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Factorization and Decomposition of Relations

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Abstract

The paper extends the factor and decomposition theorems for functions to the category ***Rel*** of sets and relations. The converse theorems are also stated and proved. We show that the (retraction,isomorphism,section)-decomposable relations form a closure system on *Mor* (***Rel***), and characterize it in terms of a closure operator.

1 Introduction

Let $(A, \mathcal{E})$ be a relational structure consisting of a set A and an equivalence $\mathcal{E}$ on A. It is well known that a function $f : A \to B$ decomposes into a canonical surjection $\nu_A : A \to A/(f^{-1} \circ f)$, bijection $\psi_f : A/(f^{-1} \circ f) \to im\, f$, and an injection (inclusion) $\iota : im\, f \to B$. The celebrated Homomorphism Theorems of E. Nöther (van der Waerden [3], Hungerford [2]) provide the corresponding decompositions for sets with operators (*e.g.*, Cohn [1]).

Recall that in the category ***Set*** of sets and functions, injections are sections and surjections are retractions. If R, S, and I are, respectively, the classes of retractions, sections, and isomorphisms, then $\nu_A \circ \psi_f \circ \iota$ is the essentially unique (R, I, S)-decomposition of an arbitrary morphism f of ***Set***. In the present paper we investigate the (R, I, S)-decomposability in a larger category ***Rel*** of sets and relations. Although ***Rel*** is not (R, I, S)-decomposable as a category, we identify the class of those morphisms which do have such a decomposition. We find, furthermore, that this class is a closure system, and characterize it by the associated closure operator on *Mor*(***Rel***). This development allows us to formulate and prove the respective converse statements for the factorization and decomposition theorems.

2 The Closure of a Relation

Recall that in the category ***Rel***, the objects are sets and morphisms the triples of sets $(A, \mathcal{R}, B)$ such that $\mathcal{R} \subseteq A \times B$. Composability of morphisms is defined as in ***Set***, and the domain and codomain functions $dom, codom : Mor\,(\boldsymbol{Rel}) \to Ob\,(\boldsymbol{Rel})$ are defined by $dom\,(A, \mathcal{R}, B) = A$ and $codom\,(A, \mathcal{R}, B) = B$. For each $S \in Ob\,(\boldsymbol{Rel})$, the morphism (S, id_S, S) with the diagonal id_S of $S \times S$ is the identity on S. The composition of morphisms is defined in a natural way in terms of the composition of the corresponding relations. Since, furthermore, $\hom(A, B) = \mathcal{P}(A \times B)$ is a set, the smallness of the morphism class condition is satisfied, and ***Rel*** is indeed a category.

Some special morphisms in ***Rel*** are natural generalizations from their counterparts in ***Set***. Below, we refer to the image of the inverse of a relation as the coimage of that relation:

Definition 2.1 *We shall say that a relation $\mathcal{R}$ is:*

(i) A correspondence *if* $\operatorname{coim} \mathcal{R} = \operatorname{dom} \mathcal{R}$.

(ii) A partial function *if for all* $x \in \operatorname{dom} \mathcal{R}$, $\operatorname{card} R[a] \leq 1$.

And dually, $\mathcal{R}$ is:

(i) A surjection *if* $\operatorname{im} \mathcal{R} = \operatorname{codom} \mathcal{R}$.

(ii) An injection *if for all* $x \in \operatorname{codom} \mathcal{R}$, $\operatorname{card} R^{-1}[a] \leq 1$.

