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Auctions with Downstream Interaction Among Buyers

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Abstract

We study auctions for an indivisble object. The outcome of the auction influences the future interaction among agents. The impact of that interaction on agent i is assumed to be a function of the agents' valuations. While agent's i valuation is private information to i, other valuations are not observable by i at the time of the auction. We derive equilibrium bidding strategies for second price auctions in which the seller may impose reserve prices or entry fees, and we point out differences between the cases where impacts (which we call externalities) are positive or negative. Finally, we study the effect of reserve prices and entry fees on the seller's revenue.

1. Introduction

In a variety of economic settings significant changes of ownership influence the nature of the interaction in the respective markets. In particular, agents that are not directly involved in an actual transaction may be affected by its outcome. If those effects are anticipated at the transaction stage, potential traders will take them into account and they will adjust their trading strategies accordingly.

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Conversely, those strategies will determine the outcome of the transaction, and hence the effect on the future interaction.

A good illustration is offered by the following quotation from *The Economist*, June 28th, 1997:

> "The good sales run at Rolls-Royce began 18 months ago, when it snatched a huge order to supply Singapore Airlines with engines for its latest twin-engined Boeing 777s. Its hard-nosed American rivals, Pratt&Whitney and General Electric, were prepared to take a loss to land such a prestigious deal. So they assumed Roll-Royce won the bid by taking an even greater loss"

We demonstrate below how precisely such a behavior arises in the equilibrium of an auction if loosing the auction has a negative impact on future expected profits.

Besides the award of major projects as in the example above, other good examples for the situation we focus on are: changes of ownership in oligopolies (through merger, privatization, etc...); the sale of patents that cover technical innovations ; the award of projects that lead to the creation of a new technological standard; the location of environmentally hazardous enterprises; the provision by a single agent of a service having a public good aspect.

Both cases of positive and negative externalities are discussed in the large IO literature on vertical and horizontal relations. We explicitly describe below how the sale of a cost-reducing innovation creates negative externalities in a simple oligopoly model. An interesting illustration for the positive externalities case is offered by Katz and Shapiro (1985): Two oligopolists offer incompatible products, and the consumers' utility increases in the size of the group that uses the same product (there are network externalities). If compatibility can be achieved by attaching an "adapter" to one of the products, then one firm will usually bear the cost of the adapter, while the increased compatibility benefits both firms. This creates a free-rider effect, and the incentives to invest in an adapter may be too low.

In Jehiel and Moldovanu (1996) we focused on participation decisions in auctions with negative externalities and heterogenous buyers having complete information. In Jehiel et.al. (1996a) we derived revenue-maximizing mechanisms when agents possess private information about externalities imposed on others, and in Jehiel et.al.(1996b) we allowed for private information about incurred externalities in case another wins the object[1]. In both papers, an externality term depends only on the identities of the actual buyer and the sufferer, but not on their types. However, a more general model of information in transactions followed by downstream interaction requires that the effect of the downstream competition on buyer i will depend both on the type of i (which is private information to i), and on the types of others (in particular on the type of the auction's winner j which is not observable by i at the transaction stage if $i \neq j$). In this paper we study symmetric settings where the impact on i if the object is sold to a competitor will depend both on i's and the competitor's types, but does not depend on the identity of that competitor.

Our main goal is to illustrate several qualitative phenomena induced by the presence of externalities. We focus on the case of two potential buyers bidding for an indivisible object in a second-price, sealed-bid auction where the seller may sometimes keep the object, and we indicate the changes needed when there are more than two bidders. The second-price auction is chosen for its analytical simplicity: it allows us to highlight the phenomena caused by the presence of externalities without getting too entangled in complex bidding mechanics. A very similar analysis can be performed for other sale mechanisms, e.g., first-price auctions, all-pay auctions, etc...

We want to note that our auction model is not a special case of the general model with affiliated valuations studied by Milgrom and Weber (1982). In that model, bidder i's valuation for the object is a function of the signals obtained by bidders and, possibly, of other variables whose realization is independent of the auction's result (some of the variables may not be observable at the time of the auction). However, in Milgrom and Weber's model, a bidder that does not get the auctioned object obtains a fixed payoff, usually normalized to be zero. In our model, when a bidder does not get the object his utility is influenced by the realized allocation (e.g., by events such as "the good is not sold" or "the good is sold to another bidder having certain characteristics". Thus, bidder i's willingness to pay depends on i's belief about possible auction outcomes if this bidder decides **not** to acquire the object. As a consequence, even in a complete information framework bidders face strategic uncertainty.

The paper is organized as follows. In Section 2 we illustrate in detail two simple settings that fit in our model: the sale of a cost-reducing innovation to one of two duopolists (negative externalities) and the location of a firm in one of

[1] The analysis there employs and further develops the optimal mechanism design methodology for multi-dimensional type spaces.

several jurisdictions competing in tax rebates (positive externalities). In Section 3 we present the economic model with externalities, and we describe the analyzed auction procedures: standard second-price auctions (where the seller uses neither a reserve price, nor an entry fee), second price auctions with reserve prices, and second-price auctions with entry fees. We then derive an equilibrium for the standard second-price auction. In this equilibrium a bidder takes into account both the expected profit if she acquires the object (i.e., her pure valuation net of externalities) and the impact she expects in case her competitor acquires the object. As in the case of affiliated valuations, the equilibrium bid of bidder i is determined by the event that the competitor j, $j \neq i$, has the same valuation as bidder i (because the setting is symmetric). We note that bids are higher than pure valuations if externalities are negative, and lower if externalities are positive.

In Section 4 we focus on auctions with a reserve price [2] . The main difficulty is that a buyer with a high enough valuation expects the good to be sold for sure, and the effective competition is provided by the other bidder (as in the case of a standard auction), while for bidders with low enough valuations the effective competition is provided by the seller's reserve price. Since the impact of a loss to the other buyer is different from the impact of the event that the seller keeps the object, we obtain alternative bidding strategies which must be somehow combined to form an overall optimal strategy. For the negative externality case, the two areas of valuations are separate, and we always find an equilibrium in pure strategies. This equilibrium displays a discontinuity at a valuation equal to the reserve price. For the positive externality case, it is impossible to have separate areas, and an equilibrium in pure strategies (if it exists !) will display a region of pooling.

In Subsection 4.1 we derive an equilibrium for the negative externalities case. The lowest relevant bid is strictly higher than the reserve price, and some types that are, in principle, willing to pay for preemption, choose nevertheless to make irrelevant bids. We next derive the seller's optimal reserve price, and we show that the seller should sometimes announce a reserve price that is strictly lower than her own valuation for the object. The intuition for this result is simple: since externalities are negative, when the seller sells more often, the bidders are more afraid that the good will fall in the hands of a competitor. Consequently, they bid more aggressively. This effect (which works in the direction of decreasing the reserve price) may well be stronger than the effect caused by the "traditional"

[2] We normalize bidders' payoffs to be zero in the case where the seller keeps the object. The seller has also a valuation for the object, which may differ from zero.

interest of a monopolist seller to restrict supply (which works in the direction of raising the reserve price).

In Subsection 4.2 we look at the case of positive externalities. The derivation of equilibria is rather complex: 1) For the case where the positive externality is decreasing in the winner's valuation we are able to prove the existence of a symmetric equilibrium in pure strategies. We find that all types in an interval (which includes types with pure valuations above the reserve price) make the same equilibrium bid, equal to the reserve price. The main difficulty is to determine the boundaries of the interval where bids are pooled ; 2) For the case where the positive externality increases in the winner's valuation, we show that equilibria in pure strategies may not exist.

In Section 5 we look at second-price auctions with entry fees. The analysis here is simpler than that for auctions with reserve prices. Once a bidder has decided to participate in the auction, it is clear that competition is against the other bidder (if any) and not against the seller (who has committed to sell the object). After deriving equilibrium strategies, we show that, in the case of negative externalities, there is a natural one-to-one correspondence between entry fees and reserve prices that achieve the same expected revenue for the seller (this is the case in the model without externalities). The situation is more complicated in the case of positive externalities. We first show that, no matter what the seller's valuation for the object is, a strictly positive measure of types is excluded from participation in the auction with the optimal entry fee. This result, which sharply contrasts the usual intuition, stems from the fact that, with positive externalities, exclusion has the additional effect of mitigating the free-rider effect among buyers. Finally, we consider a simple class of situations where the externality term does not depend on the other agents' private information. We show that, for each relevant entry fee, the seller can find a reserve price that leads to a strictly higher revenue. Hence, these two instruments are not equivalent. More generally, whether the seller prefers a reserve price or an entry fee depends on the size of the pooling interval at the optimal reserve price (assuming an equilibrium for the auction with reserve price exists).

In Section 6 we extend our model to $n > 2$ buyers, and illustrate several facts that are not immediately apparent in the $2-$buyer case. For example, we show that the optimal reserve price depends on the number of bidders.

Concluding comments are gathered in Section 7. All proofs appear in an Appendix.

