agent perceives no more than two payoff-relevant alternatives. An application is made to auctions for one indivisible good (as studied by Dasgupta and Maskin, 1998). However, generalized Groves mechanisms fail to be incentive compatible if agents perceive more than two payoff-relevant alternatives (even under appropriate conditions on marginal valuations).

Finally, we inquire whether other types of efficient, incentive compatible mechanisms generally exist if agents have one-dimensional signal spaces. To allow a more amenable differential approach (that provides a somewhat clearer insight) we assume in this part that the space of alternatives is continuous. We show that efficient incentive compatible mechanisms exist whenever the private and the social interests are congruent. In the one-dimensional model the integrability constraint does not bind at all, and the congruence condition reduces to a usual monotonicity condition that can be generically satisfied[10]

2. The Model

There are K of social alternatives, indexed by $k = 1, ...K$ and there are N agents, indexed by $i = 1, .., N$. Each agent i has a signal (or type) s^i which is drawn from a space $S^i \subseteq \Re^{K \times N}$ according to density $f_i(s^i)$, independently of other signals. Each agent i knows s^i, and the densities $\{f_j(\cdot)\}_{j=1}^N$ are common knowledge. The idea is that the coordinate s^i_{kj} of s^i influences the utility of agent j in alternative k[11]. We assume that the signal spaces S^i are bounded and convex[12].

If alternative k is chosen, and if i obtains a transfer x_i, then $i's$ utility is given by $V^i_k(s^1_{ki}, ..., s^n_{ki}) + x_i$, where $V^i_k(s^1_{ki}, ..., s^n_{ki}) = \sum_{j=1}^n a^j_{ki} s^j_{ki}$, and where the scalar parameters[13] $\{a^j_{ki}\}_{1 \leq k \leq K, 1 \leq j,i \leq N}$ are common knowledge. We assume throughout

[10] The congruence condition requires here only that some inequalities are satisfied, while it required that some equalities are satisified in the multidimensional framework of Theorem 6.5.

[11] We address below (see Example 6.4 and the discussion preceding it) situations where the signal of an agent i affecting the utility of agent j in alternative k is itself multidimensional.

[12] Convexity is assumed only for convenience. If S^i is simply connected and if it has a well behaved boundary, then all results go through unchanged (all what is needed is a condition that allows proving Stokes' Theorem)

[13] We can easily allow the valuation functions to include also a constant, i.e., $V^i_k(s^1_{ki}, ..., s^n_{ki}) =$

the paper that $\forall i, \forall k$, $a^i_{ki} \geq 0$

The special case of linear valuations is chosen because we are mainly going to prove negative results. Extensions of our impossibility results to nonlinear frameworks are straightforward. Also, we comment below on the straightforward generalization of the few positive results to general quasi-linear utility functions.

3. Applications to Auctions

Consider an auction where a set M of objects is divided among $n+1$ agents (where agent zero is the seller, and the rest are potential buyers). An alternative is a partition μ of M, $\mu = \{M_i\}_{i=0}^N$, where M_i is the set of objects allocated to bidder i, $i = 1, 2, ...N$ and M_0 is the set of unsold objects. Bidder $i's$ piece of information $s^i_{\mu j}$ (which, in general, may be by itself multidimensional) summarizes, from the point of view of i, the important aspects for j (say, attributes of the objects in M_j) given partition μ.

This framework is general enough to allow for both informational and allocative externalities in a multi-object auction with potentially asymmetric bidders. Particularly simple special cases are: 1) The private values case where $V\mu^i(\cdot)$ is only a function of $s^i_{\mu i}$; 2) The private values case without allocative externalities where $V\mu^i(\cdot)$ is only a function of $s^i_{\mu i}$, and $V\mu^i(\cdot) = V^i_{\mu'}(\cdot)$ for all partitions μ and μ' such that i receives the same set of objects , etc...It should be clear that even such simple cases generally require multidimensional signals.

3.1. Private values examples

In private values contexts one assumes that $a^j_{ki} = 0$ for all k and all $j \neq i$. Most works consider the one-dimensional case where $a^i_{ki} = 0$ for all $k \neq k^*_i$. For example k^*_i is the alternative "agent i gets the object" in a standard auction setting for an indivisible object and in the bilateral trading framework of Myerson and Satterthwaithe, or the alternative "the public good is provided" in a classic public good problem. In this case $i's$ signal space S^i is a subset of the real line, and it represents $i's$ possible valuations for the alternative k^*_i.

A multi-dimensional auction with private values is analyzed in Jehiel, Moldovanu and Stacchetti (1999): an indivisible object is sold to one of N potential buyers (a social alternative is identified with the agent that gets the object) , and $a^i_{ji}s^i_{ji} = s^i_j$

$\sum_{j=1}^n a^j_{ki}s^j_{ki} + b^i_k$. Such constants do not affect incentives, and in order to simplify notation, we assume throughout the paper that $\forall i, k$, $b^i_k = 0$.

represents the externality suffered by i if the object is sold to agent j, while $a^i_{ij} = 0$ for all $j \neq i$. Hence, if agent j gets the object, agent i's utility is given by s^i_j which is known only to i. Finally, the multiproduct monopolist model (see for example Wilson, 1993) can be viewed as a private value auction with an unique agent whose characteristics are unknown.

3.2. Examples with interdependent valuations

Maskin (1992) and Dasgupta and Maskin (1998) consider the following situation: an indivisible object is to be allocated among n agents. Each agent obtains a one-dimensional signal, and valuations for the object are functions of all the signals received. To accommodate this situation in our setting[14], we identify social alternatives with the agents themselves, and define: $\forall i, j, h,\ i \neq h,\ a^j_{hi} = 0$ and $\forall i, j,\ s^j_{ii} = s^j_{jj}$.

Jehiel, Moldovanu and Stacchetti (1996) consider the following framework: an indivisible object is sold to one of N potential buyers (a social alternative is identified with the agent who gets the object) , and $a^i_{ij} s^i_{ij} = s^i_j$ represents the externality on agent j caused by agent i if he gets the object, while $a^i_{jh} = 0$ for all $j, h \neq i$. Hence, if agent i obtains the object her utility is given by s^i_i (which is known to i), while if agent $j, j \neq i$ gets the object, i's utility is given by s^j_i (which is known only to j).

4. Direct Revelation Mechanisms

Let S denote the Cartesian product $\prod_{i=1}^N S^i$, with generic element s , and define S^{-i}, s^{-i} as usual.

A function $p : S \to \Re^K$ such that $\forall k, s,\ 0 \leq p_k(s) \leq 1$ and $\forall s,\ \sum_{k=1}^K p_k(s) = 1$ is called a *social choice rule.* A social choice rule (SCR) is said to be *efficient* if

$$\forall s,\ p_q(s) \neq 0\ \Rightarrow q \in \arg\max_k \sum_{i=1}^N V^i_k(s^1, ..s^N) = \arg\max_k \sum_{i=1}^N \sum_{j=1}^N a^j_{ki} s^j_{ki}.$$

A *direct revelation mechanism* (DRM) is defined by a pair (p, x) where p is a social choice rule, and $x : S \to \Re^N$ is a payment scheme. The term $p_k(s)$ is the probability that alternative k is chosen if the agents report signals $s = (s^1, ..., s^N)$,

[14]Dasgupta and Maskin consider more general utility functions.

and $x_i(s)$ is the transfer to agent i if the agents report signals s. A DRM is *efficient* if the associated social choice rule is efficient[15].

For a DRM (p,x) we define functions $y_i : S^i \rightarrow \Re$ and $q^i : S^i \rightarrow \Re^K$ as follows:

$$y_i(t^i) = \int_{S^{-i}} x_i(t^i, s^{-i}) f_{-i}(s^{-i}) ds^{-i}$$

$$q^i_k(t^i) = \int_{S^{-i}} p_k(t^i, s^{-i}) f_{-i}(s^{-i}) ds^{-i}$$

Assume that agent i believes that all other agents report truthfully and assume that i reports type t^i when his true type is s^i. Then, $i's$ expected utility is given by:

$$\begin{aligned} & U_i(t^i, s^i) = \\ = \ & \int_{S^{-i}} [\sum_k (p_k(t^i, s^{-i}) \cdot \sum_{j=1}^{N} a^j_{ki} s^j_{ki})] f_{-i}(s^{-i}) ds^{-i} + y_i(t^i) = \\ = \ & \sum_k a^i_{ki} s^i_{ki} q^i_k(t^i) + \sum_k \int_{S^{-i}} [(p_k(t^i, s^{-i}) \cdot \sum_{j \neq i} a^j_{ki} s^j_{ki})] f_{-i}(s^{-i}) ds^{-i} + y_i(t^i) (4.1) \end{aligned}$$

A DRM is *incentive compatible* if:

$$\forall i, \forall s^i, t^i \in S^i, \ U_i(s^i, s^i) \geq U_i(t^i, s^i).$$

Let $V_i(s^i) = U_i(s^i, s^i)$, and note that in an incentive compatible mechanism we must have$V_i(s^i) = \max_{t^i} U_i(t^i, s^i)$.

