Discussion Paper No. 1227

Quitting Games

Eilon Solan† and Nicolas Vieille‡

September 28, 1998

Math Center web site:
http://www.kellogg.nwu.edu/research/math

Abstract

Quitting games are sequential games in which, at any stage, each player has the choice between *continuing* and *quitting*. The game ends as soon as *at least* one player chooses to quit; player i then receives a payoff r^i_S, which depends on the set S of players that did choose to quit. If the game never ends, the payoff to each player is 0.

We prove the existence of cyclic ε-equilibrium under some assumptions on the payoff function (r_s). We prove on an example that our result is essentially optimal. We also discuss the relation to Dynkin's stopping games, and provide a generalization of our result to these games.

† MEDS Department, Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Rd., Evanston IL 60208.

‡ CEREMADE, Université Paris 9-Dauphine, Place de Lattre de Tassigny, 75 016 Paris, France.

The first author thanks the hospitality of the Center for Rationality and Interactive Decision Theory and of the Laboratoire d'Econométrie de l'Ecole Polytechnique, and the financial support of the Institute of Mathematics at the Hebrew University of Jerusalem.

Introduction

Quitting games are sequential games in which, at any stage, each player has the choice between *continuing* and *quitting*. The game ends as soon as *at least* one player chooses to quit; player i then receives a payoff r^i_S, which depends on the set S of players that did choose to quit. If the game never ends, the payoff to each player is 0.

We address here the problem of existence of ϵ-equilibrium in such games. In the case of *two* players, stationary ϵ-equilibria do exist. A *three*-player example was devised by Flesch, Thuijsman and Vrieze [4], where ϵ-equilibrium strategies are more complex - they have a *cyclic* structure, and the length of the cycle is at least 3. This gave the impetus to the study of the three-player case, solved by Solan [11] (for a more general class of games). We prove here the existence of cyclic ϵ-equilibria under some assumptions on the payoff function.

Quitting games form a class of stochastic games. More precisely, they are both *recursive* games (in the sense of Everett [3]) and repeated games with absorbing states. They are also deeply related to Dynkin's stopping games. The latter are two-player, zero-sum games, where the players choose stopping times τ_1 and τ_2, and the payoff is given by $X_{\tau_1} 1_{\tau_1 < \tau_2} + Y_{\tau_2} 1_{\tau_1 > \tau_2} + W_{\tau_1} 1_{\tau_1 = \tau_2}$, where $(X_n), (Y_n), (W_n)$ are processes. Dynkin [2] gave sufficient conditions on these processes for the game to have a value. Subsequently, some classes of two-player, non zero-sum games were analyzed (Morimoto [9], Ohtsubo[10]). Thus, we deal here with *constant* payoff processes, and allow for *randomized* stopping times. This enables us to deal with any number of players, and different sets of assumptions on payoffs. We also provide an extension of our result to general payoff processes.

Quitting games are a variant of the popular attrition models, first introduced in evolutionary biology, also used in auction theory and entry deterrence models (we refer to Hammerstein and Selten [6] and Wilson [12] for references), or in the analysis of strategic exit (see Ghemawat and Nalebuff [5] or Li [8]). Some minor differences are that attrition models are usually continuous-time models, in which strategic interaction lasts as long as two players at least did not quit. The major departure point is that papers on attrition models have most of the time dealt with incomplete information situations, and focused on the existence of equilibria for a *given* discounting function, whereas our cyclic ϵ-equilibria would be ϵ'-equilibria in any discounted version of the quitting game, provided the discounting be low enough.

The model is set up in section 1, and the results proved in section 2. In section 3, we study a 4-player game, and show that our result is in some sense optimal.

1 The Model and the Main Results

A *quitting game* is a pair $(N, (r_S)_{\emptyset \subset S \subseteq N})$, where (i) $N = \{1, \ldots, N\}$ is a finite set of players, and (ii) for every $\emptyset \subset S \subseteq N$, $r_S \in \mathbf{R}^N$.

The game is a sequential game, that is played as follows. The set of stages is $\mathbf{N}$. At every stage each player chooses an action, either *continue* or *quit*. Let S be the subset of the players who chose to quit. If $S \neq \emptyset$, then the game terminates, and each player i receives the payoff r^i_S. If $S = \emptyset$, the game continues to the next stage. If the game never terminates, each player gets 0.

We denote the two actions of player i by $\{c^i, q^i\}$. A *strategy* for player i is a function $\mathbf{x}^i = (x^i_n)_{i \in \mathbf{N}} : \mathbf{N} \to [0, 1]$, x^i_n being the probability that player i continues at stage n. If $x^i_n = 1$ then at stage n player i plays the pure action c^i, or *continue*, while if $x^i_n = 0$ then at stage n player i plays the pure action q^i, or *quit*. For every stage $n \in \mathbf{N}$, S_n is the set of players that quit at that stage, and a_n is the action combination that is played.

A *profile* is a vector of strategies, one for each player. A profile $\mathbf{x} = (x_n)_{n \in \mathbf{N}}$ induces a probability distribution $\mathbf{P}_{\mathbf{x}}$ over the set of plays. We denote by $\mathbf{E}_{\mathbf{x}}$ the corresponding expectation operator. If the players abide by $\mathbf{x}$, their expected payoff in the game is therefore given by

$$\gamma(\mathbf{x}) = \mathbf{E}_{\mathbf{x}}(r_{S_t} \mathbf{1}_{t < +\infty}),$$

where $t = \inf\{n, S_n \neq \emptyset\}$ is the termination stage.

We say that the profile $\mathbf{x}$ is *terminating* if $\mathbf{P}_{\mathbf{x}}(t < +\infty) = 1$: that is, if the players follow $\mathbf{x}$ then with probability 1 eventually some players quit. This is equivalent to

$$\prod_{n \in \mathbf{N}} \prod_{i \in N} x^i_n = 0.$$

We say that $\mathbf{x}$ is *cyclic* if there exists $n_0 \in \mathbf{N}$ such that $x_n = x_{n+n_0}$ for every $n \in \mathbf{N}$.

As usual, $\mathbf{x}^{-i}$ stands for $(\mathbf{x}^j)_{j\neq i}$. We shall abbreviate similarly whenever convenient. In particular, c^{-i} is the action combination where all players but player i continue.

DEFINITION 1.1 *A profile* $\mathbf{x}$ *is an* ϵ-equilibrium *if for every player* i *and every strategy* $\mathbf{y}^i$ *of player* i,

$$\gamma^i(\mathbf{x}) \geq \gamma^i(\mathbf{x}^{-i}, \mathbf{y}^i) - \epsilon.$$

It is a subgame perfect ϵ*-equilibrium if, for every* $n \in \mathbf{N}$, *the profile* $(x_n, x_{n+1}, \ldots)$ *is an* ϵ*-equilibrium. The corresponding payoff vector* $\gamma(\mathbf{x})$ *is an* ϵ-equilibrium payoff.

Our main result is:

THEOREM 1.2 *Every quitting game that satisfies:*

A.*1.* $r^i_{\{i\}} = 1$ *for every* $i \in N$.

A.*2.* $r^i_S \leq 1$ *for every* S *such that* $i \in S$.

has a cyclic subgame perfect ϵ*-equilibrium.*

Assumption **A**.1 essentially claims that any player prefers his unilateral termination to indefinite continuation. Assumption **A**.2 is somewhat restrictive and can be partially weakened (see Lemma 2.5). It claims that if some player i decides to quit, then he cannot profit if some other player also quits.

2 Existence result

This section is devoted to the proof of Theorem 1.2. It is organized as follows. For every $w \in \mathbf{R}^N$, we define in section 2.1 an associated one-shot game $G(w)$, in which player i receives w^i if termination does not occur (in one stage). Thus, w should be interpreted as continuation payoff. We define an *ad hoc* refinement of ϵ-equilibrium, which we call perfect ϵ-equilibrium.

In section 2.2, we show that there exists a profile $\mathbf{x}$ such that, for every n, x_n is a perfect ϵ-equilibrium in $G(\gamma(x_{n+1}, \ldots))$ and $(x_n, x_{n+1}, \ldots)$ is terminating, and we conclude the proof, *modulo* an additional lemma.

The core of the proof is in section 2.3. We prove there that $\gamma(\mathbf{x})$ is an ϵ'-equilibrium.

2.1 The One-Shot game

Fix a quitting game $G = (N, (r_S)_{\emptyset \subset S \subseteq N})$. Let $\rho = 2\max\{|r^i_S| \mid i \in N, \emptyset \subset S \subseteq N\}$ be twice the maximal payoff in absolute values.

For every $w \in \mathbf{R}^N$ we define a one-shot game $G(w)$ as follows. Each player has two possible actions, *continue* and *quit*. Let S be the subset of players that chose to quit. If $S = \emptyset$, the players receive the payoff w, and otherwise they receive the payoff r_S.

A profile in $G(w)$ is a vector $x \in [0,1]^N$, x^i being the probability that player i chooses *continue*. With every profile x we associate the probability of termination:

$$p(x) = 1 - \prod_{i \in N} x^i$$

and the expected payoff in the one-shot game $G(w)$:

$$\langle G(w), x\rangle = (\prod_{i \in N} x^i)w + \sum_{\emptyset \subset S \subseteq N} (\prod_{i \notin S} x^i)(\prod_{i \in S}(1 - x^i))r_S.$$

Definition 2.1 *A profile x in $G(w)$ is a* perfect *ϵ-equilibrium if it is an ϵ-equilibrium and, for every player i and every action $a^i \in$* supp(x^i)

$$|\langle G(w), x^{-i}, a^i\rangle^i - \langle G(w), x\rangle^i| \leq \epsilon;$$

that is, up to 2ϵ, each player i is indifferent between his actions.

2.2 The Proof

We prove our result, *modulo* the next lemma, whose proof is postponed to section 2.3.

Lemma 2.2 *Let $\mathbf{x} = (x_n)_n$ be a profile in G. Assume that the following properties hold for every $n \geq 1$:*

1. *$(x_n, x_{n+1}, \ldots)$ is terminating;*
2. *x_n is a perfect ϵ-equilibrium of $G(\gamma(x_{n+1}, x_{n+2}, \ldots))$.*

Then $\mathbf{x}$ is an $\epsilon^{\frac{1}{6}}$-equilibrium, or there is a stationary $\epsilon^{\frac{1}{6}}$-equilibrium.

We start with a lemma about correspondences.

