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Maxmin Expected Utility over Savage Acts with a Set of Priors*

by

RAMON CASADESUS-MASANELL[†]
PETER KLIBANOFF[‡]
EMRE OZDENOREN[§]

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[†] Department of Management and Strategy and Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208. E-mail: rcasades@nwu.edu

[‡] Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208. E-mall: peterk@nwu.edu

[§] Department of Management and Strategy and Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208. E-mail: e-ozdenoren@nwu.edu

Abstract

This paper provides an axiomatic foundation for a maxmin expected utility over a set of priors (MMEU) decision rule in an environment where the elements of choice are Savage acts. This characterization complements the original axiomatization of MMEU developed in a lottery-acts (or Anscombe-Aumann) framework by Gilboa and Schmeidler [9]. MMEU preferences are of interest primarily because they provide a natural and tractable way of modeling decision makers who display an aversion to uncertainty or ambiguity. Characterizing MMEU in a setting with Savage acts, as we do here, is of particular interest given a number of recent papers (for example, Ghirardato [6], and Sarin and Wakker [14]) that point out that there may be real differences when using uncertainty averse preferences between a two-stage lottery-acts formulation and a one-stage Savage acts setting. MMEU over Savage acts also appears prominently in related papers that examine randomization and uncertainty averse decision makers (Eichberger and Kelsey [4], Klibanoff [11]).

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1. Introduction

This paper provides an axiomatic foundation for a maxmin expected utility over a set of priors (MMEU) decision rule in an environment where the elements of choice are Savage [15] acts. This characterization complements the original axiomatization of MMEU developed in a lottery-acts (or Anscombe-Aumann [1]) framework by Gilboa and Schmeidler [9]. MMEU preferences are of interest primarily because they provide a natural and tractable way of modeling decision makers who display an aversion to uncertainty or ambiguity. A leading motivation for examining such preferences is the evidence described by Ellsberg [5] and many afterwards demonstrating that such aversion seems common and is incompatible with standard expected utility theory. A closely related representation that has also been used to capture uncertainty aversion is Choquet expected utility (CEU). CEU was first axiomatized in a lottery-acts framework (Schmeidler [16]) and later in settings with Savage acts (Gilboa [8], Wakker [17], Nakamura [13], Sarin and Wakker [14], and Chew and Karni [2]). In contrast, MMEU has never before been characterized in a setting with Savage acts.

A great attraction of settings with Savage acts is, of course, that no primitive notion of probabilities need be assumed. That probabilities nonetheless appear in a representation is then quite satisfying and provides a strong foundation. However, the interest in a Savage act characterization of CEU and MMEU is not only philosophical. It is of particular interest given a number of recent papers (for example, Ghirardato [6], and Sarin and Wakker [14]) that point out that there may be real differences when using CEU preferences between a two-stage lottery-acts formulation and a one-stage Savage acts setting. Since MMEU is closely related to CEU, this suggests value in having a foundation for MMEU in both settings. CEU and MMEU over Savage acts also appear prominently in papers that examine randomization and uncertainty averse decision makers (Eichberger and Kelsey [4], Klibanoff [11]). The latter paper in particular suggests a privileged role for MMEU over CEU in modeling uncertainty aversion and randomization simultaneously in a Savage acts setting.

The remainder of the paper presents two overlapping sets of axioms and two corresponding representation theorems proving the equivalence between these axioms and an MMEU rule. One set of axioms is appropriate when there exists an event that is perceived to be unambiguous and that is neither null nor universal (i.e., in the representation, an event that is assigned a fixed probability strictly between zero and one). The other set of axioms describes the more general case where only the existence of an event that is neither null nor universal (but not necessarily unambiguous) is assumed. In the case with an unambiguous event, the novel axioms are weakenings of Gul's ([10], axiom 2) act-independence condition. In the general case, the novel axioms are formulated using standard sequences, a measurement theory construction (see Krantz, Luce, Suppes and Tversky [12]). Results in Nakamura [13] and Gilboa and Schmeidler [9] are useful for the proofs.

2. Notation and framework

Ω is the *set of states*. A *state* in Ω is represented by ω. Σ is an algebra of subsets of Ω. *Events* are elements of Σ. $X = [m, M] \subset \mathbb{R}$, $m < M$ is the set of *prizes* or outcomes. An *act* f is a function $f : \Omega \to X$. A *simple act* is an act with only finitely many distinct values. A simple act is Σ-measurable if $\{\omega \in \Omega \mid f(\omega) \in W\} \in \Sigma$ for all $W \subseteq X$. F is the set of all Σ-measurable acts defined as the uniform closure in the supnorm of all Σ-measurable simple acts. A *constant act* f is one for which $f(\omega) = x$ for all $\omega \in \Omega$, for some $x \in X$; we denote this constant act by x^* or simply x when no confusion would result. F^* is the subset of F consisting of all constant acts; note that F^* can be identified with X. For any event $B \in \Sigma$ and $x, y \in X$, $x_B y$ denotes $f \in F$ such that $f(\omega) = x$ for $\omega \in B$ and $f(\omega) = y$ for $\omega \notin B$; such acts are referred to as B-measurable. $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$ denote, respectively, the sets of all positive and all strictly positive integers. $\mathcal{P}$ is the set of all finitely additive probability measures $P : \Sigma \to [0,1]$. Finally, $\succeq$ is a binary relation on F.

Note that this environment is similar to that in Savage [15] with the difference that we impose more structure on the prize set (X). The important aspect of this structure is that X is connected and separable. This, together with a continuity assumption on preferences that we will make below, generate a richness in preference equivalence classes that is heavily used in what follows. As a consequence of this richness in the prize set, we do not need to impose axioms that require the set of states (Ω) to be infinite as Savage does. In fact, other than the uninteresting case of Ω containing only one state, our axioms allow for Ω to be of any size, finite or infinite.

As mentioned above, we will analyze two alternative axiomatizations of MMEU over Savage acts with a set of priors. The first one assumes the existence of an unambiguous event – that is, of an event that will end up having the same weight according to every measure in the set of priors used by the decision maker. The second axiomatization does not assume the existence of such an unambiguous event, and hence, it is more general. The reason we study the less general case at all is that in a framework with such additional structure, we are able to write down somewhat simpler and arguably more attractive axioms.

3. A theory when there exists an unambiguous event

3.1. Axioms

Axiom 1. ***(Weak order)*** *$\succeq$ is complete and transitive.*

Definition 3.1. *An ordered null event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$ whenever $x \preceq y \preceq z$.*

Definition 3.2. *An ordered universal event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$ whenever $z \preceq y \preceq x$.*

Hence, an event B is *ordered non-null* if there exist x, y and z in X with $x \preceq y \preceq z$ such that $x_B z \nsim y_B z$. Likewise, an event B is *ordered non-universal* if there exist x, y and z in X with $z \preceq y \preceq x$ such that $z_B x \nsim z_B y$. Intuitively, B ordered non-null implies that the prize given on B matters sometimes; and B ordered non-universal implies that the prize on B is not always the only thing that matters to the decision maker. Note that since we impose restrictions on the ordering of x, y and z, our definitions of null and universal (borrowed from Nakamura [13]) are weaker than the corresponding notions in Savage [15]. Section 5 contains a discussion of why the ordered notion is appropriate here.

The next axiom is structural and contains two parts.

Axiom 2. ***(Structure)***
(a) $x > y \Rightarrow x^* \succ y^*$.
(b) There exists an event $A \in \Sigma$ such that A and A^c are non-null and non-universal.

Regarding part (a), note that this will require preference over prizes to be increasing in the real number ordering. This is purely a simplifying assumption which allows the easy identification of the set of equivalence classes and eliminates the trivial case where preference is never strict. Excepting the trivial case, the richness and separability of the set of equivalence classes given by this assumption is already required by the continuity axiom (Axiom 4, below) together with the assumption that the set of prizes is connected and separable. In part (b), note that because of the ordering conditions in the definitions of ordered null and ordered universal events, B ordered non-null does not imply B^c ordered non-universal, nor does B ordered non-universal imply B^c ordered non-null. Therefore it is not redundant to assume that both A and A^c are ordered non-null and ordered non-universal.

The next axiom is a weakening of the act-independence axiom introduced in Gul [10]. Specifically, it imposes act-independence only for A-measurable acts.

Axiom 3. *(A-act-independence) Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ and let $f = x_{1A}x_2$, $g = y_{1A}y_2$ and $h = z_{1A}z_2$. If f', $g' \in F$ are such that, for either* $B = A$ or $B = A^c$,

$$f'(\omega) \sim h(\omega)_B\, f(\omega) \text{ for all } \omega \in \Omega$$

and

$$g'(\omega) \sim h(\omega)_B\, g(\omega) \text{ for all } \omega \in \Omega$$

then,

$$f \succeq g \Leftrightarrow f' \succeq g'.$$

Suppose f, g and h are acts and B is an event. We will use the following terminology: If, for every state ω, $h(\omega)$ is indifferent to the act that gives $f(\omega)$ on B and $g(\omega)$ on B^c, then we say that h is a *statewise combination of f and g over the event B*. With this terminology, this last axiom reads: given A-measurable acts f, g and h and given the event $B = A$ or $B = A^c$, if f' is a statewise combination of h and f over the event B and g' is a statewise combination of h and g over the event B, then f is at least as good as g if and only if f' is at least as good as g'. As discussed in Gul [10], act-independence is analogous to the independence axiom in the theory of expected utility over lotteries. As we will see below, assuming A-act-independence will help generate an expected utility representation for A-measurable acts, thus requiring that a fixed probability be associated with the event A. It is this aspect of the theory that will be relaxed in section 4.

The next axiom is a standard continuity assumption. Together with the structural assumptions on X (crucially that X is a connected and separable set) and the weak order axiom, it guarantees the existence of a real-valued representation of preferences and a certainty equivalent of any given act. It also implies that the utility function, u, in the representation is continuous.

Axiom 4. *(Continuity) For all $f \in F$, the sets $M(f) = \{g \in F \mid g \succeq f\}$ and $W(f) = \{g \in F \mid f \succeq g\}$ are closed.*

The following axiom is a monotonicity requirement. Specifically, part (a) requires that if, in every state, the prize that f gives is at least as good as the prize that g gives, then, overall, f must be at least as good as g. Part (b) requires this monotonicity to be strict on ordered non-null and ordered non-universal events (i.e., if f gives a prize strictly better than g on an ordered non-null event or on the complement of an ordered non-universal event and is weakly better than g elsewhere then, overall, f must be strictly better than g), subject to ordering conditions as in the definitions of ordered null and ordered universal. In this sense, each ordered non-null event (and complement of an ordered non-universal event) "matter" in determining overall preference.[1]

Axiom 5. ***(Monotonicity)***

(a) For all $f, g \in F$, if $f(\omega) \succeq g(\omega)$, for all $\omega \in \Omega$ then $f \succeq g$.

(b) If $B \in \Sigma$ is ordered non-null and $z \succeq x$ and $z \succeq y$, then $x_B z \succeq y_B z$ implies $x \succeq y$. If $B \in \Sigma$ is ordered non-universal and $x \succeq z$ and $y \succeq z$, then $z_B x \succeq z_B y$ implies $x \succeq y$.

As is shown in lemma A.12, these first five axioms guarantee the existence of an expected utility representation for A-measurable acts. That is, there exists a strictly increasing and continuous function $u : X \to \mathbb{R}$ and $\rho \in (0,1)$ such that if $x, y, v, w \in X$ then

$$x_A y \succeq v_A w \Leftrightarrow \rho u(x) + (1-\rho) u(y) \geq \rho u(v) + (1-\rho) u(w).$$

Moreover, u is unique up to positive affine transformations and ρ is unique. We refer to the event A as *unambiguous* because the decision maker always assigns the same weight, ρ, to the event A when evaluating A-measurable acts.

How do preferences extend from A-measurable acts to all acts? From the above, overall preferences must be a continuous, monotonic, weak order. Furthermore they must satisfy act-independence on A-measurable acts. Where else must act-independence hold, and when it is violated, what form can the violation take? Different answers to these questions characterize different functional representations of preferences. If (in addition to the other axioms) act-independence is required to hold for all acts f, g, and h, expected utility preferences result.[2] This act-independence is incompatible, for example, with the Ellsberg Paradox. To characterize MMEU, act-independence must be required to hold on a much smaller set of acts. What is this set of acts? We know from Ghirardato, Klibanoff, and Marinacci [7] that the MMEU functional must satisfy additivity in general only on sets of functions that are *affinely related*. This concept translates to acts in the following way, making use of the utility function u as above:

Definition 3.3. *Two acts f and g are affinely related if there exist $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that either $u(f(\omega)) = \alpha u(g(\omega)) + \beta$ for all $\omega \in \Omega$ or $u(g(\omega)) = \alpha u(f(\omega)) + \beta$ for all $\omega \in \Omega$.*

[1]Since, in any state, preference between prizes is equated with preference between the corresponding constant acts, this axiom also implicitly rules out state *dependent* preferences.

[2]For finite state spaces, this is essentially Gul's [10] result, although it is closer to that of Chew and Karni [2], who, by using – as do we – results of Nakamura [13] are able to dispense with Gul's assumption of a symmetric ("half-half") event.

In other words, f and g are affinely related if either f is a constant utility act or g is a constant utility act or there is a positive affine transformation relating the utility f gives in each state with the utility g gives in each state.

The next axiom expresses a limited form of act-independence directly through preferences (i.e., without referring to the u function). The key step in formulating the axiom is the use of statewise combinations over the event A to form what will turn out to be sets of affinely related acts. This requires the following three definitions:

Definition 3.4. *A set $S \subset F$ contains all statewise combinations over the event A if $f_1, f_2 \in S$, and*

$$f(\omega) \sim f_1(\omega)_A f_2(\omega) \text{ for all } \omega \in \Omega$$

implies $f \in S$.

Recall that F^* is the set of all constant acts.

Definition 3.5. *Fix $f \in F$. We define*

$$S^f = \{f\} \cup F^*.$$

Definition 3.6. *Let $\overline{S}^f \supseteq S^f$ be the smallest closed set containing all statewise combinations over the event A.*

As will be shown in the proof of lemma A.35, given an act f, the set $\overline{S}^f$ may be constructed by the following iterative method: at each step $i = 1, 2, 3, \ldots$ we produce a set S_i^f in the following way

$$\begin{aligned} S_1^f &= \left\{f^1 \in F : f^1(\omega) \sim f_1^0(\omega)_A f_2^0(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^0, f_2^0 \in S^f\right\} \\ S_2^f &= \left\{f^2 \in F : f^2(\omega) \sim f_1^1(\omega)_A f_2^1(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^1, f_2^1 \in S_1^f\right\} \\ &\vdots \\ S_n^f &= \left\{f^n \in F : f^n(\omega) \sim f_1^{n-1}(\omega)_A f_2^{n-1}(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^{n-1}, f_2^{n-1} \in S_{n-1}^f\right\} \end{aligned}$$

Finally, $\overline{S}^f = closure\left(\cup_{i=1}^{\infty} S_i^f\right) \subseteq F$. Observe that in the first step, we take statewise combinations over the event A of either f or a constant act with either f or a constant act; so, by the expected utility representation for A-measurable acts, we end up with either a constant utility act or a positive affine transformation of the state-by-state utility of f. Either way the resulting acts are affinely related to f. In the second step, we take any two acts constructed in step 1 and combine them statewise. Since both of these acts are affinely related to the act f, the resulting act will also be affinely related to f. This argument plus continuity shows that the set $\overline{S}^f$ consists only of acts that are affinely related to f. In fact, if f is not a constant act, the set $\overline{S}^f$ contains *all* acts that are affinely related to f. (This is shown in Proposition A.37 in section A.4.)

Using these sets of statewise combinations, we are ready to state the appropriate weakening of act-independence.

Axiom 6. ***($\overline{S}$-act-independence)*** *For any f, g, $h \in F$ such that there exist l, $k \in \{f, g\}$ for which f, $h \in \overline{S}^l$ and g, $h \in \overline{S}^k$, if f', $g' \in F$ are such that, for either $B = A$ or $B = A^c$,*

$$f'(\omega) \sim h(\omega)_B f(\omega) \text{ for all } \omega \in \Omega$$

and

$$g'(\omega) \sim h(\omega)_B g(\omega) \text{ for all } \omega \in \Omega$$

then,

$$f \succeq g \Leftrightarrow f' \succeq g'.$$

$\overline{S}$-act-independence requires act-independence only for certain set of acts $f, g, h \in F$. Using the representation for A-measurable acts and the definition of $\overline{S}^f$, it can be shown that the axiom applies only if: (1) h is a constant act; or (2) h is not a constant act, but f, g and h are pairwise affinely related.

It is useful to compare $\overline{S}$-act-independence to the certainty-independence (C-independence) axiom of Gilboa and Schmeidler [9]:

(C-independence) *For any acts f and g, any constant act h, and any $\alpha \in (0, 1)$, if f', g' are such that,*

$$f' = \alpha f + (1 - \alpha)h$$

and

$$g' = \alpha g + (1 - \alpha)h$$

then,

$$f \succeq g \Leftrightarrow f' \succeq g'.$$

Note that the convex combination operation is defined statewise and is well-defined since acts in their setting are functions from states to probability distributions over prizes. C-independence relaxes the independence axiom of Anscombe and Aumann [1] so that it is only required to hold when the third act, h, is a constant act.

