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Capacity Investment under Demand Uncertainty: Price vs. Quantity Competition

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Abstract

This article shows that under uncertainty, a firm's capacity investment decision crucially depends on the mode of market approach (price-setting vs. quantity-setting) and competition that follows investment. We model an industry in which firms have to make capacity investment decisions when demand is uncertain. First each firm must decide on its capacity investment level. Then, industry capacity levels are observed and firms engage in quantity or price competition. Finally, demand and revenues are realized.

We begin by considering a monopoly and show that the monopoly price given an uncertain demand curve can be higher or lower than the reference price when the demand curve is known at the start. Moreover price and firm value may fall or rise with increasing uncertainty. We compare these results with a setting where a monopolist sets quantity instead of price. The resulting investment fundamentally differs from the price-setting investment. Moreover, the investment strategy under quantity-setting is significantly less sensitive to variability *and* more profitable than under price-setting.

Under quantity competition, these results extend to a duopoly, oligopoly and perfect competition. In addition, entry dettering investments are possible yet more difficult as variability increases and credible only at low investment costs. Under price competition, no pure equilibria exist if there is demand uncertainty.

Key Words: pricing, quantity, capacity, competition, strategy, game theory, demand uncertainty.

1 Introduction

When firms invest in productive capacity they usually face uncertainty and competition. Not only is the magnitude of potential demand uncertain, but potential actions of competitors may significantly affect the profitability of the investment. When making the investment decisions, a variety of questions are naturally posed. Should the firm invest more or less when uncertainty increases? How does uncertainty affect firm value? How does competition affect the decision? Sure, we expect firm investment to decrease while industry investment increases, but does the investment decision depend on *how* firms later compete in the market? Is it worthwhile to invest in excess capacity to deter entry by potential competitors?

The purpose of this article is to study how strategic investment in productive capacity depends on demand uncertainty, the mode of competition (price setting vs. quantity setting) and the number of competing firms. We analyze an industry in which firms have to make capacity investment decisions when demand is uncertain. First each firm must decide on its capacity investment level. Then, industry capacity levels are observed and firms engage in price or quantity competition. Finally, demand and revenues are realized. We will show that, under uncertainty, a firm's capacity investment decision crucially depends on the potential and the mode of competition that follows the investment. We will also show how increasing uncertainty affects the investment decision and the firm's value. In contrast, without uncertainty, investment decisions do not depend on the mode of competition.

In this article competitive investment in capacity is a long-term strategic decision while the pricing or quantity decision provides control in the short-term. We explicitly examine how the level of risk or variability in demand affects this decision. We begin by analyzing the joint pricing and capacity decisions of a monopolist who faces an uncertain demand curve. The monopolist first invests in capacity and then sets a price before market demand uncertainty is resolved. We analyze the capacity-constrained pricing decision in detail and show that the monopoly price given an uncertain demand curve can be higher or lower than the reference price when the demand curve is known at the start. Moreover price and firm value may fall or rise with increasing uncertainty. Contrary to classical deterministic results, we show that in the presence of uncertainty, the capacity investment level under this price setting scenario *differs* from the investment under a quantity setting scenario. Not only is quantity setting more profitable, but it also yields an investment policy that is significantly less sensitive to demand variability than the price-setting policy. It is surprising that the price-capacity decision problem which has two control variables affecting two dimensions (price and quantity via the capacity constraint) is inferior to the quantity-capacity problem which effectively controls only one dimension (quantity). The reason is that quantity-setting firms benefit from the market mechanism that sets a state-dependent market clearing price. Thus, under this mode of competition and up to medium levels of variability, no sales are ever lost due to capacity constraints nor is there ever any unused excess capacity. This perfect balance can never be achieved under price-setting: because of demand uncertainty, there always will be scenarios where the announced price was either too low, leading to excess demand and lost sales, or too high, resulting in unused excess capacity. (Notice that we show the superiority of quantity setting without charging penalties for lost sales or excess capacity; clearly, doing so would only strengthen our case.) Thus, to restore the symmetry, price-setting firms should be allowed *ex-post* capacity flexibility to increase (lease or subcontract) or decrease (sell) capacity. Or, they should be allowed to postpone the pricing decision until after uncertainty is realized. Indeed, we will show that, up to medium levels of variability, the investment decision under quantity-setting is independent of *postponement* and the corresponding *value of information* is zero. This may suggest why capacity decisions by operating managers (who typically reason in terms of quantity rather than price) can be made by deterministic reasoning if levels of variability are moderate.