LEMMA 2.3 *Let $\psi : K \to K$ be an upper-semi-continuous correspondence with non-empty values defined over a compact space K. Then there exists a sequence $k_1, k_2, \ldots \in K$ such that $k_i \in \psi(k_{i+1})$ for every $i \in \mathbf{N}$.*

Proof: Define $K_0 = K$ and $K_i = \psi(K_{i-1}) = \cup_{k \in K_{i-1}} \psi(k)$ for every $i \in \mathbf{N}$. Since ψ has non-empty values, K_i is non empty, and since ψ is upper-semi-continuous and K compact, K_i is compact. Clearly $K_i \subseteq K_{i-1}$, hence $K_\infty = \cap_{i \in N} K_i$ is non-empty.

Choose $k_1 \in K_\infty$. In particular, $k_1 \in K_i$ for every i, and therefore for every i there exists a sequence $k_1 = k^i_1, k^i_2, \ldots, k^i_i$ such that $k^i_j \in \psi(k^i_{j+1})$. By taking subsequences, we can assume that the limit $k^\infty_j = \lim_i k^i_j$ exists for every j. Since ψ is upper-semi-continuous, $k^\infty_j \in \psi(k^\infty_{j+1})$, as desired. ■

Let $W \subset \mathbf{R}^N$ be a compact set. Define the correspondence $\psi : W \to W$ by: $\psi(w)$ is the subset of all vectors $\langle G(w), x \rangle$ such that x is a perfect $\rho\epsilon$-equilibrium profile in $G(w)$ that satisfies $\langle G(w), x \rangle \in W$ and $p(x) \geq \epsilon$. Clearly, ψ is upper-semi-continuous.

The next result is an immediate corollary of Lemmas 2.2 and 2.3.

PROPOSITION 2.4 *If there exists a compact set W such that ψ has non-empty values, then the game G has an $(\rho\epsilon)^{\frac{1}{6}}$-equilibrium.*

Proof: Denote by (w_n) the sequence obtained by use of Lemma 2.3. For every $n \in \mathbf{N}$ choose $x_n \in [0, 1]^N$ such that x_n is a perfect $\rho\epsilon$-equilibrium profile in $G(w_{n+1})$ that satisfies

- $\langle G(w_{n+1}), x_n \rangle = w_n$.
- $p(x_n) \geq \epsilon$.

Since $p(x_n) \geq \epsilon$ for every n, it follows that for every $n \in \mathbf{N}$, $\gamma(x_n, x_{n+1}, \ldots) = w_n$, and that $(x_n, x_{n+1}, \ldots)$ is terminating. The proposition now follows from Lemma 2.2. ■

It is straightforward to check from the proof of Lemma 2.2 that the ϵ'-equilibrium which is built is in fact a subgame perfect ϵ'-equilibrium.

We now explain why the profile can be taken to be cyclic. There are two ways to do this:

(i) Denote the subgame perfect ϵ-equilibrium profile by $\mathbf{x} = (x_1, x_2, \ldots)$, and, for each n, set $\mathbf{x}_n = (x_n, x_{n+1}, \ldots)$. Since W is compact, there exist $n_1 \leq n_2 \in \mathbf{N}$ such that $P_{\mathbf{x}}(t \geq n_2 | t \geq n_1) < 1/\epsilon^2$ and $\| \gamma(\mathbf{x}_{n_2}) - \gamma(\mathbf{x}_{n_1}) \| < \epsilon^2$. Define a cyclic profile $\mathbf{y} = (x_{n_1}, x_{n_1+1}, \ldots, x_{n_2}, x_{n_1}, \ldots)$. It is easy to verify that $\mathbf{y}$ is a subgame perfect 2ϵ-equilibrium profile, provided ϵ is small enough.
(ii) Let $\{W_1, \ldots, W_K\}$ be a partition of W to subsets with diameter smaller than ϵ^2:

$$\| w_1 - w_2 \| < \epsilon^2 \qquad \forall k = 1, \ldots, K, \forall w_1, w_2 \in W_k.$$

For every $k = 1, \ldots, K$, choose one element $w_k \in \cup_{w \in W_k} \psi(w)$. Denote by x_k a perfect $\rho\epsilon$-equilibrium profile in $G(w_k)$, with $\langle G(w_k), x_k \rangle \in W$, and $p(x_k) \geq \epsilon$. Define for *every* $w \in W_k$, $\hat{\psi}(w) = \langle G(w_k), x_k \rangle$, and $x(w) = x_k$. Notice that x_k is a perfect $2\rho\epsilon$-equilibrium in $G(w)$, for every $w \in W_k$, and $\| \hat{\psi}(w) - \langle G(w), x(w) \rangle \| \leq \epsilon^2$.

Since the range of $\hat{\psi}$ is finite, there exist $w_1, \ldots, w_L \in W$ (where $L \leq K$) such that $w_l \in \hat{\psi}(w_{l+1})$ and $w_L \in \hat{\psi}(w_1)$. Define a cyclic profile $\mathbf{x} = (x(w_1), x(w_2), \ldots, x(w_L), x(w_1), \ldots)$. It is not difficult to show that for each n, x_n is a perfect $C\epsilon$-equilibrium in $G(\gamma(x_{n+1}, \ldots))$, where C does not depend on n. Lemma 2.2 may then be applied.

Therefore, the conclusion of Theorem 1.2 holds, provided there exists a set W, such that ψ has non-empty values, whatever be ϵ. We now exhibit such a set W, under conditions **A**.1 and **A**.2. Thus, Theorem 1.2 will be established.

LEMMA 2.5 *Define*

$$W \stackrel{\text{def}}{=} \left\{ w \in [-\rho, \rho]^N \mid \exists i \in N \text{ with } w^i \leq 1 \right\}.$$

If for every $w \in W$ and every equilibrium x in $G(w)$ such that $x \neq (1, \ldots, 1)$, there exists a player $i \in N$ such that $x^i < 1$ and $\langle G(w), x \rangle^i \leq 1$, then ψ has non-empty values.

In other words, the lemma claims that if for any continuation payoff $w \in W$ and every profile in $G(w)$, one of the players who quit with positive probability receives at most 1, then W is the desired set. It is clear that if the game satisfies **A**.1 and **A**.2 then the conditions of Lemma 2.5 hold. However, there are games that do not satisfy **A**.2 but satisfy the condition of Lemma 2.5.

Proof: Let $w \in W$ be arbitrary, and x be an equilibrium profile in $G(w)$. If $x = (1, 1, \ldots, 1)$ then let i be a player with $w^i = 1$. Otherwise, let $i \in N$ such that $x^i < 1$ and $\langle G(w), x\rangle^i \leq 1$. In the latter case, such a player exists by **A**.2. Define a profile x' in $G(w)$ by: $x' = (1-\epsilon)x + \epsilon(x^{-i}, q^i)$. Then $p(x') \geq \epsilon$ and

$$\| \langle G(w), x'\rangle - \langle G(w), x\rangle \| \leq \rho\epsilon. \tag{1}$$

By (1) and since $\langle G(w), (x^{-i}, c^i)\rangle \leq \langle G(w), (x^{-i}, q^i)\rangle$ (with equality if $x^i > 0$) it follows that x' is a perfect $\rho\epsilon$-equilibrium profile in $G(w)$. ■

2.3 Proof of the main lemma

This section contains the proof of Lemma 2.2, which we state again for convenience.

LEMMA 2.6 *Let* $\mathbf{x} = (x_n)_n$ *be a profile in* G. *Assume that the following properties hold for every* $n \geq 1$*:*

1. $(x_n, x_{n+1}, \ldots)$ *is terminating;*

2. x_n *is a perfect* ϵ*-equilibrium of* $G(\gamma(x_{n+1}, x_{n+2}, \ldots))$.

Then $\mathbf{x}$ *is an* $\epsilon^{\frac{1}{6}}$*-equilibrium, or there is a stationary* $\epsilon^{\frac{1}{6}}$*-equilibrium.*

In general, $\mathbf{x}$ needs not be an $\epsilon^{\frac{1}{6}}$-equilibrium. Indeed, consider the following game, where only the payoffs of player 1 appear:

	$1-\epsilon^2$	ϵ^2
$1-\epsilon$	*continue*	5
ϵ	1	1

Figure 1

The stationary profile depicted in Figure 1 yields player 1 a payoff $\frac{1+5\epsilon(1-\epsilon)}{1+\epsilon(1-\epsilon)}$, and hence player 1 receives, up to 6ϵ, the payoff 1 by either quitting or continuing in the one-shot game $G\left(\frac{1+5\epsilon(1-\epsilon)}{1+\epsilon(1-\epsilon)}\right)$. However, if player 1 always continue, then his payoff is 5.

Observe that in this case, player 1 quits with probability ϵ, and player 2 quits with probability ϵ^2. Therefore, if player 2 did continue with probability 1, the expected payoff for the players would not change by much, while the altered profile would be an equilibrium. This idea is fundamental to the proof.

We use the following notations. Given a profile $\mathbf{y} = (y_1, y_2, \ldots)$, we define for each n the profile $\mathbf{y}_n = (y_n, y_{n+1}, \ldots)$. For each player i, $\mathbf{c}^i$ (*resp.* $\mathbf{q}^i$) is the strategy which plays repeatedly c^i (*resp.* q^i); $\mathbf{q}^i_n$ is the strategy which plays c^i up to stage n, and q^i afterwards.

In the proof, we use repeatedly the following facts:

Fact 1. if u is a random variable on $(\Omega, \mathcal{A}, \mathbf{P})$, bounded by ρ, and $A \in \mathcal{A}$,

$$|\mathbf{E}(u)| \leq \rho \mathbf{P}(A^c) + \sup_A |u|.$$

It will mostly be applied to the difference of *two* r.v.

Fact 2. for every profile $\mathbf{y}$, and $n \in \mathbf{N}$,

$$\gamma(\mathbf{y}) = \mathbf{P}_{\mathbf{y}}(t < n)\mathbf{E}_{\mathbf{y}}[r_{S_t} | t < n] + \mathbf{P}_{\mathbf{y}}(t \geq n)\gamma(\mathbf{y}_n).$$

We fix $a, b, e \in]0, 1[$ so that: $e > a > \frac{1}{5}$, $b - e > \frac{1}{5}$, $1 - b - e > \frac{1}{5}$. We will assume that ϵ is small, so that inequalities of the form $(1 - \epsilon^a)^{\frac{1}{\epsilon^e}} \leq \epsilon$ hold.