$\overline{S}$-act-independence and C-independence are quite similar in form, with two salient differences. First, statewise combinations over A or A^c replace convex combinations. Second, as pointed out in possibility (2) above, $\overline{S}$-act-independence applies to some h which are not constant acts. In fact, the first difference leads to the second one. In an Anscombe-Aumann framework, all probabilities in the unit interval are available (see also our section 4.1 where α in the constant-independence and uncertainty aversion axioms may be any number between 0 and 1). Consequently, to express the fact that preference is preserved by homogeneous transformations (of utility),[3] one need only consider convex combinations of the act in question and the constant act that gives utility 0 in each state. In contrast, the only effective probabilities that are available through statewise combinations over A are those of A and A^c. So, for example, if the revealed probability of A happens to be $\frac{1}{3}$ and we want to show that multiplying the utility of a pair of acts by $\frac{1}{2}$ preserves the preference ordering between them, we cannot construct the "$\frac{1}{2}$-acts" through statewise combinations over A without taking combinations of two non-constant acts. This is why we cannot restrict h to be a constant act in the $\overline{S}$-act-independence axiom. This is the only place where possibility (2) above is important. (In particular, it is used in the proof of Lemma A.14).

The final axiom, act-uncertainty aversion, restricts the way that act-independence can be violated. Essentially it requires that the decision-maker weakly like to smooth utilities across states of the world, since this leaves her less exposed to any uncertainty or ambiguity about the probability of various states. In an Anscombe-Aumann framework, Gilboa and Schmeidler's uncertainty aversion axiom states that if two acts are indifferent, any convex combination of these two acts should be at least as preferred as either of these acts. The following seems to be the natural generalization of such an axiom to the Savage acts framework:

Axiom 7. ***(Act-uncertainty aversion)*** *For all $f, g, f' \in F$, if $f \sim g$ and*

$$f'(\omega) \sim f(\omega)_A\, g(\omega) \text{ for all } \omega \in \Omega$$

then

$$f' \succeq f.$$

Thus, as in the earlier axioms, statewise combinations over A play the role of mixtures. These seven axioms lead us to our first main result.

[3] This is true for an MMEU representation.

3.2. A Representation Theorem

Theorem 3.7. *Let $\succeq$ be a binary relation on F. Then $\succeq$ satisfies Axioms 1-7 if and only if there exists a continuous and strictly increasing function $u : X \longrightarrow \mathbb{R}$, and a non-empty, compact and convex set C of finitely additive probability measures on Σ such that*

$$[f \succeq g] \Leftrightarrow \left[\min_{P \in C} \int u \circ f dP \geq \min_{P \in C} \int u \circ g dP\right] \text{ for all } f \text{ and } g \in F.$$

Furthermore, there exists an event $A \in \Sigma$ and a $\rho \in (0,1)$ such that $P \in C$ implies $P(A) = \rho$. Moreover, $u(\cdot)$ is unique up to positive affine transformations and the set C is unique.

Proof. See section A.2 for the proof that the axioms are sufficient and see section A.3 for the proof that the axioms are necessary. As the sufficiency proof relies in part on the sufficiency proof of the more general representation theorem below, that theorem will be proved first. ■

4. A General Representation Theorem

In this section, we provide a more general characterization of MMEU, in that we no longer require the existence of an unambiguous event. A characterization without an unambiguous event is of interest given that ambiguity or lack thereof may truly be in the eye of the beholder. The cost of this greater generality is a somewhat more complex axiomatization. In particular, the statewise combinations over A which were so helpful in the previous section become much less useful in the absence of an unambiguous event. We work instead with the somewhat more involved, and possibly less familiar, tool of standard sequence constructions.

4.1. Axioms

Weak order, continuity, and monotonicity are taken exactly as in the previous section. Structure, A-act-independence, $\overline{S}$-act-independence, and act-uncertainty aversion of the previous section are replaced by the four axioms below.

Axiom 8. ***(General structure)***

(a) $x > y \Rightarrow x^* \succ y^*$.

(b) There exists an event $A \in \Sigma$ such that A is ordered non-null and ordered non-universal.

Note that part (b) requires less structure on the set of events than in the previous section, in that no requirements are placed on A^c.

The next axiom is a weakening of A-act-independence. We will refer to an act $f = x_A y$ as an *ordered A-measurable act* if $x \preceq y$.

Axiom 9. ***(Ordered A-act-independence)***[4] *Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ be such that $x_1 \preceq x_2$, $y_1 \preceq y_2$ and $z_1 \preceq z_2$. Let $f = x_{1A}x_2$, $g = y_{1A}y_2$ and $h = z_{1A}z_2$. Then,*

(i) if $\{x_i, y_i\} \succeq z_i$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim h(\omega)_A f(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim h(\omega)_A g(\omega) & \text{for all } \omega \in \Omega \end{cases}$ then $[f \succeq g \Leftrightarrow f' \succeq g']$;

(ii) if $z_i \succeq \{x_i, y_i\}$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim f(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim g(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \end{cases}$ then $[f \succeq g \Leftrightarrow f' \succeq g']$.

Observe that there are two differences from A-act-independence. The first is the restriction to specific orderings; that is, f, g and h are all ordered A-measurable acts. The second is that the axiom applies only when either f and g both dominate h (case *(i)*) or h dominates both f and g (case *(ii)*). Within each case, the only statewise combinations allowed are required to have the less preferred prizes on the event A (rather than on A^c). This is because, in this more general theory, we do not assume that A^c is ordered non-null or ordered non-universal. Therefore, the restriction on statewise combinations is needed to ensure that non-zero weight is placed on f and g.

Just as in the previous section, these two axioms together with the axioms weak order, continuity and monotonicity imply the existence of an expected utility representation. However, the representation here holds only for *ordered* A-measurable acts. That is, there is a strictly increasing and continuous function $u : X \to \mathbb{R}$ and a real number $\rho \in (0, 1)$ such that for all x, y, v, $w \in X$, if $x \preceq y$ and $v \preceq w$ then

$$x_A y \succeq v_A w \Leftrightarrow \rho u(x) + (1 - \rho) u(y) \geq \rho u(v) + (1 - \rho) u(w).$$

Moreover, u is unique up to positive affine transformations and ρ is unique.

[4]This is essentially a restriction of Chew and Karni's [2] comonotonic act-independence axiom to the event A and A-measurable acts. We choose the term ordered to distinguish this axiom from the one that is appropriate for the "reverse" ordering discussed in section 5.

Observe that the decision maker attaches probability ρ to the event A when evaluating ordered A-measurable acts. Thus, while A is not necessarily unambiguous, when evaluating only ordered A-measurable acts it behaves as if it were. This is the weakest version of act-independence for A-measurable acts that is compatible with MMEU.

Recall that in the $\overline{S}$-act-independence axiom, the set $\overline{S}^f$ gave us a convenient way to describe the set of acts that were affinely related to f. Now, since the event A is not necessarily an unambiguous event, the set $\overline{S}^f$ need no longer contain only affinely related acts. This occurs because statewise combinations on ambiguous events can combine utilities with differing weights depending on the ordering of the prizes involved. Therefore we will need an alternative approach to identifying affinely related acts. The approach we follow uses *standard sequences.*

Definition 4.1. *Given $B \in \Sigma$ which is neither ordered null nor ordered universal and a and $b \in X$ with $a \succ b$, we define a standard sequence with respect to a and b, denoted by $SS^B_{a,b}$, as a set $\{a_i : a_i \in X, i \in \mathbb{Z}_{++}\}$ for which either (i) $a \preceq a_i$ and $a_B a_i \sim b_B a_{i+1}$ for all $i \in \mathbb{Z}_{++}$; or (ii) $a_i \preceq b$ and $a_{iB} b \sim a_{i+1B} a$ for all $i \in \mathbb{Z}_{++}$.*

We use standard sequences as rulers with which to calibrate preferences. Any consecutive elements in a standard sequence are the same "distance" apart, in that the difference between receiving, say, a_{i+1} instead of a_i on the event B^c just compensates for the difference between receiving b instead of a on the event B. Since axioms weak order, continuity, monotonicity, general structure, and ordered A-act-independence guarantee an expected utility representation for A-measurable acts, we can translate this notion of distance into utility terms by forming standard sequences using the event A. In particular, it is not hard to see that in any standard sequence using the event A,

$$u(a_{i+1}) - u(a_i) = \frac{\rho}{1-\rho}\left[u(a) - u(b)\right].$$

Hence, having an expected utility representation for ordered A-measurable acts, is what lets us associate distance measured by standard sequences with distances measured by ratios of utility differences.[5]

Definition 4.2. *Given $n \in \mathbb{Z}_{++}$, we define $SS(n) := SS^A_{m+\frac{1}{n},m} \cup \{a_0\}$ where $a_0 = m$ and $a_1 = m + \frac{1}{n}$.*

[5]For an excellent discussion of standard sequences and their use in both preference measurement and other types of measurement, see Krantz, Luce, Suppes, and Tversky [12].

$\{SS(n)\}_{n=1}^{\infty}$ is a sequence of finer and finer grained standard sequences. As n grows large, we can use these rulers to measure more and more accurately the ratios of utility differences between any two pairs of prizes. Using this idea, we state an independence axiom that replaces $\overline{S}$-act-independence.

The easiest way to understand this axiom is by thinking about it in terms of utilities. Fix acts f, g, f', g' and a constant act h. The idea is that for all ω, if f' and g' are such that $u(f'(\omega)) = \alpha u(h) + (1-\alpha)\, u(f(\omega))$ and $u(g'(\omega)) = \alpha u(h) + (1-\alpha)\, u(g(\omega))$ for some $\alpha \in (0,1)$, then (assuming $u(h) - u(f(\omega)) \neq 0$),

$$\frac{u(f'(\omega)) - u(f(\omega))}{u(h) - u(f(\omega))} = \alpha$$

and, (assuming $u(h) - u(g(\omega)) \neq 0$),

$$\frac{u(g'(\omega)) - u(g(\omega))}{u(h) - u(g(\omega))} = \alpha.$$

In the limit, the standard sequence approximations of these ratios are correct. In this way we can "check" to see if f' and g' are a given convex combination of h with f and g. If so, the axiom requires that the preference between f and g be the same as the preference between the convex combination. Thus, this axiom is a standard sequence approach to implementing Gilboa and Schmeidler's C-Independence axiom.

Axiom 10. ***(Constant-independence)*** *Let f, g, f' and $g' \in F$, $h \in F^*$. For each $\omega \in \Omega$ and $n \in \mathbb{Z}_{++}$*

(i) *let $a_{i_\omega(n)} \in SS(n)$ be such that $a_{i_\omega(n)+1} \succeq f(\omega) \succeq a_{i_\omega(n)}$*
(ii) *let $a_{i'_\omega(n)} \in SS(n)$ be such that $a_{i'_\omega(n)+1} \succeq f'(\omega) \succeq a_{i'_\omega(n)}$*
(iii) *let $a_{j_\omega(n)} \in SS(n)$ be such that $a_{j_\omega(n)+1} \succeq g(\omega) \succeq a_{j_\omega(n)}$*
(iv) *let $a_{j'_\omega(n)} \in SS(n)$ be such that $a_{j'_\omega(n)+1} \succeq g'(\omega) \succeq a_{j'_\omega(n)}$*
(v) *let $a_{k(n)} \in SS(n)$ be such that $a_{k(n)+1} \succeq h \succeq a_{k(n)}$.*

If for some $\alpha \in (0,1)$, we have

for all $\omega \in \Omega$ such that $f(\omega) \nsim h$, $\lim_{n\to\infty} \frac{i'_\omega(n) - i_\omega(n)}{k(n) - i_\omega(n)} = \alpha$
and
for all $\omega \in \Omega$ such that $f(\omega) \sim h$, $f'(\omega) \sim h$

and

for all $\omega \in \Omega$ such that $g(\omega) \nsim h$, $\lim_{n\to\infty} \frac{j'_\omega(n) - j_\omega(n)}{k(n) - j_\omega(n)} = \alpha$
and
for all $\omega \in \Omega$ such that $g(\omega) \sim h$, $g'(\omega) \sim h$

then,

$$f \succeq g \Leftrightarrow f' \succeq g'.$$

Recalling the discussion of $\overline{S}$-act-independence, note that since α may be any element of $(0,1)$, constant-independence need only consider cases where h is a constant act.

Our final axiom is a standard sequence approach to uncertainty aversion.

Axiom 11. ***(Uncertainty aversion)*** *Let f, g, $h \in F$ and $f \sim g$. For all $\omega \in \Omega$ and $n \in \mathbb{Z}_{++}$,*

(i) let $a_{i_\omega(n)} \in SS(n)$ be such that $a_{i_\omega(n)+1} \succeq f(\omega) \succeq a_{i_\omega(n)}$
(ii) let $a_{j_\omega(n)} \in SS(n)$ be such that $a_{j_\omega(n)+1} \succeq g(\omega) \succeq a_{j_\omega(n)}$
(iii) let $a_{k_\omega(n)} \in SS(n)$ be such that $a_{k_\omega(n)+1} \succeq h(\omega) \succeq a_{k_\omega(n)}$.

If for some $\alpha \in (0,1)$ we have

for all $\omega \in \Omega$ such that $f(\omega) \nsim g(\omega)$, $\lim_{n\to\infty} \frac{k_\omega(n)-i_\omega(n)}{j_\omega(n)-i_\omega(n)} = \alpha$
and
for all $\omega \in \Omega$ such that $f(\omega) \sim g(\omega)$, $h(\omega) \sim f(\omega)$

then,

$$h \succeq f.$$

Again, the easiest way to understand this axiom is by thinking about it in terms of utilities. Here, if $f \sim g$ and h is such that for all ω, $u(h(\omega)) = \alpha u(g(\omega)) + (1-\alpha)\, u(f(\omega))$ then the axiom requires $h \succeq f$.

4.2. A Representation Theorem

Theorem 4.3. *Let $\succeq$ be a binary relation on F. Then $\succeq$ satisfies axioms weak order, continuity, monotonicity, general structure, ordered A-act-independence, constant-independence and uncertainty aversion if and only if there exists a continuous and strictly increasing function $u : X \to \mathbb{R}$, and a non-empty, compact and convex set $\mathcal{C}$ of finitely additive probability measures on Σ such that*

$$[f \succeq g] \Leftrightarrow \left[\min_{P\in\mathcal{C}} \int u \circ f dP \geq \min_{P\in\mathcal{C}} \int u \circ g dP\right] \text{ for all } f \text{ and } g \in F.$$

Furthermore, there exists an event $A \in \Sigma$ for which $0 < \max_{P\in\mathcal{C}} P(A) < 1$. Moreover, $u(\cdot)$ is unique up to positive affine transformations and the set $\mathcal{C}$ is unique.

Proof. See section A.1 for the proof that the axioms are sufficient and see section A.3 for the proof that the axioms are necessary. ■

5. Discussion and Conclusion

5.1. Discussion of null and universal events

In the above theory, we used the concepts of ordered null and ordered universal events. These concepts differ from the more familiar notions of null and universal as in Savage [15]. A natural question is why the ordered notion is appropriate in our setting and what occurs when an ordered non-null and ordered non-universal event does not exist. To address this issue formally, consider the following alternative definitions of null and universal (including ordered null and ordered universal as above):

Definition 5.1. *A Savage null event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$. An ordered null event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$ whenever $x \preceq y \preceq z$. A reverse-ordered null event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$ whenever $z \preceq y \preceq x$.*

Definition 5.2. *A Savage universal event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$. An ordered universal event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$ whenever $z \preceq y \preceq x$. A reverse-ordered universal event $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$ whenever $x \preceq y \preceq z$.*

Hence, an event B is *Savage non-null* if there exist x, y and z in X such that $x_B z \nsim y_B z$, is *ordered non-null* if there exist x, y and z in X with $x \preceq y \preceq z$ such that $x_B z \nsim y_B z$ and is *reverse-ordered non-null* if there exist x, y and z in X with $z \preceq y \preceq x$ such that $x_B z \nsim y_B z$. Likewise, an event B is *Savage non-universal* if there exist x, y and z in X such that $z_B x \nsim z_B y$, is *ordered non-universal* if there exist x, y and z in X with $z \preceq y \preceq x$ such that $z_B x \nsim z_B y$ and is *reverse-ordered non-universal* if there exist x, y and z in X with $x \preceq y \preceq z$ such that $z_B x \nsim z_B y$.

Since the Savage notions of non-null and non-universal are the most permissive, we begin by asking what can be said in the case where there does not exist an event that is Savage non-null and Savage non-universal. In such a case, an event will either always get zero weight or always get all the weight in determining preferences over acts. There will never be any trade-off between prizes on one event and prizes on another. Therefore, preferences will only be ordinally determined, and thus the utility function, u, in the representation will be determined only up to increasing transformations. There will be a unique probability measure that assigns weight 0 to all Savage null events and weight 1 to all Savage universal events.

