Discussion Paper No. 1211

Factorization and Decomposition of Relations

by

Peter S. Faynzilberg

March 1998

Factorization and Decomposition of Relations

Peter S. Faynzilberg
MEDS, 534 Leverone Hall
J.L. Kellogg Graduate School of Management
Northwestern University
2001 Sheridan Road
Evanston, Illinois 60208–2009

March 1998

Abstract

The paper extends the factor and decomposition theorems for functions to the category ***Rel*** of sets and relations. The converse theorems are also stated and proved. We show that the (retraction,isomorphism,section)-decomposable relations form a closure system on $Mor\,(\boldsymbol{Rel})$, and characterize it in terms of a closure operator.

1 Introduction

Let $(A, \mathcal{E})$ be a relational structure consisting of a set A and an equivalence $\mathcal{E}$ on A. It is well known that a function $f : A \to B$ decomposes into a canonical surjection $\nu_A : A \to A/(f^{-1} \circ f)$, bijection $\psi_f : A/(f^{-1} \circ f) \to im\, f$, and an injection (inclusion) $\iota : im\, f \to B$. The celebrated Homomorphism Theorems of E. Nöther (van der Waerden [3], Hungerford [2]) provide the corresponding decompositions for sets with operators (*e.g.*, Cohn [1]).

Recall that in the category ***Set*** of sets and functions, injections are sections and surjections are retractions. If R, S, and I are, respectively, the classes of retractions, sections, and isomorphisms, then $\nu_A \circ \psi_f \circ \iota$ is the essentially unique (R, I, S)-decomposition of an arbitrary morphism f of ***Set***. In the present paper we investigate the (R, I, S)-decomposability in a larger category ***Rel*** of sets and relations. Although ***Rel*** is not (R, I, S)-decomposable as a category, we identify the class of those morphisms which do have such a decomposition. We find, furthermore, that this class is a closure system, and characterize it by the associated closure operator on $Mor(\boldsymbol{Rel})$. This development allows us to formulate and prove the respective converse statements for the factorization and decomposition theorems.

2 The Closure of a Relation

Recall that in the category ***Rel***, the objects are sets and morphisms the triples of sets $(A, \mathcal{R}, B)$ such that $\mathcal{R} \subseteq A \times B$. Composability of morphisms is defined as in ***Set***, and the domain and codomain functions $dom\,, codom\, : Mor\,(\boldsymbol{Rel}) \to Ob\,(\boldsymbol{Rel})$ are defined by $dom\,(A, \mathcal{R}, B) = A$ and $codom\,(A, \mathcal{R}, B) = B$. For each $S \in Ob\,(\boldsymbol{Rel})$, the morphism (S, id_S, S) with the diagonal id_S of $S \times S$ is the identity on S. The composition of morphisms is defined in a natural way in terms of the composition of the corresponding relations. Since, furthermore, $\hom(A, B) = \mathcal{P}(A \times B)$ is a set, the smallness of the morphism class condition is satisfied, and ***Rel*** is indeed a category.

Some special morphisms in ***Rel*** are natural generalizations from their counterparts in ***Set***. Below, we refer to the image of the inverse of a relation as the coimage of that relation:

Definition 2.1 *We shall say that a relation $\mathcal{R}$ is:*

(i) A correspondence *if* $\operatorname{coim} \mathcal{R} = \operatorname{dom} \mathcal{R}$.

(ii) A partial function *if for all* $x \in \operatorname{dom} \mathcal{R}$, $\operatorname{card} R[a] \leq 1$.

And dually, $\mathcal{R}$ is:

(i) A surjection *if* $\operatorname{im} \mathcal{R} = \operatorname{codom} \mathcal{R}$.

(ii) An injection *if for all* $x \in \operatorname{codom} \mathcal{R}$, $\operatorname{card} R^{-1}[a] \leq 1$.