To investigate the effect of intensity of competition under uncertainty, we analyze the capacity decisions in a duopoly and oligopoly. The seminal result of Kreps and Scheinkman [11] that price-setting investment equals quantity-setting investment in a deterministic duopoly, does not hold under uncertainty. We provide an intuitive explanation of Hviid's work [9] who showed that no pure strategy equilibria exist when firms engage in price competition if demand is uncertain. Then, we turn our attention to the quantity-setting duopoly and show that a pure strategy equilibrium exists which is relatively insensitive to demand variability, but less so than in a monopoly. Following Dixit [5], we investigate the use of excess capacity investments as a credible threat to deter entry. These entry-deterrence investments where one firm invests in excess capacity so that other firms find it not profitable to invest are more difficult as variability increases and credible only at low investment costs. Finally, we show that our results generalize to an oligopoly of n firms and perfect competition ($n \to \infty$). The investment under uncertainty equals the investment with a deterministic demand curve, even with perfect quantity competition, provided variability is low or investment costs are high. Moreover, we show that the entry-deterring investment in an oligopoly equals the total industry investment under perfect quantity competition.

Curiously, competitive joint pricing and capacity decisions under uncertainty have received limited attention. There is a recent literature in operations that studies competitive capacity investment when prices are exogenous. Lipmann and McCardle [13] and Parlar [15] study competitive inventory decisions, which in their most simple form directly correspond to capacity decisions, by analyzing a "competitive newsboy" model. To analyze subcontracting and outsourcing, Van Mieghem [17] studies the competitive capacity and production decisions of a contractor and subcontractor. In most of these articles prices are exogenous so that the effect of tactical price or quantity competition on the investment decisions cannot be analyzed. However, a considerable literature considers the interaction of the pricing, production and inventory decisions in the context of a monopoly. Whitin's model [19], which was analyzed by Mills [14], appears to have been the first to consider the simultaneous choice of inventory and prices; an integrative review is provided by Petruzzi and Dada [16]. Multi-period stochastic extensions where the monopolist makes dynamic inventory, production and/or pricing decisions include Amihud and Mendelson [1, 2], Federgruen and Heching [6], Gallego and van Ryzin [7] and Li [12]. We will compare our monopoly results with these earlier results in section 3.

The outline of this article is as follows. We start with analyzing capacity decisions of a price-setting monopolist in section 2 and compare these with what a quantity-setting monopolist would do in section 3. Similarly, section 4 discusses capacity investment under price competition while section 5 focuses on quantity competition. We conclude in section 6. (All proofs, as well as explicit pricing, quantity and capacity results as a function of both investment cost and variability when demand is uniformly distributed, can be found in [18, Appendix]).

2 A Capacity and Pricing Model

Consider a monopolist who faces uncertain market demand D when the market price is p. Uncertainty in the market demand forecast is modeled by a random variable ε, also called a "shock." Specifically, the (inverse) demand curve is assumed to be linear and we can always scale[1] units such that

$$p = \varepsilon - D. \tag{1}$$

ε has mean 1 and perturbs the deterministic demand curve $p = 1 - D$ by representing uncertainty in the intercept or market size or willingness-to-pay. (We will discuss uncertainty in the slope, $p = 1 - \varepsilon D$, and more general demand curves $p = \varepsilon_1 g(D) + \varepsilon_2 h(D)$ in section 3.) Thus, ε is non-negative and to avoid technicalities we will assume that its probability measure P has a distribution $F(\varepsilon)$ with continuous[2] density $f(\varepsilon)$ over $\mathbb{R}_+$ and standard deviation σ. Clearly, if ε is unfavorably low ($\varepsilon < p$), the willingness-to-pay a price p is zero so that market demand is zero. Before uncertainty is resolved, the monopolist must make a capacity investment decision $K \geq 0$ at a cost $C(K)$ and announce a price $p \geq 0$. After uncertainty is resolved, the monopolist can produce and sell $q \leq D(p, \varepsilon)$ units at the announced price, where the production quantity q is constrained by the earlier capacity choice: $q \leq K$. For simplicity we assume linear investment costs, $C(K) = cK$, and zero production costs. (All our results directly extend to convex cost functions $C(K)$.) The operating profits[3] for the monopolist are simply

$$\pi_M^p(p, K, \varepsilon) = pq, \tag{2}$$

and its expected firm value is

$$V_M^p(p, K) = \mathrm{E}\pi_M^p - cK. \tag{3}$$

We assume that the monopolist is risk-neutral and maximizes expected firm value so that the research problem is to determine the investment K and price p that maximize V_M^p.