To explore the remaining possibilities, assume that there exists a Savage non-null and Savage non-universal event A. Since A is Savage non-null, there exist x, y, z such that $x_A z \nsim y_A z$. Let's consider the possible orderings of x, y and z:[6]
(i) If $\{x, y\} \succeq z$ then A is reverse-ordered non-null.
(ii) If $z \succeq \{x, y\}$ then A is ordered non-null.
(iii) If $x \succ z \succ y$ then, by monotonicity, part (a) and $x_A z \nsim y_A z$, we have $x_A z \succ y_A z$. Also by monotonicity, part (a), $x_A z \succeq z_A z \succeq y_A z$, with at least one of the preference relations being strict. Hence, there are two cases:

Case 1: $x_A z \succ z_A z$. Here, A is reverse-ordered non-null.

Case 2: $z_A z \succ y_A z$. This implies A is ordered non-null.

Thus, A is either ordered or reverse-ordered non-null.

Since A is Savage non-universal, there exist x, y, z such that $z_A x \nsim z_A y$. Let's consider the possible orderings of x, y and z (again assuming $x \succeq y$):
(i) If $\{x, y\} \succeq z$ then, A is ordered non-universal.
(ii) If $z \succeq \{x, y\}$ then, A is reverse-ordered non-universal.
(iii) If $x \succ z \succ y$ then, by monotonicity, part (a), $z_A x \succ z_A y$. Also by monotonicity, part (a), $z_A x \succeq z_A z \succeq z_A y$, with at least one of the preference relations being strict. Hence, there are two cases:

Case 1: $z_A x \succ z_A z$. Here, A is ordered non-universal.

Case 2: $z_A z \succ z_A y$. This implies A is reverse-ordered non-universal.

Thus, A is either ordered or reverse-ordered non-universal.

Therefore, assuming A Savage non-null and Savage non-universal implies that at least one of the following four possibilities must be true:
(1) A is ordered non-null and ordered non-universal.
(2) A is reverse-ordered non-null and reverse-ordered non-universal.
(3) A is ordered null and reverse-ordered universal.
(4) A is reverse-ordered null and ordered universal.

Case (1) is the one considered in the general axiomatization in the main body of the paper. Case (2) can be handled with a few small modifications to the axioms. Specifically,

1. replace "ordered" with "reverse-ordered" in the general structure axiom,

2. replace monotonicity, part (b) with the following:

[6]Without loss of generality, we may assume $x \succeq y$, as x and y play symmetric roles in all that follows.

If $B \in \Sigma$ is reverse-ordered non-null and $z \preceq x$ and $z \preceq y$, then $x_B z \succeq y_B z$ implies $x \succeq y$. If $B \in \Sigma$ is reverse-ordered non-universal and $x \preceq z$ and $y \preceq z$, then $z_B x \succeq z_B y$ implies $x \succeq y$. and,

3. replace ordered A-act-independence with the following:

Axiom 12. ***(Reverse-ordered A-act-independence)*** *Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ be such that $x_1 \succeq x_2$, $y_1 \succeq y_2$ and $z_1 \succeq z_2$. Let $f = x_{1A}x_2$, $g = y_{1A}y_2$ and $h = z_{1A}z_2$. Then,*

(i) if $\{x_i, y_i\} \preceq z_i$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim h(\omega)_A f(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim h(\omega)_A g(\omega) & \text{for all } \omega \in \Omega \end{cases}$ then $[f \succeq g \Leftrightarrow f' \succeq g']$;

(ii) if $z_i \preceq \{x_i, y_i\}$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim f(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim g(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \end{cases}$ then $[f \succeq g \Leftrightarrow f' \succeq g']$.

With these modifications, the axioms are equivalent to the same representation as in the general theorem except that $0 < \min_{p \in C} P(A) < 1$ instead of $0 < \max_{p \in C} P(A) < 1$.

Case (3) implies *all* weight is placed on the more preferred prize when evaluating A-measurable acts. Similarly, case (4) implies placing *all* weight on the less preferred prize when evaluating A-measurable acts. Therefore, in these cases there is no trade-off between any two events, and preferences are only ordinally determined.

In sum, if we wish to have an MMEU representation with meaningful cardinal preferences, then we need an event that is non-null and non-universal according to *either* the ordered or reverse-ordered notions. Note that the structure axiom, part (b) in the development with an unambiguous event is equivalent to requiring that A is non-null and non-universal in *both* the ordered and reverse-ordered sense. In the context of expected utility, any Savage non-null and non-universal event must be both ordered and reverse-ordered non-null and non-universal.

5.2. Conclusion

In this paper, we have provided two axiomatizations of MMEU over Savage acts. These axiomatizations require a rich prize space and continuous preferences, but place virtually no restrictions on the state space. The key axioms in the treatment with an unambiguous event were $\overline{S}$-act-independence and act-uncertainty aversion, each of which weaken the act-independence condition in Gul [10]. Another work that weakens act-independence is Chew and Karni [2]. They characterize CEU on a finite state space using a comonotonic act-independence axiom. In the presence of our other axioms, $\overline{S}$-act-independence is weaker than comonotonic act-independence.

In the treatment without an unambiguous event, we use ordered A-act-independence to give meaning to standard sequences as a way of measuring preference distances. Then, using standard sequences, we are able to state the needed independence and uncertainty aversion axioms. In both axiomatizations, the essential feature involves deriving a cardinal utility function over prizes and showing that preference between two acts is preserved under convex combination (in utility space) with a third act that is affinely related to the first two.

A. Proofs

A.1. Sufficiency of the Axioms for the General Representation Theorem

Given any $f \in F$, denote its certainty equivalent by $m(f)$. That is, $m(f)$ is an element in F^* such that $m(f) \sim f$. For $x_1, x_2 \in X$, $B \in \Sigma$, define $m^B(x_1, x_2) \equiv m(x_{1B}x_2)$. The next lemma shows that a unique certainty equivalent exists for each act.

Lemma A.1. *For each $f \in F$, $m(f)$ exists and is unique.*

Proof. By Axiom 5 (monotonicity), part(a), the sets $M(f)_c := \{x \in X | x \succeq f\}$ and $W(f)_c := \{x \in X | f \succeq x\}$ are non-empty. By continuity, both of these sets are closed. By weak order, $M(f)_c \cup W(f)_c = X$. Therefore, since X is connected, $M(f)_c \cap W(f)_c \neq \emptyset$ and there must be at least one $x \in X$ such that $x \sim f$. Suppose there are two such prizes, $x_1 < x_2$. Then, by the structure axiom, part (a), and weak order, $x_1 \prec x_2 \sim f$, a contradiction. ■

Definition A.2. *We say that $\succeq$ is bounded if, for each $f \in F$, there are $x, y \in X$ such that $x \succeq f \succeq y$.*

Consider the following axioms following Nakamura [13]:

A1. $\preceq$ on F is a bounded weak order.

A2. For $f \in F$, $B \in \Sigma$ and $x, y, z \in X$, if $x_B z \preceq f \preceq y_B z$ then $f \sim a_B z$ for some $a \in X$.

A3. If $B \in \Sigma$ is ordered non-null and $x \preceq z$ and $y \preceq z$, then $x \preceq y$ if and only if $x_B z \preceq y_B z$; if $B \in \Sigma$ is ordered non-universal and $z \preceq x$ and $z \preceq y$, then $x \preceq y$ if and only if $z_B x \preceq z_B y$.

A4. If $x \preceq y$ and $B \subseteq C$ then $x_C y \preceq x_B y$.

A5. Every strictly bounded standard sequence is finite.

A6. If $x_1 \preceq x_2$ and $y_1 \preceq y_2$ with $x_i \preceq y_i$ for $i = 1, 2$, then

$$m^B(x_1, x_2)_B \, m^B(y_1, y_2) \sim m^B(x_1, y_1)_B \, m^B(x_2, y_2).$$

The next two lemmas show that our axioms for the general case imply Nakamura's A1-A5 and A6 with $B = A$. This allows us to use several lemmas from Nakamura [13] to show that an expected utility representation for ordered A-measurable acts exists.

Lemma A.3. *Axioms weak order, continuity, monotonicity, general structure and ordered A-act-independence imply A1-A5.*

Proof.

- (Axioms$\Rightarrow$A1) This is implied by weak order and monotonicity.

- (Axioms$\Rightarrow$A2) Consider $W(f)_B = \{b \in X \mid b_B z \preceq f\}$ and $M(f)_B = \{b \in X \mid f \preceq b_B z\}$. $W(f)_B$ and $M(f)_B$ are non-empty because $x \in W(f)_B$ and $y \in M(f)_B$ by assumption. We want to show that $W(f)_B$ is closed. To do this we will show that $X - W(f)_B$ is open, where $X - W(f)_B = \{b \in X \mid b_B z \succ f\}$. Let $W(f)^c = \{g \in F \mid g \succ f\}$. Let t be in $X - W(f)_B$. (If such t does not exist then $X - W(f)_B = \emptyset$ and $W(f)_B = [m, M]$ which is closed, so we are done.) Since $t \in X - W(f)_B$, we have that $t_B z \in W(f)^c$. By continuity, we have that $W(f)^c$ is open. Hence $\exists \delta > 0$ such that $\forall g \in U_\delta(t_B z)$ we have that $g \in W(f)^c$, where $U_\delta(t_B z) = \{g \in F \mid \| g - t_B z \| < \delta\}$. Consider now the set $V_\delta(t_B z) = \{b \in X \mid \| b_B z - t_B z \| < \delta\}$. Note that $\{g \in F \mid g = b_B z \text{ for some } b \in V_\delta(t_B z)\} \subset U_\delta(t_B z)$. Since $\forall g \in U_\delta(t_B z)$ we have that $g \succ f$, we must have that $\forall b \in V_\delta(t_B z)$, $b_B z \succ f$, implying that $X - W(f)_B$ is open. Hence $W(f)_B$ is closed. Similarly $M(f)_B$ is closed. By weak order, $W(f)_B \cup M(f)_B = X$. Since X is connected, $W(f)_B \cap M(f)_B \neq \emptyset$. Therefore there exists $a \in X$ such that $a_B z \sim f$.

- (Axioms$\Rightarrow$A3) This follows from Axiom 5 (monotonicity).

- (Axioms$\Rightarrow$A4) This follows from Axiom 5 (monotonicity), part (a).

- (Axioms$\Rightarrow$A5) Let $B \in \Sigma$ be neither ordered null nor ordered universal and fix $a, b \in X$ such that $a \succ b$. Let $\{a_i\}$ be a strictly bounded standard sequence with respect to a and b. Let's assume first that $\{a_i\}$ is such that $a \preceq a_i$ for $i \in \mathbb{Z}_{++}$, and $a_B a_i \sim b_B a_{i+1}$ for $i \in \mathbb{Z}_{++}$. Since B is ordered non-null and $b \prec a \preceq a_{i+1}$, monotonicity implies $a_B a_i \sim b_B a_{i+1} \prec a_B a_{i+1}$. Thus, $a_i \prec a_{i+1}$ for $i \in \mathbb{Z}_{++}$ and $\{a_i\}$ is an increasing sequence. As $\{a_i\}$ is strictly bounded it must in particular be bounded above. Hence, if $\{a_i\}$ has infinitely many terms then it must converge. Towards a contradiction, assume $\{a_i\}$ has infinitely many terms. Let $a^* = \lim_{i \to \infty} a_i$. Consider $W(b_B a^*)$. Since for $i \in \mathbb{Z}_{++}$ we have that $a_B a_i \sim b_B a_{i+1} \prec b_B a^*$, we have that for each $i \in \mathbb{Z}_{++}$, $a_B a_i \in W(b_B a^*)$. Take limits to obtain $\lim_{i \to \infty} a_B a_i = a_B a^*$. By continuity, we have $a_B a^* \in W(b_B a^*)$. That is $a_B a^* \preceq b_B a^*$. Now, B ordered non-null, $b \prec a$ and $\{b, a\} \preceq a^*$, together with the monotonicity axiom, implies $b_B a^* \prec a_B a^*$, which is a contradiction. So $\{a_i\}$ must have only a finite number of terms. The other case where

$\{a_i\}$ is such that $b \succeq a_i$ for $i \in \mathbb{Z}_{++}$, and $a_{iB}b \sim a_{i+1B}a$ for $i \in \mathbb{Z}_{++}$ follows from a similar argument making use of the fact that B is not universal and $\{a_i\}$ is decreasing and bounded below. Note that Nakamura's definition of a standard sequence also allows for decreasing $\{a_i\}$ when $a \preceq a_i$ and increasing $\{a_i\}$ when $b \succeq a_i$. It is easy to adapt the arguments just given above to show that $\{a_i\}$ must be finite in these cases as well. ■

Lemma A.4. *Axioms weak order, continuity, monotonicity, general structure and ordered A-act-independence imply A6 with $B = A$.*

Proof. We divide the argument into two cases depending on preference between y_1 and $m^A(x_1, x_2)$. The argument in each case will require two applications of ordered A-act-independence.

Case 1: $y_1 \succeq m^A(x_1, x_2)$. Let f, g and $h \in F$ be such that:

$$\begin{aligned} f &= x_{1A}x_2 \\ g &= m^A(x_1, x_2) \\ h &= y_{1A}y_2. \end{aligned}$$

Consider f' and $g' \in F$ with, for all $\omega \in \Omega$,

$$\begin{aligned} f'(\omega) &\sim f(\omega)_A h(\omega) &= \begin{cases} x_{1A}y_1 & , \omega \in A \\ x_{2A}y_2 & , \omega \in A^c \end{cases} \\ g'(\omega) &\sim g(\omega)_A h(\omega) &= \begin{cases} m^A(x_1, x_2)_A y_1 & , \omega \in A \\ m^A(x_1, x_2)_A y_2 & , \omega \in A^c \end{cases}. \end{aligned}$$

Since $f \sim g$, ordered A-act-independence implies $f' \sim g'$. That is,

$$m^A(x_1, y_1)_A m^A(x_2, y_2) \sim m^A\left(m^A(x_1, x_2), y_1\right)_A m^A\left(m^A(x_1, x_2), y_2\right).$$

Now, let $\widehat{f}$, $\widehat{g}$ and $\widehat{h} \in F$ be such that:

$$\begin{aligned} \widehat{f} &= y_{1A}y_2 \\ \widehat{g} &= m^A(y_1, y_2) \\ \widehat{h} &= m^A(x_1, x_2). \end{aligned}$$

Consider $\widehat{f}'$ and $\widehat{g}' \in F$ with, for all $\omega \in \Omega$,

$$\begin{aligned} \widehat{f}'(\omega) &\sim \widehat{h}(\omega)_A \widehat{f}(\omega) &= \begin{cases} m^A(x_1, x_2)_A y_1 & , \omega \in A \\ m^A(x_1, x_2)_A y_2 & , \omega \in A^c \end{cases} \\ \widehat{g}'(\omega) &\sim \widehat{h}(\omega)_A \widehat{g}(\omega) &= m^A(x_1, x_2)_A m^A(y_1, y_2). \end{aligned}$$

Since $\widehat{f} \sim \widehat{g}$, ordered A-act-independence implies $\widehat{f}' \sim \widehat{g}'$. Hence,

$$m^A\left(m^A\left(x_1, x_2\right), y_1\right)_A m^A\left(m^A\left(x_1, x_2\right), y_2\right) \sim m^A\left(x_1, x_2\right)_A m^A\left(y_1, y_2\right).$$

It follows that

$m^A\left(x_1, x_2\right)_A m^A\left(y_1, y_2\right) \sim m^A\left(x_1, y_1\right)_A m^A\left(x_2, y_2\right)$, which is A6 with $B = A$.

Case 2: $m^A\left(x_1, x_2\right) \succ y_1$. Note that $y_1 \succeq x_1$ together with $m^A\left(x_1, x_2\right) \succ y_1$ implies $x_2 \succeq y_1$. Also, $x_2 \succeq y_1$ together with $x_2 \succeq x_1$ implies $x_2 \succeq m^A\left(x_1, y_1\right)$.