2. Illustrations

2.1. A case of negative externalities: The sale of a patent

Consider 2 firms in a Cournot oligopoly. Assume that the total cost to firm i of producing quantity q_i of a homogenous product is given by $c \cdot q_i$, where $c < 1$. Let $P(Q) = 1 - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2 \leq 1$. In the Nash equilibrium the firms produce $q_1 = q_2 = \frac{1-c}{3}$, and the price is $p = \frac{1+2c}{3}$. The profits are given by

$$\pi_1^{sq} = \pi_2^{sq} = \frac{(1-c)^2}{9} \tag{2.1}$$

(where sq stands for status-quo). All parameters in the status-quo are common knowledge.

Assume now that an inventor wants to sell a cost-reducing technical innovation protected by a patent. The firm that acquires the patent will be able to produce the good with marginal costs $0 \leq c_i \leq c$. (We assume that the patent can be sold only to one firm.) The new, reduced cost c_i is private information to firm i at the time where the innovation is to be sold. However, after the sale, the new structure of cost is assumed to be revealed to every competitor. To simplify the discussion, we assume below that both firms will produce positive quantities also after one of them acquires the innovation and becomes more efficient. If firm i acquires the patent it will earn a profit

$$\pi_i^{own} = \frac{(1 - 2 \cdot c_i + c)^2}{9} \geq \pi_i^{sq} \tag{2.2}$$

Firm j, $j \neq i$, that does not acquire the innovation will produce with the old technology (which is now relatively more costly), and will earn a profit

$$\pi_j^{ext} = \frac{(1 - 2 \cdot c + c_i)^2}{9} \leq \pi_j^{sq} \tag{2.3}$$

We are in the negative externalities case. Relatively to the status-quo, we obtain the following:

1. When firm i acquires the patent, its benefit from the innovation is given by

$$\pi_i = \pi_i^{own} - \pi_i^{sq} = \frac{4}{9} \cdot (1 - c_i) \cdot (c - c_i) \tag{2.4}$$

2. The non-acquiring firm j incurs a loss given by

$$\pi_j^{ext} - \pi_j^{sq} = \frac{1}{9} \cdot (c_i - c) \cdot (2 - 3c + c_i) \tag{2.5}$$

Note that the loss suffered by the non-acquiring firm is a function of the benefit of the acquiring firm[3] (which is private information to the acquiring firm.) Indeed, by equation 2.4 we obtain

$$c_i = \frac{1}{2} \cdot \left(1 + c - \sqrt{(1-c)^2 + 9\pi_i}\right) \tag{2.6}$$

Together with equation 2.5 , this allows us to express the loss of the non-acquiring firm j as:

$$\begin{aligned} \pi_j^{ext} - \pi_j^{sq} &= g_j(\pi_j, \pi_i) = g_j(\pi_i) = \\ &= \frac{c^2}{6} - \frac{c}{3} + \frac{\pi_i}{4} + \frac{c-1}{6} \cdot \left(\sqrt{(1-c)^2 + 9\pi_i}\right) \end{aligned} \tag{2.7}$$

In this example, the loss of the non-acquiring firm does not depend on its own benefit were it to obtain the patent i.e., it does not depend on π_j, but it does depend on the profit of the acquiring firm, which is not observable at the time of the auction[4].

The main question of interest is: How much should a firm, say firm 1, bid to acquire the patent ? Note that firm's 1 valuation is not well-defined since it depends on 1's beliefs about the likelihood of possible outcomes. To see that, consider two extreme cases: 1) If firm 1 believes that under no circumstance will the patent be sold to firm 2 , then its valuation is $\pi_1 = \pi_1^{own} - \pi_1^{sq}$. 2) If firm 1 believes that in case it fails to buy the patent, the seller will surely sell to firm 2, then its valuation is $\pi_1 - E_{\pi_2}[g_1(\pi_2)] > \pi_1$ (where E denotes an expectation).

In general, firm 1 must take into account the expected negative impact given its beliefs about the probability that the good is sold to firm 2, and its valuation incorporates a preemptive term. It should be clear that firm 1's belief, on which its bidding strategy will be based, depends both on the nature of the sale mechanism (the selling strategy of the seller) and on the bidding strategy of the other firm. For an equilibrium of a given sale procedure, bidding strategies must be optimal given beliefs, and beliefs must be consistent with the bidding strategies.

[3]It is this form of dependence which is consistent with the general model described below.

[4]Since $\frac{\partial g_j(\pi_j,\pi_i)}{\partial \pi_j} = 0$ and $\frac{\partial g_j(\pi_j,\pi_i)}{\partial \pi i} = \frac{1}{4} + \frac{3(c-1)}{4} \cdot \left((1-c)^2 + 9\pi_i\right)^{-\frac{1}{2}} \leq \frac{1}{4}$, we obtain that the function $G(\pi) = \pi - g(\pi, \pi)$ is strictly increasing. This is an important requirement for the general model we develop below.

2.2. A case of positive externalities: Firm location and tax rebate

Consider a firm that must locate either in community A or in community B of a country X, or abroad. The firm is indifferent between the locations per se[5], and it chooses the location that offers the highest tax rebate. It is often the case that the involved local authorities engage in a "bidding war" where tax rebates (and possibly other sweeteners) are offered.

Assume that each local authority i, $i = A, B$, has private information that concerns, for example, the share s_i of the local labor force that is adapted to the kind of work needed by the firm. Assume also that the "social value" of an employed worker in i is given by w_i. If the firm locates in i, then it pays the local tax t_i, diminished by the tax rebate rate Δt_i. Thus, the overall aggregate payoff to community i is:

$$\pi_i - \Delta t_i = s_i w_i + t_i - \Delta t_i.$$

On the other hand, when the firm locates in i, there is a possible spillover on the labor force in community $j, j \neq i$. Of course, this spillover is primarily influenced by geographical distance and transportation infrastructure, etc,...Denote by $\varphi(s_j, s_i)$ the share of the labor force in j that is effectively employed by the firm when it locates in i. A reasonable restriction is $\varphi(s_j, s_i) < s_j$, since the total employed force in j cannot exceed the capacity of j. Also $\frac{\partial \varphi}{\partial s_i} < 0$, since there is a likely substitutability effect between the labor force in the two locations from the viewpoint of the firm. The reduced form of this model is thus as follows: If the firm locates in i, then community j incurs a positive externality given by:

$$g_j(\pi_j, \pi_i) = \varphi(\frac{\pi_j - t_j}{w_j}, \frac{\pi_i - t_i}{w_i})w_j$$

Several variations on these theme are possible: for example, if the locating firm is environmentally hazardous (a nuclear reactor, say) the positive employment effects must be weighed against the environmental risk and the associated negative externalities.

3. The Model

To simplify notation and proofs, we focus below on the two-buyer case. In the symmetric setting we are considering (to be defined below), this is sufficient for

[5] Adding preferences about intrinsic characteristcs of the available locations is straightforward.

the illustration of the effects caused by the presence of externalities. In Section 6 we comment on the changes (if any) required for the cases where $n > 2$.

We consider the following situation: A seller owns an indivisible object. The seller's valuation for the object is π_S. There are 2 potential buyers. Buyer's i pure valuation for the object (i.e., his profit when it acquires the object) is given by π_i. Denote by π_{-i} the valuation of the other buyer.

If the good is sold to buyer i for a price p , the utilities of the agents are as follows: p for the seller; $\pi_i - p$ for buyer i; $g_j(\pi_j, \pi_{-j})$ for buyer $j, j \neq i$. We normalize the utilities of the buyers to be zero in case that the seller keeps the object.

The functions $g_k(\cdot,\cdot)$,which are common knowledge, are assumed to be differentiable. Note that the first argument of a function $g_k(\cdot,\cdot)$ is always the type of the sufferer k, and the second argument is the type of the other agent.

Buyers' pure valuations are private information, and they are independently drawn from the interval $[\underline{\pi}_i, \bar{\pi}_i]$ according with the density $f_i(\cdot)$. We denote by $F_i(\cdot)$ the distribution of $f_i(\cdot)$.

We consider below sales through a sealed-bid second-price auction, since this is a relatively simple mechanism that allows us to focus on the effects of the externalities. A standard second price auctions is described by the following rules: Buyers simultaneously submit bids for the object. Assume without loss of generality that the bids are $b_1 \geq b_2$. If $b_1 > b_2$, then buyer 1 gets the good and pays to the seller $p = b_2$. Other buyers pay nothing. If $b_1 = b_2 = b$ then each buyer gets the object with probability $\frac{1}{2}$. The winner pays $p = b$, and the other buyer pays nothing.

Second-price auctions with a reserve price proceed as follows: The seller announces a reserve price R. The buyers then simultaneously submit bids for the object. Assume without loss of generality that the bids are $b_1 \geq b_2$. If $R > b_1$, then he seller keeps the good and no payments are made. If $b_1 \geq R$, and $b_1 > b_2$, then buyer 1 gets the good and pays to the seller $p = \max(R, b_2)$. The other buyer pays nothing. If $b_1 = b_2 = R$, then each buyer gets the object with probability $\frac{1}{2}$. The winner pays $p = R$, and the other buyer pays nothing.

