Discussion Paper No. 1230

Job Matching and Coalition Formation with Utility or Disutility of Co-workers*

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October 5, 1998

Math Center web site:
http://www.kellogg.nwu.edu/research/math

Abstract

This paper studies the job matching market in Kelso and Crawford (1982) with one exception that co-workers may generate utility or disutility in the workplace. We provide a simple idea to show how a great number of sufficient conditions for a nonempty core in the literature can be extended to this labor market. We also provide a deterministic and a stochastic recursive dynamic system each of which converges to an efficient core outcome as long as the core is nonempty. *Journal of Economic Literature* Classification Numbers(s): D71.

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* I thank Vince Crawford for helpful discussions. I also thank the Kellogg G.S.M., especially Mike Satterthewaite, at Northwestern University for the hospitality during my visiting.

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1. INTRODUCTION

Disutility of co-workers is an important issue in the labor markets. Sexual harassment is an example of disutility of co-workers in the workplace. The costs of sexual harassment are enormous. A typical Fortune 500 corporation can expect to lose $6.7 million, in 1988 dollars, annually (*Working Woman Magazine*). The government itself had lost $189 million between 1978 and 1980 from the effects of sexual harassment and the loss jumped to $267 million for the year 1985 to 1987.[1]

Utility of co-workers is equally important. It is not uncommon for a graduating economist to accept a lower wage offer due to the benefit he expects to gain from the faculty members in the department. The benefit received often becomes a life-time asset and cannot be under-estimated. In the NBA labor market people often wonder whether Michael Jordan will play for the Chicago Bull without Scottie Pippen and Phil Jackson.

This paper studies the job matching model in Kelso and Crawford (1982) with one exception that co-workers may generate utility or disutility. In their model each firm may hire as many workers as it wishes and each worker works for only one firm. They showed that if firms' production functions satisfy the gross substitutes condition, then their model has a nonempty core. Roth (1984) provided a symmetric version of their model in which a worker can take multiple jobs. He showed that if the (strict) preferences of workers and firms are gross substitutable, the set of pairwise stable matchings is nonempty. Moreover, he found that the set of stable matchings in this general job matching model has the same surprising polarization property observed in the marriage problem in Gale and Shapley (1962).

Kelso and Crawford (1982) and Roth (1984) both assumed that each worker is indifferent over his co-workers. This leaves open the question how the utility or disutility of co-workers may affect the core or the set of stable matchings.[2] The utility or disutility of co-workers does change some important aspects of the model. We show in an example that a labor market that has a nonempty core without utility or disutility of co-workers may have an empty core with utility or disutility of co-workers. It is also true that a labor market that has an empty core without utility or disutility of co-workers may have a nonempty core with utility or disutility of co-workers. A great number of sufficient conditions (including the gross substitutes condition) in the literature for a nonempty core without utility or disutility of co-workers do not provide a sufficient condition for a nonempty core with utility or disutility of co-workers. This invites a question how to embody them (e.g. the

[1] The costs do not include the litigation costs or court-awarded damages or the damages to a company's image. The data cited here are from the article "Sexual Harassment in the Workplace: a Primer" by Barry S. Roberts and Richard A. Mann: http://www.uakron.edu/lawrev/robert1.html.

[2] Kelso and Crawford (1982) stated that "..., but each worker is allowed to work only at one firm; further, workers are indifferent about which other workers their firms hire. The analysis seems significantly more difficult without these restrictions, and we are not certain if they could be relaxed without changing the results." The symmetric version of their model studied by Roth (1984) resolved the issue a worker may take multiple jobs. But the issue of utility or disutility of co-workers is left unsolved.

gross substitutes condition) into such a model that each of them becomes a sufficient condition for a nonempty core with utility or disutility of co-workers. In this paper we use a very simple idea to show in one way how this may be done.

The approach is as follows. Suppose that a labor market consists of one firms i and three workers 1, 2 and 3. Firm i has a production function f_i which is a set function on the set of all subsets of workers $\{1, 2, 3\}$. Each worker has a utility function over his co-workers and the utility function may well depend on the firm he is hired. For example worker 1 may have a utility function $u_1(S_i, i)$ if he is hired by firm i and his co-worker in firm i is S_i, where $1 \in S_i \subset \{1, 2, 3\}$. Given such a labor market, we construct a new labor market without utility or disutility of co-workers. In the new labor market, the firm i has a new production function[3]

$$\tilde{f}_i(S_i) = f_i(S_i) + \sum_{j \in S_i} u_j(S_i, i), \quad S_i \subset \{1, 2, 3\}.$$

We assume that a worker in the new labor market only cares about his wage offer from the firm he is hired, as if there were no utility or disutility of co-workers. Thus, this constructed new labor market becomes an example of the labor market studied in Kelso and Crawford (1982). It follows from Kelso and Crawford (1982) that the new labor market has a nonempty core if the new production function $\tilde{f}_i$ satisfies the gross substitutes condition. It remains to show that the original labor market has a nonempty core as long as the new labor market does. We show that this is indeed the case.

In the new labor market the core equivalence theorem holds. That is, a core allocation is competitive and a competitive allocation is in the core; see Kelso and Crawford (1982) or Ma (1998a). With the core equivalence theorem and the new labor market, we may obtain several other sufficient conditions for a nonempty core for the original labor market. Bikhchandani and Mamer (1997) studied an exchange economy with multiple indivisible goods. They used the linear programming approach and obtained a necessary and sufficient condition for the existence of competitive equilibrium. With the linear programming approach in Bikhchandani and Mamer (1997), we obtain a necessary and sufficient condition for a nonempty core for the original labor market. Gül and Stacchetti (1996a) studied the exchange economy as in Bikhchandani and Mamer (1997) and found two new sufficient conditions, the no complementarities condition and the single improvement property, for the existence of competitive equilibrium. Applying their conditions to the new labor market, we also obtain two new sufficient conditions for a nonempty core for the original labor market. Ma (1998a) studied an exchange economy with indivisible goods in which a commodity may be initially owned and used a coalitional form game and the balancedness condition to obtain a necessary and

[3]This production function may not be just for the technique purpose. It may be the "right" production function to study a labor market. Firms are often liable for hostile working environment. For example firms are liable for sexual harassment if they knew or should have known but failed to take appropriate corrective action (Roberts and Mann cited above).

sufficient condition for the existence of competitive equilibrium. One may use the approach in Ma (1998a) to obtain a necessary and sufficient condition for a nonempty core for the original labor market in which some workers may be initially owned by firms.

The nonempty core is a fundamental issue in economics. An equally important issue is how a core outcome is retained. The generalization of the Gale and Shapley proposal algorithm in Kelso and Crawford (1982) provided an algorithm how a core outcome is obtained in discrete time. The algorithm is a version of a dynamic English auction for the sale of multiple indivisible goods. Gül and Stacchetti (1996b) provided an English auction for an exchange economy with multiple indivisible goods and without production and a double auction for the exchange economy with production. Each of their auctions converges to an efficient core outcome in finite time. These results depend on the gross substitutes condition (the no complementarities condition) and the initial state of the algorithms. In Section 4 we go one step further to study a gradient system in continuous time that converges to an efficient core outcome, as long as the core is nonempty. Further, the convergence result has the property of the Lyapunov global stability (it does not depend on the initial state). Moreover, we show that a stochastic recursive learning dynamics converges to an efficient core outcome, under quite general conditions. This last dynamics follows the trend of the deterministic gradient system but not necessarily in a precise manner because the periodic noise or new information may make it deviate from the trend constantly.

As noted above, the core equivalence theorem holds for the labor market in Kelso and Crawford (1982). We observe that this theorem fails in the labor market with utility or disutility of co-workers. Therefore, all sufficient conditions we obtained from the existing literature that provide sufficient conditions for a nonempty core may not apply to the existence of competitive equilibrium. We leave this question open for the future study. This shows that utility or disutility of co-workers does matter for the competitive equilibrium and the core.