We partition $\mathbf{N}$ into a sequence $(B_k)_{k \geq 1}$ of blocks. Set $n_1 = 0$ and

$$n_{k+1} = \inf\{n > n_k, \mathbf{P}_{\mathbf{x}}\{t < n_{k+1} | t \geq n_k\} \geq \epsilon^a\}.$$

Set $B_k = \{n_k, \ldots, n_{k+1} - 1\}$. For any $n \leq m \in \mathbf{N}$, we set $p_{\mathbf{x}}[n, m] = \mathbf{P}_{\mathbf{x}}(t \leq m | t \geq n\}(= \mathbf{P}_{\mathbf{x}_n}(t \leq m - n))$, and $p_{\mathbf{x}}[n] = p_{\mathbf{x}}[n, n](= 1 - \prod_i x^i_n)$.

Few comments are in order.

(a) The (B_k) are the "smallest" blocks on which the probability of termination is "non-negligible".

(b) $n_k < +\infty$ for every k since $\mathbf{x}_n$ is terminating for every n: each block consists of finitely many stages.

(c) $p_{\mathbf{x}}[n_k, n_{k+\frac{1}{\epsilon^e}}] \geq 1-\epsilon$, for every k: if termination has not occurred at the beginning of block labelled k, it occurs with high probability before the block labelled $k+\frac{1}{\epsilon^e}$.

(d) there is no estimate available for $p_{\mathbf{x}}[n_{k+1}-1]$. However, for every $n_k \leq n < n_{k+1}-1$, $p_{\mathbf{x}}[n_k, n] < \epsilon^a$.

Fix a player $i \in N$. A block B_k is of type I for player i if $p_{\mathbf{x}^{-i},\mathbf{c}^i}[B_k] \geq \epsilon^b$, and of type II otherwise. Denote by $(l^i_m)_{m\geq 1}$ the successive blocks of type I for i. We prove the following:

- if, for some $i \in N$, $m \in \mathbf{N}^*$, $l^i_{m+1}-l^i_m > \frac{1}{\epsilon^c}$, then $r_{\{i\}}$ is an $\epsilon^{\frac{1}{6}}$-equilibrium payoff, and there is a *stationary* $\epsilon^{\frac{1}{6}}$-equilibrium, with payoff $r_{\{i\}}$;
- otherwise, $\mathbf{x}$ is an $\epsilon^{\frac{1}{6}}$-equilibrium.

Intuitively, a block is of type II to player i if quitting in this block comes mainly from player i. Therefore, if there are many such blocks, then the overall payoff is close to $r_{\{i\}}$.

Case 1: let $i \in N$, $m \in \mathbf{N}^*$, with $l^i_{m+1}-l^i_m > \frac{1}{\epsilon^c}$. Given the time-stationarity of the properties, we may assume, by giving up the first blocks, that the first block l^i_1 of type I for i is such that $l^i_1 > \frac{1}{\epsilon^c}$. We set $l = l^i_1$.

Thus, for each $k < l$

$$p_{\mathbf{x}^{-i},\mathbf{c}^i}[B_k] < \epsilon^b. \tag{2}$$

We prove first that $\gamma(\mathbf{x})$ is close to $r_{\{i\}}$. (2) may be rephrased as

$$\mathbf{P}_{\mathbf{x}}(\exists j \neq i, n_k \leq n < n_{k+1}, a^j_n = q^j) < \epsilon^b. \tag{3}$$

Since $p_{\mathbf{x}}[B_k] \geq \epsilon^a$, one deduces from (3)

$$\mathbf{P}_{\mathbf{x}}\{\exists j \neq i, j \in S_t | t \in B_k\} < \epsilon^{b-a}.$$

Therefore $\mathbf{P}_{\mathbf{x}}\{S_t = \{i\} | t \in B_k\} \geq 1-\epsilon^{b-a}$. By summation over k, $\mathbf{P}_{\mathbf{x}}\{S_t = \{i\} | t < n_l\} \geq 1-\epsilon^{b-a}$.

On the other hand, $\mathbf{P}_{\mathbf{x}}(t \geq n_{k+1} | t \geq n_k) \leq 1-\epsilon^a$. This yields

$$\mathbf{P}_{\mathbf{x}}(t \geq n_l) \leq (1-\epsilon^a)^{\frac{1}{\epsilon^c}} \leq \epsilon.$$

Thus, $\mathbf{P}_{\mathbf{x}}(S_t = \{i\}, t < n_l) \geq 1 - \epsilon - \epsilon^{b-a}$. By Fact 1, one gets

$$\| \gamma(\mathbf{x}) - r_{\{i\}} \| \leq \rho(\epsilon + \epsilon^{b-a}).$$

We now claim that $\| \gamma(\mathbf{x}_2) - r_{\{i\}} \| \leq \rho(\epsilon + \epsilon^{b-a} + \epsilon^a)$. Indeed, we only used above $l \geq \frac{1}{\epsilon^e}$. Since $l > \frac{1}{\epsilon^e}$, $l - 1 \geq \frac{1}{\epsilon^e}$. Thus, the above proof, applied from stage n_2, yields $\| \gamma(\mathbf{x}_{n_2}) - r_{\{i\}} \| \leq \rho(\epsilon + \epsilon^{b-a})$. Now, either $n_2 = 2$, in which case the claim holds, or $n_2 > 2$; in the latter case, $p_{\mathbf{x}}[1] < \epsilon^a$, thus $\| \gamma(\mathbf{x}) - \gamma(\mathbf{x}_2) \| \leq \rho\epsilon^a$ (Fact 2), which proves the claim.

Since x_1 is an ϵ-equilibrium in $G(\gamma(\mathbf{x}_2))$, and $\| x_1 - (c^{-i}, x_1^i) \| < \epsilon^b$, (c^{-i}, x_1^i) is an $(\epsilon + \rho N \epsilon^b)$-equilibrium in $G(\gamma(\mathbf{x}_2))$, hence an η-equilibrium in $G(r_{\{i\}})$ with $\eta = \epsilon + \rho(\epsilon + N\epsilon^b + \epsilon^a + \epsilon^{b-a})$. One deduces easily that $(\mathbf{c}^{-i}, (x_1^i, x_1^i, \ldots))$ is a stationary η-equilibrium of the quitting game G.

Case 2: for every $i \in N$, $m \in \mathbf{N}$, $l_{m+1}^i - l_m^i \leq \frac{1}{\epsilon^e}$.

We prove that $\mathbf{x}$ is an $\epsilon^{\frac{1}{6}}$-equilibrium. Let $i \in N$ be fixed. For simplicity, we write l_m instead of l_m^i.

For each $k \in \mathbf{N}$, we define an auxiliary game Γ_k, played during B_k: it starts in stage n_k, and ends up after stage $n_{k+1} - 1$, with payoffs given by r_{S_t} if $t < n_{k+1}$, and by $\gamma(\mathbf{x}_{n_{k+1}})$ if termination did not occur.

The proof is organized as follows. We first prove that player i cannot profit too much by deviating from $\mathbf{x}_{n_k}^i$ in the game Γ_k. We then aggregate these estimates.

Step 1: an upper bound on the benefit from deviating in Γ_k.

It is obviously enough to deal with the case $k = 1$; estimates for a block B_k would be derived by conditioning on $\{t \geq n_k\}$ whenever appropriate. Only minor changes are needed.

We need to consider only pure strategies of player i. These are of two types: (i) strategy $\mathbf{q}_n^i$ for some $n \in \{0, \ldots, n_2 - 1\}$; (ii) continue in every stage (strategy $\mathbf{c}^i$).

Step 1.1: strategy $\mathbf{q}_n^i$

The payoffs to player i in Γ_1 under $\mathbf{x}$ and $(\mathbf{x}^{-i}, \mathbf{q}_n^i)$ may respectively be written as

$$\begin{aligned} g^i(\mathbf{x}) &= \pi u + (1-\pi)\gamma^i(\mathbf{x}_n) \\ g^i(\mathbf{x}^{-i}, \mathbf{q}_n^i) &= \pi^* u^* + (1-\pi^*)\langle G(\gamma(\mathbf{x}_{n+1})), (x_n^{-i}, q^i)\rangle^i \end{aligned}$$

with $\pi = \mathbf{P}_{\mathbf{x}}(t < n), \pi^* = \mathbf{P}_{\mathbf{x}^{-i}, \mathbf{c}^i}(t < n)$, $u = \mathbf{E}_{\mathbf{x}}[r_{S_t}^i | t < n]$, and $u^* = \mathbf{E}_{\mathbf{x}^{-i}, \mathbf{c}^i}[r_{S_t}^i | t < n]$.

By the perfect ϵ-equilibrium property,

$$\begin{aligned} \langle G(\gamma(\mathbf{x}_{n+1})), (x_n^{-i}, q^i)\rangle^i &\leq \langle G(\gamma(\mathbf{x}_{n+1})), x_n\rangle^i + \epsilon \\ &= \gamma^i(\mathbf{x}_n) + \epsilon. \end{aligned}$$

By construction of the sequence of blocks, $\pi, \pi^* < \epsilon^a$. Therefore,

$$g^i(\mathbf{x}^{-i}, \mathbf{q}_n^i) \leq g^i(\mathbf{x}) + 2\rho\epsilon^a + \epsilon. \tag{4}$$

Step 1.2: strategy $\mathbf{c}^i$

The payoffs to player i under $\mathbf{x}$ and $(\mathbf{x}^{-i}, \mathbf{c}^i)$ may respectively be written

$$\begin{aligned} g^i(\mathbf{x}) &= \pi_1 u_1 + \pi_2 u_2 + (1-\pi_1-\pi_2)\gamma^i(\mathbf{x}_{n_2-1}) \\ g^i(\mathbf{x}^{-i}, \mathbf{c}^i) &= \pi_2^* u_2^* + (1-\pi_2^*)\langle G(\gamma(\mathbf{x}_{n_2})), (x_{n_2-1}^{-i}, c^i)\rangle^i, \end{aligned}$$

with

$$\begin{aligned} \pi_1 &= \mathbf{P}_{\mathbf{x}}(t < n_2 - 1, i \in S_t) \\ u_1 &= \mathbf{E}_{\mathbf{x}}[r_{S_t}^i | t < n_2 - 1, i \in S_t] \\ \pi_2 &= \mathbf{P}_{\mathbf{x}}(t < n_2 - 1, i \notin S_t) \\ u_2 &= \mathbf{E}_{\mathbf{x}}[r_{S_t}^i | t < n_2 - 1, i \notin S_t] \\ \pi_2^* &= \mathbf{P}_{\mathbf{x}^{-i}, \mathbf{c}^i}(t < n_2 - 1) \\ u_2^* &= \mathbf{E}_{\mathbf{x}^{-i}, \mathbf{c}^i}[r_{S_t}^i | t < n_2 - 1] \end{aligned}$$

As above, $\langle G(\gamma(\mathbf{x}_{n_2})), (x_{n_2-1}^{-i}, c^i)\rangle^i \leq \gamma^i(\mathbf{x}_{n_2-1}) + \epsilon$. Thus,

$$g^i(\mathbf{x}^{-i}, \mathbf{c}^i) \leq g^i(\mathbf{x}) + \pi_2^* u_2^* - \pi_2 u_2 + (\pi_2 - \pi_2^*)\gamma^i(\mathbf{x}_{n_2-1}) + \pi_1(\gamma^i(\mathbf{x}_{n_2-1}) - u_1) + \epsilon. \tag{5}$$

Remark: it is *a priori* more natural to write $g^i(\mathbf{x})$ and $g^i(\mathbf{x}^{-i}, \mathbf{c}^i)$ using $\gamma^i(\mathbf{x}_{n_2})$ instead of $\gamma^i(\mathbf{x}_{n_2-1})$. The reason which motivates our choice is that $\pi_1 = \mathbf{P_x}(t < n_2 - 1, i \in S_t) < \epsilon^a$, while no upper bound is available on $\mathbf{P_x}(t < n_2, i \in S_t)$.