In second-price auctions with entry fees, the buyers who participate at the auction must pay an entry fee E. After the fees have been collected, the buyers participate in a standard second-price auction. We assume, of course, that buyers who choose not to pay the fee (and hence do not bid at the auction) are still affected by the outcome of the auction (i.e., suffer possible negative externalities, or enjoy possible positive ones).

We consider here a *symmetric* setting in the following sense: 1) $\underline{\pi}_1 = \underline{\pi}_2$ and $\bar{\pi}_1 = \bar{\pi}_2$. 2) There exists a function $f(\cdot) : [\underline{\pi}, \bar{\pi}] \to \Re$ such that $\forall \pi$ $f_1(\pi) = f(\pi)$ $= f_2(\pi)$. 3) There exists a function $g(\cdot) : [\underline{\pi}, \bar{\pi}] \times [\underline{\pi}, \bar{\pi}] \to \Re$ such that $\forall \pi, \pi'$, $g_1(\pi, \pi') = g(\pi, \pi') = g_2(\pi, \pi')$. Hence, we assume that the externality suffered by agent 1 with type π if agent 2 with type π' gets the object is the same as the externality suffered by agent 2 with type π if agent 1 with type π' gets the object.

Let $D_x g$ denote the derivative of the function $g(\cdot, \cdot)$ with respect to the first coordinate (i.e., the type of the sufferer), and let $D_y g$ denote the derivative of the function $g(\cdot, \cdot)$ with respect to the second coordinate (i.e., the type of the causer). Throughout the paper we assume that

$$\forall \pi, \pi' \in [\underline{\pi}, \bar{\pi}], D_x g(\pi, \pi') < 1 \tag{3.1}$$

and that

$$\forall \pi \in [\underline{\pi}, \bar{\pi}], D_x g(\pi, \pi) + D_y g(\pi, \pi) < 1 \tag{3.2}$$

The last assumption implies that the function $G(\pi) = \pi - g(\pi, \pi)$ is strictly monotonically increasing on $[\underline{\pi}, \bar{\pi}]$.

We will speak of the *negative externalities case* if $\forall \pi, \pi' \in [\underline{\pi}, \bar{\pi}]$, $g(\pi, \pi') \leq 0$, and of the *positive externalities case* if $\forall \pi, \pi' \in [\underline{\pi}, \bar{\pi}]$, $g(\pi, \pi') \geq 0$.

We analyze below symmetric equilibria of the various auction forms, and we first derive the equilibrium of a standard second-price auctions, without reserve prices or entry fees.

Proposition 3.1. *An equilibrium of the standard second-price auction is given by*

$$b_i(\pi_i) = \pi_i - g(\pi_i, \pi_i) \tag{3.3}$$

With suitable assumptions that ensure monotonicity of equilibrium strategies, the result above easily generalizes to the symmetric $n-$buyers case - see Section 6.

Note that in the simple case of pure auctions with two bidders there are only two possible physical outcomes (the good ends up in the hands of one of the bidders). A re-normalization of buyers' utilities yields a model covered by Milgrom and Weber's (1982) analysis. But, when there are strictly more than two physical outcomes (e.g., three or more bidders ; two bidders, but the seller may keep the object) the model is qualitatively different: even with complete information there is strategic uncertainty, since bidders' valuations depend on their belief about who is going to win.

4. Auctions with a Reserve Price

4.1. Negative externalities

Assume now that the seller sells through a second-price auction with a reserve price R. Consider the type $G^{-1}(R)$,which is given by the unique solution to the equation

$$G(\pi) = \pi - g(\pi, \pi) = R \tag{4.1}$$

Note that $G^{-1}(R) = R + g(G^{-1}(R), G^{-1}(R)) \leq R$. The interesting part in the determination of equilibrium is the prescription for buyers with valuations in the interval $[G^{-1}(R), R)$. Given a reserve price of R , these types are interested in the good only for preemptive reasons, i.e., they only want to avoid the negative externality created by the good falling in the hands of others. These types are, in principle, willing to pay more than R for preemption. However, as we show below, *given the equilibrium actions of the other bidder*, a buyer with valuation in the interval $[G^{-1}(R), R)$ has a chance to get the good only when the other bidder bids less than R. In this case the good will not be sold to the competitor, and preemption is therefore not necessary. Hence, bidding zero is ultimately optimal. Note, in particular, that the equilibrium will not be in dominant strategies. The lowest relevant bid is $G(R) = R - g(R, R)$ which is strictly above R if $g(R, R) < 0$.

Proposition 4.1. *Assume that externalities are negative. An equilibrium of the second-price auction with reserve price R is given by*

$$b_i(\pi_i) = \left\{ \begin{array}{ll} \pi_i - g(\pi_i, \pi_i) & \text{for } \pi_i \geq R \\ 0 & \text{for } \pi_i < R \end{array} \right\} \tag{4.2}$$

We now turn to the study of the optimal reserve price policy from the point of view of the seller. The seller's expected revenue is given by:

$$\begin{aligned} U_S(R) &= \left[F^2(R) \cdot \pi_S\right] + [2F(R) \cdot (1 - F(R)) \cdot R] + \\ & \left[2 \cdot \int_R^{\bar{\pi}} (\pi - g(\pi, \pi) \cdot (1 - F(\pi)) \cdot f(\pi) d\pi\right] \end{aligned} \tag{4.3}$$

Differentiating this expression with respect to R we obtain:

$$\frac{\partial U_S}{\partial R} = 2F(R) \cdot f(R) \cdot \left[\pi_S - R + \frac{1 - F(R)}{f(R)} + g(R, R) \cdot \frac{1 - F(R)}{F(R)}\right] \tag{4.4}$$

Comparing to the case without externalities, the thing to note is the extra term involving $g(R, R)$[6]. Assuming that the first order condition characterizes a maximum, the equation that determines the optimal reserve price is $R_{opt} - \frac{1-F(R_{opt})}{f(R_{opt})} - g(R_{opt}, R_{opt}) \cdot \frac{1-F(R_{opt})}{F(R_{opt})} = \pi_S$.

Since $g(R_{opt}, R_{opt}) \leq 0$, it may well happen that the seller optimally announce a reserve price which is strictly lower than her own valuation (even if the function $R - \frac{1-F(R)}{f(R)}$ is monotonic). The intuition is as follows: When the seller sells more often, the buyers are more afraid that the good will fall in the hands of the competitor, and they bid more aggressively. If the seller's valuation is relatively low, the gain of having higher bids fully offsets the loss suffered in cases where the good is sold at a price below valuation. Finally, note that, because of the very low reservation prices, it may well happen that the monopolist seller increases supply above the efficient level, i.e., the object is sold "too often". This is a novel phenomenon, since the usual inefficiencies created by a monopolist seller are in the opposite direction: supply is restricted below the efficient level, i.e., the good is sold "too seldom"

Example 4.2. *Let $n = 2$. Each buyer's valuation π_i is drawn from the interval $[0, 1]$ with density $f(\pi_i) = 1$. Let the externality be defined by $g(\pi, \pi') \equiv -\frac{1}{2}$. We obtain that:*

$$\frac{\partial U_S}{\partial R} = (2R) \cdot (\pi_S - 2R + 1 - \frac{1-R}{2R}) \tag{4.5}$$

The optimal reserve price R_{opt}, as a function of the seller's valuation π_S, is as follows:

$$R_{opt}(\pi_S) = \left\{ \begin{array}{ll} 0, & \text{if } \pi_S \leq 0.8094 \\ \frac{1}{4}\pi_S + \frac{3}{8} + \frac{1}{4}\sqrt{\pi_S^2 + 3\pi_S - \frac{7}{4}}, & \text{if } 0.8094 < \pi_S \leq 1 \\ 1, & \text{if } \pi_S > 1 \end{array} \right\} \tag{4.6}$$

Note that a seller with a low positive valuation prefers to set a reservation price equal to zero. At the cutoff-value $\pi_S = 0.8094$ the loss of selling below valuation becomes too high, and the optimal reserve price displays a discrete jump (it is a continuous function afterwards).

[6] Observe that in Myerson's *regular* case without externalities (i.e., where the function $R - \frac{1-F(R)}{f(R)}$ is monotonic), the optimal reserve price, R_{opt}, satisfies the equation $R_{opt} - \frac{1-F(R_{opt})}{f(R_{opt})} = \pi_S$, and hence $R_{opt} \geq \pi_S$. This confirms the usual economic intuition about the monopolist that restricts supply.

4.2. Positive Externalities

In this section we study equilibria for the case where the seller imposes a reserve price and there are positive externalities. The derivation is relatively involved because, if an equilibrium exists, it will typically involve an area of pooling that includes types with valuations that are larger than the reserve price.

Let again $G^{-1}(R)$ denote the unique solution to the equation $\pi - g(\pi, \pi) = R$. Note that $G^{-1}(R) = R + g(G^{-1}(R), G^{-1}(R)) \geq R$. The equilibrium we describe below will have the following structure:

1. There is bidder with type $\tilde{\pi}$, $R \leq \tilde{\pi} \leq G^{-1}(R)$,which is indifferent between a bid of zero and a bid equal to R, and there is a bidder with type $\tilde{\tilde{\pi}}$, $G^{-1}(R) \leq \tilde{\tilde{\pi}} \leq \bar{\pi}$, which is indifferent between any two bids in the interval $[R, G(\tilde{\tilde{\pi}})]$.