We shall further say that $\mathcal{R}$ is a bijection *if it is both surjective and injective. And dually, $\mathcal{R}$ is a* function *if it is both a partial function and a correspondence.*
Single-valuedness of a relation is, thus, in duality with bijectivity. The bijective functions are the only isomorphisms of $\boldsymbol{Rel}$, as follows from the next easily verifiable observation.
Proposition 2.2 *A morphism $(A, \mathcal{R}, B)$ is a section in $\mathbf{Rel}$ if and only if it is an injective correspondence. In this case $\mathcal{R}^{-1}$ is a left inverse of $\mathcal{R}$:*

$$\mathcal{R}^{-1} \circ \mathcal{R} = \mathrm{id}_{\mathrm{dom}\,\mathcal{R}}. \tag{2.1}$$

Dually, a morphism $(A, \mathcal{R}, B)$ is a retraction in $\mathbf{Rel}$ if and only if it is a surjective partial function. In this case $\mathcal{R}^{-1}$ is a right inverse of $\mathcal{R}$:

$$\mathcal{R} \circ \mathcal{R}^{-1} = \mathrm{id}_{\mathrm{codom}\,\mathcal{R}}. \tag{2.2}$$

Each relation $\mathcal{R}$ gives rise to an equivalence generated on its coimage by $\mathcal{R}^{-1} \circ \mathcal{R}$; that is, there exists a mapping

$$E = \begin{cases} Mor(\boldsymbol{Rel}) & \rightarrow & Mor(\boldsymbol{Rel}) \\ \mathcal{R} & \mapsto & \bigcup_{n\in\mathbf{N}} (\mathcal{R}^{-1} \circ \mathcal{R})^n . \end{cases} \tag{2.3}$$

Generalizing from the terminology for functions, we shall refer to $\ker \mathcal{R} = \mathcal{R}^{-1} \circ \mathcal{R}$ as the *kernel* of the relation $\mathcal{R}$. If $\mathcal{R}$ is a partial function, $E_{\mathcal{R}} = \ker \mathcal{R}$, since by (2.1) the kernel is an equivalence.

The mapping (2.3) allows us to define the closure operator $\langle\cdot\rangle$ on the class of relations (Definition 2.4 below). The proof of the closure property is based on two lemmata.
Lemma 2.3 *For any relation $\mathcal{R}$,*

$$\mathcal{R} \circ E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathrm{Hom}_{\,Rel}(\mathcal{R}^{-1}, \mathcal{R})\,(E_{\mathcal{R}}) = E_{\mathcal{R}^{-1}}. \tag{2.4}$$

Also, if $f_A : A \rightarrow \mathrm{coim}\,\mathcal{R}$ and $f_B : B \rightarrow \mathrm{im}\,\mathcal{R}$ are retractions, then

$$\mathrm{Hom}_{\,Rel}(f_A, f_A^{-1})\,(E_{\mathcal{R}}) = E\mathrm{Hom}_{Rel}(f_A, f_B^{-1})(\mathcal{R}). \tag{2.5}$$

Proof. The identity (2.4) follows directly from definition (2.3) and the associativity of composition: $\mathcal{R} \circ E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathcal{R} \circ \bigcup_{n\in\mathbf{N}} (\mathcal{R}^{-1} \circ \mathcal{R})^n \circ \mathcal{R}^{-1} = \bigcup_{n\in\mathbf{N}} (\mathcal{R} \circ \mathcal{R}^{-1})^{n+1} = E_{\mathcal{R}^{-1}}$, since the sequence $\{(\ker \mathcal{R})^n\}_{n\in\mathbf{N}}$ is monotone increasing in view of the reflexivity of $\ker \mathcal{R}$.

In order to prove (2.5), we may utilize (2.2) since f_A and f_B are retractions:

$$E_{\mathrm{Hom}_{Rel}(f_A, f_B^{-1})(\mathcal{R})} = \bigcup_{n\in\mathbf{N}} (f_A^{-1} \circ \mathcal{R}^{-1} \circ f_B \circ f_B^{-1} \circ \mathcal{R} \circ f_A)^n = f_A^{-1} \circ \left(\bigcup_{n\in\mathbf{N}} \left(\ker \mathcal{R}|_{im\, f_A}^{im\, f_B} \right)^n \right) \circ f_A,$$

and (2.5) follows from the surjectivity of f_A and f_B guaranteed by Proposition 2.2. Q.E.D.