We shall further say that $\mathcal{R}$ is a bijection *if it is both surjective and injective. And dually, $\mathcal{R}$ is a* function *if it is both a partial function and a correspondence.*
Single-valuedness of a relation is, thus, in duality with bijectivity. The bijective functions are the only isomorphisms of ***Rel***, as follows from the next easily verifiable observation.
Proposition 2.2 *A morphism $(A, \mathcal{R}, B)$ is a section in* **Rel** *if and only if it is an injective correspondence. In this case $\mathcal{R}^{-1}$ is a left inverse of $\mathcal{R}$:*

$$\mathcal{R}^{-1} \circ \mathcal{R} = \mathrm{id}_{\mathrm{dom}\,\mathcal{R}}. \tag{2.1}$$

Dually, a morphism $(A, \mathcal{R}, B)$ is a retraction in **Rel** *if and only if it is a surjective partial function. In this case $\mathcal{R}^{-1}$ is a right inverse of $\mathcal{R}$:*

$$\mathcal{R} \circ \mathcal{R}^{-1} = \mathrm{id}_{\mathrm{codom}\,\mathcal{R}}. \tag{2.2}$$

Each relation $\mathcal{R}$ gives rise to an equivalence generated on its coimage by $\mathcal{R}^{-1} \circ \mathcal{R}$; that is, there exists a mapping

$$E = \begin{cases} Mor(\boldsymbol{Rel}) & \rightarrow \quad Mor(\boldsymbol{Rel}) \\ \quad \mathcal{R} & \mapsto \quad \bigcup_{n \in \mathbf{N}} (\mathcal{R}^{-1} \circ \mathcal{R})^n . \end{cases} \tag{2.3}$$

Generalizing from the terminology for functions, we shall refer to $\ker \mathcal{R} = \mathcal{R}^{-1} \circ \mathcal{R}$ as the *kernel* of the relation $\mathcal{R}$. If $\mathcal{R}$ is a partial function, $E_{\mathcal{R}} = \ker \mathcal{R}$, since by (2.1) the kernel is an equivalence.

The mapping (2.3) allows us to define the closure operator $\langle \cdot \rangle$ on the class of relations (Definition 2.4 below). The proof of the closure property is based on two lemmata.
Lemma 2.3 *For any relation $\mathcal{R}$,*

$$\mathcal{R} \circ E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathrm{Hom}_{\,Rel}(\mathcal{R}^{-1}, \mathcal{R})\,(E_{\mathcal{R}}) = E_{\mathcal{R}^{-1}}. \tag{2.4}$$

Also, if $f_A : A \rightarrow \mathrm{coim}\,\mathcal{R}$ and $f_B : B \rightarrow \mathrm{im}\,\mathcal{R}$ are retractions, then

$$\mathrm{Hom}_{\,Rel}(f_A, f_A^{-1})\,(E_{\mathcal{R}}) = E_{\mathrm{Hom}_{Rel}(f_A, f_B^{-1})(\mathcal{R})}. \tag{2.5}$$

Proof. The identity (2.4) follows directly from definition (2.3) and the associativity of composition: $\mathcal{R} \circ E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathcal{R} \circ \bigcup_{n \in \mathbf{N}} (\mathcal{R}^{-1} \circ \mathcal{R})^n \circ \mathcal{R}^{-1} = \bigcup_{n \in \mathbf{N}} (\mathcal{R} \circ \mathcal{R}^{-1})^{n+1} = E_{\mathcal{R}^{-1}}$, since the sequence $\{(\ker \mathcal{R})^n\}_{n \in \mathbf{N}}$ is monotone increasing in view of the reflexivity of $\ker \mathcal{R}$.

In order to prove (2.5), we may utilize (2.2) since f_A and f_B are retractions:

$$E_{\mathrm{Hom}_{Rel}(f_A, f_B^{-1})(\mathcal{R})} = \bigcup_{n \in \mathbf{N}} (f_A^{-1} \circ \mathcal{R}^{-1} \circ f_B \circ f_B^{-1} \circ \mathcal{R} \circ f_A)^n = f_A^{-1} \circ \left(\bigcup_{n \in \mathbf{N}} \left(\ker \mathcal{R}|_{im\, f_A}^{im\, f_B} \right)^n \right) \circ f_A,$$

and (2.5) follows from the surjectivity of f_A and f_B guaranteed by Proposition 2.2. Q.E.D.