We wish to deduce from (5) an estimate of the form $g^i(\mathbf{x}^{-i}, \mathbf{c}^i) \leq g^i(\mathbf{x}) + \epsilon + \eta\pi_2^*$, with η small. It follows from elementary probability theory that $\pi_2^* - \pi_2$ is small compared to π_2^*, and that $u_2^* - u_2$ is small (see Lemma 2.7 below). We also know that π_1 is small. Therefore, the main issue will be to establish that $\gamma^i(\mathbf{x}_{n_2-1}) - u_1$ is at most of the order π_2^*.

We start with a lemma.

LEMMA 2.7 *Let $N \in \mathbf{N}$, and $(X_0, \ldots, X_N, Y_0, \ldots, Y_N)$ be independent $\{0, 1\}$-valued random variables. Let $S_1 = \inf\{n \leq N, X_n = 1\}$, $S_2 = \inf\{n \leq N, Y_n = 1\}$ be the first successes of the two sequences. Set $T = S_1$ if $S_1 < S_2$ and $T = +\infty$ otherwise.*

Assume that $\mathbf{P}(T \leq N) > 0$. Then

1. $\mathbf{P}(S_1 \leq N) - \mathbf{P}(T \leq N) \leq \mathbf{P}(S_1 \leq N)\mathbf{P}(S_2 \leq N)$;
2. $|\mathbf{P}(S_1 = n|S_1 \leq N) - \mathbf{P}(S_1 = n|T \leq N)| \leq 2\mathbf{P}(S_2 \leq N)$.

Proof: for each n, $\{T = n\} \subseteq \{S_1 = n\}$, and $\{S_1 = n\} \setminus \{T = n\} = \{S_1 = n\} \cap \{S_2 \leq n\}$. Thus

$$\mathbf{P}(S_1 = n) - \mathbf{P}(T = n) = \mathbf{P}(S_1 = n)\mathbf{P}(S_2 \leq n) \leq \mathbf{P}(S_1 = n)\mathbf{P}(S_2 \leq N). \tag{6}$$

The first claim follows by summation over n.

On the other hand,

$$|\mathbf{P}(S_1 = n|S_1 \leq N) - \mathbf{P}(S_1 \leq n|T \leq N)| = |\frac{\mathbf{P}(S_1 = n)}{\mathbf{P}(S_1 \leq N)} - \frac{\mathbf{P}(S_1 = n < S_2)}{\mathbf{P}(T \leq N)}|$$

$$= |\frac{\mathbf{P}(S_1 = n)\mathbf{P}(S_2 \leq n)}{\mathbf{P}(S_1 \leq N)} + \mathbf{P}(S_1 = n < S_2)[\frac{1}{\mathbf{P}(S_1 \leq N)} - \frac{1}{\mathbf{P}(T \leq N)}]|.$$

Now, $\frac{\mathbf{P}(S_1 = n)\mathbf{P}(S_2 \leq n)}{\mathbf{P}(S_1 \leq N)} \leq \mathbf{P}(S_2 \leq N)$, and

$$\left|\mathbf{P}(S_1 = n < S_2)[\frac{1}{\mathbf{P}(S_1 \leq N)} - \frac{1}{\mathbf{P}(T \leq N)}]\right|$$

$$= \frac{\mathbf{P}(S_1 = n < S_2)}{\mathbf{P}(S_1 \le N)\mathbf{P}(T \le N)} |\mathbf{P}(T \le N) - \mathbf{P}(S_1 \le N)|$$
$$\le \frac{\mathbf{P}(S_1 = n < S_2)}{\mathbf{P}(S_1 \le N)\mathbf{P}(T \le N)} \times \mathbf{P}(S_2 \le N)\mathbf{P}(S_1 \le N) \le \mathbf{P}(S_2 \le N),$$

using the first part of the lemma. ■

We use this lemma with $N = n_2 - 2$, $\mathbf{P} = \mathbf{P}_\mathbf{x}$, and X_n and Y_n defined as: $X_n = 1$ *iff* at least one player $j \neq i$ quits at stage n, and $Y_n = 1$ *iff* player i quits at stage n.

Thus, $\mathbf{P}(S_1 \le N) = \pi_2^*$, $\{S_2 \le N\} \subseteq \{t < n_2 - 1\}$, and $\{T \le N\} = \{t < n_2 - 1, i \notin S_t\}$. The first assertion of the lemma implies $|\pi_2 - \pi_2^*| \le \epsilon^a \pi_2^*$, while the second implies $|u_2 - u_2^*| \le 2\rho\epsilon^a$ (the lemma can only be applied if $\pi_2^* > 0$; but, when $\pi_2^* = 0$, one has $\pi_2 = 0$, and the inequality (7) below holds trivially).

Therefore,

$$(\pi_2^* - \pi_2)u_2^* + (u_2^* - u_2)\pi_2 + (\pi_2 - \pi_2^*)\gamma^i(\mathbf{x}_{n_2-1}) \le 3\rho\epsilon^a\pi_2^*. \tag{7}$$

Claim: one has

$$\gamma^i(\mathbf{x}_{n_2-1}) - u_1 \le 4\rho N \pi_2^* + \epsilon. \tag{8}$$

Proof:

1- for every n, $1 \le n < n_2$, $\| x_n^{-i} - c^{-i} \| \le \pi_2^*$. Therefore,

$$\mathbf{E}_\mathbf{x}[|r^i_{S_t} - 1| \; |a_t^i = q^i, t < n_2 - 1] \le \rho N \pi_2^*.$$

Thus, $|u_1 - 1| \le \rho N \pi_2^*$.

2- Let $\overline{n}$ be the *last* stage before $n_2 - 1$, for which $x_n^i < 1$. By the perfect ϵ-equilibrium property,

$$\langle G(\gamma(\mathbf{x}_{\overline{n}+1})), (x_{\overline{n}}^{-i}, c^i)\rangle^i \le G(\gamma(\mathbf{x}_{\overline{n}+1})), (x_{\overline{n}}^{-i}, q^i)\rangle^i + \epsilon. \tag{9}$$

Using $\| x_{\overline{n}}^{-i} - c^{-i} \| \le \pi_2^*$, one has

$$|\langle G(\gamma(\mathbf{x}_{\overline{n}+1})), (x_{\overline{n}}^{-i}, c^i)\rangle^i - \gamma^i(\mathbf{x}_{\overline{n}+1})| \le \rho N \pi_2^*,$$

and

$$|\langle G(\gamma(\mathbf{x}_{\overline{n}+1})), (x_{\overline{n}}^{-i}, q^i)\rangle^i - r^i_{\{i\}}| \le \rho N \pi_2^*.$$

Therefore, (9) yields

$$\gamma^i(\mathbf{x}_{\overline{n}+1}) \leq 1 + \epsilon + 2\rho N \pi_2^*.$$

3- By definition of $\overline{n}$, $p_{\mathbf{x}}[\overline{n}+1, n_2 - 2] = p_{\mathbf{x}^{-i},\mathbf{c}^i}[\overline{n}+1, n_2 - 2] \leq \pi_2^*$. Thus, using Fact 2 (applied with $\mathbf{y} = \mathbf{x}_{\overline{n}+1}$),

$$|\gamma^i(\mathbf{x}_{\overline{n}+1}) - \gamma^i(\mathbf{x}_{n_2-1})| \leq \rho \pi_2^*.$$

The claim follows from 1 through 3.

Conclusion: from (5), (7) and (8), one deduces

$$g^i(\mathbf{x}^{-i}, \mathbf{c}^i) \leq g^i(\mathbf{x}) + 7\rho\epsilon^a N \pi_2^* + 2\epsilon. \tag{10}$$

Step 2: global estimates

We define a sequence $(X_k)_{k \in \mathbf{N}^*}$ of random variables by

$$X_k = \begin{cases} \gamma^i(\mathbf{x}_{n_k}) \text{ if } t \geq n_k \\ r^i_{S_t} \text{ if } t < n_k \end{cases}$$

Claim 1: $\sup_k \mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_k] \leq X_1 + 2\epsilon \sum_{k=1}^{\infty} \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n_k) + 7\rho N \epsilon^a$.

Notice that $\mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_k]$ is the payoff to player i if player i follows $\mathbf{c}^i$ up to stage n_k, then $\mathbf{x}^i$, while players $-i$ use $\mathbf{x}^{-i}$.

Proof: The conclusion (10) of step 1 can be rewritten: for every $k \in \mathbf{N}^*$,

$$\mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_{k+1}|t \geq n_k] \leq \mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_k|t \geq n_k] + 7\rho N \epsilon^a \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t < n_{k+1}|t \geq n_k) + 2\epsilon.$$

Of course, $X_{k+1} = X_k$ if $t < n_k$.

Therefore,

$$\mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_{k+1}] \leq \mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_k] + 7\rho N \epsilon^a \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \in B_k) + 2\epsilon \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n_k).$$

Thus,

$$\mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_{k+1}] \leq X_1 + 7\rho N \epsilon^a + 2\epsilon \sum_{p=1}^{k} \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n_p)$$

which yields Claim 1. ■

Claim 2: $\epsilon \sum_{k=1}^{\infty} \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n_k) \leq 2\epsilon^{1-b-c}$.