We consider below several properties of incentive compatibe mechanisms. The function $V_i(\cdot)$ is the supremum of a collection of affine functions, hence it is convex[16]. Convex functions are twice differentiable almost everywhere. The convexity of $V_i(\cdot)$ implies the *cyclical monotonicity*[17] of the subdifferential map $\partial V_i(s^i)$. At

[15] We ignore here the (ex post) "budget balancedness" condition, which imposes $\sum_i x_i(s) \leq 0$, $\forall s$. In other words, we abstract from the efficiency losses due to potential external subsidies.

[16] This and all following properties of convex functions are listed in the classical text of Rockafellar, 1972.

[17] A (possibly multivalued) mapping $\Psi : S^i \rightarrow S^i$ is cyclically *monotone* if $(x_1 - x_0) \cdot x_0^* + (x_2 - x_1) \cdot x_1^* + ...(x_0 - x_r) \cdot x_r^* \leq 0$ for any set of pairs $(x_i, x_i^*), i = 0, 1, ...r$ (r arbitrary) such that $x_i^* \in \Psi(x_i)$. A *monotone* mapping needs to satisfy the above condition only for $r = 1$. Although there are monotone mappings that are not cyclically monotone, a monotone single valued mapping $\Omega : S^i \rightarrow \Re^m$ which is the gradient of a function $\omega : S^i \rightarrow R$ is also cyclically monotone. A necessary and sufficient condition for Ω to be the gradient of a function ω on S^i is $\int \gamma \Omega = 0$ for every closed curve γ in S^i. Such a Ω is called *conservative*.

all points where $V_i(\cdot)$ is differentiable (i.e., almost everywhere) the subdifferential $\partial V_i(\cdot)$ consists of a unique point, the gradient $\nabla V_i(\cdot)$. Hence, the function $\nabla V_i(\cdot)$ is well-defined, monotone and differentiable a.e. Finally, note that a convex function is (up to a constant) uniquely determined by its subdifferential, and that it can be recovered (up to a constant) by integrating its gradient.

Assuming that $V_i(\cdot)$ is differentiable at s^i we obtain by the Envelope Theorem that:

$$\forall k,\ \frac{\partial V_i}{\partial s^i_{ki}}(s^i) = \frac{\partial U_i}{\partial s^i_{ki}}(t^i, s^i)\mid_{t^i=s^i} = a^i_{ki} q^i_k(s^i) \tag{4.2}$$

$$\forall k, \forall j \neq i,\ \frac{\partial V_i}{\partial s^i_{kj}}(s^i) = \frac{\partial U_i}{\partial s^i_{kj}}(t^i, s^i)\mid_{t^i=s^i} = 0 \tag{4.3}$$

Note that the equality of cross-derivatives (which exist a.e.) implies here that:

$$\forall k, k',\ a^i_{ki}\frac{\partial q^i_k(s^i)}{\partial s^i_{k'i}} = \frac{\partial V_i}{\partial s^i_{k'i}\partial s^i_{ki}}(s^i) = \frac{\partial V_i}{\partial s^i_{ki}\partial s^i_{k'i}}(s^i) = a^i_{k'i}\frac{\partial q^i_{k'}(s^i)}{\partial s^i_{ki}} \tag{4.4}$$

Finally, equation 4.3 and the equality of cross-derivatives imply that:

$$\forall k, \forall j \neq i,\ a^i_{ki}\frac{\partial q^i_k(s^i)}{\partial s^i_{kj}} = \frac{\partial V_i}{\partial s^i_{kj}\partial s^i_{ki}}(s^i) = \frac{\partial V_i}{\partial s^i_{ki}\partial s^i_{kj}}(s^i) = 0 \tag{4.5}$$

The following proposition summarizes our observations and characterizes incentive compatible mechanisms (for analog results in multidimensional frameworks see Jehiel, Moldovanu, Stacchetti, 1996, 1999):

Theorem 4.1. *Let (p, x) be a DRM, and let $\{q^i(\cdot)\}_{i=1}^n$ be the associated conditional probability assignments. Then (p, x) is incentive compatible if and only if the following conditions hold:*

1. $\forall i$, the vector field $\{a^i_{ki} q^i_k(\cdot)\}_{k=1}^K$ is monotone and conservative.
2. $\forall i$, $\forall s^i$, $V_i(s^i) = V_i(\underline{s}^i) + \int_{\underline{s}^i}^{s^i} Q^i(t^i) \cdot dt^i$. The integral is taken on any path between $\underline{s}^i$ and s^i , and $Q^i(s^i) \in \Re^{K\times N}$ is the vector where $\forall k$, the ki^{th} coordinate is given by $a^i_{ki} q^i_k(s^i)$ and the kj^{th} coordinate, $j \neq i$, is zero.

Corollary 4.2. *Let (p, x) be an incentive compatible DRM, and let $\{q^i(\cdot)\}_{i=1}^n$ be the associated conditional probability assignments. Then the following conditions must hold:*

1. $\forall i, \forall k$, $q_k^i(\cdot)$ is non-decreasing in the variable s_{ki}^i.

2. $\forall i, j$, $j \neq i$, $\forall k$, $q_k^i(\cdot)$ is constant in the variable s_{kj}^i.

Proof. The first part follows from equation 4.2 and the the monotonicity condition in the above Theorem. The second part follows by equation 4.5[18]. ■

The above representation Theorem yields the following "Revenue Equivalence" result:

Theorem 4.3. *Let (p, x) and $(\hat{p}, \hat{x})$ be two efficient and incentive compatible DRMs. Then, there exist constants $\{c_i\}_{1 \leq i \leq n}$ such that $\forall s, \forall i$, $\hat{y}_i(s^i) = y_i(s^i) + c_i$ where $\hat{y}_i$ and y_i are the conditional expected payments associated with $(\hat{p}, \hat{x})$ and (p, x), respectively.*

Proof. By Theorem 4.1 and by equation 4.1, the conditional expected payment of agent i in any incentive compatible mechanism is solely a function of (i.e., an integral of) the associated expected probability assignment, and of the expected utility of an arbitrary type. Since any two efficient SCR coincide almost everywhere, the associated expected probability assignments are the same, and the associated conditional expected payments must be, up to a constant, the same. ■

5. Groves Mechanisms for the Private Values Case

For the private values case, a Groves mechanism is defined by: 1) a function $\hat{k}(s)$ such that $\hat{k}(s)$ is an efficient alternative for each vector of reports s. 2) An efficient SCR $\hat{p}$ such that, for all s, $\hat{p}_{\hat{k}(s)}(s) = 1$ and $\hat{p}_k(s) = 0$, for $k \neq \hat{k}(s)$. 3) Transfer functions $\hat{x}$ given by:

$$\hat{x}_i(s) = \sum_{j \neq i} V_{\hat{k}}^j(s^j) + D_i(s^{-i})$$

where $D_i(\cdot) : S^{-i} \rightarrow \Re$ is an arbitrary function. From the interim perspective what matters is the expected value of $D_i(\cdot)$, hence we can replace this function by its expectation. Note also that Groves mechanisms are, by definition, efficient, and that agent i's transfer directly depends only on the chosen alternative and on the signals reported by the other agents. It is well known that, in the private values case, a Groves mechanism is incentive compatible (no matter how the multidimensional signals are distributed).

[18] For a direct proof that does not use the symmetry of cross derivatives see Jehiel, Moldovanu and Stacchetti (1996).