The rest paper is organized as follows. Section 2 introduces the model. Section 3 introduces the new constructed labor market and a number of sufficient conditions for a nonempty core. Section 4 studies deterministic and stochastic wage-adjustment processes. Section 5 provides some discussions. Section 6 provides an example with an empty core, when money is not available in the market.

2. THE MODEL

Let $F = \{1, 2, \cdots, n\}$ denote the set of firms and $W = \{1, 2, \cdots, m\}$ denote the set of workers. Let

$$W_j = \{S \subset W : j \in S\}$$

denote the set of all subsets of W that contain worker j. The production function f_i for firm $i \in F$

is a set function on 2^W such that $f_i(\emptyset) = 0$. The profit function $\pi_i : 2^W \times R_+^{nm} \rightarrow R$ is defined by

$$\pi_i(S_i, p) = f_i(S_i) - \sum_{j \in S_i} p_{ij}.$$

Let $\pi_i(p) = \max_{S_i} \pi_i(S_i, p)$. The wage offer profile $(p_{i1}, p_{i2}, \cdots, p_{im})$ is the wage offers to all workers by firm i in the market. The surplus function $v_j : W_j \times R_+^{nm} \rightarrow R$ for worker $j \in W$ is defined by

$$v_j(S_i, p) = p_{ij} + u_j(S_i, i), S_i \in W_j$$

when he is hired by firm i. Let $v_j(p) = \max_{(S_i, i)} v_j(S_i, p)$. A worker is concerned with his wage offer p_{ij} from firm i and his co-workers S_i in firm i. His co-workers S_i (including himself) produces utility or disutility to the worker j. Therefore the wage is not the only factor in the mind of a worker when he makes a decision to accept or reject an offer. A market with utility or disutility of co-workers is denoted by $\mathcal{M} = (F, W, f, u)$, where $f = (f_1, \cdots, f_n)$ and $u = (u_1, \cdots, u_m)$.

A feasible allocation of workers to firms is a partition $(S_0, S_1, \cdots, S_n)$ of workers W over firms F, leaving the possibility that some workers S_0 may be unemployed in the market. Let $\mathcal{P}(W)$ denote the set of all feasible allocations of workers. Let

$$V = \max_{(S_0, S_1, \cdots, S_n) \in \mathcal{P}(W)} \sum_{i \in F} [f_i(S_i) + \sum_{j \in S_i} u_j(S_i, i)].$$

A feasible allocation $(S_0, S_1, \cdots, S_n)$ is optimal or efficient if it is a solution to the above optimization problem. The value V is the Mashallian aggregate surplus in the labor market, including the sum of utility or disutility generated in the workplace.

Suppose that $u_j(S, 0) = 0$ for all j and $S \in W_j$ when a worker j is unemployed. For the dummy firm 0, let $p_{0j} = \max_{i \in F} p_{ij}$ for all $j \in S_0$. A wage offer profile $p \in R_+^{nm}$ and a feasible allocation of workers $S \in \mathcal{P}(W)$ is individually rational if

$$\pi_i(S_i, p) \geq 0, \ \ \forall i \in F; \quad v_j(S_i, p) \geq 0, \ \ \forall j \in S_i, \forall S_i.$$

Such a pair (p, S) is in the core if it is individually rational and there do not exist a firm-workers coalition (k, T) and a wage offer profile $(q_{kj})_{j \in T}$ such that

$$f_k(T) - \sum_{j \in T} q_{kj} \geq \pi_k(S_k, p)$$

and for all $j \in T$ and all S_i such that $j \in S_i$,

$$q_{kj} + u_j(T, k) \geq v_j(S_i, p),$$

with strict inequality holding for at least one member in $T \cup \{k\}$. We say that the wage offer profile p is an efficient core wage offer profile and the allocation S is an efficient core allocation if the pair (p, S) is in the core.

Given a subset $T \subset F \cup W$, a feasible allocation of workers in $T \cap W$ to firms $T \cap F = \{i_1, \cdots, i_r\}$ is a partition $(S_0, S_{i_1}, \cdots, S_{i_r})$ of workers in T over firms in T, leaving the possibility that some workers S_0 in $T \cap W$ may be also unemployed. The set of all feasible allocations of workers in T over firms in T is denoted by $\mathcal{P}(T)$. Thus we may follow Shapley and Shubik (1972) to define a coalitional form game w for the labor market $\mathcal{M}$ as follows:

$$w(T) = \max_{S \in \mathcal{P}(T)} \sum_{i \in T \cap F} [f_i(S_i) + \sum_{j \in S_i} u_j(S_i, i)].$$

An outcome $(x, y) \in R_+^m \times R_+^n$ is in the core $C(w)$ of the game w if

$$\sum_{j \in W} x_j + \sum_{i \in F} y_i = w(W \cup F) = V$$

and

$$\sum_{j \in T \cap W} x_j + \sum_{i \in T \cap F} y_i \geq w(T), \forall T \subset W \cup F.$$

The last conditions can be written as

$$\sum_{j \in S} x_j + y_i \geq f_i(S) + \sum_{j \in S} u_j(S, i), \forall \{i, S\}, i \in F, S \subset W,$$

due to the bilateral nature of the labor market $\mathcal{M}$ (see Kelso and Crawford (1982) for a detailed discussion). This observation turns out to be crucial in our study of the core.

2.1. An Example

The following example is taken from Kelso and Crawford (1982) and studied in details in Ma (1998a). We use this example to show how utility or disutility of co-workers may affect the core.

EXAMPLE 1 (Empty Core) Let $F = \{i, k\}$ and $W = \{1, 2, 3\}$. The production functions for firms i and k are as follows (see Kelso and Crawford (1982)):

$$f_i(\{1\}) = 4, \quad f_i(\{2\}) = 4, \quad f_i(\{3\}) = 4 + \epsilon_1$$

$$f_i(\{1, 2\}) = 7 + \epsilon, \quad f_i(\{1, 3\}) = 7, \quad f_i(\{2, 3\}) = 7$$

$$f_i(\{1, 2, 3\}) = 9$$

$$f_k(\{1\}) = 4 + \epsilon_2, \quad f_k(\{2\}) = 4, \quad f_k(\{3\}) = 4$$

$$f_k(\{1, 2\}) = 7, \quad f_k(\{1, 3\}) = 7, \quad f_k(\{2, 3\}) = 7 + \epsilon$$

$$f_k(\{1, 2, 3\}) = 9$$

where $\epsilon \in [0, 1]$ and $\epsilon_1, \epsilon_2 \in [0, 3]$.

When utility or disutility of co-workers does not exist, the core is not empty when $\epsilon = 0.5$, $\epsilon_1 = 1$, and $\epsilon_2 = 0$. For example, the outcome (x, y) such that $x = (2.5, 3, 3)$ and $y = (2, 1.5)$ is in the core of the game w. The efficient core allocation (S_i^*, S_k^*) is such that $S_i^* = \{3\}$ and $S_k^* = \{1, 2\}$, with the efficient core wage offer profile (p_i, p_k) such that $p_i = p_k = (2.5, 3, 3)$.