A weaker than in Lemma 2.3 requirement, namely, that only f_B be a partial surjective function, permits us to conclude that

$$Hom_{\mathit{Rel}}(f_A, f_A^{-1})\,(\ker \mathcal{R}) = \ker Hom_{\mathit{Rel}}(f_A, f_B^{-1})(\mathcal{R}). \tag{2.6}$$

By composing the identity (2.4) with $\mathcal{R}$ on the right, we arrive at the following **Proposition** and **Definition 2.4** *For any $\mathcal{R}$, the diagram*

$$\begin{tikzcd} dom\,\mathcal{R} \arrow[r, "\mathcal{R}"] \arrow[d, "E_{\mathcal{R}}"] & codom\,\mathcal{R} \arrow[d, "E_{\mathcal{R}^{-1}}"] \\ dom\,\mathcal{R} \arrow[r, "\mathcal{R}"] & codom\,\mathcal{R} \end{tikzcd}$$

is commutative. We shall say that the diagonal of the diagram,

$$\langle \mathcal{R} \rangle = \mathcal{R} \circ E_{\mathcal{R}} = E_{\mathcal{R}^{-1}} \circ \mathcal{R}, \tag{2.7}$$

is the closure *of* $\mathcal{R}$.

The name of $\langle \cdot \rangle$ is justified by Proposition 2.6 below, in the proof of which we utilize **Lemma 2.5** *For any relation $\mathcal{R}$:*

$$E_{\langle \mathcal{R} \rangle} = E_{\mathcal{R}} \tag{2.8}$$

$$\langle \mathcal{R} \rangle^{-1} = \left\langle \mathcal{R}^{-1} \right\rangle. \tag{2.9}$$

Also, if $f_A : A \to \operatorname{coim} \mathcal{R}$ and $f_B : B \to \operatorname{im} \mathcal{R}$ are retractions, then

$$\left\langle \mathrm{Hom}_{\mathit{Rel}}(f_A, f_B^{-1})\,(\mathcal{R}) \right\rangle = \mathrm{Hom}_{\mathit{Rel}}(f_A, f_B^{-1})\,(\langle \mathcal{R} \rangle)\,. \tag{2.10}$$

Proof. From definitions (2.3) and (2.7), we obtain $E_{\langle \mathcal{R} \rangle} = \bigcup_{n \in \mathbf{N}} (E_{\mathcal{R}} \circ (\ker \mathcal{R}) \circ E_{\mathcal{R}})^n = E_{\mathcal{R}}$, since $E_{\mathcal{R}} \circ (\ker \mathcal{R}) = E_{\mathcal{R}}$, and $E_{\mathcal{R}}$ is idempotent; hence (2.8). Further, the unary operations $\langle \cdot \rangle$ and $()^{-1}$ commute:

$$\langle \mathcal{R} \rangle^{-1} = (\mathcal{R} \circ E_{\mathcal{R}})^{-1} = E_{\mathcal{R}} \circ \mathcal{R}^{-1} = \mathcal{R}^{-1} \circ E_{\mathcal{R}^{-1}} = \left\langle \mathcal{R}^{-1} \right\rangle.$$

Finally,

$$\left\langle Hom_{\mathit{Rel}}(f_A, f_B^{-1})\,(\mathcal{R}) \right\rangle = Hom_{\mathit{Rel}}(f_A, f_B^{-1})\,(\mathcal{R}) \circ E_{Hom_{\mathit{Rel}}(f_A, f_B^{-1})(\mathcal{R})} = \left(f_B^{-1} \circ \mathcal{R} \circ f_A\right) \circ \left(f_A^{-1} \circ E_{\mathcal{R}} \circ f_A\right)$$

by (2.5) of Lemma 2.3. The last composition is equal to $f_B^{-1} \circ \langle \mathcal{R} \rangle \circ f_A$ since f_A is, by assumption, a retraction, and (2.10) follows at once. Q.E.D.