Theorem 5.1. *(Williams, 1994) For any efficient and incentive compatible DRM (p, x) there exists a Groves mechanism $(\hat{p}, \hat{x})$ such that $\forall s, \forall i$, $\hat{p}(s) \equiv p(s)$ and $\hat{y}_i(s^i) = y_i(s^i)$ where $\hat{y}_i$ and y_i are the conditional expected payments associated with $\hat{x}$ and x , respectively.*

Proof. The equivalence result follows immediately from Theorem 4.3 since Groves mechanisms are efficient and incentive compatible. ■

6. Impossibility Results

We first show that efficient, incentive-compatible DRMs do not exist as soon as at least one agent receives a signal that does not affect her utility, but affects the utility of others.

Definition 6.1. *Let $\hat{p}$ be an efficient SCR, and let $\{\hat{q}^i(\cdot)\}_{i=1}^N$ be the associated conditional expected probability assignments. The variable $\tilde{s}_{kj}^i$ is said to be es-* sential *if there exist $s^i, t^i \in S^i$ such that:*

1. $s_{k'j'}^i = t_{k'j'}^i$ for all $(k', j') \neq (k, j)$.
2. $s_{kj}^i \neq t_{kj}^i$.
3. $\hat{q}_k^i(s^i) \neq \hat{q}_k^i(t^i)$.

Note that unless the density $f_i(\cdot)$ is degenerate (i.e., does not have full-dimensionality), *all* variables s_{kj}^i such that $a_{kj}^i \neq 0$ are essential. Moreover, since an efficient SCR is uniquely defined almost everywhere, the definition of essentiality does not depend on the specific SCR $\hat{p}$ which is used.

Theorem 6.2. *Assume that variable $\tilde{s}_{kj}^i$, where $i \neq j$, is essential. Then efficient, incentive compatible DRM's do not exist.*

Proof. Let s^i, t^i satisfy the conditions in Definition 6.1, and let (p, x) be an efficient, incentive-compatible DRM with associated conditional expected probability assignments $\{q^i(\cdot)\}_{i=1}^N$. By efficiency, we must have $q^i(u^i) = \hat{q}^i(u^i)$ for all $u^i \in S^i$. By Corollary 4.2, and the construction of s^i, t^i, we obtain that $q^i(s^i) = q^i(t^i)$. Since, by definition, $\hat{q}^i(s^i) \neq \hat{q}^i(t^i)$, we obtain a contradiction. ■

Example 6.3. *(A Credence Good)*

There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Agent 1 has a two-dimensional signal $s^1 = (s^1_A, s^1_B)$, distributed in the square $[0,1] \times [0,1]$ with density f_1. Agent 2 has a one-dimensional signal $s^2 = s^2_B$ distributed on the interval $[0,1]$ with density f_2, and associated cumulative distribution F_2. Valuations are given by: $V^1_A(s^1, s^2) = s^1_A$; $V^1_B(s^1, s^2) = 0$; $V^2_A(s^1, s^2) = 0$; $V^2_B(s^1, s^2) = s^1_B + s^2_B$.

Note that s^1_B does not affect the utility of agent 1 in alternative B, but it does affect the utility of agent 2 in that alternative.

Let (p, x) be an efficient incentive compatible DRM. Then, it must hold that:

$$p_A(s^1, s^2) = \begin{cases} 1, & \text{if } s^1_A \geq s^1_B + s^2_B \\ 0, & \text{otherwise} \end{cases}$$

Hence, we obtain that:

$$q^1_A(s^1) = \begin{cases} F_2(\ s^1_A - s^1_B), & \text{if } s^1_A \geq s^1_B \\ 0, & \text{otherwise} \end{cases}$$

On the other hand, by Theorem 4.1 we know that $q^1_A(s^1)$ cannot depend on s^1_B, which yields a contradiction. ■

The intuition behind Theorem 6.2 is very simple, but we now show that the displayed phenomenon has a deeper consequence. Till now we have assumed that s^i_{kj}, agent i' s piece of information affecting the utility of any agent j in a given alternative, is one-dimensional. We next look at an example where this dimensionality requirement is not satisfied. The resulting impossibility of efficient, incentive-compatible mechanisms in situations with this feature has been observed by Maskin (1992). What we show here is that this impossibility result is, in fact, a corollary of Theorem 6.2.

Example 6.4.

There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Signals are two-dimensional, $s^i = (s^i_1, s^i_2)$, $i = 1, 2$. Valuations are given by: $V^1_A(s^1, s^2) = s^1_1 + a(s^1_2 + s^2_2)$, $V^1_B(s^1, s^2) = 0$, $V^2_A(s^1, s^2) = 0$, $V^2_B(s^1, s^2) = s^2_1 + a(s^1_2 + s^2_2)$

The components s^i_1 , $i = 1, 2$, are the private parts of the signals (i.e., they influence only $i's$ utility, respectively), while the components s^i_2 are common parts[19] (i.e., they influence the utility of both agents). The present example does not,

[19] Compte and Jehiel (1998) look at related examples in order to study the value of competition in standard auctions.

a-priori, displays the main feature of Example 6.3 where agent i had a signal that does not affect her utility, but affects the utility of others. Nevertheless, we now show that the efficient rule cannot be implemented in the above setup: the problem is that agent $i's$ valuation in alternative i depends on $i's$ *two-dimensional* signal.

Consider the change of variables as follows:

$$t^i = (t_1^i, t_2^i) = (s_1^i + as_2^i, s_2^i)$$

Note that in the t^i type space we obtain: $V_A^1(t^1, t^2) = t_1^1 + at_2^2$, $V_B^1(t^1, t^2) = 0$, $V_B^2(t^1, t^2) = t_1^2 + at_2^1$, $V_A^2(t^1, t^2) = 0$.

Hence, agent i possesses a signal t_2^i which does not affect her utility, but affects the utility of agent $-i$. The impossibility result follows immediately from Theorem 6.2 , exactly as in Example 6.3. The example can be immediately extended to the case where $V_A^1(s^1, s^2) = s_1^1 + as_2^1 + bs_2^2$ and $V_B^2(s^1, s^2) = s_1^2 + as_2^2 + bs_2^1$. This shows that, even when the dependence of an agent's valuation on the signal of another agent is very small (i.e., b is very close to zero), efficiency cannot be attained. ■

The example above illustrates the general phenomenon: a simple matrix-inversion exercise and Theorem 6.2 can be used to prove impossibility in any generic framework where the type-matrix can be inverted, as above.

Our results so far suggest that, in order to obtain generic existence of efficient and incentive compatible mechanisms, it is necessary that $\forall i, j, i \neq j$, $\forall k$, s_{kj}^i is a strictly monotone function of s_{ki}^i, and that s_{ki}^i is one-dimensional. Since we want to remain in the linear framework, we consider below the case where $\forall i, j, i \neq j$, $\forall k$, s_{kj}^i is a linear function of s_{ki}^i. By suitably redefining s_{kj}^i we can assume without loss of generality that $\forall i, j, i \neq j$, $\forall k$, $s_{kj}^i = s_{ki}^i = s_k^i$.

Hence, we now look at $K-$ dimensional type-spaces, and we denote by s_k^i agent i's one-dimensional piece of information affecting (possibly in different ways) the utility of *all* agents in alternative k.

We assume in the sequel that all variables s_k^i are essential. Note that this requirement implies that $\forall i, \forall k$, $\sum_{j=1}^N a_{kj}^i \neq 0$.

The next Theorem shows that even the above reduction in complexity cannot generally ensure the existence of efficient and incentive compatible DRM's. The impossibility result has now deeper causes: it is entirely due to the conservativeness requirement imposed by incentive compatibility in the multidimensional case. Recall that equation 4.4 implies that the expected probability assignment generated by an incentive compatible mechanism must satisfy:

$$a^i_{ki}\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}} = a^i_{k'i}\frac{\partial q^i_{k'}(s^i)}{\partial s^i_k} \tag{6.1}$$

The question of existence of efficient, incentive compatible mechanisms boils down to the question whether an efficient social choice rule generates a monotone vector field having the above property[20].

Theorem 6.5. *Assume that (p, x) is an efficient DRM that is incentive compatible for agent i. Let k, k' be any pair of alternatives such that there exists a type t^i with $q^i_k(t^i) \neq 0$, and $q^i_{k'}(t^i) \neq 0$. Then it must be the case that*

$$\frac{a^i_{ki}}{a^i_{k'i}} = \frac{\sum_{j=1}^N a^i_{kj}}{\sum_{j=1}^N a^i_{k'j}} \tag{6.2}$$

Proof. See Appendix.