2. All bidders with types in the interval $[\tilde{\pi}, \tilde{\tilde{\pi}})$ make the **same** bid, equal to R (and this bid is strictly preferred to any other bid)

3. All types below $\tilde{\pi}$ bid zero, and, finally, all bidders with types $\pi \geq \tilde{\tilde{\pi}}$ bid $\pi - g(\pi, \pi)$.

The next Lemma characterizes the extremities of the pooling interval $[\tilde{\pi}, \tilde{\tilde{\pi}})$:

Lemma 4.3. *Assume that* $D_y g \leq 0$, *and that for all* $\pi \geq R$, $\bar{\pi} - g(\bar{\pi}, \pi) \geq R$. *The system of equations:*

$$\begin{aligned} (u - R) \cdot (F(u) + F(z)) - \int_u^z g(u, \pi) f(\pi) d\pi &= 0 \\ (z - R) \cdot (F(z) - F(u)) - \int_u^z g(z, \pi) f(\pi) d\pi &= 0 \end{aligned} \qquad (4.7)$$

has a solution $(u, z) = (\tilde{\pi}, \tilde{\tilde{\pi}})$ such that $R \leq \tilde{\pi} \leq G^{-1}(R)$ and $G^{-1}(R) \leq \tilde{\tilde{\pi}} \leq \bar{\pi}$.

Proposition 4.4. *Assume that* $D_y g \leq 0$[7] *and that, for all* $\pi \geq R$, $\bar{\pi} - g(\bar{\pi}, \pi) \geq R$. *Let* $(\tilde{\pi}, \tilde{\tilde{\pi}})$ *be a solution of the system 4.7 that satisfies* $R \leq \tilde{\pi} \leq G^{-1}(R)$ *and*

[7] Observe that this assumption fits with the positive externality example provided in Section 2.

$G^{-1}(R) \leq \tilde{\tilde{\pi}} \leq \bar{\pi}$. The strategy profile

$$b_i(\pi_i) = \left\{ \begin{array}{ll} \pi_i - g(\pi_i, \pi_i) & \text{for } \pi_i \in [\tilde{\tilde{\pi}}, \bar{\pi}] \\ R & \text{for } \pi_i \in [\tilde{\pi}, \tilde{\tilde{\pi}}) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}) \end{array} \right\} \tag{4.8}$$

constitutes a Nash equilibrium.

Remark 1. : *Assume that for all* $\pi \geq R$, *and for all* π', $R \leq \pi' \leq \pi$, $g(\pi, \pi') > \pi - R$ *(this means, roughly, that the externality in all relevant cases is higher than the gain of acquiring the object). In this case, no matter what* $u \geq R$ *is, there is no* $z \geq u$ *such that the second equation in the system 4.7 holds. This implies that the system of equations does not have a solution such that* $\tilde{\tilde{\pi}}$ *satisfies:* $\bar{\pi} \geq \tilde{\tilde{\pi}} \geq G^{-1}(R)$. *The equilibrium of the auction is then given by:*

$$b_i(\pi_i) = \left\{ \begin{array}{ll} R & \text{for } \pi_i \in [\tilde{\pi}, \bar{\pi}) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}) \end{array} \right\} \tag{4.9}$$

where $\tilde{\pi}$ *(i.e., the type which is indifferent between bidding* 0 *and bidding* R *) satisfies the equation*

$$(u - R) \cdot (F(u) + 1) - \int_u^{\bar{\pi}} g(u, \pi) f(\pi) d\pi = 0 \tag{4.10}$$

This is the instance of the first equation in the system 4.7 for $z = \bar{\pi}$. *Note that, for any* z, $\bar{\pi} \geq z \geq R$, *the equation* $(u - R) \cdot (F(u) + F(z)) - \int_u^z g(u, \pi) f(\pi) d\pi = 0$ *(viewed as an equation in* u *) has always a solution* $\tilde{\pi}$ *on the interval* $[R, z]$.

Our following result looks at the case where the externality function does not depend at all on the type of the acquirer- In this case the determination of the pooling interval is somewhat simpler, as the upper end of the pooling interval is $G^{-1}(R)$.

Corollary 4.5. *Assume that for* $i = 1, 2$ *and for all* $\pi, \pi' \in [\underline{\pi}, \bar{\pi}]$, $D_y(\pi, \pi') = 0$. *Define then* $h(\pi) \equiv g(\pi, \pi')$ *for all* $\pi, \pi' \in [\underline{\pi}, \bar{\pi}]$, *and let* $\tilde{\pi}$ *satisfy* $H(\tilde{\pi}) = 0$ *where* $H(u) = (u - R) \cdot (F(u) + F(G^{-1}(R))) - (F(G^{-1}(R)) - F(u)) \cdot h(u)$ *The strategy profile*

$$b_i(\pi_i) = \left\{ \begin{array}{ll} \pi_i - g(\pi_i, \pi_i) & \text{for } \pi_i \in [G^{-1}(R), \bar{\pi}] \\ R & \text{for } \pi_i \in [\tilde{\pi}, G^{-1}(R)) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}) \end{array} \right\} \tag{4.11}$$

constitutes a Nash equilibrium.

We next show by way of example that equilibria in pure strategies may fail to exist when the condition $D_y g \leq 0$ is not satisfied.

Proposition 4.6. *Assume that each buyer's valuation π_i is drawn from the interval $[0, 1]$ with density $f(\pi_i) = 1$. Let the externality function be given by $g(\pi, \pi') = k\pi'$ where $0 < k < 1$, and let the reserve price R be such that $R < 1 - k$*[8]*. Then there are no equilibria in pure strategies.*

5. Second-Price Auctions with an Entry Fee

Assume now that the seller sells through a second-price auction with an entry fee E. Our goal is to compare the revenue obtained by using a reserve price to the revenue obtained by using entry fees. We recall here the well-known result in the case without externalities: For each reserve price R there exists an entry fee E that yields the same revenue, and vice-versa. We show below that this standard result generalizes to the case of negative externalities, but does not hold anymore when externalities are positive. Moreover, for the case covered by Corollary 4.5, we show that a reserve price policy is superior from the point of view of the seller.

The following Proposition characterizes the equilibrium behavior in auctions with entry fees (irrespective of the sign of the externalities).

Proposition 5.1. *Let π^E be the unique solution to the equation $E = u \cdot F(u)$. The strategy profile defined by*

$$s_i(\pi_i) = \left\{ \begin{array}{ll} \textit{stay out} & \textit{for } \pi_i \in [\underline{\pi}, \pi^E) \\ \textit{enter, and bid } \pi_i - g(\pi_i, \pi_i) & \textit{for } \pi_i \in [\pi^E, \bar{\pi}] \end{array} \right\} \tag{5.1}$$

constitutes a Nash equilibrium.

[8] The condition $R < 1 - k$ above ensures that $G^{-1}(R) = \frac{R}{1-k} < 1 = \bar{\pi}$. If $G^{-1}(R) \geq 1$, then there is an equilibrium of the form $b_i(\pi_i) = \left\{ \begin{array}{ll} R & \text{for } \pi_i \in [\tilde{\pi}, \bar{\pi}) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}) \end{array} \right\}$

We now compute the seller's revenue in an auction with an entry fee E . Since there is a one-to-one correspondence between E and π^E, we write the seller's revenue as a function of π^E. (This also eases the comparison with the case where the seller imposes a reserve price). The revenue is :

$$\begin{aligned} U_S(\pi^E) &= \left[F^2(\pi^E)\cdot\pi_S\right] + \left[2\cdot F(\pi^E)\cdot(1-F(\pi^E))\cdot\pi^E\right] + \\ &\left[2\cdot\int_{\pi^E}^{\bar{\pi}}(\pi - g(\pi,\pi)\cdot(1-F(\pi))\cdot f(\pi)d\pi\right] \end{aligned} \tag{5.2}$$

Differentiating this expression with respect to π^E, we obtain:

$$\begin{aligned} \frac{\partial U_S}{\partial \pi^E} &= 2\cdot F(\pi^E)\cdot f(\pi^E)\cdot \\ &\left[\pi_S - \pi^E + \frac{1-F(\pi^E)}{f(\pi^E)} + g(\pi^E,\pi^E)\cdot\frac{1-F(\pi^E)}{F(\pi^E)}\right] \end{aligned} \tag{5.3}$$

For the case of non-positive externalities we obtain that any entry fee policy is revenue equivalent to an appropriately constructed reserve price policy, and vice-versa. Indeed, observe the analogy between the expression above and the respective expression in the reserve price policy (Equation 4.4). In particular, the optimal entry fee is given by $E_{opt} = R_{opt}\cdot F(R_{opt})$.

We now turn to the case of positive externalities, and we first illustrate a rather surprising phenomenon arising in this case .