Lemmata 2.3 and 2.5 may be summarized as the commutativity of the diagram

$$\begin{tikzcd}
& A \arrow[dl, "E_{\mathcal{R}'}"'] \arrow[dr, "\langle\mathcal{R}'\rangle"] \arrow[dd, "f_A"] & \\
A \arrow[rr, "\mathcal{R}'"] \arrow[dd, "f_A"'] & & B \\
& A' \arrow[dl, "E_{\mathcal{R}}"'] \arrow[dr, "\langle\mathcal{R}\rangle"] & \\
A' \arrow[rr, "\mathcal{R}"'] & & B' \arrow[uu, "f_B^{-1}"']
\end{tikzcd}$$

in which $\mathcal{R}' = Hom_{Rel}(f_A, f_B^{-1})(\mathcal{R})$. We are in a position to state and prove

Proposition 2.6 *Let* $<>$: $Mor(\boldsymbol{Rel}) \to Mor(\boldsymbol{Rel})$ *be the map, the action of which is defined by (2.7). Then* $\langle\cdot\rangle$ *is a closure operator on the Boolean* $(Mor(\boldsymbol{Rel}), \subseteq)$; *that is, for all* $\mathcal{R}, \mathcal{S} \in Mor(\boldsymbol{Rel})$,

$$R \subseteq \mathcal{S} \implies \langle\mathcal{R}\rangle \subseteq \langle\mathcal{S}\rangle \tag{2.11}$$
$$\mathcal{R} \subseteq \langle\mathcal{R}\rangle \tag{2.12}$$
$$\langle\langle\mathcal{R}\rangle\rangle = \langle\mathcal{R}\rangle\,. \tag{2.13}$$

Proof. The monotonicity of the map E implies that, whenever relations $\mathcal{R}$ and $\mathcal{S}$ are such that $\mathcal{R} \subseteq \mathcal{S}$, we have $\langle\mathcal{R}\rangle = \mathcal{R} \circ E_{\mathcal{R}} \subseteq \mathcal{S} \circ E_{\mathcal{S}} = \langle\mathcal{S}\rangle$, hence the monotonicity (2.11) of $< \cdot >$. The reflexivity of the equivalence $E_{\mathcal{R}}$ on $coim\,\mathcal{R}$ implies (2.12): $\langle\mathcal{R}\rangle = \mathcal{R} \circ E_{\mathcal{R}} \supseteq \mathcal{R} \circ id_{coim\,\mathcal{R}}$. Finally, (2.13) is a consequence of (2.8) and idempotency of equivalences: $\langle\langle\mathcal{R}\rangle\rangle = (\mathcal{R} \circ E_{\mathcal{R}}) \circ E_{\langle\mathcal{R}\rangle} = (\mathcal{R} \circ E_{\mathcal{R}}) \circ E_{\mathcal{R}} = \langle\mathcal{R}\rangle$. Q.E.D.

The commutativity of $\langle\cdot\rangle$ and $(\cdot)^{-1}$ (Lemma 2.5) implies that a relation is $\langle\cdot\rangle$-closed if and only if its inverse is. Furthermore, whenever $\ker\mathcal{R}$ is an equivalence, $\langle\mathcal{R}\rangle = R \circ \ker\mathcal{R} = R$. This observation and its dual prove the following

Corollary 2.7 *For a relation* $\mathcal{R}$ *to be closed it is sufficient that either (or both)* $\ker\mathcal{R}$ *or* $\ker\mathcal{R}^{-1}$ *be an equivalence.*

Thus, all equivalences, partial functions and injections are closed. In addition, identity (2.10) implies that $Hom_{Rel}(f_A, f_B^{-1})(\mathcal{R})$ is closed if and only if the relation $\mathcal{R}$ is.

3 Factorizations

In the category $\boldsymbol{Set}$, the requirement that the kernel of a function f include as its subset an equivalence $\mathcal{E}$ on $dom\,f$ is sufficient for f to be uniquely factorizable through the canonical surjection $\nu_{\mathcal{E}} : dom\, f \to dom\, f/\mathcal{E}$. The following characterization theorem extends this result to the category ***Rel*** and, in addition, provides its converse.