Now suppose that utility of co-workers is given as follows:

$$
\begin{aligned}
u_1(\{1, 2\}, i) &= 1, & u_1(S, i) &= 0, \ \forall S \in W_1 \setminus \{1, 2\} \\
u_2(\{2, 3\}, k) &= 1, & u_2(S, k) &= 0, \ \forall S \in W_2 \setminus \{2, 3\} \\
u_1(S, k) &= 0, \ \forall S \in W_1, & u_2(S, i) &= 0, \ \forall S \in W_2 \\
u_3(S, i) &= 0, \ \forall S \in W_3, & u_3(S, k) &= 0, \ \forall S \in W_3.
\end{aligned}
$$

We now show that the core of the game w is empty. Suppose, by way of contradiction, that (x, y) is a core outcome of the game w. Then

$$
\begin{aligned}
x_1 + x_2 + x_3 + y_i + y_k &= 12.5 \\
x_1 + x_2 + y_i &\geq w(\{i, 1, 2\}) = 8.5 \\
x_2 + x_3 + y_k &\geq w(\{k, 2, 3\}) = 8.5 \\
x_1 + y_i &\geq w(\{i, 1\}) = 4 \\
x_3 + y_k &\geq w(\{k, 3\}) = 4
\end{aligned}
$$

which implies that $x_1 + y_i = 4$ and $x_3 + y_k = 4$. Since $x_1 + y_k \geq w(\{k, 1\}) = 4$, this implies that $x_3 + y_i = 8 - x_1 - y_k \leq 4$. But the core condition requires that $x_3 + y_i \geq w(\{i, 3\}) = 5$. This is a contradiction. Therefore, the core of the game w is empty. It follows from Theorem 1 in Section 3 that there do not exist a wage offer profile p and an allocation of workers $\mathcal{S}$ such that $(p, \mathcal{S})$ is in the core.

The above shows how utility or disutility of co-workers may lead to an empty core even if the core is nonempty without utility or disutility of co-workers. One can also use this example to show how utility or disutility of co-workers may lead to a nonempty core even if the core is empty without utility or disutility of co-workers (this can be shown by switching the role of the above argument).

3. THE NONEMPTY CORE

We showed in Section 2.1 by an example how utility or disutility of co-workers in the workplace may cause an empty core. That example implies that a great number of sufficient conditions in the literature for a nonempty core without utility or disutility of co-workers are not sufficient for a nonempty core in the labor market $\mathcal{M}$ with utility or disutility of co-workers. This section will show how each of them (including the gross substitutes condition) can be embodied into the labor market $\mathcal{M}$ to provide a sufficient condition for a nonempty core.

The idea is as follows. Given a market $\mathcal{M} = (F, W, f, u)$, we construct a new labor market $\tilde{\mathcal{M}} = (F, W, \tilde{f}, \tilde{u})$ without utility or disutility of co-workers as follows. The new production function $\tilde{f}_i : 2^W \to R$ for the firm $i \in F$ is defined by

$$\tilde{f}_i(S_i) = f_i(S_i) + \sum_{j \in S_i} u_j(S_i, i),$$

which is the sum of its product and utility or disutility generated in the workplace when it hires a group of workers S_i. The utility function of co-workers $\tilde{u}_j$ for the worker j in the market $\tilde{\mathcal{M}}$ is a zero function, i.e.,

$$\tilde{u}_j(S_k, k) = 0, \quad \forall S_k \in W_j, \forall k \in F.$$

In the labor market $\tilde{\mathcal{M}}$ workers are concerned with wages only. But firms are concerned with their product and the total welfare of their workers. Therefore, a firm i in the market $\tilde{\mathcal{M}}$ is a profit maximizer with respect to the new production function $\tilde{f}_i$. Clearly he is not necessarily a profit maximizer with respect to f_i. This changes some important aspects of the model, especially the competitive equilibrium.

Let p^{nm}_{+} be a wage offer profile and $\mathcal{S} = (S_0, S_1, \cdots, S_n)$ be an allocation of workers. A coalition (i, T), $i \in F$ and $T \subset W$, blocks the pair $(p, \mathcal{S})$ in the market $\tilde{\mathcal{M}}$ if there exists a wage offer profile $(q_{ij})_{j \in T}$, $q_{ij} \geq 0$ for all $j \in T$, such that

$$\tilde{f}_i(T) - \sum_{j \in T} q_{ij} \geq \tilde{f}_i(S_i) - \sum_{j \in S_i} p_{ij}$$

and

$$q_{ij} \geq p_{kj}, \forall j \in T, j \in S_k$$

with strict inequality holding for at least one member in $T \cup \{i\}$.

Such a pair $(p, \mathcal{S})$ is in the core if it is individually rational in the market $\tilde{\mathcal{M}}$ and it is not blocked by any firm-workers coalition. We provide two definitions of the core: One is based on the wage offers and allocations of workers and the other is based on the game w. The reason why we introduce the game w becomes obvious from the proof of Theorem 2 below. The next result shows that the two definitions are equivalent in both markets $\mathcal{M}$ and $\tilde{\mathcal{M}}$. Next we provide a proof for the market $\mathcal{M}$ since $\tilde{\mathcal{M}}$ is a special case of $\mathcal{M}$.

THEOREM 1 *A wage offer profile $p \in R^{nm}_{+}$ and an allocation $\mathcal{S}$ in $\mathcal{P}(W)$ are in the core if and only if the core of the game w is nonempty in the market $\mathcal{M}$.*

Proof Suppose that (p, S) is in the core. Then let

$$x_j = p_{ij} + u_j(S_i, i), \text{ where } j \in S_i$$

$$y_i = \pi_i(S_i; p) = f_i(S_i) - \sum_{j \in S_i} p_{ij}, \forall i \in F.$$

Suppose that (x, y) is not in the core of the game w. Then there exists a coalition (i, T), $i \in F$ and $T \subset W$, such that

$$\sum_{l \in T} x_l + y_i < f_i(T) + \sum_{l \in T} u_l(T, i).$$

That is,

$$f_i(S_i) - \sum_{j \in S_i} p_{ij} + \sum_{l \in T: l \in S_k} [p_{kl} + u_l(S_k, k)] < f_i(T) + \sum_{l \in T} u_l(T, i).$$

Let $(q_{il})_{l \in T}$ be a wage offer profile such that

$$q_{il} + u_l(T, i) = p_{kl} + u_l(S_k, k), \forall l \in T, l \in S_k.$$

Then

$$f_i(T) - \sum_{l \in T} q_{il} = f_i(T) - \sum_{l \in T: l \in S_k} [p_{kl} + u_l(S_k, k)] + \sum_{l \in T} u_l(T, i) > f_i(S_i) - \sum_{j \in S_i} p_{ij}.$$

Now suppose that the core of the game w is nonempty. Let (x, y) be a core outcome. We show that there exists a pair (p, S) of a wage offer profile $p \in R_+^{nm}$ and an allocation of workers S such that (p, S) is in the core. Let $p_{ij} = p_{kj} = x_j - u_j(S_i)$ for all $i, k \in F$ and all $j \in W$ such that $j \in S_i$. Let S be an optimal allocation. We first show the claim that

$$\sum_{j \in S_i} x_j + y_i = f_i(S_i) + \sum_{j \in S_i} u_j(S_i, i)$$

for all $i \in F$. By the core conditions, we have

$$\sum_{j \in S_i} x_j + y_i \geq f_i(S_i) + \sum_{j \in S_i} u_j(S_i, i)$$

for all $i \in F$. Thus

$$V = \sum_{j \in W} x_j - \sum_{i \in F} y_i \geq \sum_{i \in F} [f_i(S_i) + \sum_{j \in S_i} u_j(S_i, i)] = V.$$

This shows the claim. We now show that (p, S) is in the core. Suppose, by way of contradiction, that this is not true. Then there exists a coalition (i, T) and a wage offer profile $(q_{il})_{l \in T}$ such that

$$f_i(T) - \sum_{l \in T} q_{il} \geq f_i(S_i) - \sum_{j \in S_i} p_{ij} = y_i$$

and, for all $l \in T$ and all S_k such that $l \in S_k$,

$$q_{il} + u_l(T, i) \geq p_{kl} + u_l(S_k, k) = x_l$$

with strict inequality holding for at least one member in $T \cup \{i\}$. Thus

$$y_i + \sum_{l \in T} x_l < f_i(T) + \sum_{l \in T} u_l(T, i)$$

which is a contradiction. □

THEOREM 2 *The core is nonempty in the labor market $\tilde{\mathcal{M}}$ if and only if it is nonempty in the labor market $\mathcal{M}$.*

Proof Note that the transformation of the production functions from f_i to $\tilde{f}_i$, $i \in F$, does not change the coalitional form game w. Thus Theorem 1 completes the proof. □

3.1. THE GROSS SUBSTITUTES CONDITION

With Theorem 2 we now show how the gross substitutes condition in Kelso and Crawford (1982) should be correctly imposed in the market $\mathcal{M}$ to obtain a nonempty core.