Proposition 5.2. *Assume that* $\forall\pi,\pi'$, $g(\pi,\pi') > 0$, *and that the function* $K(u) = u - \frac{1-F(u)}{f(u)} - g(u,u)\cdot\frac{1-F(u)}{F(u)}$ *is increasing*[9] *in the interval* $[\underline{\pi},\bar{\pi}]$. *Then, for any seller's valuation* π_S, *a positive measure of buyers' types is excluded from participation in the auction with the optimal entry fee.*

The standard economic intuition for the case without externalities is as follows: When the demand parameters (here the buyers' valuations) are much larger than the supply parameters (here the seller's valuation), supply restriction (here exclusion) does not make sense. The valuable sale opportunities that are lost by exclusion cannot be compensated by the higher payments obtained from types that participate.

[9]This assumption is met, for example, when $g(\cdot,\cdot)$ is constant and $f(\cdot)$ is uniform (see Example 5.3.)

With positive externalities, exclusion, however, has an additional effect: by selling less often the seller mitigates the free-rider effect among buyers. It is interesting that the free-riding mitigation effect is always stronger than the lost-opportunities effect. We illustrate the exclusion phenomenon in the following example.

Example 5.3. *Each buyer's valuation π_i is drawn from the interval $[0,1]$ with density $f(\pi_i) = 1$. Let the externality be $g(\pi_1, \pi_2) \equiv \frac{1}{2}$. We obtain that:*

$$\frac{\partial U_S}{\partial \pi^E} = (2\pi^E) \cdot (\pi_S - 2\pi^E + 1 + \frac{1-\pi^E}{2\pi^E}) \tag{5.4}$$

The optimal cut-off type $\pi_{opt}(\pi_S)$ is given by :

$$\pi_{opt}(\pi_S) = \left\{ \begin{array}{ll} \frac{1}{8} + \frac{1}{4}\pi_S + \frac{1}{8}\sqrt{(17 + 4\pi_S + 4(\pi_S)^2)}, & \text{if } \pi_S < 1 \\ 1, & \text{if } \pi_S \geq 1 \end{array} \right\}$$

Note that $\pi_{opt}(\pi_S) > 0$ for all π_S. The optimal entry fee is $E_{opt}(\pi_S) = (\pi_{opt}(\pi_S))^2$.

The general comparison of the seller's revenue when using an entry fee or a reserve price is quite difficult when externalities are positive (in some cases, we do not even know whether an equilibrium of the second price auction with reserve price exists - see Proposition 4.7). We prove below that, when the externality term does not depend on the valuation of the competitor, the seller is better off using an optimal reserve price rather than using any entry fee.

Proposition 5.4. *Assume that $\forall \pi, \pi'$, $g(\pi, \pi') > 0$, and that $\pi - g(\underline{\pi}, \underline{\pi}) \geq 0$. Moreover, assume that for all $\pi, \pi' \in [\underline{\pi}, \bar{\pi}]$, $D_y g(\pi, \pi') = 0$. For each auction with an entry fee there is an auction with reserve price that yields a strictly higher revenue for the seller.*

6. Extension to $n > 2$ Buyers

We now comment on the extension of our (symmetric) model when there are $n \geq 2$ potential buyers.

Buyers' pure valuations are private information, and they are all independently drawn from the interval $[\underline{\pi}, \bar{\pi}]$ according with the density $f(\cdot)$. We denote by $F(\cdot)$ the distribution of $f(\cdot)$.

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_n)$. We denote by $\boldsymbol{\pi}_{-ij}$ the vector obtained from $\boldsymbol{\pi}$ by deleting the coordinates $i, j, i \neq j$, and by $\pi_{-ij}^{\max}$ the largest coordinate of $\boldsymbol{\pi}_{-ij}$.

Let $\boldsymbol{\pi}$ be the vector of pure valuations. If the good is sold to buyer i for a price p , the utilities of the agents are as follows: p for the seller; $\pi_i - p$ for buyer i; $g_j^i(\pi_j, \pi_i, \boldsymbol{\pi}_{-ji})$ for buyer $j, j \neq i$. The functions g_j^i are common knowledge. A *symmetric* setting is characterized by the existence of a function $g : \Re^N \rightarrow R$, symmetric in its last $n-2$ coordinates, such that if any buyer i with type π_i obtains the object, the externality on any buyer j, $j \neq i$, with type π_j is given by $g(\pi_j, \pi_i, \boldsymbol{\pi}_{-ji})$.

With suitable assumptions that ensure monotonicity, an equilibrium in a pure second-price auction is given by:[10]

$$b_i(\pi_i) = \pi_i - E_{\{\boldsymbol{\pi}_{-ij}/\pi_{-ij}^{\max} \leq \pi_i\}}[g(\pi_i, \pi_i, \boldsymbol{\pi}_{-ij})] \tag{6.1}$$

Note that the symmetry assumption ensures that the above expression does not depend on the choice of j, $j \neq i$, and that all buyers with the same pure valuation (i.e., with the same type) make the same bid (i.e., the equilibrium is symmetric).

The equilibrium for the negative externality case is similar to the one derived for the setting with only two buyers: All types below R bid zero, and types above R bid according to expression 6.1.

A phenomenon which is not apparent for $n = 2$ is the fact that the optimal reserve price does, in general, depend on the number of buyers n[11]. For a simple illustration of this dependence, consider the case where for any number $n \geq 2$ of potential buyers, the externality if the good falls in the hands of another is constant, and equal to $\alpha \leq 0$. Then, for each buyer i , the equilibrium bidding strategy is given by $b(\pi) = 0$, for $\pi \in [\underline{\pi}, R)$ and $b(\pi) = \pi - \alpha$, for $\pi \in [R, \bar{\pi}]$. The seller's revenue is given by

$$\begin{aligned} U_S(R) &= [F^n(R) \cdot \pi_S] + \left[nF^{n-1}(R) \cdot (1 - F(R)) \cdot R\right] + \\ & \left[n \cdot (n-1) \cdot \int_R^{\bar{\pi}} (\pi - \alpha) \cdot F^{n-2}(\pi) \cdot (1 - F(\pi)) \cdot f(\pi) d\pi\right] \end{aligned} \tag{6.2}$$

[10] The event that determines the bid is that where one of the other bidders has the same valuation, and all other bidders have a lower valuation.

[11] The optimal reserve price in the symmetric independent private values case without externalities case does not depend on n. This is a somewhat surprising, but well known result (see, for example, Myerson (1981)).

Differentiating this expression with respect to R we obtain:

$$\frac{\partial U_S}{\partial R} = n \cdot F^{n-1}(R) \cdot f(R) \cdot \left[\pi_S - R + \frac{1 - F(R)}{f(R)} - (n-1) \cdot \alpha \cdot \frac{1 - F(R)}{F(R)} \right] \quad (6.3)$$

It is clear that the optimal reserve price will depend on n unless the total externality imposed by any buyer, $(n-1) \cdot \alpha$, is kept constant as n varies[12].

An equilibrium for the n-buyer case with positive externalities (whenever it exists and it is not trivial) displays a region of pooling, as before. The only significant change is the derivation of the critical types $\tilde{\pi}(n)$, $\tilde{\tilde{\pi}}(n)$. It should be clear from the argument for $n = 2$ that this derivation depends on the number of bidders, not the least through the specification of the tie-breaking rule.

Assume that for any number $n \geq 2$ of potential buyers, the externality if the good falls in the hands of another is constant, and equal to $\alpha \geq 0$. We are then in a similar case to the one covered by Corollary 4.5, and $\tilde{\tilde{\pi}}(n) \equiv R + \alpha$. By keeping the reserve price R constant, and by maintaining the symmetric tie-breaking rule, one easily obtains that $\lim_{n \to \infty} \tilde{\pi}(n) = G^{-1}(R) = R + \alpha$. The intuition is as follows: as $n \to \infty$, the probability that the good is eventually sold (even if there is a positive reserve price) goes to 1. Hence, as $n \to \infty$, a bid of zero becomes attractive for higher valuation types since, in the limit, a payoff of α is assured with probability one. On the other hand, the probability of winning the good with the minimal bid R goes to zero, and this bid is optimal for fewer and fewer types.

Finally, the equilibrium for the auction with an entry fee is analogous to the 2− buyer case. The critical type π^E is given by the unique solution to the equation $E = u \cdot (F(u))^{n-1}$. All types below π^E do not enter the auction, and all types above bid according to expression 6.1

7. Concluding Remarks

This paper has explored bidding behavior in contexts where there are externalities between bidders, and where these externalities depend on characteristics that may not be observable at the time of the auction. The main driving force is the fact that a buyer's willingness to pay (which determines her bid), depends in a complex

[12] Situations where the suffering decreases if it is shared among many is captured by the old saying: "Misery loves company".

way on the allocation of the good (which is, in turn, determined by the bids at the auction). While studying the effects of standard tools such as reserve prices and entry fees, we have illustrated several important qualitative differences between the cases where externalities are positive or negative.

It is still an open question whether Nash equilibria (possibly in mixed strategies) exist in auctions with a reserve price when the positive externality increases in the competitor's valuation (see Proposition 4.6[13])

Throughout the paper we have abstracted from the possibility that the bids at the auction may serve as signals that influence beliefs in the future interaction. This theme will be treated in subsequent work.

