DISCUSSION PAPER NO. 123

RATIONAL PRICE DISPERSION, SEARCH AND ADJUSTMENT

by

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November, 1974

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To be presented at the 1974 Meeting of the Econometrica Society, San Francisco.

The existing dual theory of competitive exchange at non-market-clearing prices and of competitive price adjustment lacks a simple consistent logical foundation comparable to that underlying the theory of equilibrium competitive trading and price determination. Anyone who has attempted to do theoretical research on these topics is fully aware of the reason for this sad state of affairs. The standard elegant theory of individual trading behavior is based on an assumption which simply cannot hold where trade takes place at non-market-clearing prices.[1] Specifically, not every trader can effect his most preferred affordable exchange in such a situation.

Of course, approaches exist which handle this problem. For example, one can either postulate a decentralized exchange process in which small groups of traders who are willing and able to exchange among themselves do so or maintain the central market place construct but introduce rules to determine who is to be rationed. However, most writers using the first approach seem only to ask whether the usual <u>ad hoc</u> price adjustment rule will find a market clearing price vector given that exchange takes place during the process.[2] To date, writers applying the second approach have restricted their concern to the effects of non-price rationing on allocation.[3] Neither group has addressed the central question. Why are competitive prices slow to adjust and why do traders exchange at disequilibrium prices?

One possible reason that the question is not central in the literature is that the answer is too obvious. Prices simply don't adjust instantaneously because the cost of finding the market clearing vector that quickly far outweighs the benefit. Equivalently, finding equilibrium in an economical manner requires time. Because there are real costs associated with "waiting to exchange", each agent finds it in his interest to trade during the adjustment process. Such reasoning has been used to justify modelling price adjustment as though there were an auctioneer who changes prices with a lag in response to demand and supply signals.

Personally, I find this story convincing. Nevertheless, the question deserves a less superficial answer - one which casts more light into the black box. Ideally, such an answer would specify the costs which are economized by lagged price adjustment and would provide a logical foundation from which the "law of supply and demand" could be derived. The purpose of this paper is to suggest one possibility.

The suggestion is anticipated in my earlier work on the market implications of price search. Indeed, the model analyzed in this paper is a dynamic generalization of one developed in [9]. Exchange is decentralized in the sense that the process takes place at meetings of many trader coalitions which form at specified "local markets." In any short interval of time those traders who happen to be at a given local market trade among themselves at prices which maximize exchange among them subject to the ability and willingness of each to participate.

In general, a single exchange ratio between any two commodities does not characterize either all the market at a point in time or a given market over time. Consequently, an incentive to speculate exists. Specifically, each trader in a particular market and period must decide whether to trade now or to wait in anticipation of a better price. Those traders who decide to search or are unable to trade carry over speculative stocks which depend on expected future prices and the costs of seeking a better price. Finally, current transaction prices are jointly determined by the composition of the coalitions and the rules by which traders decide to search rather than trade. Present and future prices and stocks are tied in these various ways.

Given a large number of heroic simplifying assumptions, the following results are obtained in the case of a single durable commodity. If traders search the local markets at random and if search costs increase as the frequency with which these markets are searched, then speculative gains are never eliminated in the sense that price dispersion across markets persists. Moreover, the distribution of prices converges over time to a steady state characterized by the equality of flow supply and demand. Indeed, the mean of the price distribution changes at a rate which increases with the difference between the aggregate flow demand and volume of transactions per period and decreases with the difference between the aggregate flow supply and the aggregate volume of transactions per period. Finally, if there are no costs of search or if they vanish as search becomes instantaneous, then the price distribution collapses and the speed with which the common price adjusts to its competitive equilibrium value tends to infinity with the search frequency. In other words, both price dispersion and lagged price adjustment can be explained in principle by transaction costs which rise with search frequency.

1) The Model

A single durable commodity is exchanged in a set of identical local markets denoted as $M = \{1, \ldots, m\}$. Let $N = \{1, \ldots, n\}$ represent the set of traders. This set is the union of two disjoint sets, the set of sellers, N_1, and the set of buyers, N_2.

All sellers are risk neutral in the sense that each acts to maximize the expected present value of a future stream of monetary benefits attributable to their activities. In the case of a seller, the flow equals the difference between the value of sales and the sum of transaction and production costs. The stream equals the net value of purchases less ordering and transactions costs in the case of a buyer.

We begin the analysis by formulating the model in discrete time. The length of the representative period has a dual interpretation. Its inverse is the average search frequency, on the one hand, and exchange takes place among the traders represented at each local market during the interval on the other.[4] The costs associated with both search and exchange, total transactions costs, presumably depend on the length of the interval. Specifically, instantaneous search and exchange are likely to be prohibitively expensive and decline with the length of the interval. Later we show that a positive length is socially optimal under these conditions.

Search is assumed to be random in the sense that each trader assigned each unit in his stock to markets with equal probability. Given these assignments, inventory and orders are matched in a manner which maximizes sales given the willingness and ability of the agents associated with the stocks to trade. Willingness to trade is determined by a trader's reservation price. In the case of a seller, we refer to the minimum price acceptable as his ask price. The maximum price that a buyer will accept is referred to as his bid price. Ability to trade in a given market is determined by the quantity of a seller's inventory assigned to the market in the course of search and by the number of orders assigned to the market in the case of a buyer.[5] The price in a market during any interval is set conditional on reservation prices and stock assignments so as to maximize sales.

Given the pricing and trading rules outlined above, the future price in every market is a function of future reservation price and future stock assignments to the market. Since the latter is a random variable by virtue of random search, so is price and quantity traded. Moreover, the distribution function characterizing these variables in any two markets are identical since the price setting and matching rules used are the same.

Each trader controls two variables. In any period the typical seller must set the ask price and production rate associated with that period. Analogously, the buyer sets his bid price and new order rate. In each case the rules by which the values of the controls are determined are optimal solutions to a stochastic control problem. One aspect of the specification of this problem concerns the manner in which traders are assumed to forcast the future outcomes of the pricing and matching process outlined above. The specific assumption used is crucial in the sense that the nature of the market solution obtained depends on the traders' strategies which in turn depend on the manner in which traders forcast.

In the paper we adopt the so called "rational expectations hypothesis" suggested by Muth [11]. Specifically, the probability distributions that traders expect to characterize future prices and sales in each market are those generated by the traders' anticipations. The reason that expectations of this type are rational is that traders would find it in their interest to forcast in such a manner if they had the information which would allow them to do so. Hence, a valid objection exists to the hypothesis which is based on the contention that traders find it uneconomical to obtain information which will allow them to make predictions which are selffulfilling in this strong sense. In spite of this objection, we maintain the hypothesis because it allows for a clear view of the potential role of trader speculation in the price adjustment process.

Finally, we remind the reader that we are concerned with competitive analysis in this paper. Specifically, each trader acts as if his decisions have no effect on the nature of the process which determines price and trading opportunities.

2) Trader Behavior

Later we will show that the price in any market in period t can be characterized at the beginning of the period by a probability distribution function, denoted as $F(p,\Omega_t)$. The argument Ω_t represents a set of parameters which completely characterizes the state of the entire system of local markets as of the beginning of period t. One implication of the rational expectations hypothesis is that the typical trader knows both the form of the function F and the past and current values of Ω_t. Moreover, a joint probability distribution on future states conditioned on the sequence of past states exists and is known to the trader. The notation $E\{\cdot|\Omega_t\}$ represents the expectation of some random variable taken with respect to this distribution. In the sections which follow, the determinants of the state of the market system and the nature of the distribution on future states are derived.