To explicitly write out operating profits, it is useful to partition the state-space for ε as follows: $\mathbb{R}_+ = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where

$$\Omega_0(p) = [0, p), \quad \Omega_1(p, K) = [p, p + K) \quad \text{and} \quad \Omega_2(p, K) = [p + K, +\infty). \tag{4}$$

The three domains represent three possible outcomes. Domain Ω_0 represents the undesired outcome where the willingness-to-pay is so low, or equivalently we priced so high, that there is no demand at the announced price p and thus $q = 0$ and $\pi_M^p(p, K, \varepsilon) = 0$. In domain Ω_1, the monopolist has sufficient (actually excess) capacity to satisfy demand so that $q = D(p, \varepsilon)$ and $\pi_M^p(p, K, \varepsilon) = pD(p, \varepsilon)$. Finally, in domain Ω_2 market demand at the announced price is so high that it exceeds capacity and some potential sales will be lost: $q = K < D$ and $\pi_M^p(p, K, \varepsilon) = pK$. The expected operating profits become

$$\mathrm{E}\pi_M^p = \int_{\Omega_1} p\,(\varepsilon - p)\,dP + \int_{\Omega_2} pK dP = \int_p^{p+K} p\,(\varepsilon - p)\,f(\varepsilon)d\varepsilon + pK\bar{F}(p + K), \tag{5}$$

where $\bar{F}$ denotes the tail distribution: $\bar{F}(\varepsilon) = 1 - F(\varepsilon)$.

[1] An arbitrary linear demand curve $p' = \varepsilon' - bD'$, where ε' has mean ε_0, reduces to (1) after scaling prices $p' = \varepsilon_0 p$ and quantities $q' = \frac{\varepsilon_0}{b} q$.

[2] Limiting arguments show that all essential results remain valid for measures with bounded support. As an example, [18, Appendix] gives explicit results for a uniform distribution on an interval $[a, b]$ with $0 \leq a \leq 1 \leq b$.

[3] For the arbitrary demand curve, the unscaled marginal costs are $c' = \varepsilon_0 c$ so that unscaled revenues and costs are $\frac{\varepsilon_0^2}{b} pq$ and $\varepsilon_0^2 cK$, respectively.

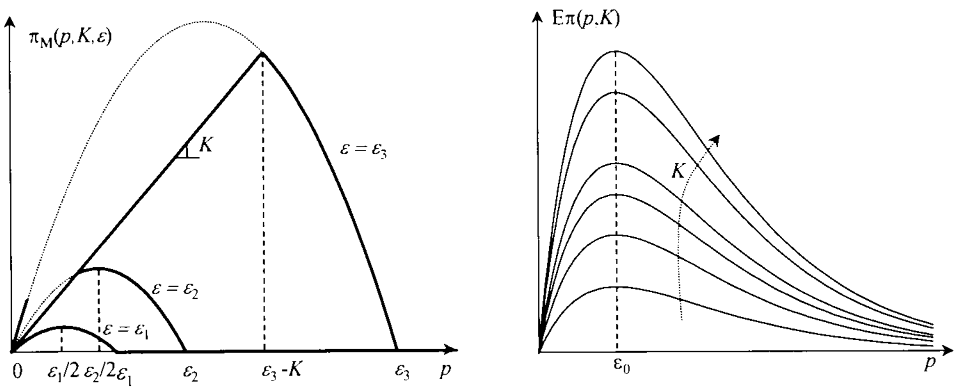

Figure 1: The sample path of $\pi_M(p, K, \varepsilon)$ as a function of p for three representative values of ε: low (ε_1), medium (ε_2) and high (ε_3). On the right, we have $\mathrm{E}\pi_M(p, K, \varepsilon)$ when ε is exponentially distributed.

2.1 Capacity-constrained Monopoly Pricing

As an initial step in determining the joint capacity-pricing decisions, it is useful to first consider the optimal pricing decision for a given capacity level K. Because investment cost is sunk, the optimal capacity-constrained price, denoted by $p_M(K)$, maximizes the expected revenue function $\mathrm{E}\pi_M^p(p, K)$ which is a weighted linear superposition of $\pi_M^p(p, K, \varepsilon)$ with weight factor $f(\varepsilon)$.