Theorem 3.1 [Factor Theorem for Relations] *Let $\mathcal{R}$ be a relation and $\mathcal{E}$ an equivalence on* $\operatorname{coim}\mathcal{R}$ *with the canonical surjection* $\nu_{\mathcal{E}} : \operatorname{coim}\mathcal{R} \to \operatorname{coim}\mathcal{R}/\mathcal{E}$. *Then a decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ exists if and only if*

$$\mathcal{R} \circ \mathcal{E} \subseteq \mathcal{R}, \tag{3.1}$$

in which case $\hat{\mathcal{R}}$ is unique and equal to $\mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$.

Proof. Suppose first that $\hat{\mathcal{R}}$ satisfying $\hat{\mathcal{R}} \circ \nu_{\mathcal{E}} = \mathcal{R}$ exists. Since the canonical surjection $\nu_{\mathcal{E}}$ is a retraction (Proposition 2.2), this yields a unique $\hat{\mathcal{R}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$. A composition of $\hat{\mathcal{R}}$ with $\nu_{\mathcal{E}}$ on the right results in $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1} \circ \nu_{\mathcal{E}} = \mathcal{R} \circ \mathcal{E}$.

Conversely, if $\mathcal{R} \subseteq \mathcal{R} \circ \mathcal{E}$, then $\mathcal{R} = \mathcal{R} \circ \mathcal{E}$ by the reflexivity of $\mathcal{E}$, and $\hat{\mathcal{R}} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$ provides the stated decomposition of $\mathcal{R}$. Q.E.D.

The next proposition presents upper bounds on $\ker \hat{\mathcal{R}}$ and $\ker \hat{\mathcal{R}}^{-1}$ for the factor $\hat{\mathcal{R}}$ in the decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ of a relation $\mathcal{R}$. It utilizes the canonical "insertion" $\rho_{\mathcal{E}} : coim\,\mathcal{R}/\mathcal{E} \to coim\,\mathcal{R}/(\mathcal{E} \vee E_R)$ defined, more specifically, as follows. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two equivalences on a set S with canonical identification maps $\nu_{\mathcal{E}_1}$ and $\nu_{\mathcal{E}_2}$, respectively. If $\mathcal{E}_1 \subseteq \mathcal{E}_2$, then $\rho_{12} = \nu_{\mathcal{E}_2} \circ \nu_{\mathcal{E}_1}^{-1}$ is a unique function that assigns to each equivalence class α of $\mathcal{E}_1$ the unique equivalence class of $\mathcal{E}_2$ that includes α as a subset (the existence and uniqueness of ρ_{12} follow immediately from Theorem 3.1 specialized to $\mathcal{R} = \nu_{\mathcal{E}}$, the conditions of which hold since functions, by Corollary 2.7, are closed; the fact that $\nu_{\mathcal{E}_i}$ is canonical for $\mathcal{E}_i$, $i = 1, 2$, ensures that ρ_{12} is a function). The kernel of the map $\rho_{\mathcal{E}} = \nu_{\mathcal{E} \vee E_{\mathcal{R}}} \circ \nu_{\mathcal{E}_1}^{-1}$ that inserts the equivalence classes of $coim\,\mathcal{R}/\mathcal{E}$ into the classes of $coim\,\mathcal{R}/(\mathcal{E} \vee E_R)$ provides an upper bound on the $\ker \hat{\mathcal{R}}$:

Proposition 3.2 *Let $\mathcal{R}$ be a relation and $\mathcal{E}$ an equivalence on* $\operatorname{coim}\mathcal{R}$ *with the canonical surjection* $\nu_{\mathcal{E}} : \operatorname{coim}\mathcal{R} \to \operatorname{coim}\mathcal{R}/\mathcal{E}$. *If $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ for some $\hat{\mathcal{R}}$, then*

$$\ker \hat{\mathcal{R}} \subseteq \ker \rho_{\mathcal{E}} \tag{3.2}$$

$$\ker \hat{\mathcal{R}}^{-1} \subseteq \ker \mathcal{R}^{-1}. \tag{3.3}$$

In particular, $\hat{\mathcal{R}}$ is: (i) injective if and only if $\mathcal{E} \supseteq E_{\mathcal{R}}$; and, (ii) a partial function if and only if $\mathcal{R}$ is a partial function.