Because each trader is one among many, the current value of his own stock has an indiscernable effect on the elements of Ω_t now and in the future. However, the value of his controls depends on the value of his own stock in general. When it does, the trader must also forecast the future values of his own stock as well as the future states of the market system. In the specification which follows we eliminate this addition complication by assuming that each trader is risk neutral in the sense that he maximized a discounted sum of a sequence of future net monetary benefits and that each element in the sequence is linear in the trader's own stock. The latter condition holds when the cost of attempting to sell a unit of inventory is independent of the size of the stock held by a seller and when the cost of attempting to fill an order is independent of the size of the buyer's unfilled order stocks. These assumptions are not unreasonable as first approximations and allow for a considerable simplification of the analysis which follows.

At the beginning of period t, the typical trader inherits a stock denoted as x_t from the past. Let q_t denote the trader's reservation price in period t. Because the matching rule in every market insures that no trader will be prevented from exchanging a unit when the price is acceptable, the trader's reservation price and the price in the market to which any unit in the stock is assigned determine whether or not it will be sold.[6] If the trader is a seller, then

$$y_i = \begin{cases} 1 & \text{if } p_i \geq q_t \\ 0 & \text{otherwise} \end{cases}$$

where $y_i = 1$ means that the i^{th} unit of the stock x_t is sold and p_i denotes the price at which the transaction takes place. Total sales made by the seller in question can, then, be expressed as $\sum_{i=1}^{x_t} y_i$. Consequently, the seller's inventory in the next period is determined by the identity

$$x_{t+1} = x_t + s_t - \sum_{i=1}^{x_t} y_i$$

where s_t denote the seller's production rate during the period.

The net cash flow generated by the seller's activities is defined as

$$\pi_t = \sum_{i=1}^{x_t} p_i y_i - \varphi_j(s_t) - c_j x_t, \quad j \in N_1$$

where $\varphi_j(\cdot)$ denotes the cost of production and c_j denotes the cost of attempting to sell any unit in inventory. These costs are incurred in storing the unit, searching for an acceptable price and engaging in the exchange process itself. Consequently, c_j represents the total transaction cost incurred per unit of inventory held. Finally, the subscript j simply distinguishes the seller's cost functions from those associated with all other elements in the set N_1.

The present value of the future cash flow is defined by

$$V_t = \sum_{\tau=t}^{T} \beta^{\tau-t} \pi_\tau, \quad \beta = {}^1/(1+r) \tag{1}$$

where r is the interest rate at which the trader can borrow and lend financial capital and T is the sellers horizon. At each date the seller is assumed to set his ask price and his production rate (q_t, s_t) according to a strategy which maximizes

$$E\{V_t | \Omega_t\}$$

subject to the identity determining the evolution of his inventory stock. For the reasons already stated his strategy is a function from the state space characterizing the market to R^2_+.

Because the i^{th} unit of inventory is sold only if it is assigned to a market in which the price is no less than his ask price, the following relationships hold:

$$E\{x_{\tau+1} | \Omega_\tau\} \equiv \bar{x}_{\tau+1} = x_\tau(1 - \int_{q_\tau}^{\infty} dF(p;\Omega_\tau)) + s_\tau, \ \forall\tau \tag{2a}$$

$$E\{\pi_\tau | \Omega_\tau\} = x_\tau \int_{q_\tau}^{\infty} p \, dF(p; \Omega_\tau) - \varphi_j(s_\tau) - c_j x_\tau, \ \forall\tau \text{ and } j \in N_1 \tag{2b}$$

Of course, $F(p; \Omega_\tau)$ represents the distribution function defined on price in any market during period τ. Consequently, the first term in (2a) represents the number of units in inventory which the sellers expect not to sell and the first term in (2b) equals the value of expected sales.

Since the expected value of the operation at date t can be expressed as

$$E\{V_t | \Omega_t\} = E\{\pi_t | \Omega_t\}$$

$$+ \beta E\{E\{\pi_{t+1} | \Omega_{t+1}\} | \Omega_t\} + \beta^2 E\{E\{V_{t+2} | \Omega_{t+2}\} | \Omega_t\},$$

appropriate substitutions suggested by the equations of (2) yield

$$E\{V_t | \Omega_t\} = x_t \int_{q_t}^{\infty} p \, dF(p; \Omega_t) - \varphi_j(s_t) - c_j x_t \tag{3}$$

$$+ \beta[\bar{x}_{t+1} E\{\int_{q_{t+1}}^{\infty} p \, dF(p;\Omega_{t+1}) | \Omega_{t+1}\} | \Omega_t\} + E\{\varphi_j(s_{t+1}) | \Omega_t\} - c_j \bar{x}_{t+1}]$$

$$+ \beta^2 E\{E\{V_{t+2} | \Omega_{t+2}\} | \Omega_t\}$$

where $\bar{x}_{t+1}$ denotes the expectations of x_{t+1} given Ω_t as defined in (2a).

Because the optimal strategy is independent of the sellers own stock and the future states of the market are independent of the seller's actions, the value of the last term is independent of the choices made with respect to the values of the current controls. However, the current controls enter the first term directly and the second term indirectly through the first two relationships in the sequence defined in (2a). Consequently, $(q_t, s_t, \bar{x}_{t+1})$ maximizes the sum of these two terms subject to the first two equations of (2a). The Lagrangian function associated with this sub-problem can be written as

$$H_t = x_t \int_{q_t}^{\infty} p \, dF(p;\Omega_t) - \varphi_j(s_t) - c_j x_t$$

$$+ \eta_t [x_t (1 - \int_{q_t}^{\infty} dF(p;\Omega_t)) + s_t - \bar{x}_{t+1}]$$

$$+ \beta[\bar{x}_{t+1} E\{\int_{q_{t+1}}^{\infty} p \, dF(p;\Omega_{t+1}) | \Omega_t\} - E\{\varphi_j(s_{t+1}) | \Omega_t\} - c_j \bar{x}_{t+1}]$$

$$+ \beta[\bar{x}_{t+1} E\{\eta_{t+1}(1 - \int_{q_{t+1}}^{\infty} dF(p;\Omega_{t+1}) | \Omega_t\} + E\{\eta_{t+1}(s_{t+1} - \bar{x}_{t+2}) | \Omega_t\}$$

where η_t and η_{t+1} are the multipliers associated with the first and second constraints respectively. These too are functions of the state of the market in their respective periods. Consequently, in period t when Ω_t is known, η_{t+1} is a random variable independent of x_{t+1}. This fact justifies the last line in the expression.