A weaker than in Lemma 2.3 requirement, namely, that only f_B be a partial surjective function, permits us to conclude that

$$Hom_{Rel}(f_A, f_A^{-1})\,(\ker \mathcal{R}) = \ker Hom_{Rel}(f_A, f_B^{-1})(\mathcal{R}). \tag{2.6}$$

By composing the identity (2.4) with $\mathcal{R}$ on the right, we arrive at the following **Proposition** and **Definition 2.4** *For any $\mathcal{R}$, the diagram*

$$\begin{tikzcd} dom\,\mathcal{R} \arrow[r, "\mathcal{R}"] \arrow[d, "E_{\mathcal{R}}"] & codom\,\mathcal{R} \arrow[d, "E_{\mathcal{R}^{-1}}"] \\ dom\,\mathcal{R} \arrow[r, "\mathcal{R}"] & codom\,\mathcal{R} \end{tikzcd}$$

is commutative. We shall say that the diagonal of the diagram,

$$\langle \mathcal{R} \rangle = \mathcal{R} \circ E_{\mathcal{R}} = E_{\mathcal{R}^{-1}} \circ \mathcal{R}, \tag{2.7}$$

is the closure *of $\mathcal{R}$.*

The name of $\langle\cdot\rangle$ is justified by Proposition 2.6 below, in the proof of which we utilize **Lemma 2.5** *For any relation $\mathcal{R}$:*

$$E_{\langle \mathcal{R} \rangle} = E_{\mathcal{R}} \tag{2.8}$$

$$\langle \mathcal{R} \rangle^{-1} = \left\langle \mathcal{R}^{-1} \right\rangle. \tag{2.9}$$

Also, if $f_A : A \to \operatorname{coim} \mathcal{R}$ and $f_B : B \to \operatorname{im} \mathcal{R}$ are retractions, then

$$\left\langle \mathrm{Hom}_{Rel}(f_A, f_B^{-1})\,(\mathcal{R}) \right\rangle = \mathrm{Hom}_{Rel}(f_A, f_B^{-1})\,(\langle \mathcal{R} \rangle)\,. \tag{2.10}$$

Proof. From definitions (2.3) and (2.7), we obtain $E_{\langle \mathcal{R} \rangle} = \bigcup_{n \in \mathbf{N}} (E_{\mathcal{R}} \circ (\ker \mathcal{R}) \circ E_{\mathcal{R}})^n = E_{\mathcal{R}}$, since $E_{\mathcal{R}} \circ (\ker \mathcal{R}) = E_{\mathcal{R}}$, and $E_{\mathcal{R}}$ is idempotent; hence (2.8). Further, the unary operations $\langle\cdot\rangle$ and $()^{-1}$ commute:

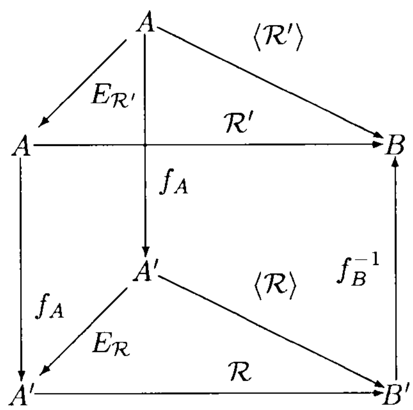
$$\langle \mathcal{R} \rangle^{-1} = (\mathcal{R} \circ E_{\mathcal{R}})^{-1} = E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathcal{R}^{-1} \circ E_{\mathcal{R}^{-1}} = \left\langle \mathcal{R}^{-1} \right\rangle.$$

Finally,

$$\langle Hom_{Rel}(f_A, f_B^{-1})\,(\mathcal{R}) \rangle = Hom_{Rel}(f_A, f_B^{-1})\,(\mathcal{R}) \circ E_{Hom_{Rel}(f_A, f_B^{-1})(\mathcal{R})} = \left(f_B^{-1} \circ \mathcal{R} \circ f_A\right) \circ \left(f_A^{-1} \circ E_{\mathcal{R}} \circ f_A\right)$$

by (2.5) of Lemma 2.3. The last composition is equal to $f_B^{-1} \circ \langle \mathcal{R} \rangle \circ f_A$ since f_A is, by assumption, a retraction, and (2.10) follows at once. Q.E.D.