Proof: by assumption, $l^i_{m+1} - l^i_m \leq \frac{1}{\epsilon^c}$, for every m. Consecutive blocks of type I are never distant by more than $\frac{1}{\epsilon^c}$ blocks. Thus, in the first $\frac{p}{\epsilon^c}$ blocks, there are at least $p-1$ blocks of type I. For such a block k, $\mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}\{t \in B_k | t \geq n_k\} \geq \epsilon^b$, *i.e.* $\mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}\{t \geq n_{k+1} | t \geq n_k\} \leq 1 - \epsilon^b$. Therefore,

$$\mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n_{\frac{p}{\epsilon^c}}) \leq (1-\epsilon^b)^{p-1},$$

from which Claim 2 follows. ■

Conclusion: we now prove that $\gamma^i(\mathbf{x}^{-i}, \mathbf{x}^i_\star) \leq \gamma^i(\mathbf{x}) + \epsilon^{\frac{1}{6}}$, for every *pure* strategy $\mathbf{x}^i_\star$ of player i. There are two cases.

- $\mathbf{x}^i_\star = \mathbf{c}^i$

Since $(\mathbf{x}^{-i}, \mathbf{c}^i)$ is terminating, $(X_k)_k$ converges, $\mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}$-a.s., to $r^i_{S_t}$. Hence

$$\gamma^i(\mathbf{x}^{-i}, \mathbf{c}^i) = \lim_k \mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_k] \leq \gamma^i(\mathbf{x}) + 4\epsilon^{1-b-c} + 7\rho N \epsilon^a.$$

- $\mathbf{x}^i_\star = \mathbf{q}^i_n$, for some $n \in \mathbf{N}^*$.

Let B_k be the block containing n. $\gamma^i(\mathbf{x}^{-i}, \mathbf{q}^i_n)$ can be written

$$\gamma^i(\mathbf{x}^{-i}, \mathbf{q}^i_n) = \mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_k 1_{t<n_k}] + \mathbf{E}_{\mathbf{x}^{-i},\mathbf{q}^i_n}[r^i_{S_t} 1_{t \geq n_k}]. \tag{11}$$

The expression $\mathbf{E}_{\mathbf{x}^{-i},\mathbf{q}^i_n}[r^i_{S_t} | t \geq n_k]$ is the analog for block k of what was called $g^i(\mathbf{x}^{-i}, \mathbf{q}^i_n)$ in step 1.1. Thus (see (4)),

$$\mathbf{E}_{\mathbf{x}^{-i},\mathbf{q}^i_n}[r^i_{S_t} | t \geq n_k] \leq \gamma^i(\mathbf{x}_{n_k}) + \epsilon + 2\rho\epsilon^a.$$

Since $\gamma^i(\mathbf{x}_{n_k})$ may be written $\mathbf{E}_{\mathbf{x}^{-i},\mathbf{q}^i_n}[X_k | t \geq n_k]$, one gets, using (11) and claims 1 and 2

$$\begin{aligned} \gamma^i(\mathbf{x}^{-i}, \mathbf{q}^i_n) &\leq \mathbf{E}_{\mathbf{x}^{-i},\mathbf{c}^i}[X_k] + \epsilon + 2\rho\epsilon^a \\ &\leq \gamma^i(\mathbf{x}) + \epsilon + 9\rho N \epsilon^a + 4\epsilon^{1-b-c}. \end{aligned}$$

■

2.4 General payoff processes

We give here a slight extension of Theorem 1.2, within the framework of non zero-sum Dynkin games. Let $\mathbf{r} = (r_n)_{n\geq 1}$ be a process over $(\Omega, \mathcal{A}, \mathbf{P})$, where $r_n = (r_{n,S})_{\emptyset\neq S\subseteq N}$ is a vector of $\mathbf{R}^N$-valued variables. The quitting game $\Gamma(\mathbf{r})$ is played as above: $r_{n,S}$ is the payoff vector if termination occurs in stage n, and the quitting coalition is S.

Set $\mathcal{H}_n = \sigma(r_p, p \leq n)$. A strategy of player i is a process $\mathbf{x}^i = (x^i_n)_{n\geq 1}$ adapted to $(\mathcal{H}_n)_{n\geq 1}$, with values in $[0,1]$. Provided expectations are well-defined, the extension of the above definitions of γ and ϵ-equilibrium is straightforward.

THEOREM 2.8 *Assume that the sequence (r_n) converges to r_∞, $\mathbf{P}$-a.s. Assume r_∞ satisfies assumptions $\mathbf{A}.1$ and $\mathbf{A}.2$ of Theorem 1.2, $\mathbf{P}$-a.s. Assume $\rho = \mathbf{E}[\sup_{n\geq 1} \parallel r_n \parallel] < +\infty$. Then, for every $\epsilon > 0$, $\Gamma(\mathbf{r})$ has an ϵ-equilibrium.*

Proof: The idea is the following. Take N_0 large enough. At stage N_0, players start using an η-equilibrium of the quitting game with constant payoffs $\Gamma(r_{N_0}, r_{N_0}, \ldots)$. Behavior in the first $N_0 - 1$ stages is then defined by backwards induction.

We address first the measurability issue. Set $m = N(2^N - 1)$. Denote by X the space of strategy profiles in a quitting game with constant payoffs. Denote by $\Delta \subset \mathbf{R}^m$ the set of vectors $r = (r_S)_{\emptyset\neq S\subseteq N}$ which satisfy $\mathbf{A}.1$ and $\mathbf{A}.2$. For $\eta > 0$, and $r \in \mathbf{R}^m$, denote by Δ_η the η-neighborhood of Δ, and by $E_\eta(r) \subseteq X$ the set of η-equilibrium of the quitting game with constant payoffs $\Gamma(r, r, \ldots)$. By Theorem 1.2, $E_\eta(r) \neq \emptyset$, for every $r \in \Delta$, $\eta > 0$.

It is clear that whenever $\parallel r - r' \parallel < \eta$, $E_\eta(r) \subseteq E_{2\eta}(r')$. Therefore, there is a measurable step function $\sigma_\eta : \mathbf{R}^m \to X$, with $\sigma_\eta(r) \in E_{2\eta}(r)$, for every $r \in \Delta_\eta$.

Choose $0 < \eta < \frac{1}{4+\rho}$ and N_0 such that

$$\mathbf{P}(\exists n \geq N_0, \parallel r_n - r_\infty \parallel > \eta) < \eta. \tag{12}$$

We now construct a profile $\mathbf{x}$. Set $\mathbf{x}_{N_0} = (x_{N_0}, x_{N_0+1}, \ldots) = \sigma_\eta(r_{N_0})$. By construction, x_n is $\mathcal{H}_{N_0}$-measurable, for $n \geq N_0$. We define $x_{N_0-1}, \ldots, x_1$

inductively: for $k \leq N_0 - 1$, define x_k to be an $\mathcal{H}_k$-measurable equilibrium in $G(\mathbf{E}[\gamma(\mathbf{x}_{k+1})|\mathcal{H}_k])$.

Such a choice for x_k exists. Indeed, fix a game form, *i.e.* a finite set N of players, and a finite set of actions to each player. Let E be the correspondence which assigns to any payoff function ($\mathbf{R}^N$-valued function defined over the product of action sets) the set of (mixed) Nash equilibria of the corresponding game. Since E is upper-semi-continuous with non-empty values, it has a measurable selection ([7]).

By (12), $\mathbf{P}(\mathbf{x}_{N_0} \in E_{3\eta}(r_{N_0}, r_{N_0+1}, \ldots)) \geq 1 - \eta$: the probability of $\mathbf{x}_{N_0}$ being a 4η-equilibrium is at least the probability that r_n will remain 2η-close to r_{N_0}, which is at least $1 - \eta$ by the choice of N_0.

This easily yields that $\mathbf{x}$ is a $(4\eta + \eta\rho)$-equilibrium of $\Gamma(\mathbf{r})$. ■

3 An Example

The above result asserts the existence of cyclic ϵ-equilibria in a class of quitting games. For two- or three-player games, "better" results are available. For two-player games, stationary ϵ-equilibria exist. For 3-player games, either a stationary ϵ-equilibrium exists, or there exists an ϵ-equilibrium in which the probability of termination in any given stage is arbitrarily small (therefore, there is an equilibrium payoff in $\text{co}\{r_{\{i\}}, i \in N\}$), or both. The purpose of this section is to show on an example that this is no longer true for 4-player games. In that sense, our result is optimal.

We study the game

4

		2				**2**
1	*continue*	4, 1, 0, 0		**1**	0, 0, 4, 1	1, 1, 0, 1
	1, 4, 0, 0	1, 1, 1, 1			1, 0, 1, 1	0, 1, 0, 0
3						
1	0, 0, 1, 4	0, 1, 1, 1		**1**	1, 1, 1, 1	0, 0, 1, 0
	1, 1, 1, 0	1, 0, 0, 0			0, 0, 0, 1	$-1, -1, -1, -1$

Figure 2

In this game player 1 chooses either the top row (continue) or the bottom row (quit), player 2 chooses the left column (continue) or the right column (quit), player 3 chooses either the top two matrices (continue) or the bottom two matrices (quit), and player 4 chooses either the left two matrices (continue) or the right two matrices (quit).

Note that there are the following symmetries in the payoff function: for every 4-tuple of actions (a, b, c, d) we have:

$$\begin{aligned} v^1(a, b, c, d) &= v^2(b, a, d, c), \\ v^1(a, b, c, d) &= v^4(c, d, b, a) \qquad \text{and} \\ v^2(a, b, c, d) &= v^3(c, d, b, a), \end{aligned}$$

where $v^i(a, b, c, d)$ is the payoff to i if the action combination is (a, b, c, d) (and $v^i(c^1, c^2, c^3, c^4) = 0$). This game satisfies conditions **A**.1 and **A**.2. We first exhibit a cyclic equilibrium. We then show that (i) there is no stationary equilibrium, and (ii) there is no equilibrium payoff in the convex hull of $\{(4, 1, 0, 0), (1, 4, 0, 0), (0, 0, 1, 4), (0, 0, 4, 1)\}$ (any limit of ϵ-equilibrium payoffs is an *equilibrium payoff*). Finally, we argue that no stationary ϵ-equilibrium exists.

3.1 A cyclic equilibrium

Define a cyclic profile $\mathbf{x}$ as follows: (i) at odd stages, players 2, 4 play c^2 and c^4, while both players 1 and 3 continue with probability $\frac{1}{\sqrt{2}}$; (ii) at even stages, players 1 and 3 continue, while both players 2 and 4 continue with probability $\frac{1}{\sqrt{2}}$.