Finally, then, a solution interior relative to any natural boundaries constraining the choice set must satisfy the following first order conditions for all $j \in N_1$.[7]

$$(\eta_t - q_t)\, dF(q_t;\ \Omega_t) = 0 \tag{4a}$$

$$\eta_t - \varphi_j'(s_t) = 0 \tag{4b}$$

$$-\eta_t - \beta c_j + \beta\, E\{ \int_{q_{t+1}}^{\infty} p\, dF(p;\ \Omega_{t+1}) + \eta_{t+1}(1 - \int_{q_{t+1}}^{\infty} dF(p;\Omega_{t+1})) \mid \Omega_t\} = 0 \tag{4c}$$

where $\varphi_j'(s_t)$ denotes the marginal cost of production incurred by seller j. The conditions hold in every period prior to the seller's horizon date. In the case of a horizon date in the indefinite future, the end point condition is

$$\lim_{T\to\infty} B^{T-t}\, \eta_T = 0. \tag{4d}$$

The first necessary condition implies that the value of inventory held over from period t to $t+1$ equals the minimum acceptable price in period t, his ask price. According to the second, the production rate must be set to equate the cost of producing the marginal addition to inventory to the imputed value of a unit in inventory. Finally, the last equation, given the first, can be interpreted as an optimal stopping rule condition. The lowest acceptable price in period t plus the present value of the cost of attempting to sell a unit in period $t+1$ equals the present value of the expectation from the viewpoint of period t of the expected outcome attributable to search in the next period. The latter is the average of the price obtained if the unit is sold and the value of the unit as inventory if not sold with weights equal to probabilities of the two events. Consequently, the cost of searching for a better price in the next period plus the current imputed value equals the expected present value of the two possible outcomes of search in the next period.

For the sake of a story, it is convenient to regard the buyers as firms. They might be viewed as "retailers" all of whom resell what they purchase at the common price μ. In other words, μ represents the value of the commodity to the ultimate consumer.[8] To obtain a determinant optimal flow of new orders we postulate the existence of an ordering cost. Let $\varphi_j(d_t)$ denote the total cost of ordering at a rate d_t faced by buyer j. Costs of search and other exchange activities are assumed to be proportional to the buyer's stock of unfilled orders. Let c_j, $j \in N_2$ denote the transaction cost per order and let x_t denote the stock of orders.

The typical order is filled only if it is assigned to a market with a price no greater than the buyer's bid price, q_t.[9] Let

$$y_i = \begin{cases} 1 & \text{if } p_i \leq q_t \\ 0 & \text{otherwise} \end{cases}$$

where $y_i = 1$ corresponds to the event that the i^{th} order in the stock x_t is sold. We have, then,

$$x_{t+1} = x_t + d_t - \sum_{i=1}^{x_t} y_i$$

and

$$\pi_t = \sum_{i=1}^{x_t} (\mu - p_i)y_i - \varphi_j(d_t) - c_j x_t .$$

The buyers controls (q_t, d_t) are determined by a strategy which maximized

$$E(V_t \mid \Omega_t)$$

where the present value of the future net cash flow is defined as in (1).

The formal equivalence between the typical buyers problem and that of the typical seller is obvious. For precisely the same reasons, the buyer's optimal strategy depends only on the state of the market. Indeed if we replace s_t by d_t. p_i by $\mu - p_i$ and take account of the fact that the buyer purchases rather than sells, the same formal argument used to derive the equations of (4) can be applied to obtain the following necessary conditions for all $j \in N_2$:

$$(\mu - q_t - \eta_t)\, dF(q_t; \Omega_t) = 0 \tag{5a}$$

$$\eta_t - \varphi_j'(d_t) = 0 \tag{5b}$$

$$-\eta_t - \beta c_j + \beta E\left\{\int_0^{q_{t+1}} (\mu - p)\, dF(p; \Omega_{t+1}) + \eta_{t+1}\left(1 - \int_0^{q_{t+1}} dF(p; \Omega_{t+1})\right) \Big|\, \Omega_t\right\} = 0 \tag{5c}$$

Of course, the multiplier η_t is to be interpreted as the inputed value of an unfilled order and $\varphi_j(d_t)$ is the marginal cost of ordering at a rate d_t. Finally,

$$\lim_{T \to \infty} \beta^{T-t} \eta_T = 0 \tag{5d}$$

is the end point condition.

In the case of a buyer, the bid price equals the difference between the value of a purchased unit and the inputed value of an unfilled order. The optimal new order rate is such that the marginal cost of ordering is equal to the value of an unfilled order. The last condition can also be given an optimal stopping rule interpretation. The buyer is indifferent between purchasing now and speculating in the hope of finding a lower price in the next period when the value of a unit to him less the highest acceptable price in the current period plus the cost of search equals the expected present value of the two possible outcomes attributable to search in the next period.

An inspection of (4) and (5) reveals that the variations in transaction costs is the only reason for differences in the reservation prices set by members of the same trader type, given that all have the same expectations. In the case of the typical seller, equations (4a) and (4c) reduce to

$$q_t + \beta c_j = \beta E\{q_{t+1} + \int_{q_{t+1}}^{\infty} (p - q_{t+1}) dF(p; \Omega_{t+1}) | \Omega_t\}, \quad j \in N_1 \tag{6a}$$

and (5a) and (5c) implying

$$(1-\beta)u + \beta c_j - q_t = \beta E\{-q_{t+1} + \int_0^{q_{t+1}} (q_{t+1} - p) \; dF(p; \Omega_{t+1} | \Omega_t\}, \; j \in N_2 \tag{6b}$$

in the case of the typical buyer. Note that the current ask price decreases with the cost of attempting to sell a unit from inventory and increases with next period's ask price. In the finite horizon case, the ask price of all sellers at the horizon date is zero. Consequently, an induction argument implies that the seller facing the highest transaction cost has the lowest ask price at every prior date. An analogous argument can be used to prove that the buyer facing the highest transaction cost is willing to pay the highest price among all the buyers.

Necessary second order conditions for optimal trader strategies requires that the marginal cost of production and ordering not be decreasing functions of their respective arguments. If we strengthen those conditions slightly, a flow supply function exists for each seller and a flow demand function exists for each buyer. Let

$$s_t = s_j(q_t), \quad j \in N_1 \tag{7a}$$

and $$d_t = d_j(u - q_t), \quad j \in N_2 \tag{7b}$$

denote the inverses of (4a) and (4b) respectively. All such functions are positive, increasing and continuous. The equations of (6) and (7) form the basis of the analysis which follows.

3) Market Equilibrium

The purpose of this section is to find the price distribution generated by the pricing and matching rules assumed to characterize exchange, by random search and by individual trader strategies which are optimal relative to the price distribution.[10] This object is an equilibrium in the sense that it is the only distribution consistent with optimal behavior when the traders know the distributions. In this sense it is a stochastic generalization of the concept of a competitive equilibrium price in the standard deterministic theory. However, the system of markets does not clear in the usual sense in a stochastic equilibrium.

As an introduction to the analysis, it is useful to show that the range of possible prices is determined solely by transactions costs and by the inputed cost of waiting to trade that are incurred because search and exchange require time. Recall that variations in reservation prices across members of either trader type are due only to differences in transactions costs when all have the same expectation. Specifically, the seller who must spend the most in his effort to sell a unit of the commodity has the lowest ask price and the buyer with the highest bid price faces the highest transaction cost per order. These two reservation prices, the lowest ask and the highest bid, obviously bound the set of possible prices at which trade can take place. Below we show that the difference between them equals the total cost per period of attempting to match an inventory-order pair held by the seller-buyer pair who face the highest combined transactions cost.