Condition 6.2 is a congruence requirement between private and social rates of information substitution. Unfortunately, the implied algebraic relations among parameters *cannot* be generically satisfied[21]. Note that condition 6.2 is trivially satisfied in two interesting and much studied non-generic cases: the private values case where $\forall i, j, i \neq j, \forall k,\ a^i_{kj} = 0$, and the symmetric case where $\forall i, j, k,\ a^i_{kj} = a^i_{ki}$.

The above Theorem has a converse: If condition 6.2 is satisfied, and if an efficient social choice rule p yields monotone vector field $(a^i_{ki}q^i_k(s^i))_{k=1,..K}$, then there exists a payment schedule $x_i(\cdot)$ such that (p, x) is incentive compatible for i.

The following 2-agent, 2-alternative example illustrates the insight which underlies the proof of Theorem 6.5.

Example 6.6.

There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Signals are two dimensional, $s^i = (s^i_A, s^i_B)$, $i = 1, 2$.

Valuations are given by:

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[20] Technically and conceptually, the problem is analogous to the celebrated *integrability* question in classical demand theory: which demand functions (for several goods) can be rationalized by some utilitiy maximization.

[21] i.e., the set of parameters satisfying the condition is closed and has Lebesgue-measure zero.

$$V_k^i(s^i, s^{-i}) = a_{ki}^i s_k^i + a_{ki}^{-i} s_k^{-i},\ i = 1, 2, ; k = A, B$$

We assume below that

$$a_{ki}^i + a_{k-i}^i \neq 0,\ i = 1, 2, ; k = A, B$$

Assume that an efficient and incentive compatible DRM exists, and denote it by (p, x). Let $q_k^i(s^i)$ denote agent's i associated interim expected probability that alternative k is chosen by the mechanism. We know that $\forall i, \forall s^i$, $q_A^i(s^i) + q_B^i(s^i) = 1$. Let $V_i(s^i)$ be the utility of agent i with type s^i in the truth-telling equilibrium of (p, x). By Theorem 4.1 we know that:

$$\frac{\partial V_i}{\partial s_k^i}(s^i) = a_{ki}^i q_k^i(s^i),\ i = 1, 2;\ k = A, B$$

By equation 6.1 we have:

$$a_{Ai}^i \frac{\partial q_A^i(s^i)}{\partial s_B^i} = a_{Bi}^i \frac{\partial q_B^i(s^i)}{\partial d s_A^i},\ i = 1, 2 \tag{6.3}$$

For the above derivations we have used only conditions imposed by incentive compatibility. We now look at the consequences of efficiency. Alternative A is chosen at reports (s^1, s^2) by an efficient DRM iff

$$\sum_{i=1}^{2} \sum_{j=1}^{2} a_{Ai}^j s_A^j \geq \sum_{i=1}^{2} \sum_{j=1}^{2} a_{Bi}^j s_B^j$$

This is equivalent to:

$$(a_{A1}^1 + a_{A2}^1) s_A^1 - (a_{B1}^1 + a_{B2}^1) s_B^1 \geq (a_{B1}^2 + a_{B2}^2) s_B^2 - (a_{A1}^2 + a_{A2}^2) s_A^2 \tag{6.4}$$

We now obtain:

$$q_A^1(s^1) = \int_{\Delta(s^1)} f_2(s^2) ds^2$$

where $\Delta(s^1) = \{s^2$ such that condition 6.4 is satisfied$\}$.
As before, we have also $q_B^1(s^1) = 1 - q_A^1(s^1) = 1 - \int_{\Delta(s^1)} f_2(s^2) ds^2$.

Fix now a type s^1 and let $(a^1_{A1}+a^1_{A2})s^1_A-(a^1_{B1}+a^1_{B2})s^1_B=C$. Then for any t^1 such that $(a^1_{A1}+a^1_{A2})t^1_A-(a^1_{B1}+a^1_{B2})t^1_B=C$ we obtain that $\Delta(t^1)=\Delta(s^1)$ and therefore that $q^1_A(t^1)=q^1_A(s^1)$. This "pooling" of types along specific lines yields the following differential condition:

$$(a^1_{A1}+a^1_{A2})\frac{\partial q^1_A(s^1)}{\partial s^1_B}+(a^1_{B1}+a^1_{B2})\frac{\partial q^1_A(s^1)}{\partial s^1_A}=0$$

Since $q^i_B(s^i)=1-q^i_A(s^i)$, we obtain that:

$$\frac{\partial q^i_B(s^i)}{\partial s^i_A}=-\frac{\partial q^i_A(s^i)}{\partial s^i_A},\ i=1,2$$

Combining the last two equations we obtain for $i=1$:

$$(a^1_{A1}+a^1_{A2})\frac{\partial q^1_A(s^1)}{\partial s^1_B}=(a^1_{B1}+a^1_{B2})\frac{\partial q^1_B(s^1)}{\partial s^1_A} \tag{6.5}$$

A similar reasoning yields an analogous condition also for $i=2$. Equations 6.3 and 6.5 yield together:

$$\frac{a^i_{Ai}}{a^i_{Bi}}=\frac{a^i_{Ai}+a^i_{A-i}}{a^i_{Bi}+a^i_{B-i}},\ i=1,2 \tag{6.6}$$

■

Theorem 6.5 sheds some light on the outcome of any multi-object auction where the objects and the agents are heterogenous in a non-trivial way. If there are informational externalities, we have shown that, whatever sale mechanism is considered (including mechanisms that allow for "combinatorial" bidding), efficiency cannot be achieved. The inefficiency has structural causes and the construction of a second-best mechanism is not at all trivial since it is, essentially, a problem of finding the monotone and conservative vector field that maximizes a certain functional. Finally, note that the exhibited inefficiency is not necessarily diminished as, say, the number of agents (or the number of agents and the number of objects) gets larger[22].

[22]For examples of market models that display limit efficiency results see Gul and Postlewaite (1992) and Pesendorfer and Swinkels (1997).

7. Generalized Groves Mechanisms

We now pursue a further reduction in complexity, and we assume that agents get one-dimensional signals (i.e., the same, one-dimensional piece of information affects all valuations in *all* alternatives).

7.1. The Pigou-Clarke-Groves approach

We first try to mimic the intuition of the Groves mechanisms, which, as we have seen before, "span" the set of efficient and incentive compatible mechanisms in the private values case. By the result of Williams and Radner (1988), there is no hope of finding a generalization that yields dominant-strategy efficient mechanisms. Nevertheless, as we shall immediately see, generalized Groves mechanisms that are efficient and (Bayes-Nash) incentive compatible may exist in some cases.

Let $s^i \in [\underline{s}^i, \bar{s}^i]$. Here signals need not be independently distributed, and the equilibrium we find will not depend in any way on the signals' distribution functions . A Generalized Groves Mechanism (p^*, x^*) is based on a function $\hat{k}(s)$ such that $\hat{k}(s)$ is an efficient alternative for each vector of reports s and an efficient SCR $\hat{p}$ such that, for all s, $\hat{p}_{\hat{k}(s)}(s) = 1$, and $\hat{p}_k(s) = 0$, for $k \neq \hat{k}(s)$.

To formulate the transfers for each agent i, consider for each vector of reported signals s^{-i} and for each alternative k the set:

$$\Psi_i(s^{-i}, k) = \{t^i / k^*(t^i, s^{-i}) = k\}$$

Hence, $\Psi_i(s^{-i}, k)$ is the set of reports that can be made by i (given reports by others) such that the efficient alternative is k. For each vector of reported signals $s = (s^1, ..., s^N)$, let $\hat{s}^i(s^{-i}, k^*(s))$ be a selection[23] out of the set $\Psi_i(s^{-i}, k^*(s))$, which is not empty because, by definition, $s^i \in \Psi_i(s^{-i}, k^*(s))$. We define transfers by :

$$\begin{aligned} x_i^*(s) &= x_i^*(s^{-i}, k^*(s)) = \sum_{j \neq i} V^j_{k^*(s)}(s^{-i}, \hat{s}^i(s^{-i}, k^*(s))) + D_i(s^{-i}) = \\ & \sum_{j \neq i} a^i_{k^*(s)j} \hat{s}^i(s^{-i}, k^*(s)) + \sum_{j \neq i} \sum_{h \neq i} a^h_{k^*(s)j} s^h + + D_i(s^{-i}) \end{aligned}$$

where $D_i(s^{-i})$ is an arbitrary function of s^{-i}.