One checks that $\gamma(\mathbf{x}) = (\sqrt{2}, 1, \sqrt{2}, 1)$ and $\gamma(\mathbf{x}_2) = (1, \sqrt{2}, 1, \sqrt{2})$. Moreover, x_1 and x_2 are respectively equilibria in the one-shot games $G(\gamma(\mathbf{x}_2))$ and $G(\gamma(\mathbf{x}))$. Since $(\mathbf{x}^{-i}, \mathbf{c}^i)$ is terminating for every i, this implies that $\mathbf{x}$ is an equilibrium in Γ.

3.2 No Stationary Equilibria

We check that there is no stationary equilibrium. We discuss according to the number of players who play both actions with positive probability.

It is immediate to check that there is no stationary equilibrium in which at least three players play pure strategies.

We shall now verify that there is no stationary equilibrium where two players play pure stationary strategies. Indeed, assume that players 3 and 4 play pure stationary strategies. If such a case arises, players 1 and 2 are playing a 2×2 game. We will see that all the equilibria in these games are pure, and therefore they cannot generate an equilibrium in the four-player game.

Case 1: Players 3 and 4 play (q^3, q^4)
The unique equilibrium is (c^1, c^2, q^3, q^4).
Case 2: Players 3 and 4 play (c^3, q^4)
The unique equilibrium is (c^1, q^2, c^3, q^4).
Case 3: Players 3 and 4 play (q^3, c^4) — symmetric to case 2.
Case 4: Players 3 and 4 play (c^3, c^4)
There are two equilibria: (q^1, c^2, c^3, c^4) and (c^1, q^2, c^3, c^4).

We shall now see that there is no stationary equilibrium where players 2 and 4 play pure actions.
Case 1: Players 2 and 4 play (c^2, c^4)
The unique equilibrium is (q^1, c^2, q^3, c^4).
Case 2: Players 2 and 4 play (q^2, c^4)
The unique equilibrium is $(\frac{1}{2}c^1 + \frac{1}{2}q^1, q^2, \frac{1}{4}c^3 + \frac{3}{4}q^3, c^4)$. In this equilibrium player 2 receives $\frac{5}{8}$, but if he plays c^2 he gets 1.
Case 3: Players 2 and 4 play (c^2, q^4)
The unique equilibrium is (q^1, c^2, c^3, q^4).
Case 4: Players 2 and 4 play (q^2, q^4)
The unique equilibrium is (c^1, q^2, q^3, q^4).

All the other cases are symmetric to these 8 cases.

Next, we check that there is no stationary equilibrium where one player, say player 4, plays a pure strategy, and all the other players play a fully mixed strategy. We denote by (x, y, z) the fully mixed stationary equilibrium in the three-player game when player 4 plays some pure stationary strategy.

Assume first that player 4 plays q^4. Then, in order to have player 2 indifferent, we should have

$$x(1-z) = z - (1-x)(1-z)$$

which implies that $z = 1/2$. In order to have player 1 indifferent, we should have

$$(1-y)z + y(1-z) = yz - (1-y)(1-z)$$

which solves to $yz = 1/2$, and therefore $y = 1$, which is pure.

Assume now that player 4 plays c^4. First we note that $x < 1/2$, otherwise player 3 prefers to play q^3 over c^3. Next, if player 2 is indifferent between his actions, then

$$\frac{(1-x)(1+3z)}{1-xz} = x + (1-x)z$$

or equivalently,

$$(1-x)(1+2z+xz^2) = (1-xz)x.$$

Since $x < 1/2$, it follows that $1-x > x$. Therefore it follows that

$$1 + 2z + xz^2 < 1 - xz$$

or equivalently $2 + xz < -x$, which is clearly false.

We prove now by contradiction that there is no fully mixed stationary equilibrium.

Let (x^*, y^*, z^*, t^*) be a fully mixed stationary equilibrium, where $0 < x^* < 1$ is the probability player 1 puts on c^1. Set $(a^*, b^*, c^*, d^*) = \gamma(x^*, y^*, z^*, t^*)$. Notice that $0 < a^*, b^*, c^*, d^* < 1$.

Let $0 < y, z, t < 1$. Assume that $a \in]0, 1[$ is the payoff of player 1 if quitting does not occur at the first stage. Then, by playing c^1 at stage 1, player 1 gets

$$\alpha(a; y, z, t) = yzt(a-2) - 2yz + 3zt - yt + y + z,$$

whereas by playing q^1 he gets

$$\beta(y, z, t) = t + (1-t)(y + z - 1).$$

By the equilibrium condition for player 1,

$$a^* = \beta(y^*, z^*, t^*) = \alpha(a^*; y^*, z^*, t^*).$$

Therefore, the polynomial

$$\Delta_1(y, z, t) = \alpha(\beta(y, z, t); y, z, t) - \beta(y, z, t)$$

vanishes at (y^*, z^*, t^*). For simplicity, we write

$$\Delta_1(y, z, t) = (a-2)yzt - 2yz + 4zt + 1 - 2t,$$

with the understanding that a stands for $\beta(y, z, t)$. $\Delta_2(x, z, t)$, $\Delta_3(x, y, t)$ and $\Delta_4(x, y, z)$ are defined in a symmetric way.

Claim: (x^*, y^*, z^*, t^*) is not on the diagonal of $[0, 1]^4$.

First, consider the sum of payoffs $\Sigma(x) = \sum_{i=1}^4 \gamma^i(x, x, x, x)(= 4\gamma^1(x, x, x, x))$. There are four cells in which the sum of payoffs is 5. Each of these cells is reached under (x, x, x, x) with probability $\frac{x^3(1-x)}{1-x^4}$. There are 2 cells, each with probability $\frac{x^2(1-x)^2}{1-x^4}$ where the sum is 4, etc. One gets

$$\begin{aligned}\Sigma(x) &= \frac{1}{1-x^4}\{4\cdot 5x^3(1-x) + 2\cdot 4x^2(1-x)^2 + 4\cdot 3x^2(1-x)^2 + 4x(1-x)^3 \\ &\quad -4(1-x)^4\} \\ &= \frac{1}{1-x^4}\{-8x^4 - 8x^3 - 16x^2 + 20x - 4\}\end{aligned}$$

We argue that $\Sigma(x) > 4$ for $x \in [\frac{3}{4}, 1[$. Indeed, this amounts to $\{-2x^4 + 2x^3 - 4x^2 + 5x - 1\} - (1 - x^4) > 0$ on that interval. After simplification by $1 - x$, this is equivalent to $x^3 - x^2 + 3x - 2 > 0$. The left-hand side is increasing in x, and equals $\frac{7}{64}$ at $x = \frac{3}{4}$.

Therefore, $\gamma^1(x, x, x, x) > 1$ for $x \in [\frac{3}{4}, 1[$. Since player 1 gets at most 1 by playing q^1, this implies that (x^*, y^*, z^*, t^*) is not on the diagonal of $[\frac{3}{4}, 1]^4$.

We now prove that $P(x) = \Delta_1(x, x, x)$ does not vanish on $[0, \frac{3}{4}]$. We use Sturm's method to count the number of roots of P (see for instance [1], section 1.1). Set

$$\begin{aligned}P_1(x) &= P(x) = -2x^5 + 4x^4 - 3x^3 + 2x^2 - 2x + 1 \\ P_2(x) &= P'(x) = -10x^4 + 16x^3 - 9x^2 + 4x - 2.\end{aligned}$$

and, for $i = 1, 2, 3, 4$, P_{i+2} as minus the remainder of the division of P_i by P_{i+1}. One gets

$$\begin{aligned}P_3(x) &= \frac{1}{25}(-2x^3 - 12x^2 + 32x - 21) \\ P_4(x) &= 625(25x^2 - 53x + 32) \\ P_5(x) &= \frac{1}{15625}(-82x + 133) \\ P_6(x) &< 0\end{aligned}$$

One checks that $P_i(0)$ and $P_i(\frac{3}{4})$ have the same sign. Therefore, P has no root in $[0, \frac{3}{4}]$. This ends the proof of the claim. ■

Without loss of generality, we assume $y^* = \min(x^*, y^*, z^*, t^*)$. We shall prove that Δ_1 and Δ_4 do not vanish simultaneously. We now point out several facts that will be used extensively:

1. $\frac{\partial a}{\partial t}(y, z, t) = 2 - y - z > 0$; $\frac{\partial a}{\partial y}(y, z, t) = \frac{\partial a}{\partial z}(y, z, t) = 1 - t > 0$;
2. $\frac{\partial \Delta_1}{\partial y}(y, z, t) = (a-2)zt + yzt(1-t) - 2z < 0$;
3. $\frac{\partial \Delta_1}{\partial z}(y, z, t) = (a-2)yt + yzt(1-t) - 2y + 4t$ is decreasing in y; therefore, on the region $y \leq t$, $\frac{\partial \Delta_1}{\partial z}(y, z, t) \geq \frac{\partial \Delta_1}{\partial z}(t, z, t) = (a-2)t^2 + t^2 z(1-t) + 2t > 0$.

Thus, on the region $y \leq t \leq z$,

$$\Delta_1(y, z, t) \geq \Delta_1(t, t, t) > 0.$$

Therefore, (x^*, y^*, z^*, t^*) belongs to the region $D = \{y \leq z \leq t\}$. The proof is divided in two steps. First, we prove that Δ_1 does not vanish on $D \cap \{z \geq \frac{1}{2}\}$. Then, we prove that Δ_4 does not vanish on $D \cap \{z \leq \frac{1}{2}\}$.

LEMMA 3.1 *$z^* \geq \frac{1}{2}$ is impossible.*

Proof: We argue by contradiction, and assume throughout this proof $z^* \geq \frac{1}{2}$. Notice that

$$y \leq \frac{1}{2} \leq z \leq t \Rightarrow \Delta_1(y, z, t) \geq \Delta_1(\frac{1}{2}, \frac{1}{2}, t) = \frac{1}{2} - \frac{t}{2} + a\frac{t}{4} > 0.$$

Thus, $y^* \geq \frac{1}{2}$.

Claim: $t^* \geq \frac{2}{3}$.

We study Δ_1 on the domain $D_1 = \{\frac{1}{2} \leq y \leq z \leq t \leq \frac{2}{3}\}$. Notice first that a is maximized at $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, where it equals $\frac{7}{9} < \frac{5}{6}$.