Let c_1 denote the maximum element in the set $\{c_j \mid j \in N_1\}$ and c_2 denote the largest element in the set $\{c_j \mid j \in N_2\}$. All sellers who face cost c_1 have the same ask price in every period which we donote as $p_1(t)$. The bid price common to all buyers who pay transaction cost c_2 is represented by $p_2(t)$. The sequences of each of these satisfy

$$p_1(t) + \beta c_1 = \beta E\{p_1(t+1) + \int_{p_1(t+1)}^{p_2(t+1)} (p - p_1(t))\, dF(p; \Omega_{t+1}) \mid \Omega_t\}$$

$$(1-\beta)\mu + \beta c_2 - p_2(t) = \beta E\{-p_2(t+1) + \int_{p_1(t+1)}^{p_2(t+1)} (p_2(t+1) - p)\, dF(p; \Omega_{t+1}) \mid \Omega_t\}$$

by virtue of (6), the argument made subsequent to (6) and the fact that the range of possible prices is the closed interval (p_1, p_2) in every period. Since the rational expectations hypothesis implies that all traders have the same expectations and know the range of possible prices given the current state of the market, the two equations above can be rewritten as

$$p_1(t) + \beta c_1 = \beta E\{\bar{p}(t+1) \mid \Omega_t\} \tag{8a}$$

$$(1-\beta)\mu + \beta c_2 - p_2(t) = -\beta E\{\bar{p}(t+1) \mid \Omega_t\} \tag{8b}$$

where

$$\bar{p}(t+1) = \int_{p_1(t+1)}^{p_2(t+1)} p \, dF(p; \Omega_{t+1}) = E\{p_{t+1} \mid \Omega_{t+1}\}. \tag{8c}$$

Since $(1-\beta)/\beta = r$, equation (8a) and (8b) implying

$$p_2(t) - p_1(t) = \beta(r\mu + c_1 + c_2). \tag{9}$$ [11]

Hence, the range of possible prices equals the present value of the total waiting cost and the maximum transaction costs incurred in the next period associated with any inventory-order pair not matched in the current period. Equivalently, the maximum value of a match in the current period, the maximum difference between the price which some buyer is willing to pay and some seller is willing to accept, equals the present value of the maximum cost of attempting to obtain the match in the next period.

In the sequel we assume that all sellers face the same transaction cost per unit of inventory and all buyers pay the same amount per order in their effort to fill orders. One might argue that sellers and buyers facing the lowest cost would specialize in the exchange activity, if such were not the cases, and the others would purchase the service from them. Equivalently, one may simply view c_1 and c_2 as the prices paid by producers and consumers to the specialized traders. In any case the derivation of the equilibrium distribution which follows can easily be generalized to account for differentials in transactions cost.

Given the assumption, $p_1(t)$ is the ask price common to all sellers and $p_2(t)$ is the common bid price. Since all possible prices are in the interval bounded by these two, they are all acceptable to the traders represented in any local market. Consequently, the number of units of the commodity traded equals the size of the stock assigned to that market at the beginning of the period representing the short side given any price in the interval $(p_2(t), p_1(t))$. Finally, let

$$y(t) = \min [x_1(t), x_2(t)]$$

where $y(t)$ is the quantity traded during period t in any local market characterized by the vector $(x_1(t), x_2(t))$ where $x_1(t)$ represents the inventory assigned to the market and $x_2(t)$ represents the number of orders assigned to the market by the search process at the beginning of period t.

If the inventories assigned to a market exceed orders, then the quantity which sellers prefer to exchange exceeds the quantity which sellers are able to purchase at every feasible price except the ask price. At a price equal to the minimum acceptable to the sellers, the sellers are indifferent between trade and further search. Because buyers are indifferent between trade and search only when the market price equals the bid price, an excess local demand exists at all prices less than the bid price when orders exceed inventory assigned by the search process. When the inventory and orders are equal in the local market any price bounded by the bid and ask price will do. It is convenient to resolve this indeterminancy by assuming that trade takes place at the average price in this case. We maintain, then,

$$p(t) = \begin{cases} p_2(t) & \text{if } x_2(t) > x_1(t) \\ \bar{p}(t) & \text{if } x_2(t) = x_1(t) \\ p_1(t) & \text{if } x_2(t) < x_1(t). \end{cases}$$

as the local market clearing price rule.[12]

In sum, a local market is characterized by the vector $(x_1(t), x_2(t))$ in the sense that both the quantity traded and the price at which they are traded in period t are determined by these given the bid and ask price. Since the process determining this vector is random, so are price and quantity prior to search. Specifically, the joint distribution of the stocks assigned to any market is a bivariate binomial since each unit in the aggregate stock of either type is assigned to each local market independently with the same probability equal to $1/m$.

$$\begin{aligned} pr\{(x_1(t), x_2(t)) \leq (x_1, x_2)\} &= G(x_1, x_2; \lambda_1(t), \lambda_2(t)) \\ &= B(x_1; \tfrac{1}{m}, m\lambda_1(t))\, B(x_2; \tfrac{1}{m}, m\lambda_2(t)) \end{aligned} \tag{10}$$

where B denotes the binomial distribution functions and $\lambda_1(t)$ and $\lambda_2(t)$ denotes the aggregate stock of inventory per market and of orders per market respectively.

Given (10), we are able to specify the nature of the equilibrium price distribution. Let

$$f_1(\lambda_1, \lambda_2) = Pr\{x_2 > x_1\} = \int_{x_2 > x_1} dG(x_1, x_2; \lambda_1, \lambda_2) \tag{11a}$$

$$f_2(\lambda_1, \lambda_2) = Pr\{x_2 < x_1\} = \int_{x_2 < x_1} dG(x_1, x_2; \lambda_1, \lambda_2). \tag{11b}$$

The distribution, then, is

$$dF(p; \Omega_t) = \begin{cases} f_1(\lambda_1(t), \lambda_2(t)) & \text{when } p = p_2(t) \\ 1 - f_1(\lambda_1(t), \lambda_2(t)) - f_2(\lambda_1(t), \lambda_2(t)) & \text{when } p = \bar{p}(t). \\ f_2(\lambda_1(t), \lambda_2(t)) & \text{when } p = p_1(t) \end{cases}$$

Consequently,

$$\bar{p}(t) = E\{p \mid \Omega_t\} = g(\lambda_1(t), \lambda_2(t))\ p_2(t) + (1 - g(\lambda_1(t), \lambda_2(t))p_1(t) \tag{12}$$

where

$$g(\lambda_1(t), \lambda_2(t)) = \frac{f_1(\lambda_1(t), \lambda_2(t))}{f_1(\lambda_1(t), \lambda_2(t)) + f_2(\lambda_1(t), \lambda_2(t))} \tag{13}$$

In words, the equilibrium expected price is a weighted average of the bid and ask prices with weights equal to the conditional own probability that that reservation price will prevail given that one or the other will prevail respectively.

The expected quantity traded in any market during period t is defined as follows

$$\begin{aligned} f(\lambda_1(t), \lambda_2(t)) &= E\{y(t) \mid \Omega_t\} \\ &= \int_{(x_1, x_2)} \min (x_1, x_2)\ dG(x_1, x_2; \lambda_1(t), \lambda_2(t)). \end{aligned} \tag{14}$$

Because (λ_1, λ_2) is the expectation of (x_1, x_2), the function $\min(x_1, x_2)$ is concave and the variances of x_1 and x_2 are both zero only when $(\lambda_1, \lambda_2) = 0$, the following inequality holds:

$$f(\lambda_1, \lambda_2) < \min(\lambda_1, \lambda_2)\ \forall\ (\lambda_1, \lambda_2) \neq 0 \tag{15}$$

In other words, the expected number of units of the commodity traded per market is strictly less than either the aggregate stocks of inventory per market or the aggregated stock of unfilled orders. Consequently, if the stocks are positive in period t, they are positive in every subsequent period.