8. Appendix

8.1. Proof of Proposition 3.1

We first assume that buyer 2 bids according to the strategy $\beta(\pi_2)$ which is monotonically increasing and differentiable, and we derive the necessary FOC for buyer 1. Buyer's 1 expected utility given that he has type π_1, and given that he makes a bid b is :

$$U(\pi_1, b) = \int_{\underline{\pi}}^{\beta^{-1}(b)} (\pi_1 - \beta(\pi_2)) f(\pi_2)\, d\pi_2 + \int_{\beta^{-1}(b)}^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2)\, d\pi_2 \tag{8.1}$$

Differentiating the above expression with respect to b we obtain:

$$\frac{\partial U(\pi_1, b)}{\partial b} = \frac{d\beta^{-1}(b)}{db} \cdot f\left(\beta^{-1}(b)\right) \cdot \left[\pi_1 - \beta(\beta^{-1}(b)) - g\left(\pi_1, \beta^{-1}(b)\right)\right] \tag{8.2}$$

By symmetry we must have in equilibrium that $\beta^{-1}(b) = \pi_1$. Hence, we obtain:

$$\frac{\partial U(\pi_1, b)}{\partial b} = 0 \Longleftrightarrow b = \pi_1 - g(\pi_1, \pi_1) \tag{8.3}$$

We now prove that the strategy $b(\pi_1) = \pi_1 - g(\pi_1, \pi_1)$ is optimal for buyer 1 , given that buyer 2 plays the strategy $b(\pi_2) = \pi_2 - g(\pi_2, \pi_2)$. Assume that buyer 2

[13] The conditions needed in order to prove existence of mixed-strategy equilibria in discontinuous games (see Dasgupta-Maskin (1986))do not apply here. We conjecture that equilibria in mixed strategies do not exist.

has type π_2 . When buyer 1 bids above $\pi_2 - g(\pi_2, \pi_2)$, he gets the good and his payoff is $\pi_1 - (\pi_2 - g(\pi_2, \pi_2))$. When he bids below $\pi_2 - g(\pi_2, \pi_2)$, buyer 2 gets the good, and buyer 1's payoff is $g(\pi_1, \pi_2)$. By the Mean Value Theorem we have that $\pi_1 - (\pi_2 - g(\pi_2, \pi_2)) - g(\pi_1, \pi_2) = (\pi_1 - \pi_2) \cdot (1 - D_x g(\tau, \pi_2))$, where τ is between π_1 and π_2. By assumption, $1 - D_x g(\tau, \pi_2) \geq 0$. Hence, bidding above $\pi_2 - g(\pi_2, \pi_2)$ is optimal if $\pi_1 \geq \pi_2$, and bidding below $\pi_2 - g(\pi_2, \pi_2)$ is optimal if $\pi_1 \leq \pi_2$. By the monotonicity of the function $G(\pi)$, the bidding function $b(\pi_1) = \pi_1 - g(\pi_1, \pi_1)$ satisfies all these optimality requirements for all π_1. ■

8.2. Proof of Proposition 4.1

Assume that buyer 2 bids according to the strategy in the statement of the Proposition. Consider now buyer 1, and assume that $\pi_1 \in [\underline{\pi}, R)$. For such a type, bidding zero (or any other bid below R) yields

$$U_1(\pi_1, 0) = \int_R^{\bar{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \tag{8.4}$$

Bidding $R \leq b \leq G(R)$ yields

$$U_1(\pi_1, b) = \int_{\underline{\pi}}^{R} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_R^{\bar{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \tag{8.5}$$

Since $\pi_1 \leq R$, the first integral is negative, and bidding zero is preferred to bidding b, $R \leq b \leq G(R)$. Finally, bidding $b \geq G(R)$, yields

$$\begin{aligned} U_1(\pi_1, b) &= \int_{\underline{\pi}}^{R} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_R^{G^{-1}(b)} (\pi_1 - \pi_2) f(\pi_2) d\pi_2 \\ & \int_R^{\bar{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \end{aligned} \tag{8.6}$$

Since $\pi_1 \leq R$, the first two integrals in the last expression are negative, and bidding zero is preferred to bidding above $R - g(R, R)$.

The proof that bidding $\pi_1 - g(\pi_1, \pi_1)$ is optimal for types $\pi_1 \in [R, \bar{\pi}]$ is analogous to the one of Proposition 3.1 and is omitted here. ■

8.3. Proof of Lemma 4.3

Fix u such that $R \leq u \leq G^{-1}(R)$, and consider the equation

$$(z - R) \cdot (F(z) - F(u)) - \int_u^z g(z, \pi) f(\pi) d\pi = 0 \tag{8.7}$$

Defining $P(z) = \int_u^z z - R - g(z,\pi) f(\pi) d\pi$, the previous equation becomes $P(z) = 0$. For $z > G^{-1}(R)$ we obtain that

$$P'(z) = \int_u^z (1 - D_x(z,\pi) f(\pi) d\pi + f(z) \cdot (z - g(z,z) - R) > 0 \qquad (8.8)$$

By the definition of $G^{-1}(R)$, and by $D_y g \leq 0$, we obtain that $P(G^{-1}(R)) \leq 0$. By the intermediate value theorem, we obtain that

$$P(\bar{\pi}) = (1 - F(u))(\bar{\pi} - g(\bar{\pi}, \theta) - R) \qquad (8.9)$$

where $u \leq \theta \leq \bar{\pi}$. By the assumption that $\forall \pi \geq R$, $\bar{\pi} - g(\bar{\pi}, \pi) \geq R$, we obtain that $P(\bar{\pi}) \geq 0$. Since the function $P(z)$ is strictly monotonically increasing on the interval $[G^{-1}(R), \bar{\pi}]$, there exists a unique z, $G^{-1}(R) \leq z \leq \bar{\pi}$, such that $P(z) = 0$.

Hence, for each $u, R \leq u \leq G^{-1}(R)$, we have found a unique $z = z(u) \geq G^{-1}(R)$ such that

$$(z(u) - R) \cdot (F(z(u)) - F(u)) - \int_u^{z(u)} g(z(u), \pi) f(\pi) d\pi = 0 \qquad (8.10)$$

By the implicit function theorem, the function $z(u)$ is continuous.

Consider now the continuous function

$$H(u) = (u - R) \cdot (F(u) + F(z(u))) - \int_u^{z(u)} g(u, \pi) f(\pi) d\pi. \qquad (8.11)$$

Note that

$$\begin{aligned} H(u) &= (u - R) \cdot (F(u) + F(z(u))) - \int_u^{z(u)} g(u,\pi) f(\pi) d\pi \\ &\quad + (u - R)(F(u) - F(z(u))) - (u - R)(F(u) - F(z(u))) \\ &= 2F(z(u)) \cdot (u - R) + \int_u^{z(u)} (u - R - g(u,\pi)) f(\pi) d\pi \qquad (8.12) \end{aligned}$$

We have $H(R) \leq 0$. By the intermediate value theorem we obtain also that

$$\begin{aligned} H(G^{-1}(R)) &= 2F(z(G^{-1}(R)) \cdot (G^{-1}(R) - R) \\ &\quad + (F(z(u)) - F(u)) \cdot (G^{-1}(R) - g(G^{-1}(R), \zeta) - R) \end{aligned}$$

where $G^{-1}(R) \leq \zeta \leq z(G^{-1}(R))$. By the definition of $G^{-1}(R)$, and by $D_y g \leq 0$, we obtain that $H(G^{-1}(R)) \geq 0$. Hence the equation $H(u) = 0$ has a solution $\tilde{\pi}$ on the interval $[R, G^{-1}(R)]$.[14].

The pair $(\tilde{\pi}, \tilde{\tilde{\pi}})$ where $\tilde{\tilde{\pi}} = z(\tilde{\pi})$ is a solution of the system, as required. ■

8.4. Proof of Proposition 4.4

Assume that buyer 2 uses the above strategy, and consider a type $\pi_1 \in [\underline{\pi}, G^{-1}(R)]$ of buyer 1. Bidding zero (or any other bid strictly below R) yields:

$$U_1(\pi_1, 0) = \int_{\tilde{\pi}}^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \tag{8.13}$$

Bidding R yields:

$$\begin{aligned} U_1(\pi_1, R) &= \int_{\underline{\pi}}^{\tilde{\pi}} (\pi_1 - R) f(\pi_2) d\pi_2 + \\ & \frac{1}{2} \cdot \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} (\pi_1 - R + g(\pi_1, \pi_2)) f(\pi_2) d\pi_2 + \\ & \int_{\tilde{\tilde{\pi}}}^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \\ &= \frac{1}{2} \cdot (\pi_1 - R) \cdot (F(\tilde{\pi}) + F(\tilde{\tilde{\pi}})) + \\ & \frac{1}{2} \cdot \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 + \int_{\tilde{\tilde{\pi}}}^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \end{aligned} \tag{8.14}$$

Type $\tilde{\pi}$ is indifferent between bidding zero and bidding R[15], and:

$$U_1(\tilde{\pi}, R) - U_1(\tilde{\pi}, 0) = 0 \tag{8.15}$$

Further, we have that

$$U_1(\pi_1, R) - U_1(\pi_1, 0) = \frac{1}{2} \cdot \left[(\pi_1 - R) \cdot (F(\tilde{\pi}) + F(\tilde{\tilde{\pi}})) - \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \right] \tag{8.16}$$

[14] The solution need not be unique. It is unique, if, for example, $D_i^j g \leq 0$. A more general sufficient condition for uniqueness is given by :$\forall z$, the function $\log[g(v, z) \cdot (1 - F(v))]$ is increasing in v.