Lemmata 2.3 and 2.5 may be summarized as the commutativity of the diagram



in which $\mathcal{R}' = Hom_{Rel}(f_A, f_B^{-1})(\mathcal{R})$. We are in a position to state and prove

Proposition 2.6 *Let $<>$: $Mor(\boldsymbol{Rel}) \to Mor(\boldsymbol{Rel})$ be the map, the action of which is defined by (2.7). Then $\langle\cdot\rangle$ is a closure operator on the Boolean $(Mor(\boldsymbol{Rel}), \subseteq)$; that is, for all $\mathcal{R}, \mathcal{S} \in Mor(\boldsymbol{Rel})$,*

$$R \subseteq \mathcal{S} \implies \langle\mathcal{R}\rangle \subseteq \langle\mathcal{S}\rangle \tag{2.11}$$

$$\mathcal{R} \subseteq \langle\mathcal{R}\rangle \tag{2.12}$$

$$\langle\langle\mathcal{R}\rangle\rangle = \langle\mathcal{R}\rangle\,. \tag{2.13}$$

Proof. The monotonicity of the map E implies that, whenever relations $\mathcal{R}$ and $\mathcal{S}$ are such that $\mathcal{R} \subseteq \mathcal{S}$, we have $\langle\mathcal{R}\rangle = \mathcal{R} \circ E_{\mathcal{R}} \subseteq \mathcal{S} \circ E_{\mathcal{S}} = \langle\mathcal{S}\rangle$, hence the monotonicity (2.11) of $< \cdot >$. The reflexivity of the equivalence $E_{\mathcal{R}}$ on $coim\,\mathcal{R}$ implies (2.12): $\langle\mathcal{R}\rangle = \mathcal{R} \circ E_{\mathcal{R}} \supseteq \mathcal{R} \circ id_{\,coim\,\mathcal{R}}$. Finally, (2.13) is a consequence of (2.8) and idempotency of equivalences: $\langle\langle\mathcal{R}\rangle\rangle = (\mathcal{R} \circ E_{\mathcal{R}}) \circ E_{\langle\mathcal{R}\rangle} = (\mathcal{R} \circ E_{\mathcal{R}}) \circ E_{\mathcal{R}} = \langle\mathcal{R}\rangle$. Q.E.D.

The commutativity of $\langle\cdot\rangle$ and $(\cdot)^{-1}$ (Lemma 2.5) implies that a relation is $\langle\cdot\rangle$-closed if and only if its inverse is. Furthermore, whenever $\ker\mathcal{R}$ is an equivalence, $\langle\mathcal{R}\rangle = R \circ \ker\mathcal{R} = R$. This observation and its dual prove the following

Corollary 2.7 *For a relation $\mathcal{R}$ to be closed it is sufficient that either (or both) $\ker\mathcal{R}$ or $\ker\mathcal{R}^{-1}$ be an equivalence.*

Thus, all equivalences, partial functions and injections are closed. In addition, identity (2.10) implies that $Hom_{Rel}(f_A, f_B^{-1})\,(\mathcal{R})$ is closed if and only if the relation $\mathcal{R}$ is.

3 Factorizations

In the category $\boldsymbol{Set}$, the requirement that the kernel of a function f include as its subset an equivalence $\mathcal{E}$ on $dom\,f$ is sufficient for f to be uniquely factorizable through the canonical surjection $\nu_{\mathcal{E}} : dom f \to dom f/\mathcal{E}$. The following characterization theorem extends this result to the category ***Rel*** and, in addition, provides its converse.