[23] We discuss later which selections, if any, are suitable.

Note that $x_i^*(s)$ depends only indirectly on s^i, namely through the efficient choice $k^*(s)$.

Proposition 7.1. *Assume that there are only two alternatives, $k = A, B$, and that, for each agent i, the following condition is satisfied:*

$$a_{ki}^i \geq a_{-ki}^i \Rightarrow \sum_{j=1}^{N} a_{kj}^i \geq \sum_{j=1}^{N} a_{-kj}^i \text{ , } k = A, B \tag{7.1}$$

For each agent i , and for each vector reports s^{-i}, let w be the solution to the equation $\sum_{j=1}^N V_A^j(s^{-i}, z) = \sum_{j=1}^N V_B^j(s^{-i}, z)$. Define

$$\hat{s}^i(s^{-i}, k^*(s)) = \hat{s}^i(s^{-i}) = \begin{cases} w, \text{ if } w \text{ exists and } w \in [\underline{s}^i, \bar{s}^i] \\ \underline{s}^i, \text{ otherwise} \end{cases}$$

Then, any generalized Groves mechanism (p^*, x^*) based on the above selection is incentive compatible.

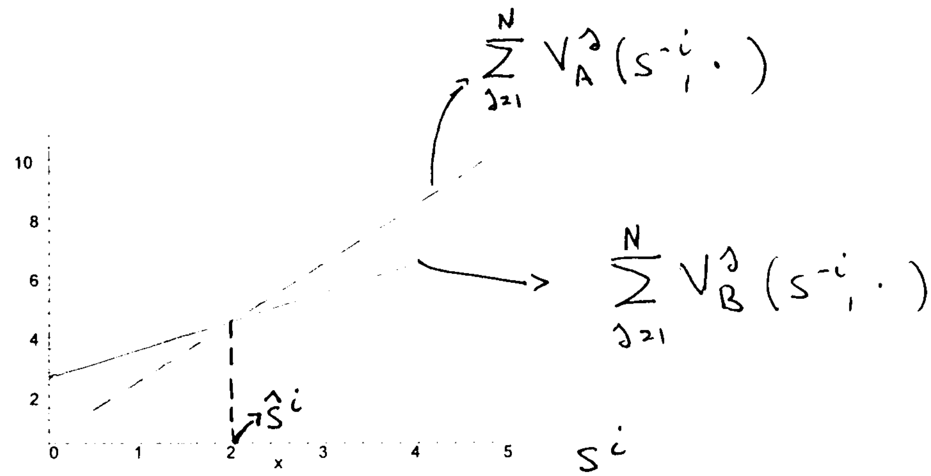
Proof. Since the arbitrary function $D_i(s^{-i})$ does not influence incentives, we assume here for simplicity that it is constantly zero.

Fix agent i and assume that the other agents truthfully report s^{-i}. Observe that if there exists $\hat{k}$ such that $\forall s^i \in [\underline{s}^i, \bar{s}^i]$, $\sum_{j=1}^N V_{\hat{k}}^j(s^{-i}, s^i) > \sum_{j=1}^N V_{-\hat{k}}^j(s^{-i}, s^i)$, then an efficient allocation rule does not depend on i's report, and we need not worry about i's incentives. In particular, given $x_i^*(s) = \sum_{j=1}^N V_{\hat{k}}^j(s^{-i}, \underline{s}^i)$ it is optimal for i to report truthfully.

Assume then that the equation $\sum_{j=1}^N V_A^j(s^{-i}, z) = \sum_{j=1}^N V_B^j(s^{-i}, z)$ has a solution $w = \hat{s}^i(s^{-i}) \in [\underline{s}^i, \bar{s}^i]$. Assume without loss of generality that $a_{Ai}^i \geq a_{Bi}^i$. By assumption, we must also have $\sum_{j=1}^N a_{Aj}^i \geq \sum_{j=1}^N a_{Bj}^i$. Because $\forall k, \sum_{j=1}^N a_{kj}^i = \sum_{j=1}^N \left(\frac{\partial V_k^j(s^{-i}, s^i)}{\partial s^i} \right)$,this implies the following:

$$\text{For all } s^i \geq \hat{s}^i(s^{-i}), \ \sum_{j=1}^{N} V_A^j(s^{-i}, s^i) \geq \sum_{j=1}^{N} V_B^j(s^{-i}, s^i)$$

$$\text{For all } s^i \leq \hat{s}^i(s^{-i}), \ \sum_{j=1}^{N} V_B^j(s^{-i}, s^i) \geq \sum_{j=1}^{N} V_A^j(s^{-i}, s^i)$$



Assume now that i has true type s^i. There are two cases:

Case I: $s^i \geq \hat{s}^i(s^{-i})$. In this case, if i reports truthfully her payoff is given by $V_A^i(s^{-i}, s^i) + \sum_{j \neq i} V_A^j(s^{-i}, \hat{s}^i(s^{-i}))$. As long as i reports $t^i \geq \hat{s}^i(s^{-i})$, her payoff does not change.

Assume then that i reports $t^i \leq \hat{s}^i(s^{-i})$. Her payoff is then $V_B^i(s^{-i}, s^i) + \sum_{j \neq i} V_B^j(s^{-i}, \hat{s}^i(s^{-i}))$. We need to show that $V_A^i(s^{-i}, s^i) + \sum_{j \neq i} V_A^j(s^{-i}, \hat{s}^i(s^{-i})) \geq V_B^i(s^{-i}, s^i) + \sum_{j \neq i} V_B^j(s^{-i}, \hat{s}^i(s^{-i}))$. Since $s^i \geq \hat{s}^i(s^{-i})$, the result follows because , by definition, $V_A^i(s^{-i}, \hat{s}^i(s^{-i})) + \sum_{j \neq i} V_A^j(s^{-i}, \hat{s}^i(s^{-i})) =$

$V_B^i(s^{-i}, \hat{s}^i(s^{-i})) + \sum_{j \neq i} V_B^j(s^{-i}, \hat{s}^i(s^{-i}))$, and because, by assumption, $a_{Ai}^i \geq a_{Bi}^i$.

Case II: $s^i \leq \hat{s}^i(s^{-i})$. In this case, if i reports truthfully her payoff is given by $V_B^i(s^{-i}, s^i) + \sum_{j \neq i} V_B^j(s^{-i}, \hat{s}^i(s^{-i}))$. As long as i reports $t^i \leq \hat{s}^i(s^{-i})$, her payoff does not change.

Assume then that i reports $t^i \geq \hat{s}^i(s^{-i})$. Her payoff is then $V_A^i(s^{-i}, s^i) + \sum_{j \neq i} V_A^j(s^{-i}, \hat{s}^i(s^{-i}))$. We need to show that $V_B^i(s^{-i}, s^i) + \sum_{j \neq i} V_B^j(s^{-i}, \hat{s}^i(s^{-i})) \geq V_A^i(s^{-i}, s^i) + \sum_{j \neq i} V_A^j(s^{-i}, \hat{s}^i(s^{-i}))$. Since $s^i \leq \hat{s}^i(s^{-i})$, the result follows because , by definition, $V_A^i(s^{-i}, \hat{s}^i(s^{-i})) + \sum_{j \neq i} V_A^j(s^{-i}, \hat{s}^i(s^{-i})) =$

$V_B^i(s^{-i}, \hat{s}^i(s^{-i})) + \sum_{j \neq i} V_B^j(s^{-i}, \hat{s}^i(s^{-i}))$, and because, by assumption, $a_{Ai}^i \geq a_{Bi}^i$. The case where $a_{Ai}^i \leq a_{Bi}^i$ is analogous. ■

For more than two alternatives, condition 7.1 becomes[24]:

$$\forall i, \forall k, k', \ a_{ki}^i \geq a_{k'i}^i \Rightarrow \sum_{j=1}^{n} a_{kj}^i \geq \sum_{j=1}^{n} a_{k'j}^i \tag{7.2}$$

Note the analogy with condition 6.2, but note also the gained slack in the one-dimensional framework. This slack (i.e., required inequalities instead of equalities)

[24] A more refined formula is needed if valuations are not linear in signals, i.e. if marginal valuations are not constant.

allows the condition to be satisfied for an open set of parameters' values.