Now, consider the case in which the number of local markets is large. Formally, this case can be approximated by the limiting model obtained by letting the number of markets tend to infinity holding average stocks per market constant. Because $B(x; \frac{1}{m}, m\lambda)$ converges to the Poisson distribution function with mean λ as $m \rightarrow \infty$,

$$dG(x_1, x_2; \lambda_1, \lambda_2) = \left(\frac{e^{-\lambda_1} \lambda_1^{y_1}}{y_1!}\right) \left(\frac{e^{-\lambda_2} \lambda_2^{y_2}}{y_2!}\right)$$

holds as an approximation in the large numbers case.

A number of simplifications result as a consequence of the fact that this function is differentiable with respect to its parameters.[13] First,

$$\frac{\partial f(\lambda_1,\lambda_2)}{\partial \lambda_1} = f_1(\lambda_1,\lambda_2) \tag{16a}$$

$$\frac{\partial f(\lambda_1,\lambda_2)}{\partial \lambda_2} = f_2(\lambda_1,\lambda_2) \tag{16b}$$

where f_1 and f_2 are the functions defined in (11). In other words, the marginal additions to the expected number of units exchanged per period per market attributable to increases in the aggregate average stocks of inventory and orders equal the probability that orders exceed inventory and the probability that inventory exceeds orders in any one of the markets. Second, it can be shown that $f_{11} < 0$, $f_{22} < 0$ and $f_{12} > 0$ where f_{ij} denotes the representative second partial derivative. These facts imply that an increase in aggregate inventory decreases the expected price while an increase in aggregate orders increases the expected price given the bid and ask prices. In particular,

$$\frac{\partial g(\lambda_1,\lambda_2)}{\partial x_1} = g_1(\lambda_1,\lambda_2) < 0 \tag{17a}$$

and

$$\frac{\partial g(\lambda_1,\lambda_2)}{\partial\lambda_2} = g_2(\lambda_1,\lambda_2) > 0 \tag{17b}$$

Finally, the average number of units traded per market equals its expectation due to the law of large numbers.

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} y_i(t) = f(\lambda_1(t),\lambda_2(t)). \tag{18}$$

We are now prepared to state the principal result of the paper. <u>The aggregate stock of inventory and orders per market are sufficient to characterize the state of the system of markets in any period; i.e.</u> $\Omega_t \equiv (\lambda_1(t), \lambda_2(t))$. <u>Morever, if the number of markets in the system is sufficiently large, the process determining the transition from one state to another is approximately deterministic</u>. The first statement asserts that optimal bid and ask prices in any period are both functions of the average stocks per market as of the beginning of the period. The second statement asserts that the stocks in each period are deterministic functions of the stocks in previous periods.

When the first assertion holds, the second is implied by (18). Define

$$s(p_1) = \frac{1}{m} \sum_{j\in N_1} s_j(p_1)$$

and

$$d(\mu-p_2) = \frac{1}{m} \sum_{j\in N_2} d_j(\mu-p_2)$$

as the aggregate flow supply functions per market and the aggregate flow demand function per market where $s_j(\cdot)$, $j \in N_1$, and $d_j(\cdot)$, $j \in N_2$, are the individual functions derived in (7).

The stock flow identities and (18) imply

$$\lambda_1(t+1) = \lambda_1(t) + s(p_1(t)) - f(\lambda_1(t),\lambda_2(t)) \tag{19a}$$

and

$$\lambda_2(t+1) = \lambda_2(t) + d(\mu-p_2(t)) - f(\lambda_1(t),\lambda_2(t)). \tag{19b}$$

Consequently, if $\Omega_t = (\lambda_1(t)_1\lambda_2(t))$, then the state in period t+1 is determined by the difference equation system (19) since both $p_1(t)$ and $p_2(t)$ are functions of Ω_t.

The equations of (8), equations (9) and (12), and the fact that (19) is a deterministic difference equation imply

$$p_1(t+1) + g(\lambda_1(t+1), \lambda_2(t+1)) \beta(r\mu + c_1 + c_2) = (1+r)p_1(t) + c_1 \tag{20a}$$

and

$$p_2(t+1) + (1 - g(\lambda_1(t+1),\lambda_2(t+1))) \beta(r\mu + c_1 + c_2) = (1 + r)p_2(t) - c_2 - r\mu \tag{20b}$$

if $\Omega_t = (\lambda_1(t),\lambda_2(t))$. To complete the proof one must show that the solution to (19) and (20) satisfying

$$\lim_{t\to\infty} p_1(t) \beta^t = 0 \tag{21a}$$

and

$$\lim_{t\to\infty} (\mu - p_2(t)) \beta^t = 0, \tag{21b}$$

the necessary optimal conditions (4d) and (5d) respectively, can be characterized as a function mapping the set of possible stocks to the set of possible reservation prices. Establishing this fact is the purpose of the next section.

A few comments concerning the assumptions underlying our market model are in order before proceeding. The local market clearing assumption is, of course, an abstraction of the outcome of what one might more realistically regard as a bargaining process. Some will object to the assumption nevertheless on the grounds that we have simply substituted an army of auctioneers for the single one common in the standard competitive market model. Although there is substance to this criticism, one should note that the information gathering, computation and matching functions implicitly attributed to each of the many auctioneers of our model are considerably simpler than those which a central acutioneer must perform. Indeed, it is not unreasonable to imagine that these functions could be profitably performed by private brokers who charge a fee but compete among one another.

As a consequence of the decentralization of exchange activities and of random search, matching is imperfect and price differentials across the local markets persists. As a result, the exchange process is "inefficient" in the standard sense; expost in each period Pareto superior trades still exist. However, the efficiency losses induced by decentralization are balanced to some extent by the saving in information gathering, computation and matching costs attributable to the decentralization. Such is our working hypothesis concerning the reason for the existence of multi-market systems of exchange for the same commodity.

Although the assumption that all traders search by selecting markets at random can be rationalized as a non-cooperative solution to the game of finding trading partners,[14] superior cooperative solutions exist. An example of such a solution is the case in which exactly $\frac{1}{m}$ of the stock held by each trader is assigned to each market. In this case there is no efficiency loss due to mismatching. However, such cooperative solutions are not in the spirit of competitive analysis. Moreover, casual observation suggests that random search is likely to be a better explanation of observed behavior.

4) A Theory of Competitive Price Adjustment

In this section we complete the proof of the assertion that the values of the inventory and order stocks completely characterize the state of the market in any period. Specifically, the solution to (19) - (21) defines a function mapping pairs of stocks to pairs of reservation prices. In the process of this proof we also derive the rules which tie prices and stocks in contiguous periods together. These rules specify the equilibrium market dynamics implied by our model. This process can be characterized as a generalization of the proposition that prices adjust in response to supply and demand pressure. We show that this statement has meaning even though "the price" is a distribution in our model.