[15] This is exactly how this type was constructed.

Note that

$$\frac{\partial\left(U_1(\pi_1, R) - U_1(\pi_1, 0)\right)}{\partial \pi_1} = F(\tilde{\pi}) + \frac{1}{2} \cdot \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} \left(1 - D_x g(\pi_1, \pi_2)\right) f(\pi_2) d\pi_2 \qquad (8.17)$$

Since $1 - D_x g(\pi_1, \pi_2) \geq 0$, the function $U_1(\pi_1, R) - U_1(\pi_1, 0)$ is increasing in π_1 . Hence $U_1(\pi_1, R) - U_1(\pi_1, 0) \geq 0$ for all types $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$, and bidding R is better than bidding zero for these types. Similarly, bidding zero is better than bidding R for types $\pi_1 \in [\underline{\pi}, \tilde{\pi}]$. In fact, it easily follows that a bid of zero is optimal for types $\pi_1 \in [\underline{\pi}, \tilde{\pi}]$. Note also that $\tilde{\pi} \geq R$ (since the first equation in system 4.7 does not admit solutions with $u < R$.)

We now show that a bid of R is optimal for **all** types $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$. We still need to consider alternative bids $b > R$. There are two cases: Assume first that $G^{-1}(b) \leq \tilde{\tilde{\pi}}$. Then bidding $b > R$ yields :

$$U_1(\pi_1, b) = \int_{\underline{\pi}}^{\tilde{\tilde{\pi}}} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_{\tilde{\tilde{\pi}}}^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2$$

Assume next that $G^{-1}(b) > \tilde{\tilde{\pi}}$. Then bidding $b > R$ yields :

$$\begin{aligned} U_1(\pi_1, b) &= \int_{\underline{\pi}}^{\tilde{\tilde{\pi}}} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_{\tilde{\tilde{\pi}}}^{G^{-1}(b)} \left(\pi_1 - (\pi_2 - g(\pi_2, \pi_2))\right) f(\pi_2) d\pi_2 \\ &\quad + \int_{G^{-1}(b)}^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \end{aligned} \qquad (8.18)$$

If $G^{-1}(b) \leq \tilde{\tilde{\pi}}$ we have then that[16]:

$$U_1(\pi_1, R) - U_1(\pi_1, b) = \frac{1}{2} \cdot \int_{\tilde{\tilde{\pi}}}^{\tilde{\tilde{\pi}}} \left(R - (\pi_1 - g(\pi_1, \pi_2))\right) f(\pi_2) d\pi_2 \qquad (8.19)$$

If $G^{-1}(b) > \tilde{\tilde{\pi}}$ we have then that:

$$\begin{aligned} U_1(\pi_1, R) - U_1(\pi_1, b) &= \frac{1}{2} \cdot \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} \left(R - (\pi_1 - g(\pi_1, \pi_2))\right) f(\pi_2) d\pi_2 + \\ &\quad \int_{\tilde{\tilde{\pi}}}^{G^{-1}(b)} \begin{pmatrix} (\pi_2 - g(\pi_2, \pi_2)) - \\ (\pi_1 - g(\pi_1, \pi_2)) \end{pmatrix} f(\pi_2) d\pi_2 \end{aligned} \qquad (8.20)$$

[16] Note that we have used the assumption that $\tilde{\tilde{\pi}} \geq G^{-1}(R)$ to derive the two expressions above.

We need to show that $U_1(\pi_1, R) - U_1(\pi_1, b) \geq 0$ for $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$. Consider first the second integral in equation 8.20. For each $\pi_2 \in [\tilde{\tilde{\pi}}, G^{-1}(b)]$ we obtain by the Mean Value Theorem that $(\pi_2 - g(\pi_2, \pi_2)) - (\pi_1 - g(\pi_1, \pi_2)) = (\pi_2 - \pi_1) \cdot (1 - D_x g(\tau, \pi_2))$, for a certain $\tau \in [\pi_1, \pi_2]$. Since $D_x g(\tau, \pi_2) \leq 1$, each term in the summation is non-negative, and therefore the integral is non-negative.

Consider now the first integral in equation 8.20 (which is also the only expression appearing in equation 8.19), and let $K(\pi_1) = \int_{\pi}^{\tilde{\tilde{\pi}}} (R - (\pi_1 - g(\pi_1, \pi_2))) f(\pi_2) d\pi_2$. Observe that $K(\tilde{\pi}) = 0$ [17]. This shows, in particular, that the type $\tilde{\pi}$ is indifferent between bidding R, and bidding any bid $b \in (R, G(\tilde{\tilde{\pi}})]$. Note also that $K'(\pi_1) = \int_{\pi}^{\tilde{\tilde{\pi}}} (-1 + D_x g(\pi_1, \pi_2)) f(\pi_2) d\pi_2 \leq 0$, and hence that $K(\pi_1) \geq 0$ for $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$. This completes the proof that a bid of R is optimal for all types in the interval $[\tilde{\pi}, \tilde{\tilde{\pi}}]$.

The proof that bidding $\pi_1 - g(\pi_1, \pi_1)$ is optimal for types $\pi_1 \in [\tilde{\tilde{\pi}}, \bar{\pi}]$ is analogous to the one in Proposition 3.1, and is omitted here. ■

8.5. Proof of Corollary 4.5

Define $h(\pi_1) = g(\pi, \pi')$ for all $\pi, \pi' \in [\underline{\pi}, \bar{\pi}]$. The function

$$H(u) = (u - R) \cdot (F(u) + F(G^{-1}(R))) - (F(G^{-1}(R)) - F(u)) \cdot h(u) \tag{8.21}$$

is continuous. Since $G^{-1}(R) \geq R$ and $h(u) \geq 0$, it holds that:

$$H(R) = -(F(G^{-1}(R)) - F(R)) \cdot h(R) \leq 0; \tag{8.22}$$

$$H(G^{-1}(R)) = 2 \cdot \Big(G^{-1}(R) - R\Big) \cdot F(G^{-1}(R)) \geq 0. \tag{8.23}$$

Hence there exists $\tilde{\pi} \in [R, G^{-1}(R)]$ such that $H(\tilde{\pi}) = 0$. The system of equations 4.7 becomes now

$$\begin{aligned} (u - R) \cdot (F(u) + F(z)) - (F(z) - F(u)) \cdot h(u) &= 0 \\ (z - R) \cdot (F(z) - F(u) - (F(z) - F(u)) \cdot h(z) &= 0 \end{aligned} \tag{8.24}$$

We now show that the pair $(u, z) = (\tilde{\pi}, G^{-1}(R))$ satisfies this system of equations. The first equality in the system holds for this pair by the definition of $\tilde{\pi}$. The second equality holds since $G^{-1}(R) - R - h(G^{-1}(R)) = G^{-1}(R) - R - g(G^{-1}(R), G^{-1}(R)) = 0$. The claim follows then by the proof of Proposition 4.4. ■

[17] This is how $\tilde{\tilde{\pi}}$ was constructed .

8.6. Proof of Proposition 4.6

By standard incentive-compatibility arguments one readily obtains that, without loss of generality, we can restrict attention to equilibria where the bidding functions are monotonic.

Lemma 8.1. *Let $b_i(\pi_i)$ be such that $b_i(\pi_i) = b^* > R$ for all π_i in an interval $[\pi_a, \pi_b]$. Then $b_i(\pi_i)$ cannot be part of an equilibrium strategy profile.*

Proof. Consider first a symmetric equilibrium $(b(\pi_1), b(\pi_2))$ such that the function b is constant on an interval as above. By monotonicity, type π_a of bidder 1 prefers bidding b^* rather than $b^* - \varepsilon$, where $\varepsilon > 0$. This yields

$$(\pi_a - b^*)(\pi_b - \pi_a) \geq \int_{\pi_a}^{\pi_b} k\pi_2 d\pi_2 \tag{8.25}$$

Analogously, type π_b prefers bidding b^* rather than $b^* + \varepsilon$, which yields:

$$(\pi_b - b^*)(\pi_b - \pi_a) \leq \int_{\pi_a}^{\pi_b} k\pi_2 d\pi_2 \tag{8.26}$$

The last two equations yield $\pi_a \geq \pi_b$, which is a contradiction[18].