On D_1, $\Delta_1(y, z, t) \geq \Delta_1(z, z, t) = f(z, t) = (a-2)z^2 t - 2z^2 + 4zt + 1 - 2t$. One has $\frac{\partial f}{\partial z}(z, t) = 2zt(a-2) + 2z^2 t(1-t) - 4z + 4t$. It is easily checked that

$\frac{\partial}{\partial t}(\frac{\partial f}{\partial z})(z,t) = 2z(a-2) + 2zt(2-y-z) + 2z^2(1-t) - 2z^2t + 4$ is positive on D_1. Therefore,

$$\frac{\partial f}{\partial z}(z,t) \leq \frac{\partial f}{\partial z}(z,\frac{2}{3}) = \frac{4}{3}z(a-2) + \frac{4}{9}z^2 - 4z + \frac{8}{3}.$$

The latter quantity is maximized at $z = \frac{1}{2}$. It is then equal to $\frac{2}{3}(a-1) + \frac{1}{9}$. Since $a < \frac{5}{6}$, this is negative.

Thus, $\frac{\partial f}{\partial z} < 0$ on D_1. Therefore,

$$\Delta_1(z,z,t) \geq \Delta_1(t,t,t) > 0.$$

The claim is established.

Claim: $z^* \geq \frac{2}{3}$ is impossible.

We shall prove that $\Delta_1 > 0$ on $D_2 = \{\frac{1}{2} \leq y, \frac{2}{3} \leq z \leq t\}$. Notice first that $a \geq \frac{2}{3}$ on D_2.

Set first $D_3 = D_2 \cap \{y < \frac{2}{3}\}$. On D_3, $\Delta_1(y,z,t) \geq \Delta_1(\frac{2}{3},\frac{2}{3},t) = (a-2)\frac{4}{9}t + \frac{1}{9} + \frac{2}{3}t$. Now, $\forall t \geq \frac{2}{3}$, $\beta(\frac{2}{3},\frac{2}{3},t) \geq \frac{7}{9} \geq \frac{3}{4}$. Therefore,

$$\Delta_1(y,z,t) \geq \Delta_1(\frac{2}{3},\frac{2}{3},t) \geq -\frac{5}{9}t + \frac{1}{9} + \frac{2}{3}t = \frac{t+1}{9} > 0.$$

Set now $D_4 = D_2 \cap \{y \geq \frac{2}{3}\}$. On D_4, one has

$$\frac{\partial \Delta_1}{\partial t} = (a-2)yz + yzt(2-y-z) + 4z - 2 \geq (a-2)yz + 4z - 2.$$

The function $(a-2)yz + 4z - 2$ is increasing in z. Therefore, it is minimized on the diagonal $\{y = z\}$, where it is at least $-\frac{5}{4}y^2 + 4y - 2$: this minorant is minimized at $y = \frac{2}{3}$; it is then equal to $\frac{1}{9}$. Therefore, Δ_1 is increasing in t, and

$$\Delta_1(y,z,t) \geq \Delta_1(z,z,z) > 0.$$

To conclude the proof of Lemma 3.1, we prove that $\Delta_1 > 0$ on $D_5 = ([\frac{1}{2},\frac{2}{3}] \times [\frac{1}{2},\frac{2}{3}] \times [\frac{2}{3},1[) \cap \{y \leq z\}$.

On D_5, $a \geq \frac{2}{3}$, thus

$$\Delta_1(y,z,t) \geq \Delta_1(z,z,t) \geq -\frac{4}{3}z^2t - 2z^2 + 4zt + 1 - 2t = h(z,t).$$

First,

$$\begin{aligned} h(\frac{1}{2},t) &= -\frac{t}{3} - \frac{1}{2} + 2t + 1 - 2t = \frac{1}{2} - \frac{t}{3} > \frac{1}{6} \\ h(\frac{2}{3},t) &= -\frac{16}{27}t - \frac{8}{9} + \frac{8}{3}t + 1 - 2t = \frac{2}{27}t + \frac{1}{9} > \frac{1}{7}. \end{aligned}$$

Now, each $z \in [\frac{1}{2}, \frac{2}{3}]$ satisfies $|z - \frac{1}{2}| \le \frac{1}{12}$, or $|z - \frac{2}{3}| \le \frac{1}{12}$. Therefore, we need only prove that $|\frac{\partial h}{\partial z}(z,t)| \le \frac{12}{7}$ on D_5. The function

$$\frac{\partial h}{\partial z}(z,t) = -\frac{8}{3}zt - 4z + 4t$$

is increasing in t and decreasing in z. Thus, it is minimal at $(\frac{2}{3}, \frac{2}{3})$, where it equals $-\frac{32}{27}$, and maximal at $(\frac{1}{2}, 1)$, where it equals $\frac{2}{3}$. ■

LEMMA 3.2 *$z^* \le \frac{1}{2}$ is impossible.*

Proof: We would otherwise have $y^* \le z^* \le \frac{1}{2}$. We prove that this would contradict $\Delta_4(x^*, y^*, z^*) = 0$. Recall that $d^* > 0$. Therefore

$$0 = \Delta_4(x^*, y^*, z^*) > -2x^*y^*z^* - 2x^*z^* + 4x^*y^* + 1 - 2y^*.$$

Hence, the polynomial $P(x,y,z) = xyz + xz - 2xy + y - \frac{1}{2}$ is positive at (x^*, y^*, z^*).

We prove now that P is negative on $D_6 = ([0,1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]) \cap \{y \le z\}$.

1. on $D_6 \cap \{x \le \frac{1}{2}\}$, $\frac{\partial P}{\partial y}(x,y,z) = xz - 2x + 1 \ge 0$; thus, P is maximized at $y = \frac{1}{2}$: it is then equal to $\frac{xz}{2} + xz - x = x(\frac{3}{2}z - 1) < 0$;
2. on $D_6 \cap \{x \ge \frac{1}{2}\}$, $\frac{\partial P}{\partial z}(x,y,z) = xy + x > 0$; thus, P is maximized at $z = \frac{1}{2}$ and equals

$$Q(x,y) = y - \frac{3}{2}xy + \frac{x}{2} - \frac{1}{2}.$$

   (a) on $\{y \le \frac{1}{3}\}$, $\frac{\partial Q}{\partial x}(x,y) = \frac{1}{2} - \frac{3}{2}y < 0$; thus, Q is maximized at $x = 1$, and then equals $y - \frac{3}{2}y < 0$;

   (b) on $\{y \ge \frac{1}{3}\}$, $\frac{\partial Q}{\partial x}(x,y) \ge 0$; thus, Q is maximized at $x = \frac{1}{2}$, and then equals $Q(\frac{1}{2}, y) = \frac{1}{4}(y - 1) < 0$.

■

3.3 No perturbed ϵ-equilibrium

In this section we prove that there is no perturbed ϵ-equilibrium in the game. That is, there is no ϵ-equilibrium profile $\mathbf{x}$ such that for every $n \in \mathbf{N}$, $\| x_n - y \| < \epsilon$ for some fixed mixed-action combination y. We first prove that it cannot be the case that $y = (1, 1, \ldots, 1)$.

We argue by contradiction. Let $\epsilon > 0$ be sufficiently small, and let $\mathbf{x}$ be an ϵ-equilibrium with $x_n^i > 1 - \epsilon$, for each $i \in N, n \in \mathbf{N}$. We refer to such a profile as a perturbed ϵ-equilibrium.

Since $\mathbf{x}$ is an ϵ-equilibrium, it follows that

$$\mathbf{P}_{\mathbf{x}}(t < +\infty) \geq 1 - 2\epsilon \text{ and } \mathbf{P}_{\mathbf{x}}(t < +\infty, |S_t| > 1) < 5\epsilon : \tag{13}$$

the probability of termination is high (otherwise, any player i would benefit by playing according to $\mathbf{x}^i$ for many stages, before switching to q^i) and the probability of the terminating coalition being a singleton is high (since $\| x_n - c \| < \epsilon$, for each n).

In the computations which follow, the estimates are not meant to be tight. Much better estimates could be obtained, but this is pointless here.

Define $p_n^i = \mathbf{P}_{\mathbf{x}}(t < n, i \in S_t)$. Clearly for every fixed i, (p_n^i) is an increasing sequence, hence $p_\infty^i = \lim_n p_n^i$ exists.

By (13), one has $1 - 2\epsilon \leq \sum_i p_\infty^i \leq 1 + 5\epsilon$, and

$$5 - 10\epsilon \leq \sum_i \gamma^i(\mathbf{x}) \leq 5. \tag{14}$$

Lemma 3.3 *For each i, $p_\infty^i \geq \frac{1}{11}$.*

Proof: We will prove the lemma for $i = 1$. Since $r_{\{1\}}^1 = 1$ and $r_{\{2\}}^1 = 4$, one deduces from (13) that

$$\begin{aligned}
\gamma^1(\mathbf{x}) &\leq \mathbf{P}_{\mathbf{x}}\{t < +\infty, S_t = \{1\}\} + 4\mathbf{P}_{\mathbf{x}}\{t < +\infty, S_t = \{2\}\} + 5\epsilon \\
&\leq p_\infty^1 + 4p_\infty^2 + 5\epsilon.
\end{aligned}$$

On the other hand, $\| x_1 - c \| < \epsilon$; thus, by playing q^i in the first stage, player i gets at least $1 - 3\epsilon$. Since $\mathbf{x}$ is an ϵ-equilibrium, $\gamma^1(\mathbf{x}) \geq 1 - 4\epsilon$.

Similar inequalities hold for $\gamma^2(\mathbf{x})$: thus,

$$1-4\epsilon \leq \gamma^2(\mathbf{x}) \leq 4p^1_\infty + p^2_\infty + 5\epsilon \tag{15}$$
$$1-4\epsilon \leq \gamma^1(\mathbf{x}) \leq p^1_\infty + 4p^2_\infty + 5\epsilon \tag{16}$$

This yields $p^1_\infty + p^2_\infty \geq \frac{2}{5} - \frac{18}{5}\epsilon$ and, by symmetry, $p^3_\infty + p^4_\infty \geq \frac{2}{5} - \frac{18}{5}\epsilon$.

Therefore,

$$p^1_\infty + p^2_\infty \leq \frac{3}{5} + \frac{18}{5}\epsilon + 5\epsilon \tag{17}$$

In particular, $p^2_\infty \leq \frac{3}{5} + 9\epsilon$. The result follows, using (15). ∎

By (14), $\gamma^i(\mathbf{x}) \geq 6/5$ for at least one i. We now proceed as follows. We first argue (steps 1 and 2 below) that it cannot be the case that $\gamma^1(\mathbf{x})$ and $\gamma^2(\mathbf{x})$ be significantly above 1. Assuming $\gamma^1(\mathbf{x}) \geq \frac{6}{5}$, we then show (step 3) that, for some n, $\mathbf{x}_n$ is a perturbed ϵ'-equilibrium such that $\gamma^3(\mathbf{x}_n)$ and $\gamma^4(\mathbf{x}_n)$ are both significantly above 1, which is ruled out by the first part.