Consider the revenue function $\pi_M^p(p, K, \varepsilon)$ for a specific realization of ε (that is, a specific sample path). For any given K, the revenue function as a function of price p has three possible shapes depending on the value of ε as shown in Figure 1. For small values of ε ($\varepsilon < K$), $\pi_M^p = p(\varepsilon - p)^+$ with a unique maximum at $p = \varepsilon/2$. For medium values of ε ($K \leq \varepsilon < 2K$), $\pi_M^p = Kp$ for $p < \varepsilon - K$ and $\pi_M^p = p(\varepsilon - p)^+$ elsewhere with a unique maximum at $p = \varepsilon/2$. For large values of ε ($2K \leq \varepsilon$), $\pi_M^p = Kp$ for $p < \varepsilon - K$ and $\pi_M^p = p(\varepsilon - p)^+$ elsewhere with a unique maximum at $p = \varepsilon - K$. Thus, each $\pi_M^p(p, K, \varepsilon)$ is unimodal concave-convex.

First consider the deterministic problem which serves as a good reference case to study the effect of uncertainty. The optimal capacity-constrained price $p_M^{\mathrm{det}}(K)$ that a monopolist should charge in the absence of uncertainty (the traditional case with deterministic demand curve $D = 1 - p$) directly follows:

$$p_M^{\mathrm{det}}(K) = \max\left\{1 - K, \frac{1}{2}\right\}. \tag{6}$$

In the presence of uncertainty, we must find the maximum of the expected revenue function $\mathrm{E}\pi_M^p(p, K)$. As a weighted linear superposition, the expected revenue function $\mathrm{E}\pi_M^p(p, K)$ may inherit some structural properties from $\pi_M^p(p, K, \varepsilon)$ such as unimodality which is a sufficient condition for the existence and uniqueness of the capacity-constrained price $p_M(K)$. For example, if ε is exponentially distributed, we have $f(\varepsilon) = e^{-\varepsilon}$ and

$$\mathrm{E}\pi_M^p = p\int_p^{p+K} (\varepsilon - p)\, e^{-\varepsilon} d\varepsilon + pK\int_{p+K}^{\infty} e^{-\varepsilon} d\varepsilon = pe^{-p}\left(1 - e^{-K}\right). \tag{7}$$

Figure 1 shows the expected revenue as a function of p and several values of K. Like the sample revenue functions, the expected revenue function is unimodal and concave-convex. To avoid technicalities, we will simply assume that the first-order conditions for our optimization problems are sufficient. This amounts to assuming that the density $f(\varepsilon)$ is such that the linear superposition $\mathrm{E}\pi_M^p(p, K)$ is also unimodal concave-convex and that the price-optimized revenue function $\mathrm{E}\pi_M^p(p_M(K), K)$ is concave. (This holds for the uniform and exponential distributions and for the deterministic case [18, Appendix].) The optimal capacity-constrained price sets expected marginal revenues equal to marginal production cost, which was assumed to be zero. Marginal revenues are zero in domain Ω_0 where demand is zero, $\varepsilon - 2p$ in domain Ω_1 and K in the capacity-constrained domain Ω_2. Thus we have (all proofs are relegated to the Appendix):