Theorem 3.1 [Factor Theorem for Relations] *Let $\mathcal{R}$ be a relation and $\mathcal{E}$ an equivalence on* coim $\mathcal{R}$ *with the canonical surjection $\nu_{\mathcal{E}}$:* coim $\mathcal{R} \to$ coim $\mathcal{R}/\mathcal{E}$. *Then a decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ exists if and only if*

$$\mathcal{R} \circ \mathcal{E} \subseteq \mathcal{R}, \tag{3.1}$$

in which case $\hat{\mathcal{R}}$ is unique and equal to $\mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$.

Proof. Suppose first that $\hat{\mathcal{R}}$ satisfying $\hat{\mathcal{R}} \circ \nu_{\mathcal{E}} = \mathcal{R}$ exists. Since the canonical surjection $\nu_{\mathcal{E}}$ is a retraction (Proposition 2.2), this yields a unique $\hat{\mathcal{R}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$. A composition of $\hat{\mathcal{R}}$ with $\nu_{\mathcal{E}}$ on the right results in $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1} \circ \nu_{\mathcal{E}} = \mathcal{R} \circ \mathcal{E}$.

Conversely, if $\mathcal{R} \subseteq \mathcal{R} \circ \mathcal{E}$, then $\mathcal{R} = \mathcal{R} \circ \mathcal{E}$ by the reflexivity of $\mathcal{E}$, and $\hat{\mathcal{R}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$ provides the stated decomposition of $\mathcal{R}$. Q.E.D.

The next proposition presents upper bounds on $\ker \hat{\mathcal{R}}$ and $\ker \hat{\mathcal{R}}^{-1}$ for the factor $\hat{\mathcal{R}}$ in the decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ of a relation $\mathcal{R}$. It utilizes the canonical "insertion" $\rho_{\mathcal{E}}$: $coim\, \mathcal{R}/\mathcal{E} \to coim\, \mathcal{R}/(\mathcal{E} \vee E_R)$ defined, more specifically, as follows. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two equivalences on a set S with canonical identification maps $\nu_{\mathcal{E}_1}$ and $\nu_{\mathcal{E}_2}$, respectively. If $\mathcal{E}_1 \subseteq \mathcal{E}_2$, then $\rho_{12} = \nu_{\mathcal{E}_2} \circ \nu_{\mathcal{E}_1}^{-1}$ is a unique function that assigns to each equivalence class α of $\mathcal{E}_1$ the unique equivalence class of $\mathcal{E}_2$ that includes α as a subset (the existence and uniqueness of ρ_{12} follow immediately from Theorem 3.1 specialized to $\mathcal{R} = \nu_{\mathcal{E}}$, the conditions of which hold since functions, by Corollary 2.7, are closed; the fact that $\nu_{\mathcal{E}_i}$ is canonical for $\mathcal{E}_i$, $i = 1, 2$, ensures that ρ_{12} is a function). The kernel of the map $\rho_{\mathcal{E}} = \nu_{\mathcal{E} \vee E_{\mathcal{R}}} \circ \nu_{\mathcal{E}_1}^{-1}$ that inserts the equivalence classes of $coim\, \mathcal{R}/\mathcal{E}$ into the classes of $coim\, \mathcal{R}/(\mathcal{E} \vee E_R)$ provides an upper bound on the $\ker \hat{\mathcal{R}}$:

Proposition 3.2 *Let $\mathcal{R}$ be a relation and $\mathcal{E}$ an equivalence on* coim $\mathcal{R}$ *with the canonical surjection $\nu_{\mathcal{E}}$:* coim $\mathcal{R} \to$ coim $\mathcal{R}/\mathcal{E}$. *If $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ for some $\hat{\mathcal{R}}$, then*

$$\begin{aligned} \ker \hat{\mathcal{R}} &\subseteq \ker \rho_{\mathcal{E}} & (3.2)\\ \ker \hat{\mathcal{R}}^{-1} &\subseteq \ker \mathcal{R}^{-1}. & (3.3) \end{aligned}$$

In particular, $\hat{\mathcal{R}}$ is: (i) injective if and only if $\mathcal{E} \supseteq E_{\mathcal{R}}$; and, (ii) a partial function if and only if $\mathcal{R}$ is a partial function.