For a firm $i \in F$ his demand correspondence $\tilde{D}_i : R_+^{nm} \to 2^W$ in the market $\tilde{\mathcal{M}}$ is defined by

$$\tilde{\mathcal{D}}_i(p) = \{S_i \subset W : \tilde{\pi}_i(S_i, p) \geq \tilde{\pi}_i(T_i, p), \forall T_i \subset W\}.$$

Let p_i and $\tilde{p}_i$ be two wage offer profiles from firm i. Let $S_i \in \tilde{D}_i(p)$ and $T_i(S_i) = \{j \mid j \in S_i, p_{ij} = \tilde{p}_{ij}\}$. A production function $\tilde{f}_i$ satisfies the gross substitutes condition if the following is true:

$$\text{if } S_i \in \tilde{\mathcal{D}}_i(p) \text{ and } \tilde{p}_i \geq p_i, \text{ then } \exists \tilde{S}_i \in \tilde{\mathcal{D}}_i(\tilde{p}) \text{ such that } T_i(S_i) \subset \tilde{S}_i.$$

Moreover we need two more natural assumptions on $\tilde{f}_i$: (a). $\tilde{f}_i(\emptyset) = 0$; (b). $\exists(S_1, S_2, \cdots, S_n)$ such that each S_i is a solution to the problem $max_{S_i} \tilde{f}_i(S_i)$ and $\cup_{i \in F} S_i = W$. These two conditions determine the initial condition how the salary-adjustment process in Kelso and Crawford (1982) should start in the market $\tilde{\mathcal{M}}$.

The MP (marginal product) condition in Kelso and Crawford (1982) satisfies condition (b) when the minimum wages are normalized to zero. The MP condition requires that $\forall i \in F, j \in W$ and all $S \subset W$ such that $j \notin S$,

$$\tilde{f}_i(S \cup \{j\}) - \tilde{f}_i(S) \geq 0$$

where the minimum wages are all zero. Therefore, $\tilde{f}_i(W) \geq \tilde{f}_i(S)$ for all $S \subset W$ and all $i \in F$.

THEOREM 3 (Gross Substitutes) *Suppose that $\tilde{f}_i$ satisfies (a), (b), and the gross substitutes condition for all $i \in F$. Then the core is nonempty in the labor market $\mathcal{M}$.*

Proof Apply the salary-adjustment process R1-R5 in Kelso and Crawford (1982) to the labor market $\tilde{\mathcal{M}}$, with R1 modified as follows:

$R1'$. Firms start with zero wage offer profile $0 \in R_+^{nm}$. Permitted wages at round t, $p_{ij}(t)$, remain constant, except as noted below. In round zero, each firm i makes offers to a set of workers S_i, where S_i is a solution of the optimization problem $\max_{S \subset W} \tilde{f}_i(S)$ such that $\cup_{i \in F} S_i = W$. For the sake of completeness, we provide the R2-R5 of the salary-adjustment process in Kelso and Crawford (1982) as follows, with some changes in notation only in order to be consistent with this paper:

R2. On each round, each firm makes offers to the members of one of its favorite sets of workers, given the wage offer profile $p(t)$. That is, firm i makes offers to the members of S_i, where $S_i \in \tilde{D}_i(p(t))$. Firms may break ties between sets of workers however they like, with the following exception: Any offer made by firm i in round $t-1$ that was not rejected must be repeated in round t. By the gross substitutes condition, the firm sacrifices no profits in doing so, since (by R4) other workers' permitted wages cannot have fallen, and the wage of a worker who did not reject an offer remains constant.

R3. Each worker who receives one or more offers rejects all but his or her favorite (taking wages into account), which he or she tentatively accepts. Workers may break ties at any time however they like.

R4. Offers not rejected in previous periods remain in force. If worker j rejected an offer from firm i in round $t-1$, $p_{ij}(t) = p_{ij}(t-1) + 1$; otherwise $p_{ij}(t) = p_{ij}(t-1)$. Firms continue to make offers to their favorite sets of workers taking into account their permitted wages.

R5. The process stops when no rejections are issued in some period. Workers then accept the offers that remain in force from the firms they have not rejected.

Theorem 1 in Kelso and Crawford (1982) shows that the salary-adjustment process R1$'$-R5 converges in finite time to a discrete core. Theorem 2 in Kelso and Crawford (1982) shows that the continuous labor market $\tilde{\mathcal{M}}$ has a nonempty core. The proof is complete by Theorem 2. □

We may use an alternative approach to show that the salary-adjustment process in Kelso and Crawford (1982) when it applies to the labor market $\tilde{\mathcal{M}}$ converges to a discrete core in finite time, by means of a Lyapunov function. The following was introduced in Ma (1998c). Given a wage offer profile $p \in R_+^{nm}$, we say that a wage offer profile has the excess demand property if

$$\mathcal{O}_j(p) = \mid \{i \in F : j \in S_i\} \mid -1 \geq 0$$

for some collection $(S_1, S_2, \cdots, S_n)$ such that $S_i \in \tilde{D}_i(p)$ for all i. A sequence of wage offer profiles

$\{p(t)\}$ has the excess demand property if each wage offer profile $p(t)$ in the sequence has the excess demand property. The salary-adjustment process R1′-R5 generates a sequence of wage offer profiles that has the excess demand property as long as the algorithm does not close. The excess demand property follows from the condition (b) and the gross substitutes condition since every worker has at least one offer from the very beginning and keeps one offer at each step. A worker who does not reject any offer will remain in the same firm. Note that the sequence of wage offer profiles obtained from the algorithm is ascending, since a worker who is in the excess demand $\mathcal{O}(p(t)) > 1$ rejects all offers but one he prefers the most. Each rejected offer is increased in wage by 1. Thus the algorithm generates a sequence of ascending wage offer profiles that has the excess demand property, as long as the algorithm remains open. Following Ma (1998c), we can show that any such a sequence converges to an efficient core outcome in the market $\tilde{\mathcal{M}}$.

Theorem 3 also shows that if the production function $\tilde{f}_i$ for all firm $i \in F$ satisfies the no complementarities condition or has the single improvement property in Gül and Stacchetti (1996a), then the labor market $\mathcal{M}$ has a nonempty core, since the no complementarities condition or the single improvement property is equivalent to the gross substitutes condition. Under the gross substitutes or the no complementarities condition or the single improvement property, the core has a nice lattice structure (Gül and Stacchetti (1996a)). When workers can take multiple jobs, the polarization property of the set of pairwise stable matchings was shown in Roth (1984), when firms and workers have strict preferences that have the property of gross substitutability.