Now, let f, g and $h \in F$ be such that:

$$\begin{aligned} f &= x_{1A}y_1 \\ g &= m^A\left(x_1, y_1\right) \\ h &= x_{2A}y_2. \end{aligned}$$

Consider $f', g' \in F$ with, for all $\omega \in \Omega$,

$$\begin{aligned} f'(\omega) &\sim f(\omega)_A h(\omega) = \begin{cases} x_{1A}x_2 & , \omega \in A \\ y_{1A}y_2 & , \omega \in A^c \end{cases} \\ g'(\omega) &\sim g(\omega)_A h(\omega) = \begin{cases} m^A\left(x_1, y_1\right)_A x_2 & , \omega \in A \\ m^A\left(x_1, y_1\right)_A y_2 & , \omega \in A^c \end{cases}. \end{aligned}$$

Since $f \sim g$, ordered A-act-independence implies $f' \sim g'$. That is,

$$m^A\left(x_1, x_2\right)_A m^A\left(y_1, y_2\right) \sim m^A\left(m^A\left(x_1, y_1\right), x_2\right)_A m^A\left(m^A\left(x_1, y_1\right), y_2\right).$$

Finally, let $\widehat{f}$, $\widehat{g}$ and $\widehat{h} \in F$ be such that:

$$\begin{aligned} \widehat{f} &= x_{2A}y_2 \\ \widehat{g} &= m^A\left(x_2, y_2\right) \\ \widehat{h} &= m^A\left(x_1, y_1\right). \end{aligned}$$

Consider $\widehat{f}', \widehat{g}' \in F$ with, for all $\omega \in \Omega$,

$$\begin{aligned} \widehat{f}'(\omega) &\sim \widehat{h}(\omega)_A \widehat{f}(\omega) = \begin{cases} m^A\left(x_1, y_1\right)_A x_2 & , \omega \in A \\ m^A\left(x_1, y_1\right)_A y_2 & , \omega \in A^c \end{cases} \\ \widehat{g}'(\omega) &\sim \widehat{h}(\omega)_A \widehat{g}(\omega) = m^A\left(x_1, y_1\right)_A m^A\left(x_2, y_2\right), \omega \in \Omega. \end{aligned}$$

Since $\widehat{f} \sim \widehat{g}$, ordered A-act-independence implies $\widehat{f}' \sim \widehat{g}'$. Hence,

$$m^A\left(m^A\left(x_1, y_1\right), x_2\right)_A m^A\left(m^A\left(x_1, x_2\right), y_2\right) \sim m^A\left(x_1, y_1\right)_A m^A\left(x_2, y_2\right).$$

It follows that

$m^A\left(x_1, x_2\right)_A m^A\left(y_1, y_2\right) \sim m^A\left(x_1, y_1\right)_A m^A\left(x_2, y_2\right)$, which is A6 with $B = A$. ■

Lemma A.5. *There is a strictly increasing, continuous function $u : X \to \mathbb{R}$ and a real number $\pi(A) \in (0,1)$ such that for all x, y, v, $w \in X$, if $x \preceq y$ and $v \preceq w$ then*

$$\begin{aligned} & x_A y \succeq v_A w \\ \Leftrightarrow \quad & \pi(A)\, u(x) + (1-\pi(A))\, u(y) \geq \pi(A)\, u(v) + (1-\pi(A))\, u(w)\,. \end{aligned} \tag{A.1}$$

Moreover, u is unique up to positive affine transformations and $\pi(A)$ is unique.

Proof. Existence of a function u and a real number $\pi(A)$ satisfying all the conditions of the lemma other than strict monotonicity and continuity, follows from Lemmas A.3 and A.4 and Nakamura [13], Lemmas 1, 2 and 3.

To see that u is strictly increasing, assume $x > v$. By part (a) of the general structure axiom, we have $x \succ v$. Apply the already proven part of the lemma to x, $y = x$, v, $w = v$ to obtain $u(x) > u(v)$. Continuity of u follows from the following argument: Since u is strictly increasing, the only discontinuities can be (an at most countable number of) jumps up. Therefore, limits from above and from below exist at each point in X. Suppose there is a jump of height $\delta > 0$ at $\hat{x} \in X$. Consider the case where $u(\hat{x}) = \lim_{y \to \hat{x}^+} u(y)$. By definition of δ, $u(\hat{x}) - \delta = \lim_{y \to \hat{x}^-} u(y)$. By (A.1), if $y \prec \hat{x}$, then $y_A \hat{x} \sim x$ only if $\pi(A)\, u(y) + (1-\pi(A))\, u(\hat{x}) = u(x)$. Since $u(\hat{x}) - \delta = \lim_{y \to \hat{x}^-} u(y)$, for any $\varepsilon > 0$, there exists $\hat{y} \prec \hat{x}$ such that $u(\hat{y}) > u(\hat{x}) - \delta - \varepsilon$. Then, fixing $\varepsilon < \frac{1-\pi(A)}{\pi(A)}\delta$,

$$\begin{aligned} u(\hat{x}) &> \pi(A)\, u(\hat{y}) + (1-\pi(A))\, u(\hat{x}) \\ &> \pi(A)\,(u(\hat{x}) - \delta - \varepsilon) + (1-\pi(A))\, u(\hat{x}) \\ &= u(\hat{x}) - \pi(A)\,(\delta + \varepsilon) \\ &> u(\hat{x}) - \delta \\ &= \lim_{y \to \hat{x}^-} u(y)\,. \end{aligned}$$

But this implies that $\hat{y}_A \hat{x}$ has no certainty equivalent, contradicting Lemma A.1. The case where $u(\hat{x}) = \lim_{y \to \hat{x}^-} u(y)$ generates a contradiction by a similar argument. This shows u can have no jumps. ■

Remark 1. *u^{-1} is continuous.*

Proof. Since u is strictly increasing and continuous, so is u^{-1}. ■

Now we use this u function to show the implications, in utility terms, of the constant-independence and uncertainty aversion axioms.

Lemma A.6. *Let $f, g \in F$, $h \in F^*$ and $\alpha \in (0,1)$. If*

(i) $f' \in F$ is such that, for all $\omega \in \Omega$,

$$u(f'(\omega)) = \alpha u(h) + (1-\alpha)u(f(\omega))$$

and

(ii) $g' \in F$ is such that, for all $\omega \in \Omega$,

$$u(g'(\omega)) = \alpha u(h) + (1-\alpha)u(g(\omega))$$

then

$$f \succeq g \Leftrightarrow f' \succeq g'.$$

Proof. Observe that if $a_{i_\omega(n)}, a_{i_\omega(n)+1} \in SS(n)$ and $i_\omega(n) \geq 1$, then

$$\pi(A)\, u(m + \frac{1}{n}) + (1 - \pi(A))u(a_{i_\omega(n)}) = \pi(A)\, u(m) + (1 - \pi(A))u(a_{i_\omega(n)+1}).$$

Therefore,

$$u(a_{i_\omega(n)+1}) - u(a_{i_\omega(n)}) = \frac{\pi(A)}{1 - \pi(A)} \left[u(m + \frac{1}{n}) - u(m) \right].$$

Define $\Delta(n) \in \mathbb{R}$ by $\Delta(n) := \frac{\pi(A)}{1-\pi(A)} \left[u(m + \frac{1}{n}) - u(m) \right]$.
For each $\omega \in \Omega$ and $n \in \mathbb{Z}_{++}$

(i) let $a_{i_\omega(n)} \in SS(n)$ be such that $a_{i_\omega(n)+1} \succeq f(\omega) \succeq a_{i_\omega(n)}$
(ii) let $a_{i'_\omega(n)} \in SS(n)$ be such that $a_{i'_\omega(n)+1} \succeq f'(\omega) \succeq a_{i'_\omega(n)}$
(iii) let $a_{j_\omega(n)} \in SS(n)$ be such that $a_{j_\omega(n)+1} \succeq g(\omega) \succeq a_{j_\omega(n)}$
(iv) let $a_{j'_\omega(n)} \in SS(n)$ be such that $a_{j'_\omega(n)+1} \succeq g'(\omega) \succeq a_{j'_\omega(n)}$
(v) let $a_{k(n)} \in SS(n)$ be such that $a_{k(n)+1} \succeq h \succeq a_{k(n)}$.

If $f(\omega) \neq m$, for n large enough, this implies that $(i_\omega(n)-1)\Delta(n) \leq u(f(\omega)) - u(m + \frac{1}{n}) \leq i_\omega(n)\Delta(n)$. If $f(\omega) = m$ then $u(f(\omega)) = u(m)$ and $i_\omega(n) = 0$. In either case since $\lim_{n\to\infty} \Delta(n) = 0$ and u is continuous, these in turn imply that $\lim_{n\to\infty} i_\omega(n)\Delta(n) = u(f(\omega)) - u(m)$.

We can show using similar arguments that $\lim_{n\to\infty} i'_\omega(n)\Delta(n) = u(f'(\omega)) - u(m)$, $\lim_{n\to\infty} j_\omega(n)\Delta(n) = u(g(\omega)) - u(m)$, $\lim_{n\to\infty} j'_\omega(n)\Delta(n) = u(g'(\omega)) - u(m)$ and $\lim_{n\to\infty} k(n)\Delta(n) = u(h) - u(m)$.

First, let's look at the case where $u(h) \neq u(f(\omega))$. Observe that $\frac{i'_\omega(n) - i_\omega(n)}{k(n) - i_\omega(n)} = \frac{i'_\omega(n)\Delta(n) - i_\omega(n)\Delta(n)}{k(n)\Delta(n) - i_\omega(n)\Delta(n)}$. But then $\lim_{n\to\infty} \frac{i'_\omega(n) - i_\omega(n)}{k(n) - i_\omega(n)} = \lim_{n\to\infty} \frac{i'_\omega(n)\Delta(n) - i_\omega(n)\Delta(n)}{k(n)\Delta(n) - i_\omega(n)\Delta(n)} =$

$\frac{u(f'(\omega))-u(f(\omega))}{u(h)-u(f(\omega))} = \alpha$. Second, look at the case where $u(h) = u(f(\omega))$. In this case $f'(\omega) \sim f(\omega) \sim h$.

Similarly, let's look at the case where $u(h) \neq u(g(\omega))$. Observe that $\frac{j'_\omega(n)-j_\omega(n)}{k(n)-j_\omega(n)} = \frac{j'_\omega(n)\Delta(n)-j_\omega(n)\Delta(n)}{k(n)\Delta(n)-j_\omega(n)\Delta(n)}$. But then $\lim_{n\to\infty} \frac{j'_\omega(n)-j_\omega(n)}{k(n)-j_\omega(n)} = \lim_{n\to\infty} \frac{j'_\omega(n)\Delta(n)-j_\omega(n)\Delta(n)}{k(n)\Delta(n)-j_\omega(n)\Delta(n)} = \frac{u(g'(\omega))-u(g(\omega))}{u(h)-u(g(\omega))} = \alpha$. Finally, look at the case where $u(h) = u(g(\omega))$. In this case $g'(\omega) \sim g(\omega) \sim h$.

Applying the constant-independence axiom yields $f \succeq g \Leftrightarrow f' \succeq g'$. ■

Lemma A.7. *Let $f, g \in F$ and $\alpha \in (0,1)$. Suppose $f \sim g$. If $h \in F$ is such that, for all $\omega \in \Omega$,*

$$u(h(\omega)) = \alpha u(g(\omega)) + (1-\alpha)u(f(\omega))$$

then

$$h \succeq f.$$

Proof. By the arguments in the proof of Lemma A.6, the hypotheses of the uncertainty aversion axiom are satisfied for such $f, g, h \in F$. Applying uncertainty aversion yields $h \succeq f$. ■

We next construct a real-valued representation of preferences over acts by fixing u and assigning each act the utility of its certainty equivalent.

Lemma A.8. *Given a $u : X \to \mathbb{R}$ from lemma A.5, there is a unique $J : F \to \mathbb{R}$ such that:*

(i) for all f and $g \in F$, $f \succeq g$ if and only if $J(f) \geq J(g)$;

(ii) for any constant act $f = x \in X$, $J(f) = u(x)$.

Proof. For constant acts, we uniquely define $J(\cdot)$ by (ii). For general acts $f \in F$, let $J(f) = u(m(f))$. Clearly, $J(\cdot)$ satisfies (i) and is unique. ■

Remark 2. *For any x, $y \in X$ such that $x \preceq y$,*

$$J(x_A y) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y)$$

where $\pi(A)$ is given by Lemma A.5.

Proof. Let u be the utility function used in the construction of J. By Lemma A.5, $x_A y \sim m^A(x, y)$ implies $u\left(m^A(x, y)\right) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y)$. Note that J has been constructed so that $J(x_A y) = u\left(m^A(x, y)\right)$. Hence, $J(x_A y) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y)$. ■

Let $K = u(X)$. Since u is continuous, K is a closed interval in $\mathbb{R}$. We normalize u such that $K = [-2, 2]$. Let B be the space of bounded (in the supnorm), Σ-measurable, real valued functions on Ω. For $\gamma \in \mathbb{R}$, we denote by γ^* the element of B that assigns γ to every ω. Let $B(K)$ be the subset of functions in B with values in K. Observe that for $f \in F$, $u \circ f \in B(K)$, and for $d \in B(K)$ there exists $f \in F$ such that $u \circ f = d$. Now we use this observation to construct a functional on $B(K)$ that represents preferences.

Definition A.9. *For $f \in F$ we define the functional $I : B(K) \to \mathbb{R}$ by $I(u \circ f) = J(f)$.*

Since J represents preferences, it is clear that I does as well. The next lemma shows that I satisfies several important properties, and that these properties may be preserved when extending I from $B(K)$ to all of B.

Lemma A.10. *$I : B(K) \to \mathbb{R}$ may be extended to all of B in such a way that:*
(i) $I(1^*) = 1$;
(ii) (I is monotonic) For all a, $b \in B$, $a \geq b$ implies $I(a) \geq I(b)$;
(iii) (I is homogeneous of degree 1) For all $b \in B$, $\alpha \geq 0$, $I(\alpha b) = \alpha I(b)$;
(iv) (I is C-independent) For all $b \in B$, $\gamma \in \mathbb{R}$, $I(b + \gamma^*) = I(b) + I(\gamma^*)$; and,
(v) (I is superadditive) For all a, $b \in B$, $I(a + b) \geq I(a) + I(b)$.

Proof. First note that there exists $x \in X$ such that $u(x) = 1$. By construction then, $I(1^*) = J(x^*) = u(x) = 1$. Also, monotonicity of I on $B(K)$ follows directly from Axiom 5. We will now show that I is homogeneous of degree 1 on $B(K)$.

It suffices to prove homogeneity for $\alpha \in [0, 1]$, as $\alpha > 1$ then follows by considering the reciprocal. First note that there exists $z \in X$ such that $u(z) = 0$. Suppose for a, $b \in B(K)$, $a = \alpha b$ for some $\alpha \in (0, 1)$. (The cases $\alpha = 0$ and $\alpha = 1$ are trivial.) Let f, $g \in F$ be such that $u \circ f = a$ and $u \circ g = b$. Then for all $\omega \in \Omega$, $u(f(\omega)) = \alpha u(g(\omega)) + (1 - \alpha)u(z)$. Now let $y \in X$ be such that $y \sim g$. Also let $x \in X$ be such that $u(x) = \alpha u(y) + (1 - \alpha)u(z)$. By Lemma A.6, $g \sim y$ implies $f \sim x$. Thus, $I(a) = J(u \circ f) = u(x) = \alpha u(y) = \alpha J(u \circ g) = \alpha I(b)$.

This shows that I is homogeneous of degree 1 on $B(K)$. Next, we extend I to all of B by homogeneity. Such an extension preserves homogeneity and monotonicity. It remains to be shown that I is C-independent and superadditive.

We now demonstrate C-independence of I. Consider $a \in B$ and $\gamma \in \mathbb{R}$. By homogeneity, we may assume without loss of generality that $\max\left(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)}\right) a \in$

$B(K)$ and $\max\left(\frac{1}{1-\pi(A)},\frac{1}{\pi(A)}\right)\gamma^{*}\in B(K)$. Note that by the structure of $B(K)$ (in particular the fact that K is an interval around 0), it follows that $\frac{1}{1-\pi(A)}a\in B(K)$ and $\frac{1}{\pi(A)}\gamma^{*}\in B(K)$. Define $\beta=I\left(\frac{1}{1-\pi(A)}a\right)$. By homogeneity, $\beta=\frac{1}{1-\pi(A)}I(a)$. Let $f\in F$ be such that $u\circ f=\frac{1}{1-\pi(A)}a$. Let $y,z\in X$ satisfy $u(y)=\beta$ and $u(z)=\frac{1}{\pi(A)}\gamma$. By construction of I, $J(f)=\beta$ and $J(y)=u(y)=\beta$, implying $f\sim y$. Now, let $g'\in F^{*}$ be the constant act such that, for all $\omega\in\Omega$,

$$u\left(g'(\omega)\right)=\pi(A)\,u(z)+(1-\pi(A))\,u(y).$$

Thus, $u\left(g'(\omega)\right)=\gamma+(1-\pi(A))\beta=I(\gamma^{*})+I(a)$.

Now, let $f'\in F$ be an act such that, for all $\omega\in\Omega$,

$$u\left(f'(\omega)\right)=\pi(A)\,u(z)+(1-\pi(A))\,u(f(\omega)).$$

By Lemma A.6 and the previously noted fact that $f\sim y$, we have $f'\sim g'$. Therefore, $I(a+\gamma^{*})=J(f')=J(g')=I(a)+I(\gamma^{*})$ and I is C-Independent.