Proof. Let $(\alpha', \alpha'') \in \ker \hat{\mathcal{R}} = \nu_{\mathcal{E}} \circ \ker \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$ (if $\mathcal{R} = \emptyset$, what follows is vacuously true), which implies that there exist $a' \in \alpha'$ and $a'' \in \alpha''$ such that $(a', a'') \in \ker \mathcal{R} \subseteq E_{\mathcal{R}}$, or $a' \sim_{E_{\mathcal{R}}} a''$. Thus, both α' and α'' intersect the same equivalence class of $E_{\mathcal{R}}$. They both and the intersected class of $E_{\mathcal{R}}$ are subsets, therefore, of the same equivalence class of $\mathcal{E} \vee E_{\mathcal{R}}$: $\rho_{\mathcal{E}}(\alpha') = \rho_{\mathcal{E}}(\alpha'')$, or $(\alpha', \alpha'') \in \ker \rho_{\mathcal{E}}$. Since the pair $(\alpha', \alpha'') \in \ker \hat{\mathcal{R}}$ is arbitrary, we conclude that (3.2) holds.

Suppose further that $E_{\mathcal{R}} \subseteq \mathcal{E}$. Then $\mathcal{E} \vee E_{\mathcal{R}} = \mathcal{E}$, and $\rho_{\mathcal{E}}$ is the identity on $coim\, \mathcal{R}/\mathcal{E}$. In this case, the property (3.2) reads $\ker \hat{\mathcal{R}} \subseteq id_{\,coim\, \mathcal{R}/\mathcal{E}}$ and implies that $\hat{\mathcal{R}}$ is injective, as stated.

Conversely, if $\hat{\mathcal{R}}$ is injective, then $id_{\,coim\,\mathcal{R}/\mathcal{E}} = \ker\hat{\mathcal{R}} = \nu_{\mathcal{E}} \circ \ker\mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$. That is, whenever $\alpha', \alpha'' \in coim\,\mathcal{R}/\mathcal{E}$ are such that there exist $a' \in \alpha'$ and $a'' \in \alpha''$ and $a'(ker\mathcal{R})a''$, then $\alpha' = \alpha''$. This may take place if and only if, for each $a \in coim\,\mathcal{E}$, $\ker\mathcal{R}[a] \subseteq \mathcal{E}[a]$ and, consequently, $E_{\mathcal{R}} \subseteq \mathcal{E}$.

The kernel $\ker\hat{\mathcal{R}}^{-1} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1} \circ \nu_{\mathcal{E}} \circ \mathcal{R}^{-1} = \ker\mathcal{R}^{-1}$ since $\ker\nu_{\mathcal{E}} = \mathcal{E}$ and, by Theorem 3.1, the condition (3.1) holds. In particular, the relations $\mathcal{R}$ and $\hat{\mathcal{R}}$ are single- or multi-valued simultaneously. Q.E.D.

The condition (3.1), which in view of the reflexivity of $\mathcal{E}$ is equivalent to $\mathcal{R} \circ \mathcal{E} = \mathcal{R}$, may be restated to say that the inverse image sets of $\mathcal{R}$ are unions of equivalence classes of $\mathcal{E}$. This, of course, is also true for all positive powers of $\ker\mathcal{R}$. As a consequence, $E_{\mathcal{R}} \supseteq \mathcal{E}$ is a necessary condition for $\mathcal{R}$ to factor through a surjective function. The case when $\mathcal{R}$ is a function is special in that this condition is also sufficient:

Corollary 3.3 [Factor Theorem for Functions] *Let $f : A \to B$ be a function and $\mathcal{E}$ an equivalence on A with the canonical surjection $\nu_{\mathcal{E}} : A \to A/\mathcal{E}$. Then a decomposition $f = \hat{f} \circ \nu_{\mathcal{E}}$ exists if and only if* $\ker f \supseteq \mathcal{E}$, *in which case $\hat{f}$ is unique and is a function.*