3.2. A Necessary and Sufficient Condition

Let $S_0, S_1, \cdots, S_{2^m-1}$, where $S_0 = \emptyset$, be an enumeration of all the subsets of W. Let A be a $m \times (2^m - 1)$ matrix such that $a_{kl} = 1$ if $k \in S_l$; otherwise $a_{kl} = 0$. Following Bikhchandani and Mamer (1997), we may use the following integer linear programming to compute the Marshallian aggregate surplus V:

$$\text{IP} \qquad V = \max_{x_1, x_2, \cdots, x_n} \sum_{i=1}^{n} \sum_{l=1}^{2^m-1} [f_i(S_l) + \sum_{j=1}^{m} u_j(S_l, i) a_{jl}] x_{il}$$

$$s.t. \qquad \sum_{l=1}^{2^m-1} a_{jl} \sum_{i=1}^{n} x_{il} \leq 1, \forall j = 1, 2, \cdots, m \tag{1}$$

$$\sum_{l=1}^{2^m-1} x_{il} \leq 1, \forall i = 1, 2, \cdots, n \tag{2}$$

$$x_{il} = 0 \text{ or } 1, \forall i, l. \tag{3}$$

A relaxation of the linear programming IP without the integer constraints is as follows:

$$\text{LP} \qquad \bar{V} = \max_{x_1, x_2, \cdots, x_n} \sum_{i=1}^{n} \sum_{l=1}^{2^m-1} [f_i(S_l) + \sum_{j=1}^{m} u_j(S_l, i) a_{jl}] x_{il}$$

$$\text{s.t.} \quad \sum_{l=1}^{2^m-1} a_{jl} \sum_{i=1}^{n} x_{il} \leq 1, \forall j = 1, 2, \cdots, m \tag{4}$$

$$\sum_{l=1}^{2^m-1} x_{il} \leq 1, \forall i = 1, 2, \cdots, n \tag{5}$$

$$x_{il} \geq 0, \forall i, l. \tag{6}$$

According to the dual theorem, the linear programming LP has the dual as follows:

$$\text{DLP} \quad \bar{V} = \min_{x_j, y_i} \sum_{j=1}^{m} x_j + \sum_{i=1}^{n} y_i \tag{7}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} a_{jl}[x_j - u_j(S_l, i)] + y_i \geq f_i(S_l), \forall i = 1, \cdots, n, l = 1, \cdots, 2^m - 1 \tag{8}$$

$$x_j \geq 0, y_i \geq 0, \forall j, i \tag{9}$$

THEOREM 4 (Necessary and Sufficient) *The labor market $\mathcal{M}$ has a nonempty core if and only if $V = \bar{V}$.*

Proof We first show the sufficient part. Let (x, y) be a solution to the dual problem DLP. Suppose that

$$\sum_{j \in W} x_j + \sum_{i \in F} y_i = V.$$

We show that the core is nonempty. Let $(S_0, S_1, S_2, \cdots, S_n)$ be an optimal allocation of workers. It follows from constraints (8) that

$$\sum_{j \in S_i} x_j + y_i \geq f_i(S_i) + \sum_{j \in S_i} u_j(S_i)$$

for all $i \in F$. Since $\sum_{j \in W} x_j + \sum_{i \in F} y_i = V$, it follows that

$$y_i = f_i(S_i) + \sum_{j \in S_i} u_j(S_i) - \sum_{j \in S_i} x_j = \tilde{f}_i(S_i) - \sum_{j \in S_i} x_j$$

for all $i \in F$. It follows from constraints (8) again that

$$\sum_{j \in T} x_j + y_i \geq f_i(T) + \sum_{j \in T} u_j(T, i)$$

for all $T \subset W$. We now show that $x_i = 0$ for all $i \in S_0$. This follows from the fact that

$$\bar{V} = \sum_{j \in W} x_j + \sum_{i \in F} y_i = \sum_{j \in S_0} x_j + \sum_{i \in F} \tilde{f}_i(S_i) = \sum_{j \in S_0} x_j + V.$$

This shows the sufficiency part.

Suppose that the core is nonempty in the market $\mathcal{M}$. Thus Theorem 1 shows that the core of the game w is nonempty. Let (x, y) be a core outcome of the game w. Thus (x, y) satisfies the constraints (8) in DLP, by the core conditions. It follows that $V \geq \bar{V}$, since $\sum_j x_j + \sum_i y_i = V$. Since DLP is a dual of LP, it follows from the dual theorem that $\bar{V} \geq V$. This shows that $\bar{V} = V$. The proof is complete by Theorem 2. □

The idea in the above proof follows from Ma (1998b) that provides some economic implication of the main theorem in Bikhchandani and Mamer (1997) by means of the core. The original proof in Bikhchandani and Mamer (1997) is based on the complementary slackness conditions in the linear programming. Note that Theorem 4 is slightly more general than their result in the market $\tilde{\mathcal{M}}$ since we do not assume that $\tilde{f}_i$ is weakly monotone, a condition that may not satisfy in the labor market $\tilde{\mathcal{M}}$.

The economic implication of Theorem 4 is quite important. Given a solution $(x, y) \in R^m_+ \times R^n_+$ to the problem DLP, there may not exist an allocation $(S_0, S_1, \cdots, S_n)$ of workers that is compatible with (x, y) (i.e., $\bar{V} > V$). In such a situation Theorem 4 shows that the core is empty in both markets $\mathcal{M}$ and $\tilde{\mathcal{M}}$. If there exists an allocation that is compatible with a solution (x, y) of the problem DLP (i.e., $\bar{V} = V$), then the core must be nonempty in both markets. What behind Theorem 4 is the elegance of the duality in linear programming and the core.

In the assignment problem, each buyer has a unit demand of goods. Thus the core conditions in (8) can be written as (Shapley and Shubik (1972)):

$$x_j + y_i \geq f_i(\{j\}) - c_j(\{j\}), \forall i, j.$$

where $c_j(\{j\})$ is the cost to seller j. In the assignment problem, we always have $V = \bar{V}$. Thus it follows from Theorem 4 that

Corollary 5 (Shapley and Shubik (1972)) *The assignment problem has a nonempty core.*

In the exchange economy with multiple indivisible goods $\mathcal{E}$ in Bikhchandani and Mamer (1997) and Gül and Stacchetti (1996a), there are a finite number of indivisible goods Ω and a finite number of agents. Each agent i can consume as many items as he wishes and has a weakly monotone utility function f_i over the set of all commodity bundles. The core conditions (8) in this exchange economy are

$$\sum_{j \in X} x_j + y_i \geq f_i(X), \forall X \subset \Omega$$

since a commodity bundle is assumed not to generate utility or disutility to a commodity in that bundle. Thus, it follows from Theorem 4 that

Corollary 6 (Bikhchandani and Mamer (1997)) *The core is nonempty in the exchange economy with indivisibilities $\mathcal{E}$ if and only if $V = \bar{V}$.*

In the labor market $\mathcal{M}$ workers are not initially owned by firms. If firms have the initial endowments of workers (say, the NBA, NFL labor markets), then one can follow the approach in Ma (1998a) to obtain a necessary and sufficient condition for a nonempty core in the market $\mathcal{M}$ with the balancedness condition of a coalitional form game. Under such a labor market one needs to pay a special attention to the definition of a "right" coalitional form game. Otherwise the balancedness condition of a coalitional form game like w defined in the standard manner may not provide sufficient information for a nonempty core in the labor market $\mathcal{M}$.

4. Continuous Wage-Adjustment Processes

The salary-adjustment process in Kelso and Crawford (1982) is a version of a dynamic English auction in discrete time that converges to an efficient core outcome in finite time. When the time is discrete, Gül and Stacchetti (1996b) provided an English auction for an exchange economy with multiple indivisible goods and without production. Their English auction converges to an efficient core outcome within finite time that is the least favorite to workers (the minimum competitive equilibrium wage core outcome). When production is possible, Gül and Stacchetti (1996b) provided a double auction that converges to an efficient core outcome within finite time. While these auctions are quite general in most aspects, there are some drawbacks. The first is that they are sensitive to the initial state. The second is that the convergence to an efficient core outcome depends on the crucial gross substitutes condition or the no complementarities condition. In discrete time, Ma (1998c) provided sufficient conditions for every sequence of ascending (descending) wage offer profiles to converge to an efficient core outcome in finite time. The convergence in Ma (1998c) does not depend on the gross substitutes condition. But it depends on the initial state too. In this section we take one step further to study a dynamic system in continuous time that converges to an efficient core outcome as long as the core is nonempty. Moreover, this dynamic system has the Lyapunov global stability property since its convergence does not depend on the initial state. Further, we will provide a stochastic recursive learning dynamics that converges to an efficient core outcome, almost surely, under quite general conditions.