Proof. Let $(\alpha', \alpha'') \in \ker \hat{\mathcal{R}} = \nu_{\mathcal{E}} \circ \ker \mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$ (if $\mathcal{R} = \emptyset$, what follows is vacuously true), which implies that there exist $a' \in \alpha'$ and $a'' \in \alpha''$ such that $(a', a'') \in \ker \mathcal{R} \subseteq E_{\mathcal{R}}$, or $a' \sim_{E_{\mathcal{R}}} a''$. Thus, both α' and α'' intersect the same equivalence class of $E_{\mathcal{R}}$. They both and the intersected class of $E_{\mathcal{R}}$ are subsets, therefore, of the same equivalence class of $\mathcal{E} \vee E_{\mathcal{R}} : \rho_{\mathcal{E}}(\alpha') = \rho_{\mathcal{E}}(\alpha'')$, or $(\alpha', \alpha'') \in \ker \rho_{\mathcal{E}}$. Since the pair $(\alpha', \alpha'') \in \ker \hat{\mathcal{R}}$ is arbitrary, we conclude that (3.2) holds.

Suppose further that $E_{\mathcal{R}} \subseteq \mathcal{E}$. Then $\mathcal{E} \vee E_{\mathcal{R}} = \mathcal{E}$, and $\rho_{\mathcal{E}}$ is the identity on $coim\,\mathcal{R}/\mathcal{E}$. In this case, the property (3.2) reads $\ker \hat{\mathcal{R}} \subseteq id_{\,coim\,\mathcal{R}/\mathcal{E}}$ and implies that $\hat{\mathcal{R}}$ is injective, as stated.

Conversely, if $\hat{\mathcal{R}}$ is injective, then $id_{coim\,\mathcal{R}/\mathcal{E}} = \ker\hat{\mathcal{R}} = \nu_{\mathcal{E}} \circ \ker\mathcal{R} \circ \nu_{\mathcal{E}}^{-1}$. That is, whenever $\alpha', \alpha'' \in coim\,\mathcal{R}/\mathcal{E}$ are such that there exist $a' \in \alpha'$ and $a'' \in \alpha''$ and $a'(ker\mathcal{R})a''$, then $\alpha' = \alpha''$. This may take place if and only if, for each $a \in coim\,\mathcal{E}$, $\ker\mathcal{R}[a] \subseteq \mathcal{E}[a]$ and, consequently, $E_{\mathcal{R}} \subseteq \mathcal{E}$.

The kernel $\ker\hat{\mathcal{R}}^{-1} = \mathcal{R} \circ \nu_{\mathcal{E}}^{-1} \circ \nu_{\mathcal{E}} \circ \mathcal{R}^{-1} = \ker\mathcal{R}^{-1}$ since $\ker\nu_{\mathcal{E}} = \mathcal{E}$ and, by Theorem 3.1, the condition (3.1) holds. In particular, the relations $\mathcal{R}$ and $\hat{\mathcal{R}}$ are single- or multi-valued simultaneously. Q.E.D.

The condition (3.1), which in view of the reflexivity of $\mathcal{E}$ is equivalent to $\mathcal{R} \circ \mathcal{E} = \mathcal{R}$, may be restated to say that the inverse image sets of $\mathcal{R}$ are unions of equivalence classes of $\mathcal{E}$. This, of course, is also true for all positive powers of $\ker\mathcal{R}$. As a consequence, $E_{\mathcal{R}} \supseteq \mathcal{E}$ is a necessary condition for $\mathcal{R}$ to factor through a surjective function. The case when $\mathcal{R}$ is a function is special in that this condition is also sufficient:

Corollary 3.3 [Factor Theorem for Functions] *Let $f : A \rightarrow B$ be a function and $\mathcal{E}$ an equivalence on A with the canonical surjection $\nu_{\mathcal{E}} : A \rightarrow A/\mathcal{E}$. Then a decomposition $f = \hat{f} \circ \nu_{\mathcal{E}}$ exists if and only if* $\ker f \supseteq \mathcal{E}$, *in which case $\hat{f}$ is unique and is a function.*