Proof. Suppose that the factorization exists. Then, by Theorem 3.1, the condition (3.1) holds, and consequently, $E_f = \ker f \supseteq \mathcal{E}$. By Proposition 3.2, $\hat{f}$ is a function. Conversely, suppose that $E_f = \ker f \supseteq \mathcal{E}$. The composition with f^{-1} on the right and inversion then yield $f \supseteq f \circ \mathcal{E}$ — the condition (3.1) of Theorem 3.1. Hence, f may be factored as $\hat{f} \circ \nu_{\mathcal{E}}$. Q.E.D.

The sufficiency part of the preceding corollary is well known. The corollary indicates, in addition, that the condition (3.1) is also necessary: even if one is willing to forego the requirement that $\hat{f}$ be single-valued, the factorization $f = \hat{f} \circ \nu_{\mathcal{E}}$ is not possible unless (3.1) is satisfied.

4 (R, I, S)-Decompositions

Turning to (R, I, S)-decompositions in ***Rel***, we observe that Theorem 3.1 and Proposition 3.2 allow us to concentrate only on relations that are closed:

Theorem 4.1 *Let $\mathcal{R}$ be a relation and $\mathcal{E}$ an equivalence on* coim $\mathcal{R}$ *with the canonical surjection $\nu_{\mathcal{E}}$:* coim $\mathcal{R} \to$ coim $\mathcal{R}/\mathcal{E}$*. Then a decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ in which $\hat{\mathcal{R}}$ is injective exists if and only if $\mathcal{R}$ is closed,* i.e., $\mathcal{R} \circ E_{\mathcal{R}} = \mathcal{R}$.

Under the conditions of the preceding characterization theorem, $\hat{\mathcal{R}}$ is closed and, in view of Lemma 2.5, its inverse is closed as well. The duality considerations suggest, then, the (injective) inverse of $\mu_{\mathcal{R}^{-1}} : im\,\mathcal{R} \to im\,\mathcal{R}/E_{\mathcal{R}^{-1}}$ as a natural candidate for $\hat{\mathcal{R}}$ in the decomposition of $\mathcal{R}$.

Theorem 4.2 [Decomposition of Relations] *A relation $\mathcal{R}$ is (R, I, S)-decomposable if and only if it is closed. In this case $\mathcal{R} = \mu_{\mathcal{R}^{-1}}^{-1} \circ \psi_{\mathcal{R}} \circ \mu_{\mathcal{R}}$, where $\mu_{\mathcal{R}}$:* coim $\mathcal{R} \to$ coim $\mathcal{R}/E_{\mathcal{R}}$ *and $\mu_{\mathcal{R}^{-1}}$:* im$\mathcal{R} \to$ im$\mathcal{R}/E_{\mathcal{R}^{-1}}$ *are canonical surjections, and $\psi_{\mathcal{R}}$:* coim$\mathcal{R}/E_{\mathcal{R}} \to$ im$\mathcal{R}/E_{\mathcal{R}^{-1}}$ *is a unique isomorphism.*

Proof. Suppose that $\mathcal{R}$ is (R, I, S)-decomposable: $\mathcal{R} = \nu_{\mathcal{F}}^{-1} \circ \psi_{\mathcal{R}} \circ \nu_{\mathcal{E}}$ where $\nu_{\mathcal{E}}, \nu_{\mathcal{F}}$ are surjective functions, and f a bijection. Then $\hat{\mathcal{R}}$ is injective and, by Theorem 4.1, $\mathcal{R}$ is closed.