For our purpose it is convenient to transform the equations of (20) into a single equation involving the means of the sequence of price distributions. By multiplying (20a) by $1 - g(\lambda_1(t), \lambda_2(t))$ and (20b) by $g(\lambda_1(t), \lambda_2(t))$ and then adding the two results, we obtain

$$\overline{p}(t+1) = (1+r)\overline{p}(t) + c_1 - g(\lambda_1(t), \lambda_2(t))\ (r\mu + c_1 + c_2) \tag{22a}$$

by virtue of equation (12) and the fact that the left sides of both (20a) and (20b) equal $\overline{p}(t+1)$. Then, given the latter fact, we can eliminate $p_1(t)$ in (19a) using (21a) and $p_2(t)$ in (19b) using (21b). The results can be written as

$$\lambda_1(t+1) = \lambda_1(t) + s(\beta(\overline{p}(t+1) - c_1)) - f(\lambda_1(t)_1\lambda_2(t)) \tag{22b}$$

and

$$\lambda_2(t+1) = \lambda_2(t) + d(\beta(\mu - c_2 - \overline{p}(t+1)) - f(\lambda_1(t)_1\lambda_2(t)) \tag{22c}$$

Since both reservation prices are functions of the expected price and the two stocks by virtue of (12) and (9), it suffices to show that the solution to these three equations satisfying

$$\lim_{t\to\infty} \beta^t \bar{p}(t) = 0 \tag{22d}$$

can be characterized as a function θ such that

$$\bar{p}(t) = \theta(\lambda_1(t), \lambda_2(t)). \tag{23}$$

We demonstrate that such a function exists with the aid of the phase diagrams in Figure 1. The slopes of the singular curves depicted and the directions of motion indicated in the diagrams are implied by the following sign restrictions on the various partial derivatives of the system (22):

$$\frac{\partial \Delta \bar{p}}{\partial \bar{p}} = r > 0 \tag{24a}$$

$$\frac{\partial \Delta \lambda_1}{\partial \bar{p}} = s'/(1+r) > 0 \tag{24b}$$

$$\frac{\partial \Delta \lambda_2}{\partial \bar{p}} = -d'/(1+r) < 0 \tag{24c}$$

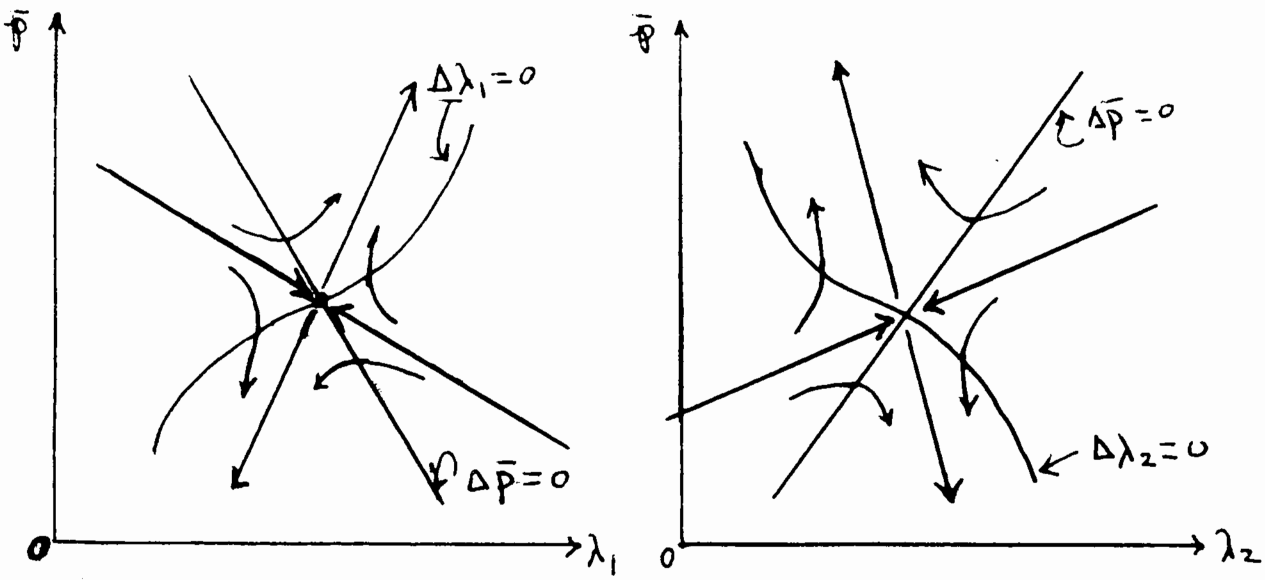
$$\left.\frac{\partial \bar{p}}{\partial \lambda_1}\right|_{\Delta\lambda_1 = 0} = \frac{f_1}{s'} + \left(\frac{r}{1+r}\right)\left(\frac{\alpha g_1}{r}\right) > \frac{\alpha g_1}{r} = \left.\frac{\partial \bar{p}}{\partial \lambda_1}\right|_{\Delta\bar{p} = 0} < 0 \tag{24d}$$

$$\left.\frac{\partial \bar{p}}{\partial \lambda_2}\right|_{\Delta\lambda_2 = 0} = -\frac{f_2}{d'} + \left(\frac{r}{1+r}\right)\left(\frac{\alpha g_2}{r}\right) < \frac{\alpha g_2}{r} = \left.\frac{\partial \bar{p}}{\partial \lambda_2}\right|_{\Delta\bar{p} = 0} > 0 \tag{24e}$$

where $\alpha = (r\mu + c_1 + c_2)$ and Δ is the first difference operator. The sign restrictions follow from the fact that α and r are positive, the aggregate flow supply and demand functions are both increasing, the partial derivatives of the function $f(\cdot)$ are both probabilities and the partials of the function $g(\cdot)$ are negative and positive respectively.

Obviously, the singular curves in each diagram intersect once. For now we will assume that simultaneous solutions to $\Delta\bar{p} = 0$ and $\Delta\lambda_1 = 0$, on the one hand and to $\Delta\bar{p} = 0$ and $\Delta\lambda_2 = 0$, on the other, exist. Later we derive the condition which insures existence in both cases.

Figure 1a Figure 1b



Because both phase diagrams exhibit a "saddle point" configuration, a two dimensional surface exists in the $\bar{p} x \lambda_1 x \lambda_2$ space which corresponds in the diagrams to the "stable" trajectories. This surface has the property that all solutions to the systems (22a) - (22c) that originate on the surface remain on it in all subsequent periods. Moreover, if a steady state solution, a point $(p^*, \lambda_1^*, \lambda_2^*)$ satisfying $\Delta\lambda_1 = \Delta\lambda_2 = \Delta\bar{p} = 0$, exists, the solution converges to it. Consequently, (22d) is satisfied by all such solutions. This surface is the sought for function θ in equation (23).

The stable trajectories in Figure 1a represents the curve made by intersecting the surface $\bar{p} = \theta(\lambda_1, \lambda_2)$ with a vertical plane oriented along the λ_1 direction. Similarly the stable trajectories in Figure 1b is the one dimensional projection of the surface on a plane lying in the λ_2 direction. Therefore,

$$\frac{\partial\theta(\lambda_1, \lambda_2)}{\partial\lambda_1} = \theta_1(\cdot) \tag{25c}$$

$$\frac{\partial\theta(\lambda_1, \lambda_2)}{\partial\lambda_2} = \theta_2(\cdot) \tag{25b}$$

We find, then, that the expected market price decreases with aggregate inventory, holding orders constant, and increases with the stock of aggregate orders holding inventory constant. This fact, of course, reflects the behavior of the individual traders. Since stocks adjust with a lag, the relative magnitudes of the two stocks in the current period will persist into the near future. This fact is known to the traders given rational expectations, and is used by them in making their future price forcasts. When future aggregate inventory stocks are large relative to unfilled order stocks, future prices will be depressed given the nature of the local market clearing rule. But expectations of low prices in the near future induce traders acting optimally to set low reservation prices in the current period. Since the expected price in the current period is a weighted average of the current bid and ask price, it too is lower than it would be if the relative relationship between the current values of the two stocks were reversed.