The result of Proposition 7.1 easily generalizes if agents have general valuations and if there are only two alternatives. Moreover, the result generalizes to the case where there are several alternatives, but each agent perceives only two alternatives as different from her point of view. As an illustration, consider the following auction model, taken from Dasgupta and Maskin (1998):

There are N agents who bid for an indivisible good. We identify alternatives with the agents, so that alternative i is the alternative "the good is allocated to agent i". Signals are one dimensional, $s^i \in [\underline{s}^i, \bar{s}^i]$. For all i, j and for all $k \neq i$, $a^j_{ki} = 0$. To simplify notation, we denote a term a^j_{ii} by a^j_i. Hence, the valuation of agent i is $V^i_i(s^1, .., s^N) = V^i(s^1, .., s^N) = \sum_{j=1}^N a^j_i s^j$ if i gets the object, and $V^i_j(s^1, .., s^N) = 0$ if agent $j \neq i$ gets the object (as stated above, linearity is not necessary for the result).

Observe that each agent i actually perceives only two distinct payoff-relevant alternatives (i.e., only two alternatives that offer potentially different payoffs): "i gets the object" (call that alternative i) and "i does not get the object" (call that alternative $-i$). Although there might be $K = N > 2$ alternatives, this observation enables the use of the technique of Proposition7.1. Since in this model $\forall j, j \neq i, \frac{\partial V^i_i(s)}{\partial s^i} \geq \frac{\partial V^i_j(s)}{\partial s^i} = 0$, condition 7.2 generally translates into:

$$\forall i, \forall s, \forall j, j \neq i, \frac{\partial V^i_i(s)}{\partial s^i} \geq \frac{\partial V^i_j(s)}{\partial s^i}$$

This is the condition used by Dasgupta and Maskin (1998)[25].

A Generalized Groves Mechanism (p^*, x^*) is now defined as follows: For each vector of reported signals $s = (s^1, ..., s^N)$, let $i^*(s) \in \arg\max_i V^i(s^1, ..., s^N)$, and let $p^*_{i^*(s)}(s) = 1$, and $p^*_j(s) = 0$, for $j \neq i^*(s)$. Assume that s is the vector of reports. If the alternative "i gets the object" is chosen, then all $j \neq i$ obtain a payoff of zero, and hence $i's$ transfer in a Generalized Groves Mechanism should be $0 + D_i(s^{-i}) = D_i(s^{-i})$. From the point of view of i, if alternative "i does not get the object" is chosen, then, in an efficient mechanism, the object is allocated to the agent $j \neq i$ such that $j \in \arg\max_{e \neq i} V^e_e(s)$, and all agents $e \neq j$ have a payoff of zero. In this case $i's$ transfer should be $\max_{j \neq i} V^j_j(s^{-i}, \hat{s}(s^{-i})) + D_i(s^{-i})$. Exactly as before, let $\hat{s}(s^{-i})$ be equal to the solution of the equation $V^i_i(s^{-i}, z) = \max_{j \neq i} V^j_j(s^{-i}, z)$, if the equation has a solution in the interval $[\underline{s}^i, \bar{s}^i]$, and let $\hat{s}(s^{-i}) = \underline{s}^i$ otherwise. By Proposition

[25] This condition has been often used in the literature. For example, Gresik (1991) uses it in the context of a bilateral market. His work focuses on *ex-ante* efficient trade mechanisms.

7.1, any such mechanism is incentive compatible and efficient. Taking $D_i(s^{-i}) = -(\max_{j\neq i} V_j^j(s^{-i}, \hat{s}(s^{-i})))$, we obtain the direct mechanism corresponding to the English auction studied by Maskin (1992). Dasgupta and Maskin offer a Vickrey mechanism that yields efficient outcomes, but whose rules do not depend on the valuation functions as our direct mechanisms do (however, knowledge of valuations functions is required from the agents).

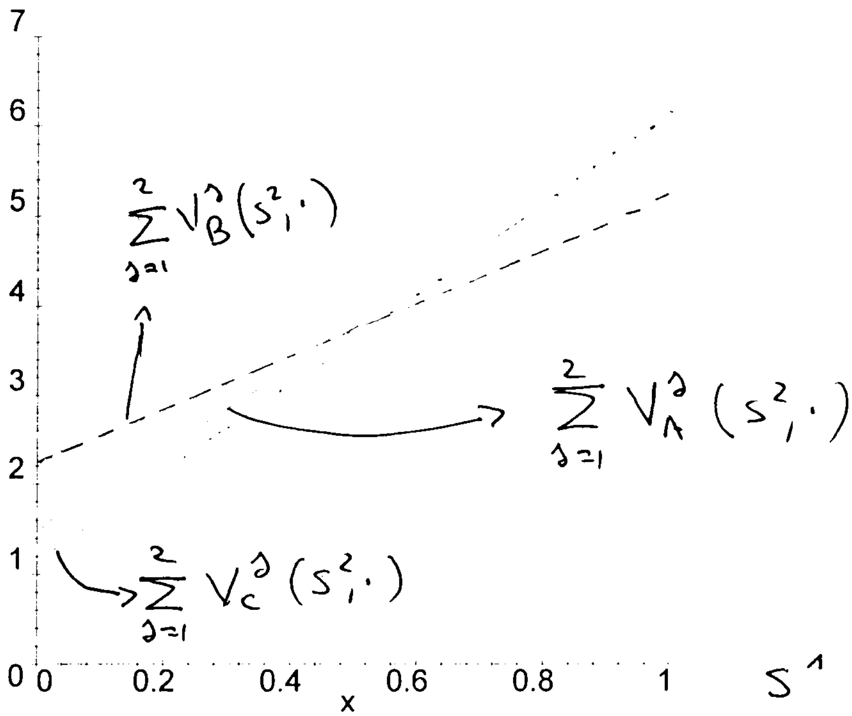
We conclude this section with a (generic) example showing that Generalized Groves Mechanisms as defined above may not be incentive compatible if agents perceive more than two payoff-relevant alternatives.

Example 7.2. *There are two agents , denoted by* 1 *and* 2, *and three alternatives, denoted by* A, B, C. *Signals* s^1, s^2 *are distributed in* $[0, 1]$. *Valuations are as follows:*

$V_A^1(s^1, s^2) = 3s^1 + s^2 \quad V_A^2(s^1, s^2) = 2s^2 + \frac{3}{2}s^1$
$V_B^1(s^1, s^2) = 2s^1 + \frac{3}{2}s^2 \quad V_B^2(s^1, s^2) = 3s^2 + s^1$
$V_C^1(s^1, s^2) = 4s^1 + \frac{3}{4}s^2 \quad V_C^1(s^1, s^2) = \frac{3}{2}s^2 + s^1$

Note first that condition 7.2 is satisfied.

Fix $s^2 = \frac{1}{2}$. We obtain the following: Alternative B is efficient for $s^1 \in [0, \frac{1}{2}]$, i.e., $\Psi_1(\frac{1}{2}, B) = [0, \frac{1}{2}]$; Alternative A is efficient for $s^1 \in [\frac{1}{2}, \frac{3}{4}]$, i.e., $\Psi_1(\frac{1}{2}, A) = [\frac{1}{2}, \frac{3}{4}]$; Alternative C is efficient for $s^1 \in [\frac{3}{4}, 1]$, i.e., $\Psi_1(\frac{1}{2}, C) = [\frac{3}{4}, 1]$.



Note also that $\sum_{j=1}^{2}(\frac{\partial V_B^j(s^1,s^2)}{\partial s^1}) \geq \sum_{j=1}^{2}(\frac{\partial V_A^j(s^1,s^2)}{\partial s^1}) \geq \sum_{j=1}^{2}(\frac{\partial V_C^j(s^1,s^2)}{\partial s^1})$ .Consider a Generalized Groves mechanism based on selections $\hat{s}_k^i = \hat{s}^i(\frac{1}{2},k)$, $k = A,B,C$, out of the sets $\Psi_1(\frac{1}{2},k)$, $k = A,B,C$, respectively, and assume that truth-telling is a Bayes-Nash equilibrium (for any distribution of agents' signals).