Now we introduce some notation. For a firm i, recall that the profit function $\tilde{\pi}_i : 2^W \times R_+^{nm} \to R$ in the labor market $\tilde{\mathcal{M}}$ is defined by

$$\tilde{\pi}_i(S_i, p) = \tilde{f}_i(S_i) - \sum_{j \in S_i} p_{ij}.$$

For a worker $j \in W$, his 'supply' correspondence $\tilde{S}_j : R_+^{nm} \to W_j \times F$ is defined by

$$\tilde{S}_j(p) = \{i \in F : p_{ij} \geq p_{kj} \ \ \forall k \in F\}.$$

Define

$$\tilde{\pi}_i(p) = \tilde{\pi}_i(S_i, p), \qquad S_i \in \tilde{\mathcal{D}}_i(p),$$

$$\tilde{v}_j(p) = p_{ij}, \qquad i \in \tilde{\mathcal{S}}_j(p),$$

and

$$\tilde{V}(p) = \sum_{i \in F} \tilde{\pi}_i(p) + \sum_{j \in W} \tilde{v}_j(p).$$

LEMMA 7 *$\tilde{V}(p) \geq V$ for all $p \in R_+^{nm}$. Moreover, a wage offer profile $p \in R_+^{nm}$ is an efficient core wage offer profile in the market $\tilde{\mathcal{M}}$ if and only if $\tilde{V}(p) = V$.*

The proof of Lemma 7 follows from Ma (1998c) and thus omitted. Note that $\tilde{V}(p)$ is convex. Thus the following is well-defined almost everywhere:

$$z_{ij}(p) = \frac{\partial \tilde{V}(p)}{\partial p_{ij}} = \frac{\partial \tilde{v}_j(p)}{\partial p_{ij}} + \frac{\partial \tilde{\pi}_i(p)}{\partial p_{ij}}.$$

The term $z_{ij}(p)$ is a measure of the price effect of the profit of firm i and the wage payment of worker j to the wage offer p_{ij} between i and j. Then we consider the following wage-adjustment process (a gradient system) in continuous time:

$$\frac{dp}{dt} = -z(p). \tag{10}$$

One may compare this last dynamic system with a dynamic system in the general equilibrium framework that satisfies the Walrasian hypothesis:

$$\frac{dp}{dt} = \mathcal{Z}(p).$$

where $\mathcal{Z}(p)$ is the excess demand function in the general equilibrium model (Arrow, Block and Hurwicz (1959)). But this comparison is quite misleading. In the Walrasian hypothesis a price adjustment process depends on the aggregate demand and supply. The true aggregate demand and supply are hardly informed for individual buyers and sellers or the market mechanism. Even if the information is known to the market mechanism, Saari (1985) showed that in a general equilibrium model with at least two commodities, no price adjustment dynamic system, which may use any information of the excess demand functions (including their derivatives) as one wishes, can always promise convergence to a price equilibrium (see Saari (1995) for a survey).

So why does the above dynamic system under (10) make so much differences in our labor market? First, our theory of wage-adjustment processes does not depend on the excess demand functions. It depends on the "value function" $\tilde{V}(p)$, the profits for firms and the wage payments for workers. Second, our labor market is a Marshallian economy and the gain to study such a relative small world labor market enables us to find a nice Lyapunov function for many dynamic systems.

Third, our dynamic system (10) follows a different hypothesis, which we call it *the diminishing principle of opportunity costs.* To see what this means, we have to go back to examine what $z(p)$ represents. Let us examine what $\tilde{V}(p)$ is first. The first term in $\tilde{V}(p)$ is the sum of the profits for all firms with the production functions $\tilde{f}_i$ and the second term is the sum of all accepted wage offer payments. Since a market is not necessarily clearing at the wage offer profile p_+^{nm} in the market $\tilde{\mathcal{M}}$, some of these values are unrealized in actual hiring transactions. Therefore, we say that $\tilde{V}(p)$ is the economic rents in the labor market when the wage offer profile is $p \in R_+^{nm}$. The term of economic rents is taken from Smith (1962, 1965) who studied a number of double auction markets and defined such a function for an economy with demand and supply curves of a single commodity. Since the economic rents $\tilde{V}(p)$ are different from the sum of actual realized profits and actual accepted wage offer payments, there are some firms or workers who have unrealized profits or wage payments, given the wage offer profile p. The unrealized profits or wage payments are the opportunity costs to the firms and workers at p. The term $z_{ij}(p)$ provides a measure how the opportunity costs change when the wage offer p_{ij} changes. The diminishing principle of opportunity costs says that a market mechanism should follow an adjustment path in wage offers that minimizes the opportunity costs in the market.

For example, suppose that $z_{ij}(p) > 0$. Should the new wage offer p'_{ij} between worker j and firm i be higher or lower? We discuss two cases: worker j is hired by firm i at p_{ij} and worker j is not hired at p_{ij} by firm i. If worker j is not hired at p_{ij}, then worker j is not hired at higher p'_{ij}. Thus the wage offer p_{ij} should be lower if worker j wishes to be hired by firm i. Now suppose that worker j is hired at p_{ij} by firm i and the new wage offer p'_{ij} is slightly higher than p_{ij}. Let $j \in S_i \in \tilde{D}_i(p)$ be the group of workers hired by firm i at p. Let q be the new wage offer profile in which p_{ij} is replaced by p'_{ij}. If $S_i \in \tilde{D}_i(q)$, then we have $z_{ij}(p) = 0$, as long as the increase in p_{ij} is small. Since $z_{ij}(p) > 0$, this implies that S_i is not in $\tilde{D}_i(q)$. Let $T_i \in \tilde{D}_i(q)$. If $T_i \in \tilde{D}_i(p)$, then j is not in T_i. Thus, the unrealized wage payment for worker j increases by p'_{ij}, since worker j is willing to work for firm j at p'_{ij} but firm i does not like to hire worker j with the new wage offer. Now if T_i is not in $\tilde{D}_i(p)$ but $j \in T_i$, then we have

$$\tilde{f}_i(T_i) - \sum_{k \in T_i \setminus \{j\}} p_{ik} - p'_{ij} \geq \tilde{f}_i(S_i) - \sum_{k \in S_i \setminus \{j\}} p_{ik} - p'_{ij}$$

since $T_i \in \tilde{D}_i(q)$. But this implies that

$$\tilde{f}_i(T_i) - \sum_{k \in T_i} p_{ik} \geq \tilde{f}_i(S_i) - \sum_{k \in S_i} p_{ik}.$$

This is a contradiction, since $S_i \in \tilde{D}_i(p)$ and $T_i \not\in S_i(p)$. This shows that $j \not\in T_i$. Therefore, worker j is not hired by firm j at the new wage offer p'_{ij}, as long as $z_{ij}(p) > 0$ and p'_{ij} is higher than p_{ij}. Thus, worker j has a unrealized wage payment p'_{ij}. This shows that if $z_{ij}(p) > 0$, then the new wage offer p'_{ij} should be lower not higher. Otherwise the opportunity costs for worker j will be higher.[4] The next question is whether the dynamic system has the (Lyapunov) global stability property. The next result provides an answer.

THEOREM 8 (Global Stability) *Suppose that the core is nonempty in the market $\mathcal{M}$. Then the dynamic system $\{p(t)\}$ under (10) converges to an efficient core wage offer profile in the market $\mathcal{M}$.*

Proof Define a function $\mathcal{F} : R^{nm}_{+} \to R$ by

$$\mathcal{F}(p) = \tilde{V}(p) - V.$$

Lemma 7 shows that $\mathcal{F}(p) \geq 0$ for all $p \in R^{nm}_{+}$ and $\mathcal{F}(p) = 0$ for all efficient core wage offer profiles $p \in R^{nm}_{+}$. Moreover

$$\begin{aligned} \frac{d\mathcal{F}(p(t))}{dt} &= \frac{\partial \mathcal{F}(p(t))}{\partial p(t)} \frac{dp(t)}{dt} \\ &= -\sum_{i,j} z_{ij}^2(p(t)) \\ &\leq 0. \end{aligned}$$

The above inequality is strict for all $z_{ij}(p(t))$ such that $z_{ij}(p(t)) \neq 0$. Therefore, the function $\mathcal{F}(p)$ is the Lyapunov function for the dynamic system under (10). Thus, the wage-adjustment process $\{p(t)\}_{t\geq 0}$ under (10) converges to a wage offer profile $p^* \in \{p : z(p) = 0\}$. Since $\mathcal{F}(p)$ is convex, this implies that $\mathcal{F}(p^*) = 0$. It follows from Lemma 7 that p^* is an efficient core wage offer profile in the market $\tilde{\mathcal{M}}$. The proof is complete by Theorem 2. □

At a core outcome in the markets $\mathcal{M}$ and $\tilde{\mathcal{M}}$, the followings are satisfied:

$$\frac{\partial \tilde{v}_j(p)}{\partial p_{ij}} = -\frac{\partial \tilde{\pi}_i(p)}{\partial p_{ij}}, \forall i, j.$$

These equations provide a system to determine all core outcomes.