Consider now the case of an asymmetric bidding profile, and assume that bidder's 1 strategy exhibits pooling at a level b^*. If b^* is not in the range of $b_2(\cdot)$, then 1's bidding strategy can be changed such that no pooling occurs at b^*, without further consequences. Assume then that b^* is in the range of $b_2(\cdot)$. If bidder's 2 strategy also requires pooling of types at the bid b^*, then the same argument as in the symmetric case works. Otherwise, the optimal bid for each type $\pi_1$in the interval $[\pi_a, \pi_b]$ is $\pi_1 - kb_2^{-1}(b^*)$, which cannot be a constant . ■

By Lemma 8.1, and the proof of Proposition 3.1 the only candidate for a symmetric equilibrium in pure strategies must have the form

$$b_i(\pi_i) = \left\{ \begin{array}{cc} \pi_i - g(\pi_i, \pi_i) & \text{for } \pi_i \in [\tilde{\tilde{\pi}}_i, \bar{\pi}] \\ R & \text{for } \pi_i \in [\tilde{\pi}_i, \tilde{\tilde{\pi}}_i) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}_i) \end{array} \right\} \tag{8.27}$$

We now show that no such equilibrium exists. Consider first the case of a symmetric equilibrium having the above form. The system of equations 4.7 reads now

[18] Note that this argument does not work for $b^* = R$, since then a bid $b^* - \varepsilon$ has other consequences.

$$\begin{aligned}(u-R)(u+z)-\int_u^z k\pi_2 d\pi_2 &= 0\\ (z-R)(z-u)-\int_u^z k\pi_2 d\pi_2 &= 0\end{aligned} \tag{8.28}$$

The only relevant solution of the system is given by: $u = \tilde{\pi} = \frac{2R}{2-k^2}$, $z = \tilde{\tilde{\pi}} = \frac{2R(k+1)}{2-k^2}$. As in the proof of Proposition 4.4, it is necessary for an equilibrium that $\tilde{\tilde{\pi}} \geq G^{-1}(R) = \frac{R}{1-k}$. However, $\frac{2R(k+1)}{2-k^2} < \frac{R}{1-k}$ for all $k > 0$. This concludes the proof that no symmetric equilibrium in pure strategies exists.

Consider now the possibility of an asymmetric equilibrium having the form given in formula 8.27. Type $\tilde{\tilde{\pi}}_1$ must be indifferent between bidding R and bidding $R+\varepsilon$, yielding:

$$(\tilde{\tilde{\pi}}_1 - R)(\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1) = \int_{\tilde{\pi}_2}^{\tilde{\tilde{\pi}}_2} k\pi_2 d\pi_2 = \frac{1}{2}k(\tilde{\tilde{\pi}}_2 - \tilde{\pi}_2)^2 \tag{8.29}$$

For type $\tilde{\tilde{\pi}}_2$ we obtain analogously:

$$(\tilde{\tilde{\pi}}_2 - R)(\tilde{\tilde{\pi}}_2 - \tilde{\pi}_2) = \int_{\tilde{\pi}_1}^{\tilde{\tilde{\pi}}_1} k\pi_1 d\pi_1 = \frac{1}{2}k(\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1)^2 \tag{8.30}$$

Combining equations 8.29 and 8.30 we obtain:

$$\tilde{\tilde{\pi}}_1 - R = \frac{k^3}{8} \cdot \frac{(\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1)^3}{(\tilde{\tilde{\pi}}_2 - R)^2} \tag{8.31}$$

Assume without loss of generality that $\tilde{\tilde{\pi}}_1 < \tilde{\tilde{\pi}}_2$ (if these are equal than it immediately follows that $\tilde{\pi}_1 = \tilde{\pi}_2$, and we are in the case of a symmetric equilibrium candidate). Equation 8.31 yields then

$$\tilde{\tilde{\pi}}_1 - R < \frac{k}{2} \cdot (\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1) \tag{8.32}$$

Since $k < 1$, and since $\tilde{\pi}_1 \geq R$ (see the proof of Proposition 4.4) we obtain a contradiction. ■

8.7. Proof of Proposition 5.1

All types that decide to pay the fee face a second-price auction without reserve price. The fact that the bid $\pi_i - g(\pi_i, \pi_i)$ is optimal for a type π_i that enters the auction follows in the same manner as in Proposition 3.1.

It remains to show that the respective entry/non-entry decisions are optimal. Consider the type π^E of buyer 1, and assume that buyer 2 plays according to strategy $s_2(\cdot)$. By staying out, the payoff of type π^E is given by

$$\int_{\pi^E}^{\bar{\pi}} g(\pi^E, \pi_2) f(\pi_2) d\pi_2 \tag{8.33}$$

By entering and bidding $\pi^E - g(\pi^E, \pi^E)$, his payoff[19] is given by

$$-E + F(\pi^E) \cdot \pi^E + \int_{\pi^E}^{\bar{\pi}} g(\pi^E, \pi_2) f(\pi_2) d\pi_2 = \int_{\pi^E}^{\bar{\pi}} g(\pi^E, \pi_2) f(\pi_2) d\pi_2 \tag{8.34}$$

Hence type π^E is indifferent between entering and staying out[20]. It is then straightforward to show that all types $\pi_1 > \pi^E$ strictly prefer to enter the auction. ■

8.8. Proof of Proposition 5.2

The claim is obvious if $\pi_S \geq \bar{\pi}$. Assume then that $\pi_S \leq \bar{\pi}$. Since $g(u,u) > 0$, we obtain that $\lim_{u \downarrow \underline{\pi}} K(u) = -\infty$. Observe also that $\lim_{u \uparrow \bar{\pi}} K(u) = \bar{\pi}$. Then, no matter how small π_S is, (possibly negative !) the equation $\pi_S = K(u)$ has a unique solution $\pi_{opt}(\pi_S) > \underline{\pi}$. By Equation 5.3, the seller's revenue is maximized at $\pi_{opt}(\pi_S)$ when the seller's valuation is π_S. If the seller uses the optimal entry fee $E_{opt}(\pi_S) = \pi_{opt}(\pi_S) \cdot F(\pi_{opt}(\pi_S))$, all types in the interval $[\underline{\pi}, \pi_{opt}(\pi_S))$ do not pay the fee and stay out. ■

8.9. Proof of Proposition 5.4

Consider an auction with entry fee E, and let π^E be the unique solution to the equation $E = uF(u)$. For each buyer i, types in the interval $[\underline{\pi}, \pi^E)$ do not pay the fee and stay out, whereas types in the interval $[\pi^E, \bar{\pi}]$ pay the fee and bid $\pi_i - g(\pi_i, \pi_i)$.

[19] Note that this type never (i.e., with probability zero) gets the good.

[20] The equality in the expression above follows by the definition of π^E.

We now construct[21] an auction with a reserve price R^E where the set of active types (i.e., types that bid at least the reserve price) is exactly $[\pi^E, \bar{\pi}]$. Define

$$\begin{aligned} H(R) &= \left(\pi^E - R\right) \cdot \left(F(\pi^E) + F(G^{-1}(R)\right) \\ &\quad - \left(F(G^{-1}(R) - F(\pi^E)\right) \cdot h(\pi^E) \end{aligned} \tag{8.35}$$

Note that $H(R)$ is well-defined and continuous in the interval $[G(\pi^E), \pi^E]$. We obtain that[22]:

$$H(G(\pi^E)) = 2 \cdot F(\pi^E) \cdot h(\pi^E) > 0; \tag{8.36}$$

$$H(\pi^E) = -\left(F(G^{-1}(\pi^E)) - F(\pi^E)\right) \cdot h(\pi^E) < 0 \tag{8.37}$$

Hence, the equation $H(R) = 0$ has a solution R^E in the interval $[G(\pi^E), \pi^E]$.

By the construction of R^E , and by the proof of Corollary 4.5, equilibrium behavior in an auction with reserve price R^E is given by

$$b_i(\pi_i) = \left\{ \begin{array}{ll} \pi_i - g(\pi_i, \pi_i) & \text{for } \pi_i \in \left[G^{-1}(R^E), \bar{\pi}\right] \\ R^E & \text{for } \pi_i \in \left[\pi^E, G^{-1}(R^E)\right) \\ 0 & \text{for } \pi \in \left[\underline{\pi}, \pi^E\right) \end{array} \right\} \tag{8.38}$$

If all types π_i in the interval $[\pi^E, G^{-1}(R^E))$ were to bid $\pi_i - g(\pi_i, \pi_i)$ in the auction with reserve price R^E, then this auction would be revenue equivalent to the auction with entry fee $E = \pi^E F(\pi^E)$. However, equilibrium behavior in the auction with reserve price R^E requires that all types $\pi_i \in [\pi^E, G^{-1}(R^E))$ bid instead R^E. Since $G(\pi_i) = \pi_i - g(\pi_i, \pi_i) < R$ for $\pi_i < G^{-1}(R)$, the seller's revenue in the auction with reserve price R^E is strictly higher than the revenue in the auction with entry fee E (although both auctions induce the same interval of active types. ■

9. References

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[21] The construction is the converse of the one used to determine the type $\tilde{\pi}$ in Corollary 4.5..

[22] For the second inequality, note that $G^{-1}(\pi^E) > \pi^E$.

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