Finally, we show that I is superadditive. Consider $a,b\in B$. As above, by homogeneity we may assume without loss of generality that $\max\left(\frac{1}{1-\pi(A)},\frac{1}{\pi(A)}\right)a\in B(K)$ and $\max\left(\frac{1}{1-\pi(A)},\frac{1}{\pi(A)}\right)b\in B(K)$. Specifically, this implies $\frac{1}{1-\pi(A)}a\in B(K)$ and $\frac{1}{\pi(A)}b\in B(K)$. Let acts $f,g\in F$ be such that $u\circ f=\frac{1}{\pi(A)}b$ and $u\circ g=\frac{1}{1-\pi(A)}a$. The argument proceeds by considering the possible orderings of $I\left(\frac{1}{1-\pi(A)}a\right)$ and $I\left(\frac{1}{\pi(A)}b\right)$.

Case 1: $I\left(\frac{1}{1-\pi(A)}a\right)=I\left(\frac{1}{\pi(A)}b\right)$. Then $f\sim g$. Define the act f' by, for all $\omega\in\Omega$,

$$u(f'(\omega))=(1-\pi(A))u(g(\omega))+\pi(A)u(f(\omega)).$$

Thus, $u\circ f'=\pi(A)(u\circ f)+(1-\pi(A))(u\circ g)=b+a$ and $J(f')=I(u\circ f')=I(a+b)$. By Lemma A.7, we have that $f'\succeq f$. Therefore, $I(a+b)=J(f')\geq J(f)=\frac{1}{\pi(A)}I(b)=\left(\frac{1-\pi(A)+\pi(A)}{\pi(A)}\right)I(b)=\left(\frac{1-\pi(A)}{\pi(A)}\right)I(b)+I(b)=I(a)+I(b)$, since $I(a)=(1-\pi(A))\,I\left(\frac{1}{\pi(A)}b\right)$.

Case 2: $I\left(\frac{1}{\pi(A)}b\right)>I\left(\frac{1}{1-\pi(A)}a\right)$. Let $\gamma=I\left(\frac{1}{\pi(A)}b\right)-I\left(\frac{1}{1-\pi(A)}a\right)>0$. Let $\frac{1}{1-\pi(A)}c=\frac{1}{1-\pi(A)}a+\gamma^{*}$. By C-independence of I, $I\left(\frac{1}{1-\pi(A)}c\right)=I\left(\frac{1}{1-\pi(A)}a\right)+\gamma=I\left(\frac{1}{\pi(A)}b\right)$. By case 1, $I(c+b)\geq I(c)+I(b)$. But $I(c+b)=I(a+(1-\pi(A))\gamma^{*}+b)=$

$I(a+b) + (1-\pi(A))\gamma$ by C-independence. Similarly, $I(c) = I(a + (1-\pi(A))\gamma^*) = I(a) + (1-\pi(A))\gamma$. Thus, $I(a+b) + (1-\pi(A))\gamma = I(c+b) \geq I(c) + I(b) = I(a) + I(b) + (1-\pi(A))\gamma$. Thus, $I(a+b) \geq I(a) + I(b)$. The third and final case, where $I\left(\frac{1}{\pi(A)}b\right) < I\left(\frac{1}{1-\pi(A)}a\right)$, is proved similarly. This shows that I is superadditive and completes the proof of the lemma. ■

The importance of Lemma A.10 is made clear by the next result which states that such an I may be written as the minimum expectation over a compact and convex set of finitely additive probability measures.

Lemma A.11. *Let $I : B \to \mathbb{R}$ be a functional satisfying:*
(i) $I(1^*) = 1$;
(ii) $I(a) \geq I(b)$ if $a \geq b$ for all $a, b \in B$;
(iii) $I(a+b) \geq I(a) + I(b)$ for all $a, b \in B$;
(iv) $I(\alpha a + \beta 1^*) = \alpha I(a) + \beta$ for all $a \in B$, $\alpha \geq 0$ and $\beta \in \mathbb{R}$.
Then there exists a unique convex and w-compact set $\mathcal{C} \subseteq \mathcal{P}$ such that*

$$I(a) = \min_{P \in \mathcal{C}} \int a dP \quad \text{for all } a \in B.$$

Proof. See Gilboa and Schmeidler [9], Lemma 3.5. ■

Observe that *(i)* - *(v)* in Lemma A.10 imply that I satisfies *(i)* - *(iv)* of Lemma A.11. Therefore, we may represent $\succeq$ on F by $J(f) = I(u \circ f) = \min_{P \in \mathcal{C}} \int u \circ f dP$ with $\mathcal{C}$ unique, convex and w*-compact and u strictly increasing, continuous and unique up to positive affine transformations. This representation together with the representation (A.1) in Lemma A.5 imply that $\max_{P \in \mathcal{C}} P(A) = \pi(A)$ and $0 < \max_{P \in \mathcal{C}} P(A) < 1$. This proves sufficiency of the axioms in Theorem 4.3.

A.2. Sufficiency of the axioms for the case when there exists an unambiguous event

The proof will proceed by showing that the axioms in this case imply the axioms for the more general case.

Remark 3. Observe that the structure axiom implies the general structure axiom and A-act-independence implies ordered A-act-independence. Therefore, the proofs of Lemmas A.1-A.8 apply to the unambiguous event case as well and we have a strictly increasing, continuous u and a unique J defined from u and an I defined from u and J such that $I(1^*) = 1$ and I is monotonic.

The next lemma shows that there exists an expected utility representation not only for preferences over ordered A-measurable acts (as in the general theorem), but for preferences over *all* A-measurable acts. This will mean that the event A will always be assigned the same probability in the representation, and thus will be our unambiguous event.

Lemma A.12. *Axioms weak order, structure, continuity, monotonicity and A-act-independence imply the representation (A.1) holds for all* $x, y, v, w \in X$.

Proof. By the previous lemma, (A.1) holds for all x, y, v, w such that $y \succeq x$ and $w \succeq v$. Note however that A-act-independence implies that a version of ordered A-act-independence with A^c replacing A holds as well. Thus the following representation holds: If $x \preceq y$ and $v \preceq w$, then

$$\begin{aligned} & x_{A^c}y \succeq v_{A^c}w \\ \Leftrightarrow \quad & \pi\left(A^c\right)u\left(x\right)+\left(1-\pi\left(A^c\right)\right)u\left(y\right) \geq \pi\left(A^c\right)u\left(v\right)+\left(1-\pi\left(A^c\right)\right)u\left(w\right). \end{aligned} \tag{A.2}$$

Combining this representation with that in (A.1), we can represent preferences over all A-measurable acts. If $y \succeq x$ then in evaluating $x_A y$ we use weights $\pi(A), 1-\pi(A)$. If $x \succeq y$ then in evaluating $x_A y$ we use weights $1-\pi(A^c), \pi(A^c)$. What remains to be shown is that these two sets of weights are equal, i.e., $\pi(A) = 1-\pi(A^c)$. We use A-act-independence to show this.

Let $z \succ y \succ x$. Define $f, g, h, \overline{f}, \overline{g}, \overline{h} \in F$ by

$$\begin{aligned} f &= y_A x, \\ g &= m^A(y,x), \\ h &= y_A z, \\ \overline{f} &= y_A z, \\ \overline{g} &= m^A(y,z), \\ \overline{h} &= y_A x. \end{aligned}$$

By monotonicity, $y \succeq m^A(y,x)$. Define $f', g', \overline{f}', \overline{g}' \in F$ by

$$\begin{aligned} f'(\omega) &\sim f(\omega)_A h(\omega) && \text{for all } \omega \in \Omega, \\ g'(\omega) &\sim g(\omega)_A h(\omega) && \text{for all } \omega \in \Omega, \\ \overline{f}'(\omega) &\sim \overline{f}(\omega)_A \overline{h}(\omega) && \text{for all } \omega \in \Omega, \\ \overline{g}'(\omega) &\sim \overline{g}(\omega)_A \overline{h}(\omega) && \text{for all } \omega \in \Omega. \end{aligned}$$

Observe that $f \sim g$ and $\overline{f} \sim \overline{g}$. A-act-independence then implies that $f' \sim g'$ and $\overline{f}' \sim \overline{g}'$. Writing these indifferences explicitly yields,

$$y_A m^A(x,z) \sim m^A(m^A(y,x),y)_A m^A(m^A(y,x),z)$$

and

$$m^A(m^A(y,x),y)_A m^A(m^A(y,x),z) \sim m^A(y,x)_A m^A(y,z),$$

which together imply,

$$y_A m^A(x,z) \sim m^A(y,x)_A m^A(y,z). \tag{A.3}$$

We will now show that (A.3) along with the representations (A.1) and (A.2) imply $\pi(A) = 1 - \pi(A^c)$.

There are two cases to consider.

Case 1: $m^A(x,z) \succeq y$. Then (A.3) implies,

$$\begin{aligned}
&\pi(A)u(y) + (1-\pi(A))u(m^A(x,z)) = \\
&\pi(A)u(y) + (1-\pi(A))(\pi(A)u(x) + (1-\pi(A))u(z)) = \\
&\pi(A)u(m^A(y,x)) + (1-\pi(A))u(m^A(y,z)) = \\
&\pi(A)(\pi(A^c)u(x) + (1-\pi(A^c))u(y)) + (1-\pi(A))(\pi(A)u(y) + (1-\pi(A))u(z)).
\end{aligned}$$

But this implies

$$\pi(A)(1-\pi(A)-\pi(A^c))u(x) = \pi(A)(1-\pi(A)-\pi(A^c))u(z).$$

Since $u(x) < u(z)$ and $\pi(A) > 0$, this equality can hold only when $\pi(A) = 1 - \pi(A^c)$.

Case 2: $y \succ m^A(x,z)$. Then (A.3) implies,

$$\begin{aligned}
&(1-\pi(A^c))u(y) + \pi(A^c)u(m^A(x,z)) = \\
&(1-\pi(A^c))u(y) + \pi(A^c)(\pi(A)u(x) + (1-\pi(A))u(z)) = \\
&\pi(A)u(m^A(y,x)) + (1-\pi(A))u(m^A(y,z)) = \\
&\pi(A)(\pi(A^c)u(x) + (1-\pi(A^c))u(y)) + (1-\pi(A))(\pi(A)u(y) + (1-\pi(A))u(z)).
\end{aligned}$$

But this implies

$$(1-\pi(A))(-1+\pi(A)+\pi(A^c))u(y) = (1-\pi(A))(-1+\pi(A)+\pi(A^c))u(z).$$

Since $u(y) < u(z)$ and $1 - \pi(A) > 0$, this equality can hold only when $\pi(A) = 1 - \pi(A^c)$. ■

It remains to be shown that the axioms in the unambiguous case imply the constant-independence and uncertainty aversion axioms. To do this, we use the fact that the I functional is homogeneous of degree 1 and that $\overline{S}^f$ contains all acts that are multiples of f in terms of utility. We prove these facts in the next lemma.

Lemma A.13. *Axioms 1-6 imply I is homogeneous of degree 1 on $B(K)$, i.e. $I(\alpha b) = \alpha I(b)$ for all $b, \alpha b \in B(K)$, $\alpha \geq 0$. Moreover if some $g \in F$ is such that $u \circ g = b$ then there exists $f' \in \overline{S}^g$ such that $u \circ f' = \alpha b$.*

Proof.

It suffices to prove homogeneity for $\alpha \in [0,1]$, as $\alpha > 1$ then follows by considering the reciprocal. First we need five lemmas:

The first lemma shows that the statewise combination over A of any two acts in the same $\overline{S}$ set is indifferent to the statewise combination over A of their certainty equivalents. Two applications of the $\overline{S}$-act-independence axiom are used to show this. This result is then used in Lemma A.15.

Lemma A.14. *If $h \in F$, $f_1, f_2 \in \overline{S}^h, x_1, x_2 \in X, x_1 \sim f_1, x_2 \sim f_2$ then*

$$f(\omega) \sim f_1(\omega)_A f_2(\omega), \tag{A.4}$$

and

$$x \sim x_{1A} x_2, \tag{A.5}$$

implies

$$f \sim x.$$

Proof. Let g be defined as follows:

$$g(\omega) \sim f_1(\omega)_A x_2, \tag{A.6}$$

Since $f_1, x_1, x_2 \in \overline{S}^h$, (A.5) and (A.6) imply (by $\overline{S}$-act-independence),

$$g \sim x \Leftrightarrow f_1 \sim x_1$$

To finish the proof we need to show that $f \sim g$. To see this note that $f_1, f_2, x_2 \in \overline{S}^h$. Then (A.4) and (A.6) will imply (again by $\overline{S}$-act-independence), $f_2 \sim x_2 \Leftrightarrow f \sim g$. ■

The following lemma provides a single step in the homogeneity argument. Specifically, it shows that if I satisfies homogeneity with respect to two specific coefficients, then I must also be homogeneous with respect to two specific weighted averages of these coefficients.

Lemma A.15. *Let $b \in B(K)$ and $s, t \in [0,1]$. Let $\underline{a} = sb$, $\overline{a} = tb$, $s' = \pi(A)s + (1 - \pi(A))t$ and $t' = (1 - \pi(A))s + \pi(A)t$. Suppose*
(i) there exists $f_1, f_2 \in \overline{S}^g$ for some $g \in F$ such that $u \circ f_1 = \underline{a}$ and $u \circ f_2 = \overline{a}$, and

(ii) $I(sb) = sI(b)$ *and* $I(tb) = tI(b)$.
Then,
(iii) there exists $h_1, h_2 \in \overline{S}^g$ *such that* $u \circ h_1 = s'b$ *and* $u \circ h_2 = t'b$,
and
(iv) $I(s'b) = s'I(b)$ *and* $I(t'b) = t'I(b)$.

Proof. Let $x_1 \sim f_1$ and $x_2 \sim f_2$ where $x_1, x_2 \in X$. Let h_1 be defined as follows: For all $\omega \in \Omega$,

$$h_1(\omega) \sim f_1(\omega)_A f_2(\omega). \tag{A.7}$$

This implies $u \circ h_1 = \pi(A)\underline{a} + (1 - \pi(A))\overline{a} = s'b$.

Let $x \in X$ be defined as

$$x \sim x_{1A}x_2. \tag{A.8}$$

This implies $u(x) = \pi(A)u(x_1) + (1 - \pi(A))u(x_2)$.

By Lemma A.14, $h_1 \sim x$. So, $I(s'b) = J(h_1) = u(x) = \pi(A)u(x_1) + (1 - \pi(A))u(x_2) = \pi(A)J(f_1) + (1 - \pi(A))J(f_2) = \pi(A)I(\underline{a}) + (1 - \pi(A))I(\overline{a}) = (\pi(A)s + (1 - \pi(A))t)I(b) = s'I(b)$. Note that $h_1 \in \overline{S}^g$.

The argument for h_2, defined by, for all $\omega \in \Omega$,

$$h_2(\omega) \sim f_2(\omega)_A f_1(\omega) \tag{A.9}$$

proceeds similarly. ■

In proving homogeneity, we will need to apply Lemma A.15 iteratively. In fact, we will want to be able to take the limit of an infinite sequence of iterations. To ensure that homogeneity is preserved in the limit, we must show that $I(\alpha b)$ is continuous in α. To this end, the next lemma shows that J is continuous.

Lemma A.16. $J : F \rightarrow \mathbb{R}$ *is continuous.*

Proof. Let A be an arbitrary open set in $[u(m), u(M)] \subset \mathbb{R}$. Since u continuous and strictly increasing, J is onto $[u(m), u(M)]$ by construction. Let O_i, $i = 1, 2, ...$ be a collection of disjoint open intervals such that $\cup_{i\in\Lambda} O_i = A$, where $\Lambda \subset \mathbb{Z}_{++}$. Let $O_i = (\underline{o}_i, \overline{o}_i)$. Fix $g_i \in J^{-1}(\underline{o}_i)$and $g'_i \in J^{-1}(\overline{o}_i)$. Then, $J^{-1}((\underline{o}_i, \overline{o}_i)) = \{f \in F | f \succ g_i\} \cap \{f \in F | g'_i \succ f\}$. By the continuity axiom each of the sets in this intersection is open, therefore their intersection is also open. Note now that $J^{-1}(A) = \cup_{i\in\Lambda} J^{-1}((\underline{o}_i, \overline{o}_i))$, and thus it is an open set. Hence, we have that $J^{-1}(\cdot)$ maps open sets to open sets, therefore $J(\cdot)$ is continuous. ■

Lemma A.17. $I(\alpha b)$ *is continuous in* α *for* $b \in B(K)$, $\alpha \in (0, 1]$.