The wage-adjustment process $\{p_{ij}(t)\}$ under the dynamic system (10) is deterministic. The determination of each wage offer $p_{ij}(t)$ in each period of time t may be accomplished in (pairwise) negotiation. It is often the case that the negotiation process is full of "noise" or new information. Under such a situation, we can consider a dynamic system that provides a learning mechanics of

[4] A worker may learn from this argument when he should ask for a raise in wages from his employer. The condition for such a request is that $z_{ij}(p) < 0$. A faculty member may also learn from this. Suppose that you have an offer from the other university and the offer is higher than your current wage. If you want to stay in your current university and to be paid with the new offer, then it is better to consider whether $z_{ij}(p) < 0$ first before you submit the new offer to your dean.

the periodic new information:

$$p(t+1) - p(t) = \frac{1}{t}(-z(p(t)) + \epsilon(t+1)), \tag{11}$$

where $\{\epsilon(t)\}_{t\geq 0}$ is a sequence of R^{nm}-valued random variables (representing noise or new information) defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume that $p(t)$ is measurable to the sigma-field $\mathcal{F}_t$ generated by $(\epsilon(t-1), \epsilon(t-2), \cdots, \epsilon(0))$.

THEOREM 9 (Asymptotic Stability) *Assume that the core is nonempty in the market $\mathcal{M}$. Suppose that $\partial\tilde{\pi}_i(p)$ is a Lipschitz continuous function in p for all $i \in F$. Moreover the noise term $\epsilon(t)$ satisfies the following*

$$E[\| \epsilon(t+1) \|^2 | \mathcal{F}_t] \leq \infty, \quad E[\epsilon(t+1) \mid \mathcal{F}(t)] = 0.$$

Then a wage-adjustment process $\{p(t)\}$ under the stochastic recursive dynamic system (11) converges to an efficient core wage offer profile in the market $\mathcal{M}$, almost surely.

Proof Since no wage offer p_{ij} is unbounded[5], we may assume that the dynamic system $\{p(t)\}$ under (11) is in fact constrained in a compact convex set (this is why we use the asymptotic stability). Since $z(p)$ is a Lipschitz continuous map, the Picard-Lindelöf theorem shows that the dynamic system under (10) has a unique solution $\{p(t)\}$ that satisfies

$$\frac{dp(t)}{dt} = -z(p(t))$$

for every initial state $p(0) \in R_+^{nm}$. With the Robbins-Siegmund theorem, Theorem 9.3.12 in Duflo (1996, pp. 337) shows that, almost surely,

$$\tilde{V}(p(t)) \to \tilde{V}(p^*)$$

for some $p^* \in \{p : z(p) = 0\}$.

Since $\tilde{V}(p)$ is convex, Lemma 7 shows the set $\{p : z(p) = 0\}$ coincides with the set of all efficient core wage offer profiles $p \in R_+^{nm}$, i.e.,

$$\{p : z(p) = 0\} = \{p : \tilde{V}(p) = V\}.$$

By the continuity of the function $\tilde{V}(p)$, we conclude that $p(t) \to p^*$, almost surely. The proof is complete by Theorem 2. □

Theorem 9 shows that even though the negotiation of wage offers may be full of noise, the actual trend of the wage-adjustment process follows the trend under the dynamic system (10), as long as

[5] No movie stars or athletes have unbounded annual income. Leave alone the ordinary workers like you and me.

the noise term is properly constrained and discounted with an appropriate rate. The "discounted" sequence $\frac{1}{t}$ is not that essential. In fact Theorem 9 is true if the sequence $\{\frac{1}{t}\}$ is replaced by any positive decreasing sequence $\{\gamma(t)\}$ such that

$$\sum \gamma(t) = \infty, \quad \sum \gamma(t)^2 < \infty.$$

5. COMPETITIVE EQUILIBRIUM

So far we only studied the core in the two markets. A closely related concept is the competitive equilibrium. A wage offer profile $p \in R_+^{nm}$ and an allocation of workers $S = (S_0, S_1, \cdots, S_n)$ are a competitive equilibrium in the market $\mathcal{M}$ if it satisfies the following

(a). For all $i \in F$, each S_i is a solution to the problem $\max_{S \subset W} \pi_i(S_i, p)$;

(b). For all $j \in W$ such that $j \in S_k$, (S_k, k) is a solution to the problem $\max_{(S_i,i) \in W_j \times F} p_{ij} + u_j(S_i, i)$;

(c). For all $j \in S_0$, $p_{0j} = 0$ (market clearing condition).

In the labor market $\tilde{\mathcal{M}}$, it follows from Kelso and Crawford (1982) that the core equivalence theorem holds: a core allocation is competitive and vice versa. When there is utility or disutility of co-workers, we find that the core equivalence theorem fails.

EXAMPLE 10 Suppose in Example 1 that $\epsilon = \epsilon_1 = \epsilon_2 = 0$. The utility or disutility of co-workers is follows

$$u_1(\{1,2\}, i) = 0.3 \quad u_2(\{1,2\}, i) = 0.2$$

$$u_2(\{2,3\}, k) = 0.3 \quad u_3(\{2,3\}, k) = 0.2$$

$$u_1(S, i) = 0, \ \forall S \in W_1 \setminus \{1,2\}; \quad u_2(S, i) = 0, \ \forall S \in W_2 \setminus \{1,2\}$$

$$u_1(S, k) = 0, \ \forall S \in W_1; \quad u_2(S, k) = 0, \ \forall S \in W_2 \setminus \{2,3\}$$

$$u_3(S, i) = 0, \ \forall S \in W_3; \quad u_3(S, k) = 0, \ \forall S \in W_3 \setminus \{2,3\}.$$

We show that the core is nonempty. Consider the payoff vector (x, y) such that $x = (3, 3.5, 3)$ and $y = (1, 1)$. It is easy to check that for every coalition (i, S) and (k, S), $S \subset W$, the following holds

$$y_i + \sum_{j \in S} x_j \geq f_i(S) + \sum_{j \in S} u_j(S, i)$$

and

$$y_k + \sum_{j \in S} x_j \geq f_k(S) + \sum_{j \in S} u_j(S, k).$$

Now consider the competitive equilibrium. Note that the optimal allocations in this market are $(\{1,2\},\{3\})$ and $(\{1\},\{2,3\})$. Let $p \in R_+^{2\times 3}$ be a wage offer profile. Suppose that p is competitive. Then we must have that (for worker 2)

$$p_{i2} + u_2(\{1,2\},i) = p_{k2} + u_2(\{2,3\},k)$$

which implies that $p_{i2} - p_{k2} = u_2(\{2,3\},k) - u_2(\{1,2\},i) = 0.1$. Now consider firm i, we have that

$$f_i(\{1\}) - p_{i1} = f_i(\{1,2\}) - p_{i1} - p_{i2}$$

which implies that $p_{i2} = 3$. Also consider firm k, we have that

$$f_k(\{3\}) - p_{k3} = f_k(\{2,3\}) - p_{k2} - p_{k3}$$

which implies that $p_{k2} = 3$. This shows that a competitive equilibrium does not exist in this market. Thus the core equivalence theorem fails.