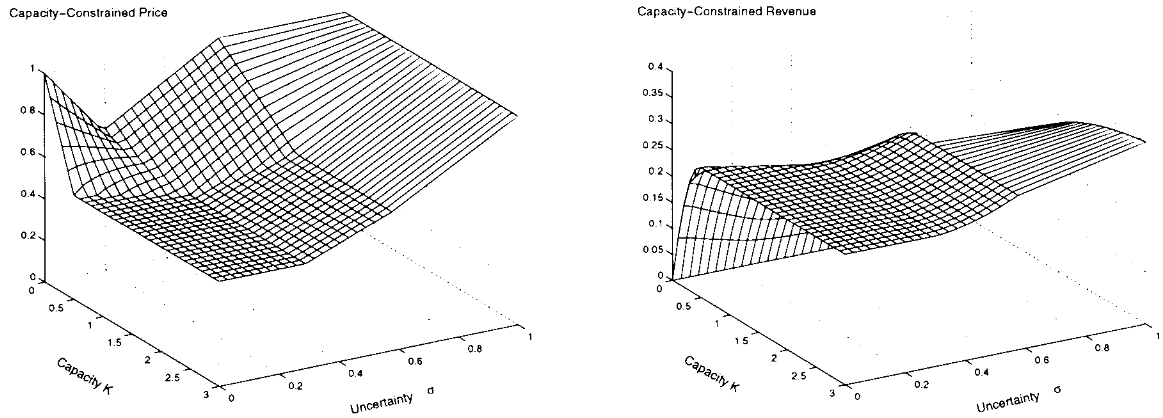

Figure 2: The capacity-constrained price $p_M(K)$ (left) and revenue $\mathrm{E}\pi^p_M(p_M(K), K)$ (right) as a function of capacity K and uncertainty σ.

Proposition 1 *The optimal capacity-constrained monopoly price $p_M(K)$ is unique, strictly positive and non-increasing in K and satisfies*

$$\int_{\Omega_1(p_M,K)} (\varepsilon - 2p_M)\, dP + \int_{\Omega_2(p_M,K)} K dP = 0. \tag{8}$$

Corollary 1 *The expected revenue at the optimal capacity-constrained monopoly price is*

$$\mathrm{E}\pi^p_M(p_M(K), K) = p^2_M(K) P\left(\Omega_1(p_M(K), K)\right) \leq p^2_M(K). \tag{9}$$

As an illustration reconsider the example of the exponential distribution. The optimality equation (8) for $p_M(K)$ becomes

$$(1 - p_M)\, e^{-p_M}\left(1 - e^{-K}\right) = 0 \Leftrightarrow p_M(K) = 1, \tag{10}$$

and the resulting revenue function is concave increasing in K:

$$\mathrm{E}\pi^p_M(p_M(K), K) = e^{-1}\left(1 - e^{-K}\right). \tag{11}$$

By comparison, the deterministic reference case yields

$$\pi^p_M(p_M(K), K) = \begin{cases} (1-K)K & \text{if} \quad K < 1/2, \\ 1/4 & \text{if} \quad K \geq 1/2. \end{cases} \tag{12}$$

Thus, with uncertainty distributed exponentially, the monopolist charges a capacity-constrained price $p = 1$ that is independent of the installed capacity (!) and that is larger than the deterministic monopoly price, reflecting a risk-premium of up to 100%.

While the elegant explicit solutions for the exponential distribution give us some first insights in the effects of variability on price and capacity, this distribution has the disadvantage that we cannot change the level of variability. The exponential distribution has only one parameter, its mean which equals its standard deviation. Analytic comparative statics on the first order optimality equations as a function of variability yield very complex equations that cannot be signed in general. Therefore, to investigate the impact of various levels of variability on the pricing and investment decisions, we explicitly solved the capacity-pricing problem for the family of uniform distributions with mean $\varepsilon_0 = 1$ and standard deviation σ and explicit solutions are given in the Appendix. Specifying the two parameters mean and standard deviation of a uniform random variable defines its support as the interval $[\varepsilon_0 - \sqrt{3}\sigma, \varepsilon_0 + \sqrt{3}\sigma]$. The fact that ε is non-negative bounds its relative amount of variability when distributed uniformly: $\frac{\sigma}{\varepsilon_0} \leq 3^{-1/2}$. For comparison, the one-parameter exponential distribution has a fixed amount of relative variability: $\frac{\sigma}{\varepsilon_0} = 1$.

Figure 2 shows the capacity-constrained monopoly price $p_M(K)$ (left) and the associated revenue function $E\pi_M^p(p_M(K), K)$ (right) as a function of capacity K and the level of uncertainty, measured by the mean-preserving spread σ. The figure shows the results for the deterministic case ($\sigma = 0$), the uniform distribution ($0 \leq \sigma \leq 3^{-1/2}$) and the exponential distribution ($\sigma = 1$). Let us first fix a given level of variability and investigate the role of capacity K. The capacity-constrained monopoly price $p_M(K)$ is decreasing almost linearly in capacity K up to some point K_0, after which the price remains constant. Initially when capacity is tight (K very small), the monopolist charges a high price. This effectively results in a form of market segmentation where the least price-sensitive customers are targeted. When capacity increases ($K \uparrow K_0$), the monopolist reduces price. This is consistent with the elasticity property of the deterministic linear demand curve $p = \varepsilon_0 - D$: if the price p is above $\varepsilon_0/2$, demand is elastic and reducing price increases revenues. This argument extends to the case of uncertainty, and when $K < K_0$, there is a probability of being capacity constrained ($P(\Omega_2) > 0$) and p_M is a function of capacity K. As K increases beyond K_0, excess capacity exists and there is no benefit of price reduction below $p(K_0)$ (lowering price below $p(K_0) = \varepsilon_0/2$ would bring us in the inelastic zone of the demand curve where reducing prices would reduce revenues). In this zone, under moderate levels of uncertainty, K is sufficiently large to exclude a risk of being capacity constrained ($P(\Omega_2) = 0$) and p_M is independent of K. Clearly, for a given K, the risk of being capacity constrained increases as uncertainty increases, so that the transition point K_0 increases when uncertainty increases. With the exponential distribution, uncertainty is so large that $K_0 = \infty$.