Conversely, let $\mathcal{R}$ be closed. Then, also by Theorem 4.1, $\mathcal{R}$ is factorizable: $\mathcal{R} = \hat{\mathcal{R}} \circ \mu_{\mathcal{R}}$ for some $\hat{\mathcal{R}}$. Since $\hat{\mathcal{R}}$ is injective, both $\mathcal{R}$ and its inverse are closed (Corollary 2.7). Hence, the conditions of the preceding theorem hold also for $\hat{\mathcal{R}}^{-1}$. Applying the dual restatement of Theorem 3.1 to $\hat{\mathcal{R}}^{-1}$, we conclude that there exists a relation $\psi_{\mathcal{R}}^{-1}$ such that $\hat{\mathcal{R}}^{-1} = \psi_{\mathcal{R}}^{-1} \circ \mu_{\mathcal{R}^{-1}}$ and, consequently, $\mathcal{R} = \mu_{\mathcal{R}^{-1}}^{-1} \circ \psi_{\mathcal{R}} \circ \mu_{\mathcal{R}}$.

Further, $\psi_{\mathcal{R}}$ is unique, as may be deduced by composing $\mathcal{R} = \mu_{\mathcal{R}^{-1}}^{-1} \circ \psi_{\mathcal{R}} \circ \mu_{\mathcal{R}}$ with $\mu_{\mathcal{R}^{-1}}$ on the left and $\mu_{\mathcal{R}}^{-1}$ on the right to yield

$$\psi_{\mathcal{R}} = \mu_{\mathcal{R}^{-1}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1}. \tag{4.1}$$

Finally, the above factorizations guarantee that both $\psi_{\mathcal{R}}$ and $\psi_{\mathcal{R}}^{-1}$ are injective. Alternatively, this follows from the following computation: from (4.1),

$$\ker \psi_{\mathcal{R}} = \mu_{\mathcal{R}} \circ \mathcal{R}^{-1} \circ E_{\mathcal{R}^{-1}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1} = \mu_{\mathcal{R}} \circ (\ker \mathcal{R}) \circ E_{\mathcal{R}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1}$$

in view of (2.7). Since $(\ker \mathcal{R}) \circ E_{\mathcal{R}} \subseteq E_{\mathcal{R}}$, we have $\ker \psi_{\mathcal{R}} \subseteq \mu_{\mathcal{R}} \circ E_{\mathcal{R}} \circ \mu_{\mathcal{R}}^{-1} = id_{\,coim\,\mathcal{R}/E_{\mathcal{R}}}$, so $\psi_{\mathcal{R}}$ is injective. Similarly, $\ker \psi_{\mathcal{R}}^{-1} \subseteq \mu_{\mathcal{R}^{-1}} \circ E_{\mathcal{R}^{-1}} \circ \mu_{\mathcal{R}^{-1}}^{-1} = id_{\,im\,\mathcal{R}/E_{\mathcal{R}}^{-1}}$, so $\psi_{\mathcal{R}}$ is a partial function. Q.E.D.

The fact that the sets $coim\,\mathcal{R}/E_{\mathcal{R}}$ and $im\,\mathcal{R}/E_{\mathcal{R}^{-1}}$ are isomorphic is well known (*e.g.*, Cohn [1]). To establish this fact, it suffices to merely define $\psi_{\mathcal{R}}$ as in (4.1), which leads , as we have seen, to $\ker \psi_{\mathcal{R}} = id_{\,coim\,\mathcal{R}/E_{\mathcal{R}}}$ and $\ker \psi_{\mathcal{R}}^{-1} = id_{\,im\,\mathcal{R}/E_{\mathcal{R}^{-1}}}$. This choice of $\psi_{\mathcal{R}}$ is not, however, sufficient for $\mathcal{R}$ to be decomposable: $\mathcal{R}$ must closed. Thus, besides yielding the form (4.1) of $\psi_{\mathcal{R}}$, Theorem 4.2 characterizes the (R, I, M)-decomposable relations in terms of the closure operator $\langle\cdot\rangle$, and establishes the uniqueness of such decomposition.

References

[1] P. M. Cohn. *Universal Algebra. Harper's Series in Modern Mathematics*, Harper & Row, New York, 1965.

[2] T.W. Hungerford. *Algebra.* Volume 73 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1974.

[3] B.L. van der Waerden. *Algebra I: Achte Auflage der Modernen Algebra.* Springer-Velag, Berlin, 1971.