Given (23), equations (22b) and (22c) imply that

$$\Delta\bar{p} = \theta_1 \Delta\lambda_1 + \theta_2 \Delta\lambda_2 = \theta_1(s - f) + \theta_2(d - f) \tag{26}$$

holds as an approximation for small changes in the stocks. Consequently, the values of the partial derivatives of the function θ represent the direction and speed with which the average price in the system of markets changes in response to flow supply and demand signals. Specifically, an increase in the difference between the flow supply rate per market and the average rate at which trade is taking place in the local markets dampens the rate at which the average price is changing. An increase in the difference between the flow demand rate per market and the average transaction rate per market increases the rate of price change. Finally, (26) reduces to the special case in which the rate of change in average price is proportional to excess flow demand when the partial derivatitives of θ are equal in absolute value. In this sense (26) is a generalization of the "law of supply and demand."

Since $\Delta p = \Delta\lambda_1 = \Delta\lambda_2 = 0$ and (22) imply that the steady state expected price is such that

$$s(\beta(\bar{p}^* - c_1)) = d(\beta(\mu - c_2 - \bar{p}^*)), \tag{27}$$

it is market clearing in the sense of the standard competitive market model. Because s(.) and d(.) are both continuous increasing functions, a unique market clearing expected price exists satisfying $0 < \bar{p}^* < \mu$ if and only if

$$\mu > c_1 + c_2 \tag{28}$$

Condition (28) requires that the value of the commodity to the ultimate consumer be sufficient to cover transaction costs. It is a relatively simple matter to show that a unique positive steady state stock pair $(\lambda_1^*, \lambda_2^*)$ exists given (28).

As S. Grossman [7] points out, the market is solving a control problem.[15] In the special case of an indefinitely large number of traders and markets the aggregate control problem reduces to a simple deterministic dynamic programming problem even though each individual trader cannot know the prices at which he will be able to trade in the future with certainty. In the more realistic case of a large but finite number of traders and markets, the stocks in period $t + 1$ from the viewpoint of period t are random because the average number of units traded per market is random. But, the distribution describing the latter depends only on the stocks. Therefore, the market solution is a first order Markov process, the simplest stochastic generalization of our deterministic difference equation. However, if external stochastic shocks are also added, the market must solve a more complicated stochastic control problem. Nevertheless, the dynamic properties of our special case underlies all of these more general versions of the model.

5) Conclusion: An Answer to the Central Question

We have argued that speculative behavior on the part of individual traders generates a process that eventually finds market clearing prices if trading is viewed as a sequence of more or less random meetings among traders, if exchange among those that meet satisfies some rather reasonable conditions and if the traders have sufficient information about the manner in which the process functions as well as its state at each stage. Moreover, price adjustments made during the process are in response to supply and demand signals even though there is no central auctioneer who performs the explicit act of setting prices according to such a rule. Underlying this aggregate characterization is the fact that current prices at which trade takes place are determined by the individual traders' reservation prices, the reservation prices depend on the current levels of speculation stocks and the changes in stocks reflect supply and demand decisions as well as the current rate at which exchange is taking place.

The complication which we have added relative to the standard approach to modelling competitive markets is that decentralized exchange over time generates a distribution of trading prices rather than a single price for each commodity. Because price dispersion is necessary as a motive for speculative behavior, this fact plays a crucial role in the derivation of the price adjustment process. Finally, the existence of price dispersion and, therefore, lagged price adjustment is due to the fact that search and exchange activities require time. Equivalently, both exist if and only if instantaneous search and exchange are too expensive. We conclude by presenting a formal illustration of the sense in which this last statement is true.

Let h denote the length of a market period, which we now regard as a variable. As before, it is both the inverse of the frequency of search and the interval during which exchange activities occur in each local market. Because the value of the commodity to the ultimate consumer, μ, has no time dimension but the interest rate does, the difference between the bid and ask price can be expressed as

$$p_2 - p_1 = \frac{1}{1+rh}\left(rh\mu + c_1(h) + c_2(h)\right) \tag{29}$$

by virtue of (12) where r denotes the interest rate per unit time period and $c_1(h)$ and $c_2(h)$ are the transactions cost per market searched incurred by the typical seller and buyer respectively expressed as function of the time required to search and trade in a market.

Consider first the case in which there are no transactions cost; i.e., $c_1(h) + c_2(h) \equiv 0$. Then, as search becomes instantaneous $(h \to 0)$, price dispersion vanishes by virtue of (29). One can establish that both the bid and ask price converge to the competitive equilibrium, the solution to

$$s(p^*) = d(\mu - p^*).$$

with the following argument. The average number of units traded per local market, $f(\lambda_1, \lambda_2)$, is independent of the length of the market interval and is positive if stocks are positive. Since both the flow supply and demand per market period are proportional to the length of the market interval, the rate of change in both stocks tends to negative infinity as search becomes instantaneous. In other words, no stocks are held in the limit which implies that the supply and demand flows are instantaneously matched at a price which equates the two.

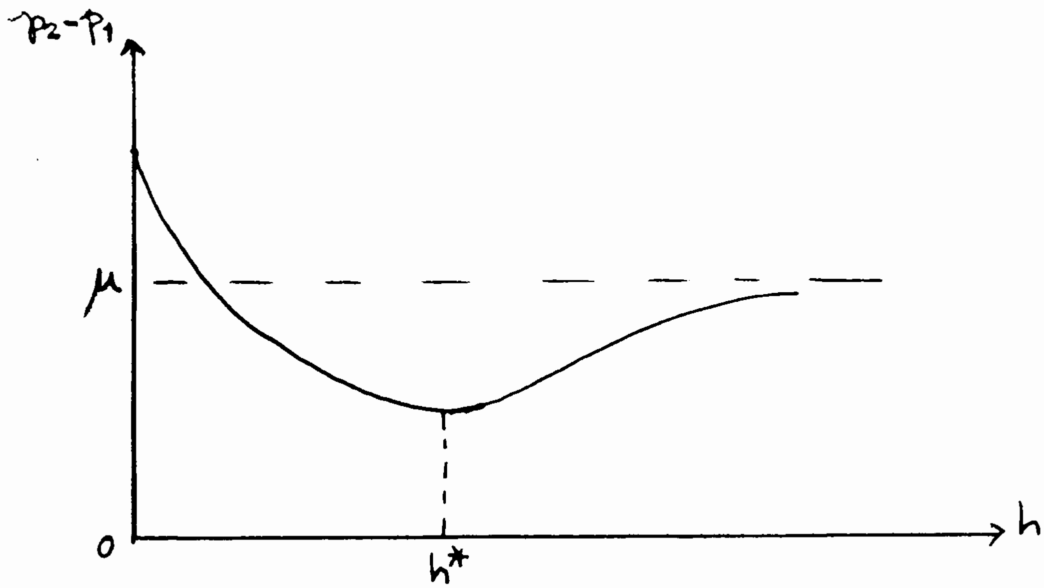
Of course, no formal distinction exists between a decentralized exchange system and one in which all traders exchange simultaneously when search is instantaneous. Consequently, no price differential for the same commodity can exist. Moreover, as the interval during which prices are fixed vanishes, so does the lag in the price adjustment process. For these reasons the model reduces to the standard one in which a single price continuously equates flow supply and demand in the absense of transactions costs.