For any $s^1 \in \Psi_1(\frac{1}{2},B) = [0,\frac{1}{2}]$ it must hold that:

$$\begin{aligned} V_B^1(s^1,\frac{1}{2}) + V_B^1(\hat{s}_B^i,\frac{1}{2}) &\geq V_A^1(s^1,\frac{1}{2}) + V_A^2(\hat{s}_A^i,\frac{1}{2}) \Leftrightarrow \\ s^1 + \frac{3}{2}\hat{s}_A^i - \hat{s}_B^i - \frac{3}{4} &\leq 0 \end{aligned}$$

For any $s^1 \in \Psi_1(\frac{1}{2},A) = [\frac{1}{2},\frac{3}{4}]$ it must hold that:

$$\begin{aligned} V_A^1(s^1,\frac{1}{2}) + V_A^2(\hat{s}_A^i,\frac{1}{2}) &\geq V_B^1(s^1,\frac{1}{2}) + V_B^1(\hat{s}_B^i,\frac{1}{2}) \Leftrightarrow \\ s^1 + \frac{3}{2}\hat{s}_A^i - \hat{s}_B^i - \frac{3}{4} &\geq 0 \end{aligned}$$

The only selection which is consistent with the above inequalities is $\hat{s}^i_A = \hat{s}^i_B = \frac{1}{2}$.

For any $s^1 \in \Psi_1(\frac{1}{2}, A) = [\frac{1}{2}, \frac{3}{4}]$ it must hold that:

$$\begin{aligned} V_A^1(s^1, \frac{1}{2}) + V_A^2(\hat{s}^i_A, \frac{1}{2}) &\geq V_C^1(s^1, \frac{1}{2}) + V_C^1(\hat{s}^i_C, \frac{1}{2}) \Leftrightarrow \\ s^1 - \frac{3}{2}\hat{s}^i_A + \hat{s}^i_C - \frac{3}{8} &\leq 0 \end{aligned}$$

For any $s^1 \in \Psi_1(\frac{1}{2}, C) = [\frac{3}{4}, 1]$ it must hold that:

$$\begin{aligned} V_C^1(s^1, \frac{1}{2}) + V_C^1(\hat{s}^i_C, \frac{1}{2}) &\geq V_A^1(s^1, \frac{1}{2}) + V_A^2(\hat{s}^i_A, \frac{1}{2}) \Leftrightarrow \\ s^1 - \frac{3}{2}\hat{s}^i_A + \hat{s}^i_C - \frac{3}{8} &\geq 0 \end{aligned}$$

The only selection which is consistent with the above inequalities is $\hat{s}^i_A = \hat{s}^i_C = \frac{3}{4}$. This yields the contradiction $\frac{1}{2} = \hat{s}^i_B = \hat{s}^i_A = \hat{s}^i_C = \frac{3}{4}$. ■

It can be shown that, provided that condition 7.2 holds, there exist other, more sophisticated, mechanisms fulfilling both efficiency and incentive compatibility. The basic intuition for the case of more than two alternatives has been first illustrated by Dasgupta and Maskin (1998), while a general condition allowing implementation with more than two alternatives was first identified in an earlier version of our paper.

To gain some general insight about efficient mechanisms in the one-dimensional case, we consider below a framework with a continuum of outcomes (or alternatives) z - this allows here a simpler differential approach.

7.2. Efficient mechanisms

Let s^j be the one-dimensional signal of agent j. Agent i' s payoff in outcome z is given by

$$V^i(s^{-i}, s^i; z) = \sum_{j=1}^{N} a_i^j(z) s^j \tag{7.3}$$

where $a_i^j(\cdot)$ are regular functions of z. Observe that this representation precisely fits the linear framework we had before: for each outcome z , the payoff function is linear in signals. We impose some additional structure, and require that, for each i, $a_i^i(\cdot)$ is strictly increasing in z.

If z is to be interpreted as a provision of public good, then the model can be enriched by a cost function $C(\cdot)$ where $C(z)$ is the cost of providing the public good z[26]. An efficient mechanism is such that for each signal draw $s = (s^1, s^2, ..., s^N)$ the outcome

$$\widehat{z}(s) = \arg\max_z \sum_{i=1}^N V^i(s; z) - C(z)$$

is chosen. For other interpretations, assume that $C(z) \equiv 0$.

In what follows, we assume that, for every s, the function $z \rightarrow \sum_{i=1}^N V^i(s; z) - C(z)$ is concave, regular and has a unique maximum[27].

Since $a_i^i(\cdot)$ is increasing, and since $\frac{\partial V^i(s;z)}{\partial s^i} = a_i^i(z)$, the analog of Condition 7.2 in this setup is as follows:

$$\text{For each } i, \text{ the function } z \rightarrow \sum_{j=1}^N a_j^i(z) \text{ is increasing.} \tag{7.4}$$

We refer to this condition as the *congruence condition.*

Proposition 7.3. *Assume that the congruence condition is satisfied. Then there exists an efficient, Bayesian incentive compatible mechanism. Moreover, the associated transfers do not depend on the distribution of signals*[28].

Proof. We prove the existence of efficient, incentive compatible mechanisms in which truth-telling is a Nash equilibrium (no matter what the distributions governing agents' signals are). Consider agent i, and assume that the other agents $j, j \neq i$ obtain signals s^{-i} and report truthfully. By a standard argument (see Chapter 23 in Mas-Colell, Whinston and Green, 1995), every function $s^i \rightarrow z(s^i, s^{-i})$ that is increasing in s^i can be Bayes-Nash implemented. It suffices thus to show that, under the congruence condition, the efficient allocation function $\widehat{z}(s)$ is increasing in s^i. By definition, we know that:

$$\widehat{z}(s) = \arg\max_z \left[s^i(\sum_{j=1}^N a_j^i(z)) + H(s^{-i}, z) \right] ,$$

[26] In the special case where $a_i^j(z) \equiv 0$ this model is the one studied by Clarke and Groves, and the Clarke-Groves mechanisms allows to implement the efficient level of public good.

[27] This is for example the case if all $a_i^j(z)$ are constant, $g_i(z) \equiv z$, and the cost function $C(\cdot)$ is strictly convex.

[28] A similar property has been observed by Dasgupta and Maskin (1998) in their auction model with discrete alternatives.

where $H(\cdot,\cdot)$ is a function that does not depend on s^i. The first order condition yields for $z = \widehat{z}$:

$$s^i \frac{d}{dz}[\sum_{j=1}^{N} a_j^i(z)] + \frac{\partial}{\partial z} H(s^{-i}, z) = 0.$$

Total differentiation with respect to s^i yields:

$$\frac{d}{dz}[\sum_{j=1}^{N} a_j^i(z)] + \frac{\partial \widehat{z}(s)}{\partial s^i} \cdot \frac{\partial^2}{\partial z^2} \left[s^i (\sum_{j=1}^{N} a_j^i(z)) + H(s^{-i}, z) \right] = 0.$$

The term $\frac{d}{dz}[\sum_{j=1}^{N} a_j^i(z)]$ is positive by the congruence assumption.

The term $\frac{\partial^2}{\partial z^2} \left[s^i (\sum_{j=1}^{N} a_j^i(z)) + H(s^{-i}, z) \right]$ is negative because of the second order condition characterizing the maximand $\widehat{z}(s)$. It follows that $\frac{\partial \widehat{z}}{\partial s^i}(\cdot) > 0$ as desired.

Given that other agents report truthfully, truthful reporting is optimal for agent i with type s^i if the following condition is satisfied:

$$\forall s^{-i}, \; s^i \in \arg\max_{t^i} [V^i(t^i, s^{-i}, \widehat{z}(t^i, s^{-i})) + x_i(t^i, s^{-i})]$$

This yields the first order condition:

$$\forall s^{-i}, \; \frac{\partial V^i(s^i, s^{-i}, \widehat{z}(s))}{\partial z} \cdot \frac{\partial \widehat{z}(s^i, s^{-i})}{\partial t^i} + \frac{\partial x_i(s^i, s^{-i})}{\partial t^i} = 0$$

By the definition of $\widehat{z}(s)$ we know that :

$$\forall s, \; \left. \frac{\partial [\sum_{j \neq i} V^j(s, z) - C(z)]}{\partial z} \right|_{z = \widehat{z}(s)} = 0$$

Combining the two equations above, we obtain that:

$$\forall s^{-i}, \; \frac{\partial x_i(s^i, s^{-i})}{\partial s^i} = \frac{\partial [\sum_{j \neq i} V^j(s, \widehat{z}(s)) - C(\widehat{z}(s))]}{\partial z} \cdot \frac{\partial \widehat{z}(s^i, s^{-i})}{\partial s^i} \qquad (7.5)$$

It is clear that incentive constraints continue to be satisfied also if we take the expectation over s^{-i}. The transfers $x_i(\cdot,\cdot)$ do not depend on the distribution of signals because the densities $f_i(\cdot)$ do not appear at all in the first-order conditions . ■

In order to better understand the additional complexity, consider the private value case. In this case, equation 7.5 simplifies to:

$$\forall s^{-i},\ \frac{\partial x_i(s^i,s^{-i})}{\partial s^i}=\frac{\partial[\sum_{j\neq i}V^j(s^{-i},\widehat{z}(s))-C(\widehat{z}(s))]}{\partial s^i}$$

Hence, we can implement the efficient allocation (even in dominant strategies) by choosing the transfers $x_i(s^i,s^{-i})=\sum_{j\neq i}V^j(s^{-i},\widehat{z}(s))-C(\widehat{z}(s))$. This is exactly the Groves insight.