Proof. Let $g \in F$ be such that $u \circ g = b$. Let α, $\alpha' \in (0,1]$. Define f', $f \in F$ such that, for all $\omega \in \Omega$,

$$u(f'(\omega)) = \alpha' u(g(\omega)) \text{ and,}$$
$$u(f(\omega)) = \alpha u(g(\omega)).$$

Thus,

$$f'(\omega) = u^{-1}(\alpha' u(g(\omega))) \text{ and,}$$
$$f(\omega) = u^{-1}(\alpha u(g(\omega))).$$

By Remark 1, $u^{-1}(\cdot)$ is continuous. Therefore, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\alpha' u(g(\omega)) - \alpha u(g(\omega))| < \delta$ then $\left|u^{-1}(\alpha' u(g(\omega))) - u^{-1}(\alpha u(g(\omega)))\right| = |f'(\omega) - f(\omega)| < \varepsilon$. That is, if $|\alpha' - \alpha| < \frac{\delta}{|u(g(\omega))|}$ then $|f'(\omega) - f(\omega)| < \varepsilon$. Note that $|u(g(\omega))| \leq 2$ by our normalization of u. Therefore, if $|\alpha' - \alpha| < \frac{\delta}{2}$ then $|f'(\omega) - f(\omega)| < \varepsilon$. Since this is true for any ω, if $|\alpha' - \alpha| < \frac{\delta}{2}$ then $\|f' - f\| < \varepsilon$.

Fix $\gamma > 0$. By continuity of J, there exists $\psi > 0$ such that if $\|f' - f\| < \psi$ then $|J(f') - J(f)| = |I(\alpha' b) - I(\alpha b)| < \gamma$. By continuity of u^{-1} and the above argument, there exists $\lambda > 0$ such that if $|\alpha' - \alpha| < \lambda$ then $\|f' - f\| < \psi$. Therefore, $I(\alpha b)$ is continuous in α for $b \in B(K)$, $\alpha \in (0,1]$. ■

Now we show that if I satisfies homogeneity with respect to two specific coefficients, then I must also be homogeneous with respect to the average of these coefficients. To do this we take limits of the convex combinations used in Lemma A.15. Notice that this limiting argument is needed because a "half-half" combination may only be able to be reached in the limit if the weight on A or A^c is not a power of $\frac{1}{2}$.

Lemma A.18. *Let $b \in B(K)$ and $s_o, t_o \in [0,1]$. Let $\underline{a} = s_0 b$ and $\overline{a} = t_0 b$. Suppose*
(i) there exists $f_1^0, f_2^0 \in \overline{S}^g$ for some $g \in F$ such that $u \circ f_1^0 = \underline{a}$ and $u \circ f_2^0 = \overline{a}$ and,
(ii) $I(s_0 b) = s_0 I(b)$ and $I(t_0 b) = t_0 I(b)$.
Then
(iii) $I(\frac{s_o + t_o}{2} b) = \frac{s_o + t_o}{2} I(b)$ and there exists $f \in \overline{S}^g$ such that $u \circ f = \frac{s_o + t_o}{2} b$.

Proof. Suppose w.l.o.g. that $s_0 < t_0$. Define $\{s_i\}$ and $\{t_i\}$ by $s_i = \pi(A) s_{i-1} + (1 - \pi(A)) t_{i-1}$ and $t_i = (1 - \pi(A)) s_{i-1} + \pi(A) t_{i-1}$ for $i = 1, 2, 3...$

We will show that $\lim_{i \to \infty} s_i = \lim_{i \to \infty} t_i = \frac{s_o + t_o}{2}$. The proof will proceed in three cases.

Case 1: $\pi(A) = \frac{1}{2}$.
If $\pi(A) = \frac{1}{2}$, then $s_i = t_i = \frac{s_0 + t_0}{2}$ for all i. So $\lim_{i \to \infty} s_i = \lim_{i \to \infty} t_i = \frac{s_o + t_o}{2}$.
Case 2: $1 > \pi(A) > \frac{1}{2}$.

We will use an induction argument to prove this case. Observe that $s_1 < \frac{s_0+t_0}{2} < t_1$ and $s_1 + t_1 = s_0 + t_0$. Suppose for all $k \leq n$, $s_k < \frac{s_0+t_0}{2} < t_k$ and $s_k + t_k = s_0 + t_0$. We now show $s_{n+1} < \frac{s_0+t_0}{2} < t_{n+1}$ and $s_{n+1} + t_{n+1} = s_0 + t_0$. Since $s_n + t_n = s_0 + t_0$, s_n and t_n are equidistant from $\frac{s_0+t_0}{2}$. Thus $s_{n+1} = \pi(A)s_n + (1 - \pi(A))t_n < \frac{s_0+t_0}{2} < (1 - \pi(A))s_n + \pi(A)t_n = t_{n+1}$. Also, $s_{n+1} + t_{n+1} = (\pi(A) + 1 - \pi(A))s_n + (\pi(A) + 1 - \pi(A))t_n = s_n + t_n = s_0 + t_0$. So for any $j = 0, 1, 2, ...$, $s_j < \frac{s_0+t_0}{2} < t_j$ and $s_j + t_j = s_0 + t_0$. Observe that $s_j > s_{j-1}$ and $t_j < t_{j-1}$ for all $j = 1, 2, 3, ...$ Since $\{s_i\}$ and $\{t_i\}$ are monotone bounded sequences, $\lim_{i\to\infty} s_i$ and $\lim_{i\to\infty} t_i$ exist. Furthermore $\lim_{i\to\infty} s_i \leq \frac{s_0+t_0}{2} \leq \lim_{i\to\infty} t_i$. Let $\overline{s} = \lim_{i\to\infty} s_i$ and $\overline{t} = \lim_{i\to\infty} t_i$. Suppose $\overline{s} < \overline{t}$. Fix any $\varepsilon > 0$. There exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, $\overline{s} - s_n < \varepsilon$ and $t_n - \overline{t} < \varepsilon$. Consider $s_{n+1} = \pi(A)s_n + (1 - \pi(A))t_n > \pi(A)(\overline{s} - \varepsilon) + (1 - \pi(A))\overline{t} = \pi(A)\overline{s} + (1 - \pi(A))\overline{t} - \varepsilon\pi(A)$. But for ε small enough $s_{n+1} > \overline{s}$, a contradiction. Therefore $\overline{s} = \overline{t} = \frac{s_0+t_0}{2}$.

Case 3: $\frac{1}{2} > \pi(A) > 0$.

Observe that $t_1 < \frac{s_0+t_0}{2} < s_1$, $s_1 + t_1 = s_0 + t_0$, $s_2 < \frac{s_0+t_0}{2} < t_2$ and $s_2 + t_2 = s_0 + t_0$. Using arguments similar to those in case 2 one can show:

$$\begin{array}{lll} s_j < \frac{s_0+t_0}{2} < t_j, & s_j + t_j = s_0 + t_0, & j \text{ even,} \\ s_j > \frac{s_0+t_0}{2} > t_j, & s_j + t_j = s_0 + t_0, & j \text{ odd.} \end{array}$$

Then by the argument in case 2 applied to even and odd subsequences, $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = \frac{s_o+t_o}{2}$ for j even and, $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = \frac{s_o+t_o}{2}$ for j odd. Thus $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = \frac{s_o+t_o}{2}$.

By lemma A.15 we know that if $s_{i-1}, t_{i-1} \in [0, 1]$, if there exists $f_1^{i-1}, f_2^{i-1} \in \overline{S}^g$such that $u \circ f_1^{i-1} = s_{i-1}b$ and $u \circ f_2^{i-1} = t_{i-1}b$ and if $I(s_{i-1}b) = s_{i-1}I(b)$ and $I(t_{i-1}b) = t_{i-1}I(b)$ then there exists $f_1^i, f_2^i \in \overline{S}^g$such that $u \circ f_1^i = s_i b$ and $u \circ f_2^i = t_i b$ and $I(s_i b) = s_i I(b)$ and $I(t_i b) = t_i I(b)$. But then by induction for any $i \in \{0, 1, 2, ...\}$ we have $I(s_i b) = s_i I(b)$ and $I(t_i b) = t_i I(b)$. By lemma A.17 we have homogeneity for the limit so $I(\frac{s_0+t_0}{2}b) = \frac{s_0+t_0}{2}I(b)$. Now let $f = \lim_{i\to\infty} f_1^i$. Since $f_1^i \in \overline{S}^g$ for all i and $\overline{S}^g$ is closed, then $f \in \overline{S}^g$. Furthermore u continuous implies $u \circ f = \lim_{i\to\infty} u \circ f_1^i = \lim_{i\to\infty} s_i b = \frac{s_0+t_0}{2}b$. ■

Now we complete the proof of Lemma A.13. Let $b \in B(K)$. Let $g \in F$ be such that $u \circ g = b$ and $z \in X$ be such that $u(z) = 0$. We want to show that for any $\alpha \in [0, 1]$, if $a = \alpha b$ then $I(a) = \alpha I(b)$. First, we will show this for all $\alpha = \frac{t}{2^k}$ where $k = 0, 1, 2, ...$ and $t = 0, 1, 2, 3, ..., 2^k - 1$. We will also show that for any such α there exists $f \in \overline{S}^g$ such that $u \circ f = \alpha b$. The proof will be by induction. First we will show that the statement is true for $k = 0$. This is true since $I(0^*) = J(u^{-1}(0)^*) = u(u^{-1}(0)) = 0$. Assume that the statement is true for some $k-1 \in \{1,2,3,..\}$, in other words for all $t \in \{0,1,2,...,2^{k-1}-1\}$, assume that $I(\frac{t}{2^{k-1}}b) = \frac{t}{2^{k-1}}I(b)$ and there exists $f_t \in \overline{S}^g$ such that $u \circ f_t = \frac{t}{2^{k-1}}b$. Now we want to show that the statement is true for k, or for all $t \in \{0,1,2,...,2^k-1\}$, we want to show that $I(\frac{t}{2^k}b) = \frac{t}{2^k}I(b)$ and there exists $f_t \in \overline{S}^g$ such that $u \circ f_t = \frac{t}{2^k}b$. But by the induction hypothesis we need to only show this for $t \in \{1,3,5,...,2^k-1\}$ since for t even $\frac{t}{2^k} = \frac{t/2}{2^{k-1}}$. Since $\frac{t}{2^k}$, t odd, is exactly halfway between $\frac{t-1}{2^k}$ and $\frac{t+1}{2^k}$ lemma A.18 shows that homogeneity for the even t implies homogeneity for the odd t. Furthermore the same lemma also implies that for all t there exists $f_t \in \overline{S}^g$ such that $u \circ f_t = \frac{t}{2^k}b$.

Now that we have we have homogeneity for all $\alpha \in \{\frac{t}{2^k} : k = 0,1,2,...$ and $t = \{0,1,2,...,2^k-1\}$, we want to extend this to all $\alpha \in [0,1]$. Since for any α we can find $\alpha_j \in \{\frac{t}{2^k} : k = 0,1,2,...$ and $t = 0,1,2,3,...,2^k-1\}$ such that $\lim_{j\to\infty} \alpha_j = \alpha$, homogeneity follows for the limit by lemma A.17. Furthermore, for each α_j we know that there exists $f_j \in \overline{S}^g$ such that $u \circ f_j = \alpha_j b$. Since $\overline{S}^g$ is closed $f' = \lim_{j\to\infty} f_j \in \overline{S}^g$ and $u \circ f' = \alpha b$ since u is continuous.

This shows that I is homogeneous of degree 1 on $B(K)$. Next, we extend $I(\cdot)$ to all of B by homogeneity. Such an extension preserves homogeneity and monotonicity. ■

The next two lemmas use homogeneity of I and $\overline{S}$-act-independence to show that the axioms in the unambiguous case imply constant-independence.

Lemma A.19. *Axioms 1-6 (weak order - $\overline{S}$-act-independence) imply Lemma A.6.*

Proof. Suppose $f, g \in F$, $h \in F^*$ and $\alpha \in (0,1)$. Assume that

(i) $f' \in F$ is such that for all $\omega \in \Omega$

$$u(f'(\omega)) = \alpha u(h) + (1-\alpha)u(f(\omega))$$

and

(ii)$g' \in F$ is such that for all $\omega \in \Omega$

$$u(g'(\omega)) = \alpha u(h) + (1-\alpha)u(g(\omega)).$$

There exists $\varepsilon > 0$ such that $\varepsilon \max\{\frac{1-\alpha}{1-\pi(A)}, \frac{\alpha}{\pi(A)}\} < 1$. By Lemma A.13 there exists $l \in \overline{S}^f$ and $k \in \overline{S}^g$ such that for all $\omega \in \Omega$, $u(l(\omega)) = \varepsilon\frac{1-\alpha}{1-\pi(A)}u(f(\omega))$ and $u(k(\omega)) = \varepsilon\frac{1-\alpha}{1-\pi(A)}u(g(\omega))$. Let $t \in F^*$ be such that $u(t) = \varepsilon\frac{\alpha}{\pi(A)}u(h)$.

Now since $l, t \in \overline{S}^f$ and $k, t \in \overline{S}^g$, we can apply $\overline{S}$-act-independence. In particular let $f'', g'' \in F$ be such that $f''(\omega) \sim t_A l(\omega)$ and $g''(\omega) \sim t_A k(\omega)$ for all $\omega \in \Omega$. Note that by our representation for A-measurable acts $u(f''(\omega)) = \pi(A)u(t) + (1 - \pi(A))u(l(\omega)) = \varepsilon u(f'(\omega))$ for all $\omega \in \Omega$. Similarly $u(g''(\omega)) = \pi(A)u(t) + (1 - \pi(A))u(k(\omega)) = \varepsilon u(g'(\omega))$ for all $\omega \in \Omega$.

Now observe that

$$\begin{aligned} f' \succeq g' \quad &\Leftrightarrow J(f') \geq J(g') \\ &\Leftrightarrow I(u \circ f') \geq I(u \circ g') \\ &\Leftrightarrow \varepsilon I(u \circ f') \geq \varepsilon I(u \circ g') \\ &\Leftrightarrow I(\varepsilon u \circ f') \geq I(\varepsilon u \circ g') \\ &\Leftrightarrow I(u \circ f'') \geq I(u \circ g'') \\ &\Leftrightarrow J(f'') \geq J(g'') \\ &\Leftrightarrow f'' \succeq g'' \end{aligned}$$

where the middle equivalence follows from homogeneity of I on $B(K)$. By $\overline{S}$-act-independence, $f'' \succeq g'' \Leftrightarrow l \succeq k$. Finally, using homogeneity again, $l \succeq k \Leftrightarrow I(\varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ f) \geq I(\varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ g) \Leftrightarrow \varepsilon \frac{1-\alpha}{1-\pi(A)} I(u \circ f) \geq \varepsilon \frac{1-\alpha}{1-\pi(A)} I(u \circ g) \Leftrightarrow f \succeq g$. Therefore $f' \succeq g' \Leftrightarrow f \succeq g$. ■

Lemma A.20. *Axioms 1-6 (weak order - $\overline{S}$-act-independence) imply Axiom 10 (constant-independence).*

Proof. By the arguments in the proof of Lemma A.30, we know that if $f, g, f', g' \in F$, $h \in F^*$ are as in the constant-independence axiom, then there exists an $\alpha \in (0, 1)$ such that

(i) $f' \in F$ is such that for all $\omega \in \Omega$

$$u(f'(\omega)) = \alpha u(h) + (1 - \alpha)u(f(\omega))$$

and

(ii) $g' \in F$ is such that for all $\omega \in \Omega$

$$u(g'(\omega)) = \alpha u(h) + (1 - \alpha)u(g(\omega)).$$

The

Since Axioms 1-6 imply Lemma A.6, $f' \succeq g' \Leftrightarrow f \succeq g$. This shows constant-independence holds. ■

The next two lemmas use act-uncertainty aversion in addition to the other axioms to show the uncertainty aversion axiom holds.

Lemma A.21. *Axioms 1-7 (weak order - act-uncertainty aversion) imply Lemma A.7.*

Proof. Suppose $f, g \in F$ and $\alpha \in (0,1)$, $f \sim g$. Assume that $h \in F$ is such that for all $\omega \in \Omega$

$$u(h(\omega)) = \alpha u(g(\omega)) + (1-\alpha)u(f(\omega)).$$

There exists $\varepsilon > 0$ such that $\varepsilon \max\{\frac{1-\alpha}{1-\pi(A)}, \frac{\alpha}{\pi(A)}\} < 1$. The proof will proceed in three cases.

Case 1: $\alpha = \pi(A)$. This implies that $I\left(\varepsilon\frac{\alpha}{\pi(A)}u \circ g\right) = I\left(\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f\right)$. Let $l, k \in F$ be such that for all $\omega \in \Omega$, $u(l(\omega)) = \varepsilon\frac{1-\alpha}{1-\pi(A)}u(f(\omega))$ and $u(k(\omega)) = \varepsilon\frac{\alpha}{\pi(A)}u(g(\omega))$. By assumption $l \sim k$. Let $h' \in F$ be such that $h'(\omega) \sim k(\omega)_A l(\omega)$ for all ω. By act-uncertainty aversion, $h' \succeq k$. Observe that $\varepsilon I(u \circ h) = I(u \circ h') \geq I(u \circ k) = \varepsilon I(u \circ f)$. So $h \succeq f$.