Step 1: Let $a > 0$, and assume that $\gamma^1(\mathbf{x}) \geq 1 + a$. We show that before player 1 quit with non-negligable probability, player 2 must have already quit with non-negligable probability.

Let n_1 be any stage such that $\gamma^1(\mathbf{x}_n) \geq 1 + \sqrt{\epsilon}$, for all $n < n_1$. We argue that $p^1_{n_1} \leq \sqrt{\epsilon}$. Denote by $\mathbf{y}^1$ the strategy which coincides with $\mathbf{c}^1$ up to stage n_1, and with $\mathbf{x}^1$ from then on. $\gamma^1(\mathbf{x})$ may be written

$$\gamma^1(\mathbf{x}) = \mathbf{P}_\mathbf{x}(t < n_1, 1 \in S_t)\{\mathbf{E}_\mathbf{x}[r^1_{S_t} | t < n_1, 1 \in S_t] - \gamma^1(\mathbf{x}_{n_1})\} \tag{18}$$

$$+\mathbf{P}_\mathbf{x}(t < n_1, 1 \notin S_t)\mathbf{E}_\mathbf{x}[r^1_{S_t} | t < n_1, 1 \notin S_t] + (1 - \mathbf{P}_\mathbf{x}(t < n_1, 1 \notin S_t))\gamma^1(\mathbf{x}_{n_1}).$$

Consider the first term. Since $r^1_S \leq 1$ whenever $1 \in S$, one has $\mathbf{E}_\mathbf{x}[r^1_{S_t} | t < n_1, 1 \in S_t] \leq 1$. On the other hand, $\gamma^1(\mathbf{x}_{n_1}) \geq 1 + \sqrt{\epsilon}$ by assumption. Therefore, the first term is at most $-\sqrt{\epsilon}p^1_{n_1}$.

The sum of the last two terms is $\gamma^1(\mathbf{x}^{-1}, \mathbf{y}^1)$. Since $\gamma^1(\mathbf{x}) \geq \gamma^1(\mathbf{x}^{-1}, \mathbf{y}^1) - \epsilon$, (18) yields $-\epsilon \leq -\sqrt{\epsilon}p^1_{n_1}$, therefore $p^1_{n_1} \leq \sqrt{\epsilon}$.

Since $p^1_\infty \geq \frac{1}{11}$, the stage $N_1 = \min\{n, \gamma^1(\mathbf{x}_n) < 1 + \sqrt{\epsilon}\}$ is well-defined and $p^1_{N_1} \leq \sqrt{\epsilon}$.

One has

$$1+a \leq \gamma^1(\mathbf{x}) \leq p^1_{N_1} + 4p^2_{N_1} + 5\epsilon + (1 - p^1_{N_1} - p^2_{N_1} - p^3_{N_1} - p^4_{N_1} + 5\epsilon)\gamma^1(\mathbf{x}_{N_1}). \tag{19}$$

Since $\gamma^1(\mathbf{x}_{N_1}) < 1 + \sqrt{\epsilon}$, the inequality (19) yields

$$1 + a \leq p^1_{N_1} + 4p^2_{N_1} + (1 - p^1_{N_1} - p^2_{N_1}) + \sqrt{\epsilon} + 5\epsilon,$$

thus

$$a \leq 3p^2_{N_1} + 2\sqrt{\epsilon}. \tag{20}$$

Step 2: We prove that, provided ϵ is small enough, it cannot be the case that $\gamma^1(\mathbf{x}), \gamma^2(\mathbf{x}) \geq 1+a$. We argue by contradiction. Assume that $\gamma^1(\mathbf{x}), \gamma^2(\mathbf{x}) \geq 1 + a$.

As in step 1, the stage $N_2 = \inf\{n, \gamma^2(\mathbf{x}_n) < 1 + \sqrt{\epsilon}\}$ is well-defined and $p^2_{N_2} \leq \sqrt{\epsilon}$.

Due to the symmetry between players 1 and 2, we may assume *w.l.o.g.* that $N_1 \leq N_2$.

Since $(p^2_n)_n$ is non-decreasing, one deduces from (20) that $p^2_{N_2} \geq \frac{a}{3} - \sqrt{\epsilon}$, which contradicts $p^2_{N_2} \leq \sqrt{\epsilon}$.

Step 3: Assume *w.l.o.g.* that $\gamma^1(\mathbf{x}) \geq \frac{6}{5}$. As above, the stage $N_1 = \min\{n, \gamma^1(\mathbf{x}_n) < 1+\sqrt{\epsilon}\}$ is finite, $p^1_{N_1} \leq \sqrt{\epsilon}$ and (see (20)) $p^2_{N_1} \geq \frac{1}{3} \times \frac{1}{5} - \sqrt{\epsilon} \geq \frac{1}{16}$. Therefore, there exists a stage $n_2 \leq N_1$ with $\gamma^2(\mathbf{x}_{n_2}) < 1+\sqrt{\epsilon}$. We denote by $N_2 - 1$ the last such stage.

Since $p^1_{N_2} \leq p^1_{N_1} \leq \sqrt{\epsilon}$ and $p^1_\infty \geq \frac{1}{11}$, one has $\mathbf{P}_\mathbf{x}(t < N_2) \leq \frac{10}{11} + \sqrt{\epsilon} \leq \frac{11}{12}$. Since $\mathbf{x}$ is an ϵ-equilibrium, $\mathbf{x}_{N_2}$ is a 12ϵ-equilibrium.

Since $\gamma^2(\mathbf{x}_n) \geq 1 + \sqrt{\epsilon}$ for every $N_2 \leq n \leq N_1$, one gets, as in step 1, that $\mathbf{P}_\mathbf{x}(t < N_1, 2 \in S_t | t \geq N_2) \leq \sqrt{12\epsilon}$ hence, *a fortiori*, $p^2_{N_1} - p^2_{N_2} \leq 4\sqrt{\epsilon}$. Therefore, $p^2_{N_2} \geq \frac{1}{16} - 4\sqrt{\epsilon}$.

We prove now that $\mathbf{x}_{N_2}$ is a 12ϵ-equilibrium with $\gamma^3(\mathbf{x}_{N_2}), \gamma^4(\mathbf{x}_{N_2}) \geq 1 + \frac{1}{18}$, which is ruled out by step 2.

As in step 1,

$$1 - 2\epsilon \leq \gamma^2(\mathbf{x}) \leq 4p^1_{N_2} + p^2_{N_2} + 5\epsilon + (1 - \sum_i p^i_{N_2} + 5\epsilon)\gamma^2(\mathbf{x}_{N_2}). \tag{21}$$

By definition of N_2, $\gamma^2(\mathbf{x}_{N_2}) \leq 1 + \sqrt{\epsilon}$. Since $p^1_{N_2} \leq p^1_{N_1} \leq \sqrt{\epsilon}$, one deduces from (21)

$$p^3_{N_2} + p^4_{N_2} \leq 5\sqrt{\epsilon}.$$

On the other hand,

$$1 - 2\epsilon \leq \gamma^3(\mathbf{x}) \leq 4p^4_{N_2} + p^3_{N_2} + 5\epsilon + (1 - \sum_i p^i_{N_2} + 5\epsilon)\gamma^3(\mathbf{x}_{N_2}). \qquad (22)$$

Since $p^2_{N_2} \geq \frac{1}{17}$, (22) yields $\gamma^3(\mathbf{x}_{N_2}) \geq 1 + \frac{1}{18}$. Similarly, $\gamma^4(\mathbf{x}_{N_2}) \geq 1 + \frac{1}{18}$. By step 2, this contradicts the fact that $\mathbf{x}_{N_2}$ is a 12ϵ-equilibrium. ■

It is now easy to conclude that there is no stationary ϵ-equilibrium for ϵ sufficiently small. Indeed, assume that for every ϵ there exists a stationary ϵ-equilibrium $\mathbf{x}_\epsilon$. Let $\mathbf{x}_\star$ be an accumulation point of $\{\mathbf{x}_\epsilon\}$ as $\epsilon \to 0$. If $\mathbf{x}_\star$ is terminating then it is a stationary equilibrium, which is ruled out by section 3.2. Otherwise, $\mathbf{x}_\star = \mathbf{c}$, and then, for ϵ sufficiently small, there is an ϵ-perturbed equilibrium, which is ruled out by section 3.3.

References

[1] R. Benedetti and J.J. Risler. *Real Algebraic and Semi-algebraic Sets.* Hermann, 1990.

[2] E.G. Dynkin. Game variant of a problem on optimal stopping. *Soviet Math. Dokl.*, 10:270–274, 1969.

[3] H. Everett. Recursive games. In *Contributions to the Theory of Games, 3*, pages 47–78, Princeton, N.J., 1957. Annals of Mathematical Studies, 39, Princeton University Press.

[4] J. Flesch, F. Thuijsman, and O.J. Vrieze. Cyclic Markov equilibria in stochastic games. *International Journal of Game Theory*, 26:303–314, 1997.

[5] P. Ghemawat and B. Nalebuff. Exit. *RAND Journal of Economics*, 16(2):184–194, 1985.

[6] P. Hammerstein and R. Selten. Game theory and evolutionary biology. In R. J. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, volume 2, pages 929–993. Elsevier Science Publishers B.V., 1995.

[7] K. Kuratowski and C. Ryll-Nardzewski. A general theorem on selectors. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, 13:397-403, 1965.

[8] L. Li. Equilibrium exit in stochastically declining industries. *Games and Economics Behavior*, 1(1):40–59, 1989.

[9] H. Morimoto. Non zero-sum discrete parameter stochastic games with stopping times. *Probab. Th. Rel. Fields*, 72:155–160, 1986.

[10] Y. Ohtsubo. A non zero-sum extension of Dynkin's stopping problem. *Math. Oper. Res.*, 12(2):277–296, 1987.

[11] E. Solan. 3-person repeated games with absorbing states. D.P. #128, Center for Rationality and Interactive Decision Theory, Hebrew University of Jerusalem, 1997.

[12] R. Wilson. Strategic models of entry deterrence. In R. J. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, volume 1, pages 305–354. Elsevier Science Publishers B.V., 1992.