We conclude the subsection by providing an example.

Example 7.4.

Consider a public good provision framework with two agents, $i=1,2$. For a level z of the public good, valuations are given by $V^1(s^2,s^1;z)=(s^1+\lambda_1 s^2)z$ and $V^2(s^1,s^2;z)=(s^2+\lambda_2 s^1)z$, respectively. The cost function is $C(z)=\frac{z^2}{2}$. Note that $\forall i$, $a^i_i(z)\equiv z$, and that the congruence condition says here: $1+\lambda_1>0$ and $1+\lambda_2>0$. The efficient outcome rule is given by $\widehat{z}(s)=(1+\lambda_2)s^1+(1+\lambda_1)s^2$, which, under the congruence condition, is obviously increasing in both s^1 and s^2. Let $x_i(s^{-i},s^i)$ be a transfer such that

$$\frac{\partial x_i(s^{-i},s^i)}{\partial s^i}=-(1+\lambda_{-i})(s^i+\lambda_i s^{-i})$$

For example, $x_i(s^{-i},s^i)$ can be taken to be $-(1+\lambda_{-i})(\frac{(s^i)^2}{2}+\lambda_i s^{-i}s^i)$, or the expectation of this term with respect to s^{-i}. These transfers implement the efficient allocation. Note that the second order conditions for maximization are satisfied whenever the congruence condition is satisfied - this is precisely the essence of Proposition 7.3.

Consider now the private values case where $\lambda_i=0, i=1,2$. We now need that $\frac{\partial x_i(s^{-i},s^i)}{\partial s^i}=-s^i$. Observe that $V^{-i}(s;\widehat{z}(s))-C(\widehat{z}(s))=\left[(s^{-i})^2-(s^i)^2\right]/2$, and hence that $\frac{\partial[V^{-i}(s;\widehat{z}(s))-C(\widehat{z}(s))]}{\partial s^i}=-s^i$. In other words, the transfer $x_i(s^{-i},s^i)=V^{-i}(s;\widehat{z}(s))-C(\widehat{z}(s))$ implements the efficient allocation in this special case. ■

8. Conclusions

We have shown that efficient, incentive compatible mechanisms can exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multi-dimensional, the congruence condition is determined by a complex integrability constraint, and it can be satsfied only in non-generic cases such as the private value case or the symmetric case. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied.

The impossibility results in the multi-dimensional case suggest a quest for the second-best (or constrained efficient) mechanisms. It is straightforward to construct second-best mechanisms if the inefficiency is purely due to the causes illustrated in Theorem 6.2 and Example 6.4 : basically, incentive compatibility implies that we are able to "extract" at most a scalar piece of information per agent and per alternative. If, after performing these reductions, it is still the case that the payoff-relevant information depends in a non-trivial way on the chosen alternative (as it is the case, say, in a general multi-object auction), we are left in the framework covered by Theorem 6.5: unfortunately, a straightforward further dimension reduction (say, to a one-dimensional model) cannot be usually performed[29]. The problem of constructing a second-best mechanism boils down to the problem of finding the monotone and conservative vector field that maximizes a certain functional. This will be the subject of future work.

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Appendix- Proof of Theorem 6.5:

Let (p, x) be an efficient,incentive compatible DRM, and let $(q^i_k(s^i))_{k=1}^K$ be the associated vector field of interim expected probabilities for agent i. By equation 4.4 we know that:

$$\forall k, k', \; a^i_{ki}\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}} = a^i_{k'i}\frac{\partial q^i_{k'}(s^i)}{\partial s^i_k} \tag{9.1}$$

Fix now two alternatives k, k'. Since p is efficient, we obtain:

$$\begin{aligned} q^i_k(s^i) &= \text{Prob}\,\{\sum_{j=1}^N\sum_{g=1}^N a^j_{kg}s^j_k = \max_{k^*}\sum_{j=1}^N\sum_{g=1}^N a^j_{k^*g}s^j_k\} = \\ &\int_{\Delta_k(s^1)} f_{-i}(s^{-i})ds^{-i} \end{aligned}$$

where $\Delta_k(s^i) = \{s^{-i} \mid \sum_{j=1}^N\sum_{g=1}^N a^j_{kg}s^j_k = \max_{k^*}\sum_{j=1}^N\sum_{g=1}^N a^j_{k^*g}s^j_k\}$
An analogous expression holds for $q^i_{k'}(s^i)$.
Define now the set

$$\Omega_{k,k'}(s^i) = \{s^{-i} \mid \sum_{j=1}^N\sum_{g=1}^N a^j_{kg}s^j_k = \sum_{j=1}^N\sum_{g=1}^N a^j_{k'g}s^j_{k'} = \max_{k^*}\sum_{j=1}^N\sum_{g=1}^N a^j_{k^*g}s^j_k\}$$

and note that $\Omega_{k,k'}(s^i) = \partial\Delta_k(s^i)\cap\partial\Delta_{k'}(s^i)$, where ∂A denotes the boundary of a set A. It should be intuitively clear that $\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}}$ involves only an integral over

$\Omega_{k,k'}(s^i)$ multiplied by the "rate of change" of this set with respect to $s^i_{k'}$. More precisely, let

$$x = \sum_{j\neq i}\sum_{g=1}^{N} a^j_{kg} s^j_k - \sum_{j\neq i}\sum_{g=1}^{N} a^j_{k'g} s^j_{k'}.$$

Hence, x is a linear combination of $\{s^j\}_{j\neq i}$, and we assume it to be nondegenerate in the sense that it is not identically equal to tero. Consider a change of variable in the $\{s^j\}_{j\neq i}$ space , where x is one of the new variables, and denote by $s^{-i,x}$ the set of the other variables. Denote by $J(s^{-i})$ the Jacobian induced by the change of variable. We obtain that:

$$\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}} = -(\sum_{g=1}^{N} a^i_{k'g}) \int_{\Omega_{k,k'}(s^i)} f_{-i}(s^{-i}) J(s^{-i}) ds^{-i,x}. \tag{9.2}$$

To see this observe that $\Delta_k(s^i) = \{s^{-i} \mid x \geq -(\sum_{g=1}^N a^i_{kg})s^i_k + (\sum_{g=1}^N a^i_{k'g})s^i_{k'}$ and $\sum_{j=1}^N \sum_{g=1}^N a^j_{kg} s^j_k \geq \sum_{j=1}^N \sum_{g=1}^N a^j_{k''g} s^j_{k''}$ for $k'' \neq k'\}$. The result follows because $s^i_{k'}$ appears only in the first inequality and because the area in $\Delta_k(s^i)$ where $x = -(\sum_{g=1}^N a^i_{kg})s^i_k + (\sum_{g=1}^N a^i_{k'g})s^i_{k'}$ is precisely $\Omega_{k,k'}(s^i)$.

The term $\frac{\partial q^i_{k'}(s^i)}{\partial s^i_k}$ is analogously computed:

$$\frac{\partial q^i_{k'}(s^i)}{\partial s^i_k} = -(\sum_{g=1}^{N} a^i_{kg}) \int_{\Omega_{k,k'}(s^i)} f_{-i}(s^{-i}) J(s^{-i}) ds^{-i,x}. \tag{9.3}$$

Combining equations 9.2 and 9.3 , we obtain that:

$$\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}} (\sum_{g=1}^{N} a^i_{kg}) = \frac{\partial q^i_{k'}(s^i)}{\partial s^i_k} (\sum_{g=1}^{N} a^i_{k'g}) \tag{9.4}$$

Equations 9.1 and 9.4 yield together the wished result. ■