Case 2: $\alpha > \pi(A)$. This implies that $I\left(\varepsilon\frac{\alpha}{\pi(A)}u \circ g\right) > I\left(\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f\right)$. Let $\gamma = I\left(\varepsilon\frac{\alpha}{\pi(A)}u \circ g\right) - I\left(\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f\right) > 0$. There exists $0 < \delta < 1$ such that $I(\delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f + (\delta\gamma)^*) \in B(K)$. Let $m \in F$ be such that for all $\omega \in \Omega$, $u(m(\omega)) = \delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u(f(\omega)) + \delta\gamma$. Since Axioms 1-6 imply Axioms 9 (ordered A-act-independence) and 10 (constant-independence), the proof of Lemma A.10 part (iv) implies that I is C-independent on $B(K)$. Thus $I(u \circ m) = I(\delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f + (\delta\gamma)^*) = I(\delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f) + \delta\gamma = I\left(\delta\varepsilon\frac{\alpha}{\pi(A)}u \circ g\right)$. Let $n \in F$ be such that for all $\omega \in \Omega$, $u(n(\omega)) = \delta\varepsilon\frac{\alpha}{\pi(A)}u(g(\omega))$. Let $h'' \in F$ be such that $h''(\omega) \sim n(\omega)_A m(\omega)$ for all ω. By act-uncertainty aversion, $h'' \succeq m$. Observe that

$$\begin{aligned} I(u \circ h') &= I(\pi(A)u \circ n + (1-\pi(A))u \circ m) \\ &= I(\delta\varepsilon\alpha u \circ g + \delta\varepsilon(1-\alpha)u \circ f + ((1-\pi(A))\delta\gamma)^*) \\ &= I(\delta\varepsilon u \circ h) + (1-\pi(A))\delta\gamma \\ &= \delta\varepsilon I(u \circ h) + (1-\pi(A))\delta\gamma \end{aligned}$$

and,

$$I(u \circ m) = I(\delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f) + \delta\gamma.$$

These imply that $\delta\varepsilon I(u \circ h) + (1-\pi(A))\delta\gamma \geq I(\delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f) + \delta\gamma$. Substitute $\gamma = I\left(\varepsilon\frac{\alpha}{\pi(A)}u \circ g\right) - I\left(\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f\right)$ in the previous expression and use homogeneity to obtain $I(u \circ h) \geq I(u \circ f)$. So $h \succeq f$.

Case 3: $\alpha < \pi(A)$. This case can be proved using an argument that is similar to the one used in proving Case 1. ■

Lemma A.22. *Axioms 1-7 (weak order - act-uncertainty aversion) imply Axiom 11 (uncertainty aversion).*

Proof. By the arguments in the proof of lemma A.31, we know that if $f, g, h \in F$ are as in the uncertainty aversion axiom, then there exists an $\alpha \in [0,1)$ such that $h \in F$ is such that for all $\omega \in \Omega$

$$u(h(\omega)) = (1-\alpha)u(f(\omega)) + \alpha u(g(\omega)).$$

Since axioms 1-7 imply lemma A.7, $f' \succeq g' \Leftrightarrow f \succeq g$. This shows uncertainty aversion holds. ■

Lemma A.23. *Axioms 1-7 imply that $\succeq$ may be represented on F by $J(f) = I(u \circ f) = \min_{P \in \mathcal{C}} \int u \circ f dP$ with $\mathcal{C}$ unique, convex and w*-compact and u strictly increasing, continuous and unique up to positive affine transformations. Also $0 < \pi(A) < 1$ and $P(A) = \pi(A)$ for all $P \in \mathcal{C}$.*

Proof. Axioms 1-7 imply Axioms 1, 4, 5, 8, 9, 10 and 11. Therefore, by Theorem 4.3, we may represent $\succeq$ on F by $J(f) = I(u \circ f) = \min_{P \in \mathcal{C}} \int u \circ f dP$ with $\mathcal{C}$ unique, convex and w*-compact, u unique up to positive affine transformations, $0 < \max_{P \in \mathcal{C}} P(A) < 1$ and $0 < \max_{P \in \mathcal{C}} P(A^c) < 1$. By Lemma A.12 $\pi(A) = \max_{P \in \mathcal{C}} P(A)$, $\pi(A^c) = \max_{P \in \mathcal{C}} P(A^c)$ and $\pi(A) + \pi(A^c) = 1$. But this implies $P(A) = \pi(A)$ for all $P \in \mathcal{C}$. ■

This completes the proof of sufficiency of the axioms in Theorem 3.7. ■

A.3. Necessity of the Axioms

Lemmas A.24-A.31 together prove necessity in Theorem 4.3. Note that any real-valued representation implies Axiom 1 (weak order).

Lemma A.24. *The representation in Theorem 4.3 ⇒Axiom 8 (general structure), part (a) and Axiom 2 (structure), part (a).*

Proof. Suppose $x > y$. Since u is strictly increasing, $u(x) > u(y)$. Therefore, $\min_{P \in \mathcal{C}} \int u(x)\, dP = u(x) > u(y) = \min_{P \in \mathcal{C}} \int u(y)\, dP$, which implies $x^* \succ y^*$. ■

Lemma A.25. *The representation in Theorem 4.3 ⇒Axiom 8 (general structure), part (b).*

Proof. Consider the event A referred to in the representation theorem. We will show that such event is ordered non-null and ordered non-universal.

Recall that an event B is ordered non-null if there exist x, y and $z \in X$ with $x \preceq y \preceq z$ such that $x_B z \prec y_B z$.

Pick x, y, $z \in X$ such that $x \prec y \prec z$, so that $u(x) < u(y) < u(z)$. Consider the following ordered binary acts: $f = x_A z$ and $g = y_A z$. Now,

$$\min_{P \in \mathcal{C}} \int u \circ f dP = \max_{P \in \mathcal{C}} P(A) u(x) + \left(1 - \max_{P \in \mathcal{C}} P(A)\right) u(z)$$

and

$$\min_{P \in \mathcal{C}} \int u \circ g dP = \max_{P \in \mathcal{C}} P(A) u(y) + \left(1 - \max_{P \in \mathcal{C}} P(A)\right) u(z)$$

Since $0 < \max_{P \in \mathcal{C}} P(A)$, we have $\min_{P \in \mathcal{C}} \int u \circ f dP < \min_{P \in \mathcal{C}} \int u \circ g dP$. But this implies that $x_A z \prec y_A z$. Hence A is ordered non-null.

Recall that an event B is ordered non-universal if there exist x, y and $z \in X$ with $z \preceq y \preceq x$ such that $z_B y \prec z_B x$.

Pick x, y, $z \in X$ such that $z \prec y \prec x$, so that $u(z) < u(y) < u(x)$. Consider the following ordered binary acts: $f = z_A y$ and $g = z_A x$. Now,

$$\min_{P \in \mathcal{C}} \int u \circ f dP = \max_{P \in \mathcal{C}} P(A) u(z) + \left(1 - \max_{P \in \mathcal{C}} P(A)\right) u(y)$$

and

$$\min_{P \in \mathcal{C}} \int u \circ g dP = \max_{P \in \mathcal{C}} P(A) u(z) + \left(1 - \max_{P \in \mathcal{C}} P(A)\right) u(x)$$

Since $\max_{P \in \mathcal{C}} P(A) < 1$, we have $\min_{P \in \mathcal{C}} \int u \circ f dP < \min_{P \in \mathcal{C}} \int u \circ g dP$. But this implies that $z_A y \prec z_A x$. Hence A is ordered non-universal. ■

Lemma A.26. *The representation in Theorem 4.3 ⇒Axiom 9 (ordered A-act-independence).*

Proof. Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ be such that $x_1 \preceq x_2$, $y_1 \preceq y_2$ and $z_1 \preceq z_2$. Let $f = x_{1A} x_2$, $g = y_{1A} y_2$ and $h = z_{1A} z_2$.

Case (i): Suppose $\{x_i, y_i\} \succeq z_i$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim h(\omega)_A f(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim h(\omega)_A g(\omega) & \text{for all } \omega \in \Omega \end{cases}$.

Let $\alpha = \max_{P \in \mathcal{C}} P(A)$. Since the weights are chosen form the set $\mathcal{C}$ to minimize the expected utility,

$$\begin{aligned}
& f' \succeq g' \\
& \text{iff } \alpha u\left(m^A(z_1, x_1)\right) + (1-\alpha) u\left(m^A(z_2, x_2)\right) \geq \alpha u\left(m^A(z_1, y_1)\right) + (1-\alpha) u\left(m^A(z_2, y_2)\right) \\
& \text{iff } \alpha\left[\alpha u(z_1) + (1-\alpha) u(x_1)\right] + (1-\alpha)\left[\alpha u(z_2) + (1-\alpha) u(x_2)\right] \\
& \qquad \geq \alpha\left[\alpha u(z_1) + (1-\alpha) u(y_1)\right] + (1-\alpha)\left[\alpha u(z_2) + (1-\alpha) u(y_2)\right] \\
& \text{iff } \alpha u(x_1) + (1-\alpha) u(x_2) \geq \alpha u(y_1) + (1-\alpha) u(y_2) \\
& \text{iff } f \succeq g.
\end{aligned}$$

Case (ii): Suppose $z_i \succeq \{x_i, y_i\}$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim f(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim g(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \end{cases}$.
Let $\alpha = \max_{P \in \mathcal{C}} P(A)$. Again, since the weights are chosen from $\mathcal{C}$ to minimize the expected utility,

$$
\begin{aligned}
& f' \succeq g' \\
& \text{iff } \alpha u\left(m^A(x_1, z_1)\right) + (1-\alpha)\, u\left(m^A(x_2, z_2)\right) \geq \alpha u\left(m^A(y_1, z_1)\right) + (1-\alpha)\, u\left(m^A(y_2, z_2)\right) \\
& \text{iff } \alpha\left[\alpha u(x_1) + (1-\alpha)u(z_1)\right] + (1-\alpha)\left[\alpha u(x_2) + (1-\alpha)u(z_2)\right] \\
& \qquad\qquad \geq \alpha\left[\alpha u(y_1) + (1-\alpha)u(z_1)\right] + (1-\alpha)\left[\alpha u(y_2) + (1-\alpha)u(z_2)\right] \\
& \text{iff } \alpha u(x_1) + (1-\alpha)\, u(x_2) \geq \alpha u(y_1) + (1-\alpha)\, u(y_2) \\
& \text{iff } f \succeq g. \blacksquare
\end{aligned}
$$

Lemma A.27. *The representation in Theorem 4.3 ⇒Axiom 4 (continuity).*

Proof. Let $f \in F$. We want to show that the sets $M(f) = \{g \in F | \, g \succeq f\}$ and $W(f) = \{g \in F | \, f \succeq g\}$ are closed. Let $\{g_n\}_{n=1}^{\infty} \in M(f)$ such that $g_n \to g$. Want to show that $g \in M(f)$. Note that $\int u \circ g_n dP \geq \min_{P \in \mathcal{C}} \int u \circ f dP$ for all $P \in \mathcal{C}$ and all n.

Since $g_n \to g$, $g_n(\omega) \to g(\omega)$ and since $u(\cdot)$ is continuous $u(g_n(\omega)) \to u(g(\omega))$. Hence, $u \circ g_n \to u \circ g$. Now, since $|u(g_n(\omega))| \leq \max_{x \in X} |u(x)|$ for all $\omega \in \Omega$, the dominated convergence theorem ([3], pp.124-125) implies $\lim_{n \to \infty} \int u \circ g_n dP = \int u \circ g dP$ for all $P \in \mathcal{C}$. Hence, $\min_{P \in \mathcal{C}} \int u \circ g dP \geq \min_{P \in \mathcal{C}} \int u \circ f dP$. To show that $W(f)$ is closed, assume to the contrary that the limiting g is strictly preferred to f while the sequence belongs to $W(f)$. By continuity of u, there is a neighborhood of g such that for any h in that neighborhood, $\int u \circ h dP > \min_{P \in \mathcal{C}} \int u \circ f dP$ for all $P \in \mathcal{C}$. But then the convergence theorem yields a contradiction. $\blacksquare$

Lemma A.28. *The representation in Theorem 4.3 ⇒Axiom 5 (monotonicity), part (a).*

Proof. Suppose for some $f, g \in F$, $f(\omega) \succeq g(\omega)$, for all $\omega \in \Omega$. Then, $u(f(\omega)) \geq u(g(\omega))$, for all $\omega \in \Omega$. Thus, $\min_{P \in \mathcal{C}} \int u \circ g dP \leq \int u \circ g d\overline{P} \leq \int u \circ f d\overline{P}$, for all $\overline{P} \in \mathcal{C}$. Hence, $\min_{P \in \mathcal{C}} \int u \circ g dP \leq \min_{P \in \mathcal{C}} \int u \circ f dP$. $\blacksquare$

Lemma A.29. *The representation in Theorem 4.3 ⇒Axiom 5 (monotonicity), part (b).*

Proof. First of all, note that if $B \in \Sigma$ is ordered non-null, then there is at least one probability measure $\widehat{P} \in \mathcal{C}$ with $\widehat{P}(B) > 0$. To see this, suppose $B \in \Sigma$ is ordered non-null but that for all $P \in \mathcal{C}$, $P(B) = 0$. Then, given any $x, y, z \in X$

such that $x \preceq y \preceq z$ we have $\min_{P \in C} \int u \circ f dP = \min_{P \in C} \int u \circ g dP$. This implies that the acts $f = x_A z$ and $g = y_A z$ are indifferent. But this contradicts B being ordered non-null. Let $\alpha = \max_{P \in C} P(B)$. We have just shown that $\alpha > 0$. Let x, y, $z \in X$ be such that $z \succeq x$ and $z \succeq y$. Now,

$$\begin{aligned}
& x_B z \succeq y_B z \\
& \Leftrightarrow \alpha u(x) + (1-\alpha) u(z) \geq \alpha u(y) + (1-\alpha) u(z) \\
& \Leftrightarrow u(x) + \tfrac{(1-\alpha)}{\alpha} u(z) \geq u(y) + \tfrac{(1-\alpha)}{\alpha} u(z) \\
& \Leftrightarrow u(x) \geq u(y) \\
& \Leftrightarrow x \succeq y.
\end{aligned}$$

The case in which $B \in \Sigma$ is ordered non-universal is proved by a similar argument noting that B ordered non-universal implies $\max_{P \in C} P(B) < 1$. ■

Lemma A.30. *The representation in Theorem 4.3 ⇒Axiom 10 (constant-independence).*

Proof. Let $\beta = \max_{P \in C} P(A)$. Observe that if $a_{i_\omega(n)}, a_{i_\omega(n)+1} \in SS(n)$ and $i_\omega(n) \geq 1$, then

$$\beta u(m + \frac{1}{n}) + (1-\beta) u(a_{i_\omega(n)}) = \beta u(m) + (1-\beta) u(a_{i_\omega(n)+1}).$$

Therefore,

$$u(a_{i_\omega(n)+1}) - u(a_{i_\omega(n)}) = \frac{\beta}{1-\beta}\left[u(m + \frac{1}{n}) - u(m)\right].$$

Define $\Delta(n) \in \mathbb{R}$ by $\Delta(n) := \frac{\beta}{1-\beta}\left[u(m + \frac{1}{n}) - u(m)\right]$.

Let f, g, f' and $g' \in F$, $h \in F^*$. For each $\omega \in \Omega$ and $n \in \mathbb{Z}_{++}$

(i) let $a_{i_\omega(n)} \in SS(n)$ be such that $a_{i_\omega(n)+1} \succeq f(\omega) \succeq a_{i_\omega(n)}$
(ii) let $a_{i'_\omega(n)} \in SS(n)$ be such that $a_{i'_\omega(n)+1} \succeq f'(\omega) \succeq a_{i'_\omega(n)}$
(iii) let $a_{j_\omega(n)} \in SS(n)$ be such that $a_{j_\omega(n)+1} \succeq g(\omega) \succeq a_{j_\omega(n)}$
(iv) let $a_{j'_\omega(n)} \in SS(n)$ be such that $a_{j'_\omega(n)+1} \succeq g'(\omega) \succeq a_{j'_\omega(n)}$
(v) let $a_{k(n)} \in SS(n)$ be such that $a_{k(n)+1} \succeq h \succeq a_{k(n)}$.

If $f(\omega) \neq m$, for n large enough, this implies that $(i_\omega(n)-1)\Delta(n) \leq u(f(\omega)) - u(m + \frac{1}{n}) \leq i_\omega(n)\Delta(n)$. If $f(\omega) = m$ then $u(f(\omega)) = u(m)$ and $i_\omega(n) = 0$. In either case since $\lim_{n\to\infty} \Delta(n) = 0$ and u is continuous, these in turn imply that $\lim_{n\to\infty} i_\omega(n)\Delta(n) = u(f(\omega)) - u(m)$.

We can show using similar arguments that $\lim_{n\to\infty} i'_\omega(n)\Delta(n) = u(f'(\omega)) - u(m)$, $\lim_{n\to\infty} j_\omega(n)\Delta(n) = u(g(\omega)) - u(m)$, $\lim_{n\to\infty} j'_\omega(n)\Delta(n) = u(g'(\omega)) - u(m)$ and $\lim_{n\to\infty} k(n)\Delta(n) = u(h) - u(m)$.