Proof. Suppose that the factorization exists. Then, by Theorem 3.1, the condition (3.1) holds, and consequently, $E_f = \ker f \supseteq \mathcal{E}$. By Proposition 3.2, $\hat{f}$ is a function. Conversely, suppose that $E_f = \ker f \supseteq \mathcal{E}$. The composition with f^{-1} on the right and inversion then yield $f \supseteq f \circ \mathcal{E}$ — the condition (3.1) of Theorem 3.1. Hence, f may be factored as $\hat{f} \circ \nu_{\mathcal{E}}$. Q.E.D.

The sufficiency part of the preceding corollary is well known. The corollary indicates, in addition, that the condition (3.1) is also necessary: even if one is willing to forego the requirement that $\hat{f}$ be single-valued, the factorization $f = \hat{f} \circ \nu_{\mathcal{E}}$ is not possible unless (3.1) is satisfied.

4 (R, I, S)-Decompositions

Turning to (R, I, S)-decompositions in ***Rel***, we observe that Theorem 3.1 and Proposition 3.2 allow us to concentrate only on relations that are closed:

Theorem 4.1 *Let $\mathcal{R}$ be a relation and $\mathcal{E}$ an equivalence on* coim $\mathcal{R}$ *with the canonical surjection $\nu_{\mathcal{E}}$:* coim $\mathcal{R} \rightarrow$ coim $\mathcal{R}/\mathcal{E}$. *Then a decomposition $\mathcal{R} = \hat{\mathcal{R}} \circ \nu_{\mathcal{E}}$ in which $\hat{\mathcal{R}}$ is injective exists if and only if $\mathcal{R}$ is closed,* i.e., $\mathcal{R} \circ E_{\mathcal{R}} = \mathcal{R}$.

Under the conditions of the preceding characterization theorem, $\hat{\mathcal{R}}$ is closed and, in view of Lemma 2.5, its inverse is closed as well. The duality considerations suggest, then, the (injective) inverse of $\mu_{\mathcal{R}^{-1}} : im\,\mathcal{R} \rightarrow im\,\mathcal{R}/E_{\mathcal{R}^{-1}}$ as a natural candidate for $\hat{\mathcal{R}}$ in the decomposition of $\mathcal{R}$.

Theorem 4.2 [Decomposition of Relations] *A relation $\mathcal{R}$ is (R, I, S)-decomposable if and only if it is closed. In this case $\mathcal{R} = \mu_{\mathcal{R}^{-1}}^{-1} \circ \psi_{\mathcal{R}} \circ \mu_{\mathcal{R}}$, where $\mu_{\mathcal{R}}$:* coim $\mathcal{R} \rightarrow$ coim $\mathcal{R}/E_{\mathcal{R}}$ *and $\mu_{\mathcal{R}^{-1}}$:* im$\mathcal{R} \rightarrow$ im$\mathcal{R}/E_{\mathcal{R}^{-1}}$ *are canonical surjections, and $\psi_{\mathcal{R}}$:* coim$\mathcal{R}/E_{\mathcal{R}} \rightarrow$ im$\mathcal{R}/E_{\mathcal{R}^{-1}}$ *is a unique isomorphism.*

Proof. Suppose that $\mathcal{R}$ is (R, I, S)-decomposable: $\mathcal{R} = \nu_{\mathcal{F}}^{-1} \circ \psi_{\mathcal{R}} \circ \nu_{\mathcal{E}}$ where $\nu_{\mathcal{E}}, \nu_{\mathcal{F}}$ are surjective functions, and f a bijection. Then $\hat{\mathcal{R}}$ is injective and, by Theorem 4.1, $\mathcal{R}$ is closed.