This implies that given a core outcome $(p,\mathcal{S})$ a worker may still have incentive to search for a new job and some firm may also have incentive to hire this worker. This new hired worker will induce some existing worker in the firm to search for a new job since there exists at least one existing worker who will get worse off due to this new hiring practice. This will create a chain of job search and may be a chain of new hirings. As long as the wage-adjustment process follows the dynamics under (10), this chain of job search and new hirings will eventually converge to an efficient core outcome (that may well be different from the previous one when the chain starts). If the core outcome is unique, the worker who deviates the first in the chain will eventually come back to the same firm.

This may be compared with a labor market $\mathcal{M}$ that has a competitive equilibrium. When a competitive equilibrium exists, then the dynamic system under (10) will converge to a competitive equilibrium, where $z_{ij}(p)$ in the market $\mathcal{M}$ should be defined by

$$z_{ij}(p) = \frac{\partial V(p)}{\partial p_{ij}} = \frac{\partial v_j(p)}{\partial p_{ij}} + \frac{\partial \pi_i(p)}{\partial p_{ij}},$$

and

$$V(p) = \sum_{i\in F} \pi_i(p) + \sum_{j\in W} v_j(p).$$

At a competitive equilibrium, the followings are satisfied:

$$\frac{\partial v_j(p)}{\partial p_{ij}} + \frac{\partial \pi_i(p)}{\partial p_{ij}} = 0, \forall i,j.$$

This system provides a means to find all competitive equilibria in the market $\mathcal{M}$. It should be compared with the system of equations in Section 4 to find all core outcomes for the market $\mathcal{M}$.

When the market $\mathcal{M}$ has an equilibrium, the chain of job search and new hirings discussed above will not happen. This is because once the dynamic system reaches an equilibrium, the dynamic system sinks at that equilibrium. No workers and firms will have (unilateral) incentive to deviate from the equilibrium point. Certainly, this argument depends on what is the 'right' production function in a labor market with utility or disutility of co-workers. If firms are liable for utility or disutility as in the sexual harassment case, the production function $\tilde{f}_i$ may be the right one. And then a core outcome is also an equilibrium. Under such a situation, the chain of job search and new hirings above will not happen either under the dynamics (10).

There is an interesting issue how a market operates if the core is empty. The dynamics under (10) will converge to a wage offer profile p^* such that p^* is a minimum point of $\tilde{V}(p)$. Since p^* is not a core wage offer profile, a coalition of firm-workers exists to negotiate a new wage offer q (restricted to the workers in the coalition) such that the new wage offer q will make at least one member in the coalition better off. But the dynamics under (10), once it jumps out of the basin of attraction (all minima of $\tilde{V}(p)$) because of the new offer q, will start to come back to the basin again, monotonely. But the job search and new hirings will likely form a complicated cycle. It is hard to characterize the cycle. Nevertheless, at the basin of attraction, the opportunity costs of unrealized profits and wage payments are always minimized. The fact that the dynamic system will monotonely move towards to its basin of attraction captures the idea that a market mechanism operates in a way to lower the opportunity costs. If the core is nonempty, then the opportunity costs in the market are zero. If the core is empty, there are always some positive opportunity costs left in the market (e.g., unemployment or tight labor markets). These are the major differences between the market with an empty core and the market with a nonempty core.

6. LABOR MARKETS WITHOUT MONEY

Money or transferable utility plays an important role in our study of the two labor markets $\mathcal{M}$ and $\tilde{\mathcal{M}}$. When a labor market does not have money and workers are indifference in their co-workers, the core is nonempty when firms' preferences have the property of substitutability: see Proposition 6.4 in Roth and Sotomayor (1990). One may wonder whether the same nonempty core result holds for the market with utility or disutility of co-workers and without money. The following example shows that this is not the case. In this example firms have preferences that have the property of gross substitutability (Blair (1984)). But the core is empty.

EXAMPLE 11 (Empty Core) *Let $F = \{F_1, F_2, F_3\}$ and $W = \{W_1, W_2, W_3\}$. The preferences are as follows*

$$P_{F_1} = (\{F_1, W_1, W_2\}, \{F_1, W_2, W_3\}, \{F_1, W_1\}, \{F_1, W_2\}, \{F_1, W_3\}, \{F_1\})$$

$$
\begin{aligned}
P_{F_2} &= (\{F_2, W_2, W_3\}, \{F_2, W_1, W_3\}, \{F_2, W_2\}, \{F_2, W_1\}, \{F_2, W_3\}, \{F_2\}) \\
P_{F_3} &= (\{F_3, W_1, W_3\}, \{F_3, W_1, W_2\}, \{F_3, W_3\}, \{F_3, W_1\}, \{F_3, W_2\}, \{F_3\}) \\
P_{W_1} &= (\{F_1, W_2, W_1\}, \{F_2, W_3, W_1\}, \{F_3, W_2, W_1\}, \{F_3, W_3, W_1\}, \{F_3, W_1\}, \{F_2, W_1\}, \{F_1, W_1\}, \{W_1\}) \\
P_{W_2} &= (\{F_1, W_3, W_2\}, \{F_2, W_3, W_2\}, \{F_1, W_1, W_2\}, \{F_3, W_1, W_2\}, \{F_1, W_2\}, \{F_2, W_2\}, \{F_3, W_2\}, \{W_2\}) \\
P_{W_3} &= (\{F_3, W_1, W_3\}, \{F_1, W_2, W_3\}, \{F_2, W_2, W_3\}, \{F_2, W_1, W_3\}, \{F_2, W_3\}, \{F_1, W_3\}, \{F_3, W_3\}, \{W_3\})
\end{aligned}
$$

Clearly any core allocation of workers must leave no worker unemployed. We show that any core allocation of workers μ must also leave no firms 'staying alone'. The following are the possible allocations that are individually rational and leave some firm 'staying alone'. The coalition after each allocation is the coalition that dominates that allocation.

$$
\begin{aligned}
\mu_1 &= [(F_1; W_1, W_2), (F_2; W_3), (F_3; \emptyset)] && \{F_2, W_2, W_3\} \\
\mu_2 &= [(F_1; W_1, W_2), (F_2; \emptyset), (F_3; W_3)] && \{F_2, W_3\} \\
\mu_3 &= [(F_1; W_2, W_3), (F_2; W_1), (F_3; \emptyset)] && \{F_3, W_1\} \\
\mu_4 &= [(F_1; W_2, W_3), (F_2; \emptyset), (F_3; W_1)] && \{F_3, W_1, W_3\} \\
\mu_5 &= [(F_1; W_1), (F_2; W_2, W_3), (F_3; \emptyset)] && \{F_3, W_1\} \\
\mu_6 &= [(F_1; \emptyset), (F_2; W_2, W_3), (F_3; W_1)] && \{F_1, W_2, W_3\} \\
\mu_7 &= [(F_1; W_2), (F_2; W_1, W_3), (F_3; \emptyset)] && \{F_1, W_1, W_2\} \\
\mu_8 &= [(F_1; \emptyset), (F_2; W_1, W_3), (F_3; W_2)] && \{F_1, W_2\} \\
\mu_9 &= [(F_1; W_2), (F_2; \emptyset), (F_3; W_1, W_3)] && \{F_1, W_1, W_2\} \\
\mu_{10} &= [(F_1; \emptyset), (F_2; W_2), (F_3; W_1, W_3)] && \{F_1, W_2\}
\end{aligned}
$$

Thus we show that any core allocation must assign every firm at least one worker and every worker at least one firm. But any such allocation is dominated by at least one coalition with two workers and one firm. This shows that the core is empty in this example.

This example leaves open a question when the core is nonempty in a labor market with utility or disutility of co-workers and without money. For example the college admissions problem has a nonempty core when students are not concerned with the size of their classes; see Roth and Sotomayor (1990). Our example above shows that this may not be true if students or their parents prefer small size of classes. Therefore, a great number of questions remain open.

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