First consider those $\omega \in \Omega$ for which $f(\omega) \nsim h$. Observe that $\frac{i'_\omega(n)-i_\omega(n)}{k(n)-i_\omega(n)} = \frac{i'_\omega(n)\Delta(n)-i_\omega(n)\Delta(n)}{k(n)\Delta(n)-i_\omega(n)\Delta(n)}$. The constant-independence axiom assumes $\lim_{n\to\infty} \frac{i'_\omega(n)-i_\omega(n)}{k(n)-i_\omega(n)} = \alpha$, which implies $\frac{u(f'(\omega))-u(f(\omega))}{u(h)-u(f(\omega))} = \alpha$. Second, consider those $\omega \in \Omega$ for which $f(\omega) \sim h$. Here, constant-independence assumes that $f'(\omega) \sim h$, hence $u(f'(\omega)) = u(f(\omega)) = u(h)$. Therefore, for all $\omega \in \Omega$,

$$u(f'(\omega)) = (1-\alpha)\, u(f(\omega)) + \alpha u(h).$$

Similarly, under the assumptions of the constant-independence axiom,

$$u(g'(\omega)) = (1-\alpha)\, u(g(\omega)) + \alpha u(h) \text{ for all } \omega \in \Omega.$$

Note now that h a constant act implies that $\int u \circ h dP = \int u \circ h dP'$ for all $P' \in \mathcal{C}$. Now,

$f' \succeq g'$
iff $\min_{P\in\mathcal{C}} \int u \circ f' dP \geq \min_{P\in\mathcal{C}} \int u \circ g' dP$
iff $\min_{P\in\mathcal{C}} \int ((1-\alpha)(u \circ f) + \alpha (u \circ h))\, dP \geq \min_{P\in\mathcal{C}} \int ((1-\alpha)(u \circ f) + \alpha (u \circ h))\, dP$
iff $\min_{P\in\mathcal{C}} \int (1-\alpha)(u \circ f)\, dP + \int \alpha (u \circ h)\, dP$
$\geq \min_{P\in\mathcal{C}} \int (1-\alpha)(u \circ g)\, dP + \int \alpha (u \circ h)\, dP$
iff $\min_{P\in\mathcal{C}} \int (1-\alpha)(u \circ f)\, dP \geq \min_{P\in\mathcal{C}} \int (1-\alpha)(u \circ g)\, dP$
iff $f \succeq g$. This proves constant-independence. ■

Lemma A.31. *The representation in Theorem 4.3 ⇒Axiom 11 (uncertainty aversion).*

Proof. By the arguments in the proof of Lemma A.30, under the assumptions of the uncertainty aversion axiom,

$$u(h(\omega)) = (1-\alpha)\, u(f(\omega)) + \alpha u(g(\omega)) \text{ for all } \omega \in \Omega.$$

Now, $f \sim g \Rightarrow \min_{P\in\mathcal{C}} \int u \circ f dP = \min_{P\in\mathcal{C}} \int u \circ g dP$. This implies

$\min_{P\in\mathcal{C}} \int u \circ f dP = (1-\alpha)\left(\min_{P\in\mathcal{C}} \int u \circ f dP\right) + \alpha\left(\min_{P\in\mathcal{C}} \int u \circ g dP\right)$
$\leq (1-\alpha)\left(\int u \circ f d\overline{P}\right) + \alpha\left(\int u \circ g d\overline{P}\right)$ for all $\overline{P} \in \mathcal{C}$
$= \int ((1-\alpha)(u \circ f) + \alpha (u \circ g))\, d\overline{P}$ for all $\overline{P} \in \mathcal{C}$
$= \int u \circ h d\overline{P}$ for all $\overline{P} \in \mathcal{C}$.

Hence, $\min_{P\in\mathcal{C}} \int u \circ f dP \leq \min_{P\in\mathcal{C}} \int u \circ h dP$. Therefore, by the representation, we have $h \succeq f$. ■

We now provide the additional results needed to show necessity in Theorem 3.7.

Lemma A.32. *The representation in Theorem 3.7 ⇒ Axiom 2 (structure), part (b).*

Proof. Consider the event A referred to in the representation theorem. We will show that A and A^c are ordered non-null and ordered non-universal.

Observe that $0 < P(A) < 1$ for all $P \in \mathcal{C}$ implies $0 < \max_{P \in \mathcal{C}} P(A) < 1$ and $0 < \max_{P \in \mathcal{C}} P(A^c) < 1$. Therefore, lemma A.25 applies to A and to A^c and each must be ordered non-null and ordered non-universal. ■

Lemma A.33. *The representation in Theorem 3.7 ⇒ Axiom 10 (A-act-independence).*

Proof. Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ and let $f = x_{1A}x_2$, $g = y_{1A}y_2$ and $h = z_{1A}z_2$. Suppose f', $g' \in F$ are such that,

$$f'(\omega) \sim h(\omega)_A f(\omega) \text{ for all } \omega \in \Omega$$

and

$$g'(\omega) \sim h(\omega)_A g(\omega) \text{ for all } \omega \in \Omega.$$

By the representation $P(A) = \rho \in (0,1)$ for all $P \in \mathcal{C}$. Then,

$$\begin{aligned}
& f' \succeq g' \\
& \Leftrightarrow \rho u(m^A(z_1, x_1)) + (1-\rho)u(m^A(z_2, x_2)) \geq \rho u(m^A(z_1, y_1)) + (1-\rho)u(m^A(z_2, y_2)) \\
& \Leftrightarrow \rho^2 u(z_1) + \rho(1-\rho)u(x_1) + \rho(1-\rho)u(z_2) + (1-\rho)^2 u(x_2) \geq \\
& \quad \rho^2 u(z_1) + \rho(1-\rho)u(y_1) + \rho(1-\rho)u(z_2) + (1-\rho)^2 u(y_2) \\
& \Leftrightarrow \rho u(x_1) + (1-\rho)u(x_2) \geq \rho u(y_1) + (1-\rho)u(y_2) \\
& \Leftrightarrow f \succeq g.
\end{aligned}$$

Since $P(A^c) = 1 - \rho \in (0,1)$ for all $P \in \mathcal{C}$, the above argument may be repeated replacing A with A^c. This proves A-act-independence. ■

Lemma A.34. *The representation in Theorem 3.7 ⇒ Axiom 6 ($\overline{S}$-act-independence).*

Proof. We will first prove the following lemma which states that the representation evaluates all acts in $\overline{S}^l$ using the same probability measure.

Lemma A.35. *Let f, $l \in F$ where $f \in \overline{S}^l$, if $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ l dP$, then $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ f dP$.*

Proof. Given S^l, let S_n^l be the set of acts obtained after n iterations in the statewise combination process. That is,

$$\begin{aligned}
S_1^l &= \left\{ f^1 \in F : f^1(\omega) \sim f_1^0(\omega)_A f_2^0(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^0, f_2^0 \in S^l \right\} \\
S_2^l &= \left\{ f^2 \in F : f^2(\omega) \sim f_1^1(\omega)_A f_2^1(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^1, f_2^1 \in S_1^l \right\} \\
&\vdots \\
S_n^l &= \left\{ f^n \in F : f^n(\omega) \sim f_1^{n-1}(\omega)_A f_2^{n-1}(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^{n-1}, f_2^{n-1} \in S_{n-1}^l \right\}
\end{aligned}$$

First we will show that the representation evaluates any act $f \in \cup_{n=1}^{\infty} S_n^l$ using the measure P^l. This is shown by induction.

Suppose $f \in S_1^l$. Then we can find acts f_1^0, $f_2^0 \in S^l$ such that for each $\omega \in \Omega$, $f(\omega) \sim f_1^0(\omega)_A f_2^0(\omega)$. By the representation, $u(f(\omega)) = \rho u\left(f_1^0(\omega)\right) + (1-\rho) u\left(f_2^0(\omega)\right)$. Hence, for all $P \in \mathcal{C}$

$$\int u \circ f dP = \rho \int u \circ f_1^0 dP + (1-\rho) \int u \circ f_2^0 dP.$$

Note that the right hand side of this expression is minimized by choosing $P = P^l$. Thus, $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ f dP$ for all $f \in S_1^l$.

Fix $k \geq 1$. Assume $f \in S_n^l$, for some $n \leq k$, implies $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$. Consider an act $f \in S_{k+1}^l$. Then there exist acts f_1^k, $f_2^k \in S_k^l$ such that for all $\omega \in \Omega$, $f(\omega) \sim f_1^k(\omega)_A f_2^k(\omega)$. By the representation, $u(f(\omega)) = \rho u\left(f_1^k(\omega)\right) + (1-\rho) u\left(f_2^k(\omega)\right)$. Hence, for all $P \in \mathcal{C}$

$$\int u \circ f dP = \rho \int u \circ f_1^k dP + (1-\rho) \int u \circ f_2^k dP.$$

By the induction hypothesis, P^l minimizes the right hand side. Therefore, $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$ for all $f \in \cup_{n=1}^{k+1} S_n^l$. This completes the induction argument.

To complete the proof of the lemma it must be shown that if $f \in \overline{S}^l \setminus \cup_{n=1}^{\infty} S_n^l$, then $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$.

We will first show the following: (i) $\cup_{n=1}^{\infty} S_n^l \supseteq S^l$, (ii) $\cup_{n=1}^{\infty} S_n^l$ contains all statewise combinations over A, and (iii) if $S \supseteq S^l$ and S contains all statewise combinations over A then $\cup_{n=1}^{\infty} S_n^l \subseteq S$.

To see (i), note that $S^l = \left\{f^1 \in F : f^1(\omega) \sim f_1^0(\omega)_A f_2^0(\omega) \text{ where } f_1^0 = f_2^0 \in S^l\right\} \subseteq S_1^l$. To see (ii), consider $f, g \in \cup_{n=1}^{\infty} S_n^l$. We need to show that the statewise combination over A of any such f and g is also in $\cup_{n=1}^{\infty} S_n^l$. Since $S_n^l \subseteq S_{n+1}^l$, there exists a N such that for all $n \geq N$, $f, g \in S_n^l$. But then h such that $h(\omega) \sim f(\omega)_A g(\omega)$ for all $\omega \in \Omega$, must be an element of S_{N+1}^l.

Finally, to prove (iii), consider a set S as in (iii). Fix $f \in \cup_{n=1}^{\infty} S_n^l$. If $f \notin S$ then S cannot contain all statewise combinations over A since f can be reached in a finite number of statewise combination operations starting from elements of S^l.

By (iii) and the definition of $\overline{S}^l$, we have that the closure of $\cup_{n=1}^{\infty} S_n^l$ is $\overline{S}^l$. Fix $f \in \overline{S}^l \setminus \cup_{n=1}^{\infty} S_n^l$. There must exist a sequence of acts $\{f_i\}$ converging to f with $f_i \in \cup_{n=1}^{\infty} S_n^l$. Since u is continuous, for all $P \in \mathcal{C}$, $\left\{\int u \circ f_i dP\right\}$ converges to $\int u \circ f dP$. Let $P^f \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$. As $\int u \circ f_i dP^l \leq \int u \circ f_i dP^f$ for all i,

$\int u \circ f dP^l > \int u \circ f dP^f$ would contradict continuity of preferences. This proves $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ f dP$ for all $f \in \overline{S}^l \setminus \cup_{n=1}^{\infty} S_n^l$. ■

Now we complete the proof of Lemma A.34. Assume we have $f,\ g,\ h \in F$ such that there exist $l,\ k \in F$ for which $f,\ h \in \overline{S}^l$ and $g,\ h \in \overline{S}^k$. If $f',\ g' \in F$ are such that

$$f'(\omega) \sim h(\omega)_A f(\omega) \text{ for all } \omega \in \Omega$$

and

$$g'(\omega) \sim h(\omega)_A g(\omega) \text{ for all } \omega \in \Omega,$$

we have that for all $P \in \mathcal{C}$

$$\int u \circ f' dP = \rho \int u \circ h dP + (1-\rho) \int u \circ f dP$$

and

$$\int u \circ g' dP = \rho \int u \circ h dP + (1-\rho) \int u \circ g dP.$$

Since $f,\ h,\ f' \in \overline{S}^l$, $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ f dP$, $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ h dP$ and $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ f' dP$. Similarly, since $g,\ h,\ g' \in \overline{S}^k$, $P^k \in \arg\min_{P \in \mathcal{C}} \int u \circ g dP$, $P^k \in \arg\min_{P \in \mathcal{C}} \int u \circ h dP$ and $P^k \in \arg\min_{P \in \mathcal{C}} \int u \circ g' dP$. Therefore,

$$\min_{P \in \mathcal{C}} \int u \circ f' dP = \rho \int u \circ h dP^l + (1-\rho) \int u \circ f dP^l$$

and

$$\min_{P \in \mathcal{C}} \int u \circ g' dP = \rho \int u \circ h dP^k + (1-\rho) \int u \circ g dP^k.$$

But since $\int u \circ h dP^l = \int u \circ h dP^k$, we have $\min_{P \in \mathcal{C}} \int u \circ f' dP \geq \min_{P \in \mathcal{C}} \int u \circ g' dP$ if and only if $\int u \circ f dP^l = \min_{P \in \mathcal{C}} \int u \circ f dP \geq \min_{P \in \mathcal{C}} \int u \circ g dP = \int u \circ g dP^k$. Thus, $f \succeq g$ if and only if $f' \succeq g'$. This proves $\overline{S}$-act-independence. ■

Lemma A.36. *The representation in Theorem 3.7 ⇒Axiom 7 (act-uncertainty aversion).*

Proof. Let $f,\ g \in F$. Suppose $f \sim g$ and

$$f'(\omega) \sim f(\omega)_A g(\omega) \text{ for all } \omega \in \Omega.$$

By the representation, we have that $u(f'(\omega)) = \rho u(f(\omega)) + (1-\rho) u(g(\omega))$ for all $\omega \in \Omega$. Now, $f \sim g$ implies $\min_{P \in \mathcal{C}} \int u \circ f dP = \min_{P \in \mathcal{C}} \int u \circ g dP$. Note,

$$\min_{P \in \mathcal{C}} \int u \circ f dP \leq \int u \circ f d\overline{P} \text{ for all } \overline{P} \in \mathcal{C}$$

and

$$\min_{P\in\mathcal{C}}\int u\circ gdP \leq \int u\circ gd\overline{P} \text{ for all } \overline{P}\in\mathcal{C}.$$

This implies

$$\begin{aligned}&\min_{P\in\mathcal{C}}\int u\circ fdP=\rho\left(\min_{P\in\mathcal{C}}\int u\circ fdP\right)+(1-\rho)\left(\min_{P\in\mathcal{C}}\int u\circ gdP\right)\\ &\leq\rho\left(\int u\circ fd\overline{P}\right)+(1-\rho)\left(\int u\circ gd\overline{P}\right)\\ &=\int\left(\rho\left(u\circ f\right)+(1-\rho)\left(u\circ g\right)\right)d\overline{P}\\ &=\int u\circ f'd\overline{P} \text{ for all } \overline{P}\in\mathcal{C}.\end{aligned}$$

Hence, $\min_{P\in\mathcal{C}}\int u\circ fdP\leq\min_{P\in\mathcal{C}}\int u\circ f'dP$ and therefore, $f'\succeq f$. ■

The above lemmas together prove necessity of the axioms in Theorem 3.7. ■

A.4. Proof that $\overline{S}^f$ contains all affinely related acts

Proposition A.37. *If $f\in F$ is not a constant act, then $\overline{S}^f$ contains all acts that are affinely related to f.*

Proof. By definition, all constant acts are in $\overline{S}^f$. Fix any non-constant act $g\in F$ that is affinely related to f. There must exist an $\alpha>0$ and $\beta\in\mathbb{R}$ such that, for all $\omega\in\Omega$,

$$u(g(\omega))=\alpha u(f(\omega))+\beta.$$

There exists $\varepsilon>0$ such that $\frac{\varepsilon}{\pi(A)}\alpha(u\circ f)\in B(K)$ and $\frac{\varepsilon}{1-\pi(A)}\beta^*\in B(K)$. Let $h\in F$ be such that $u\circ h=\frac{\varepsilon}{\pi(A)}\alpha(u\circ f)$ and h' be such that $u\circ h'=\frac{\varepsilon}{1-\pi(A)}\beta^*$. Since h' is a constant act, it is in $\overline{S}^f$. By Lemma A.13, $h\in\overline{S}^f$ as well.

Let $g'(\omega)\sim h(\omega)_A h'(\omega)$ for all $\omega\in\Omega$. Since $\overline{S}^f$ contains all statewise combinations over A, $g'\in\overline{S}^f$. Note that by the representation in (A.1), $u\circ g'=\varepsilon(\alpha(u\circ f)+\beta^*)$. Another application of Lemma A.13 yields $g\in\overline{S}^{g'}$. Since $\overline{S}^{g'}=closure\left(\cup_{i=1}^{\infty}S_i^{g'}\right)\subseteq F$ (by Lemma A.35) and $closure\left(\cup_{i=1}^{\infty}S_i^{g'}\right)\subseteq\overline{S}^f$, we have $g\in\overline{S}^f$. ■

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