Conversely, let $\mathcal{R}$ be closed. Then, also by Theorem 4.1, $\mathcal{R}$ is factorizable: $\mathcal{R} = \hat{\mathcal{R}} \circ \mu_{\mathcal{R}}$ for some $\hat{\mathcal{R}}$. Since $\hat{\mathcal{R}}$ is injective, both $\mathcal{R}$ and its inverse are closed (Corollary 2.7). Hence, the conditions of the preceding theorem hold also for $\hat{\mathcal{R}}^{-1}$. Applying the dual restatement of Theorem 3.1 to $\hat{\mathcal{R}}^{-1}$, we conclude that there exists a relation $\psi_{\mathcal{R}}^{-1}$ such that $\hat{\mathcal{R}}^{-1} = \psi_{\mathcal{R}}^{-1} \circ \mu_{\mathcal{R}^{-1}}$ and, consequently, $\mathcal{R} = \mu_{\mathcal{R}^{-1}}^{-1} \circ \psi_{\mathcal{R}} \circ \mu_{\mathcal{R}}$.

Further, $\psi_{\mathcal{R}}$ is unique, as may be deduced by composing $\mathcal{R} = \mu_{\mathcal{R}^{-1}}^{-1} \circ \psi_{\mathcal{R}} \circ \mu_{\mathcal{R}}$ with $\mu_{\mathcal{R}^{-1}}$ on the left and $\mu_{\mathcal{R}}^{-1}$ on the right to yield

$$\psi_{\mathcal{R}} = \mu_{\mathcal{R}^{-1}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1}. \tag{4.1}$$

Finally, the above factorizations guarantee that both $\psi_{\mathcal{R}}$ and $\psi_{\mathcal{R}}^{-1}$ are injective. Alternatively, this follows from the following computation: from (4.1),

$$\ker \psi_{\mathcal{R}} = \mu_{\mathcal{R}} \circ \mathcal{R}^{-1} \circ E_{\mathcal{R}^{-1}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1} = \mu_{\mathcal{R}} \circ (\ker \mathcal{R}) \circ E_{\mathcal{R}} \circ \mathcal{R} \circ \mu_{\mathcal{R}}^{-1}$$

in view of (2.7). Since $(\ker \mathcal{R}) \circ E_{\mathcal{R}} \subseteq E_{\mathcal{R}}$, we have $\ker \psi_{\mathcal{R}} \subseteq \mu_{\mathcal{R}} \circ E_{\mathcal{R}} \circ \mu_{\mathcal{R}}^{-1} = id_{\,coim\,\mathcal{R}/E_{\mathcal{R}}}$, so $\psi_{\mathcal{R}}$ is injective. Similarly, $\ker \psi_{\mathcal{R}}^{-1} \subseteq \mu_{\mathcal{R}^{-1}} \circ E_{\mathcal{R}^{-1}} \circ \mu_{\mathcal{R}^{-1}}^{-1} = id_{\,im\,\mathcal{R}/E_{\mathcal{R}}^{-1}}$, so $\psi_{\mathcal{R}}$ is a partial function. Q.E.D.

The fact that the sets $coim\,\mathcal{R}/E_{\mathcal{R}}$ and $im\,\mathcal{R}/E_{\mathcal{R}^{-1}}$ are isomorphic is well known (*e.g.*, Cohn [1]). To establish this fact, it suffices to merely define $\psi_{\mathcal{R}}$ as in (4.1), which leads , as we have seen, to $\ker \psi_{\mathcal{R}} = id_{\,coim\,\mathcal{R}/E_{\mathcal{R}}}$ and $\ker \psi_{\mathcal{R}}^{-1} = id_{\,im\,\mathcal{R}/E_{\mathcal{R}^{-1}}}$. This choice of $\psi_{\mathcal{R}}$ is not, however, sufficient for $\mathcal{R}$ to be decomposable: $\mathcal{R}$ must closed. Thus, besides yielding the form (4.1) of $\psi_{\mathcal{R}}$, Theorem 4.2 characterizes the (R, I, M)-decomposable relations in terms of the closure operator $\langle\cdot\rangle$, and establishes the uniqueness of such decomposition.

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