Now, let us consider the more realistic case in which transactions costs per market interval increase with search frequency. Instantaneous search and exchange are prohibitively expensive in this case when they are such that no trade takes place given a market period of infinitesmal length. Since in the steady state, equation (27) implies

$$s(p_1^*) = d(\mu - p_2^*) \qquad (30)$$

such a situation obtains when $\mu < c_1(0) + c_2(0)$ if $s(0) = d(0) = 0$. If the sum of transactions costs $c_1(h) + c_2(h)$ fall with the length of the market interval, then the total costs of failing to match, the right side of (29), first decreases as h increases from zero and then eventually limits to μ. If in some range trade is possible; i.e. $\mu < c_1(h) + c_2(h)$ for some h, then the relationship between the range of the equilibrium price distribution, $p_2 - p_1$, and the length of the market period, h, is that depicted in Figure 2.

Figure 2



Note that the particular interval h* minimizes the maximum possible price differential between markets and, equivalently, the total cost of waiting and trading. By virtue of (30), h* also maximizes the steady state rate at which trade takes place. Of course, at h* the addition to the cost of waiting attributable to lengthening the market period is just balanced by the saving in transaction cost. The fact that the optimal length of the period is positive suggests the following answer to the principal question.

The need to economize on the cost of waiting to trade, on the one hand, and on the cost of trading, on the other, can explain both exchange at disequilibrium prices and lagged price adjustment in a competitive market context.

APPENDIX

The equations of (16) and (17) are based on the following fact:

Lemma: If x is an n vector of non-negative random integers distributed according to the multivariate Poisson with mean vector equal to λ, then for and real valued function $\varphi(x)$

$$\frac{\partial E\varphi(x)}{\partial \lambda_i} = E\Delta_i \varphi(x)$$

where λ_i is the expectation of x_i and $\Delta_i\varphi(x) \equiv \varphi(x_1, \ldots, x_{i+1}, \ldots x_n) - \varphi(x_1, \ldots, x_i, \ldots, x_n)$.

Proof: Let $\hat{x}$ denote the vector x formed by deleting x_i so that $x = (\hat{x}, x_i)$. Then,

$$p(x) = \prod_{j=1}^{n} \frac{e^{-\lambda_j} \lambda_j^{x_j}}{x_j!} = \frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!} \cdot p(\hat{x})$$

is the probability of x under the hypothesis. Consequently,

$$E\varphi(x) = \sum_x f(x) p(x) = \sum_{\hat{x}} \sum_{x_i=0}^{\infty} \varphi(\hat{x}, x_i) \frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!} p(\hat{x})$$

so that

$$\frac{\partial E\varphi(x)}{\partial \lambda_i} = \sum_{\hat{x}} \sum_{x_i=0}^{\infty} \varphi(\hat{x}, x_i) \left[\frac{x_i e^{-\lambda_i} \lambda_i^{x_i - 1}}{x_i!} - \frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!}\right] p(\hat{x})$$

$$= \sum_{\hat{x}} \sum_{x_i=0}^{\infty} \varphi(\hat{x}, x_i+1) \left[\frac{(x_i+1) e^{-\lambda_i} \lambda_i^{x_i}}{(x_i+1)!}\right] p(\hat{x}) - \sum_{\hat{x}} \sum_{x_i=0}^{\infty} \varphi(\hat{x}, x_i) \left[\frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!}\right] p(\hat{x})$$

$$= \sum_{\hat{x}} \sum_{x_i=0}^{\infty} [\varphi(\hat{x}, x_i+1) - \varphi(\hat{x}, x_i)] \; p(\hat{x}, x_i)$$

$$= \sum_x \Delta_i \varphi(x) \; p(x). \quad \text{Q.E.D.}$$

In the case of our model, $f(\lambda_1, \lambda_2) = E\varphi(x_1, x_2)$ where

$\varphi(x_1, x_2) = \min(x_1, x_2)$.

Since x_1 and x_2 are both integers,

$$\Delta_1\varphi(x) = \min(x_1+1, x_2) - \min(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \geq x_2 \\ 1 & \text{if } x_1 < x_2 \end{cases}.$$

Therefore,

$$\frac{\partial f}{\partial \lambda_1} = \frac{\partial E\varphi(x)}{\partial \lambda_1} = \sum_{x_1 < x_2} p(x) = pr\{x_1 < x_2\} = f_1(\lambda_1, \lambda_2)$$

as asserted in (16a). By virtue of the symmetry, (16b) holds as well.

To obtain the second partial derivatives of $f(\lambda_1, \lambda_2)$, we use the lemma again to obtain

$$\frac{\partial f_i}{\partial \lambda_j} = \frac{\partial E\Delta_i\varphi(x)}{\partial \lambda_j} = E\,\Delta_i\Delta_j\varphi(x).$$

Since $\Delta_1\Delta_1\varphi(x) = \begin{cases} -1 & \text{if } x_1+1 = x_2 \\ 0 & \text{otherwise} \end{cases}$

$$\Delta_1\Delta_2\varphi(x) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_2\Delta_2\varphi(x) = \begin{cases} -1 & \text{if } x_2+1 = x_1 \\ 0 & \text{otherwise,} \end{cases}$$

$f_{11}(\lambda_1, \lambda_2) < 0$, $f_{22}(\lambda_1, \lambda_2) < 0$ and $f_{12}(\lambda_1, \lambda_2) = f_2(\lambda_1, \lambda_2) > 0$ as asserted in the text. These signs and (13) imply the results in (18).

FOOTNOTES

1/ See Arrow [1] for a discussion of this point.

2/ See Arrow and Hahn [2], Chapter 13, for a review of this literature.

3/ This literature includes [3], [4], [5] and [6].

4/ In general, the interval required to search a market and the interval between exchange agreements in each market need not be equal as we have assumed. If the latter is longer than the former, each trader can choose among a set of known prices rather than simply choosing to accept or reject a single price. The derivation of the distribution of prices is much more complicated in this more realistic formulation.

5/ An order is viewed as a promise to purchase one unit at any price less or equal to the buyer's own bid price. In the exchange process a unit of "inventory" can be analogously viewed as a promise to sell at any price equal to or greater than the seller's own ask price. We require that these promises be backed by an ability to fulfill them in each case.

6/ As we shall see later, rationing does occur in local markets. However, the assumed pricing rule is such that any rationed trader is indifferent between exchange and further price search. Consequently, the individual trader's optimal decision rule is unaffected by rationing.

7/ If H_t is differentiable, the equations of (4) can be obtained directly. A more complicated argument is needed to obtain the conditions is general.

8/ In many applications it is reasonable to suppose that the used stock of the commodity is a determinate of μ; e.g. automobiles or any other consumer durable. If one were to take account of this complication, another stock-flow process would enter the model.

9/ The comment in footnote 6 applies here as well.

10/ The definition of market equilibrium used in this section is analogous to that introduced by Lucas and Prescott [8].

11/ If the two traders also face different interest rates, this condition must be modified slightly.

12/ These rules can be generalized in the obvious way to account for differences either in bid prices across buyers or in ask prices across sellers without affecting the validity of the comment in footnote 6.

13/ See the Appendix for the derivation of (16) and (17).

14/ See Mortensen [10].

15/ In our particular case, the expected present value of future sales to the ultimate consumers less transactions, ordering and production costs is maximum given the assumptions regarding the nature of the search process and the rules of exchange in each market.

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