Now fix a capacity level and notice that the capacity-constrained monopoly price $p_M(K)$ is not monotone in the level of variability σ for low levels of K, whereas it is non-decreasing in σ for high levels of K (say $K \geq 1.5$). When capacity is sufficiently high increasing variability increases price reflecting a "risk-premium" to protect against uncertainty. However, no "risk-premium" is charged (price may even decrease in variability) when capacity is low and there is little variability. We believe this effect for low capacity levels may be explained as follows. When variability is low and we operate with small capacity, we will price high to skim the market. Now as variability increases, the probability that not all capacity is used increases: $P(\Omega_2) \downarrow$. Since at this high price demand is likely to be elastic, lowering price increases demand and revenues, mitigating some of the risk of capacity under-utilization. However if price becomes sufficiently low, there is higher likelihood to be in the inelastic part of the demand curve so that one should increase price as variability increases in order not to decrease revenues. Finally, notice that for high capacity levels and low levels of variability, the probability of being capacity constrained or of having priced too high is zero $P(\Omega_2) = P(\Omega_0) = 0$ and the optimal price equals the deterministic price: $p_M(K) = \frac{1}{2}$.

Now consider the price-optimized revenue function (Figure 2 on the right). Up to the transition point K_0, price is decreasing and revenue increasing (conforming with the inelastic zone argument), after which both are constant. Finally, notice that uncertainty does not necessarily result in decreased expected capacity-constrained revenues. Indeed, for high capacity levels, increased variability increases revenues. Effectively, the demand distribution is truncated and the effective mean demand is thus increasing in variability. Indeed, domain Ω_0 act as censoring demand equal to zero, while in domain Ω_1 the conditional mean demand is increasing in variability (and K is sufficiently high so that $P(\Omega_2) = 0$). Thus, the effective mean demand and associated mean revenues increase in variability.

2.2 Optimal Capacity & Pricing Strategy for a Monopolist

Given our assumptions on the distribution function of ε, the price-optimized revenue function $E\pi_M^p(p_M(K), K)$ is concave non-decreasing in K while costs $C(K) = cK$ are convex increasing. Thus, $V(K)$ is concave and the optimal investment strategy follows a critical number $\bar{c}$ policy. If $c \geq \bar{c}$ then capacity is too expensive to invest so that $K = 0$. For $c < \bar{c}$, optimal investment sets marginal revenue $\frac{d}{dK}E\pi_M^p(p_M(K), K)$ equal to marginal cost c which yields the familiar critical fractile solution:

$$P(\varepsilon \geq p_M(K) + K) = c/p_M(K). \tag{13}$$

Concavity directly shows that $\bar{c} = \frac{d}{dK}E\pi_M^p(p_M(K), K)|_{K=0}$.

Proposition 2 *The optimal capacity-price monopoly strategy is unique. There exists a threshold $\bar{c}(\sigma) > 0$ such that $K = 0$ and p is arbitrary if $c \geq \bar{c}(\sigma)$, otherwise $K > 0$ and $p > 0$ satisfy:*

$$pP(\Omega_2(p, K)) = c = \frac{p}{K} \int_{\Omega_1(p,K)} (2p - \varepsilon) dP, \tag{14}$$

and K is decreasing in c. If in addition the density f of P satisfies $P(\Omega_2(p,K)) > pf(p+K)$, then p is increasing in c. The optimal firm value V is decreasing in capacity costs:

$$\frac{\partial V}{\partial c} = -K. \tag{15}$$

The proposition shows that under the optimal capacity-price strategy there will always be a positive probability of having insufficient capacity leading to lost sales ($0 < P(\Omega_2) = \frac{c}{p} < 1$), and of having excess capacity ($0 < P(\Omega_1) \leq 1 - P(\Omega_2) < 1$). While this is a familiar result of the newsvendor model, we will show in the next section that this is not true when the firm is engaged in quantity competition. The fact that firm value and capacity levels K are decreasing in marginal investment costs, while p is increasing, is not surprising and conforms with results from the deterministic base-case which has a threshold cost $\bar{c} = \frac{d}{dK}\pi_M^p(p_M(K),K)|_{K=0} = (1-2K)|_{K=0} = 1$ and if $c < 1$:

$$K^{\det} = \frac{1-c}{2}, p^{\det} = \frac{1+c}{2} \text{ and } V^{\det} = \frac{(1-c)^2}{4} \leq \frac{1}{4}. \tag{16}$$

More interesting is the effect of uncertainty. As a start, exponential uncertainty has a threshold $\bar{c} = \frac{d}{dK}\mathrm{E}\pi_M^p(p_M(K),K)|_{K=0} = e^{-(1+K)}|_{K=0} = e^{-1}$ and if $c < e^{-1}$:

$$K = -(1+\ln c),\ p = 1 \text{ and } V = e^{-1} + c\ln c \leq e^{-1}. \tag{17}$$

Thus, the threshold $\bar{c}$ depends on the level of variability σ and is decreasing: as uncertainty increases, the firm requires a cost break before it is willing to invest. Figure 3 shows that the capacity investment level decreases almost linearly in c, where the slope is steeper as uncertainty increases. Thus, the threshold cost is monotone decreasing in σ, first linearly and then concavely. Also, for moderate and high capacity costs c, the investment level K decreases as uncertainty increases. Thus, the concavity of the problem induces risk-aversion onto risk-neutral firm. However, the reverse is true for low capacity costs: capacity is so inexpensive that one invests in more excess capacity as uncertainty increases. Notice that as capacity costs approach zero ($c \to 0$), capacity investment levels under exponential uncertainty go to infinity. Obviously, there is no justification to do this in the uniform case because demand is always finite and bounded by $\max \varepsilon = 1 + \sqrt{3}\sigma$. The optimal price is not monotone in uncertainty: for low variability levels, the optimal price is actually below the deterministic reference price while for high variability levels it is increasing in variability. With exponential uncertainty, the monopolist charges a price $p = 1$ independent of the investment cost. This price includes a risk premium of $\frac{1-c}{2}$ compared to the deterministic monopoly price of $\frac{1+c}{2}$. Interestingly, this risk premium is decreasing in cost, perhaps because the total risk exposure has decreased (K is decreasing in cost). Finally, because K is almost linearly decreasing in c, the optimal firm value falls quadratically. While at moderate and high costs, the firm is worse of as variability increases, more uncertainty is actually better when investment costs are low.

3 A Capacity and Quantity-Setting Model

Consider the counterpart of the pricing model where a firm now sets quantity. First, the firm invests in capacity K. Then, it announces the quantity q (constrained by its earlier investment: $q \leq K$) that it will bring to the market. Finally, uncertainty is resolved and the market mechanism determines the market clearing price $p = \varepsilon - q$ for the supplied market quantity q. If q exceeds the market size ε, the market clearing price is $p = 0$. As before, the operating profits for the monopolist are simply $\pi_M^q = pq$ and the expected firm value is $V_M^q = \mathrm{E}\pi_M^q - cK$.

To explicitly write out expected revenues, we must distinguish two possible outcomes. Again domain $\Omega_0(q) = [0,q)$ represents the undesired outcome where the willingness-to-pay is so low that the market clearing price is zero p and thus $\pi_M^q(p,K,\varepsilon) = 0$. The other outcome is positive price and revenues, represented by domain $\Omega_{1+2}(q) = [q, +\infty)$. The expected operating profits become

$$\mathrm{E}\pi_M^q(q,K) = \int_{\Omega_{1+2}} (\varepsilon - q)\, q dP = \int_q^\infty (\varepsilon - q)\, q f(\varepsilon) d\varepsilon \text{ with } q \leq K, \tag{18}$$

which is a simpler expression than its price-setting counterpart. This fact makes quantity-setting problems significantly easier to analyze than price-setting problems.

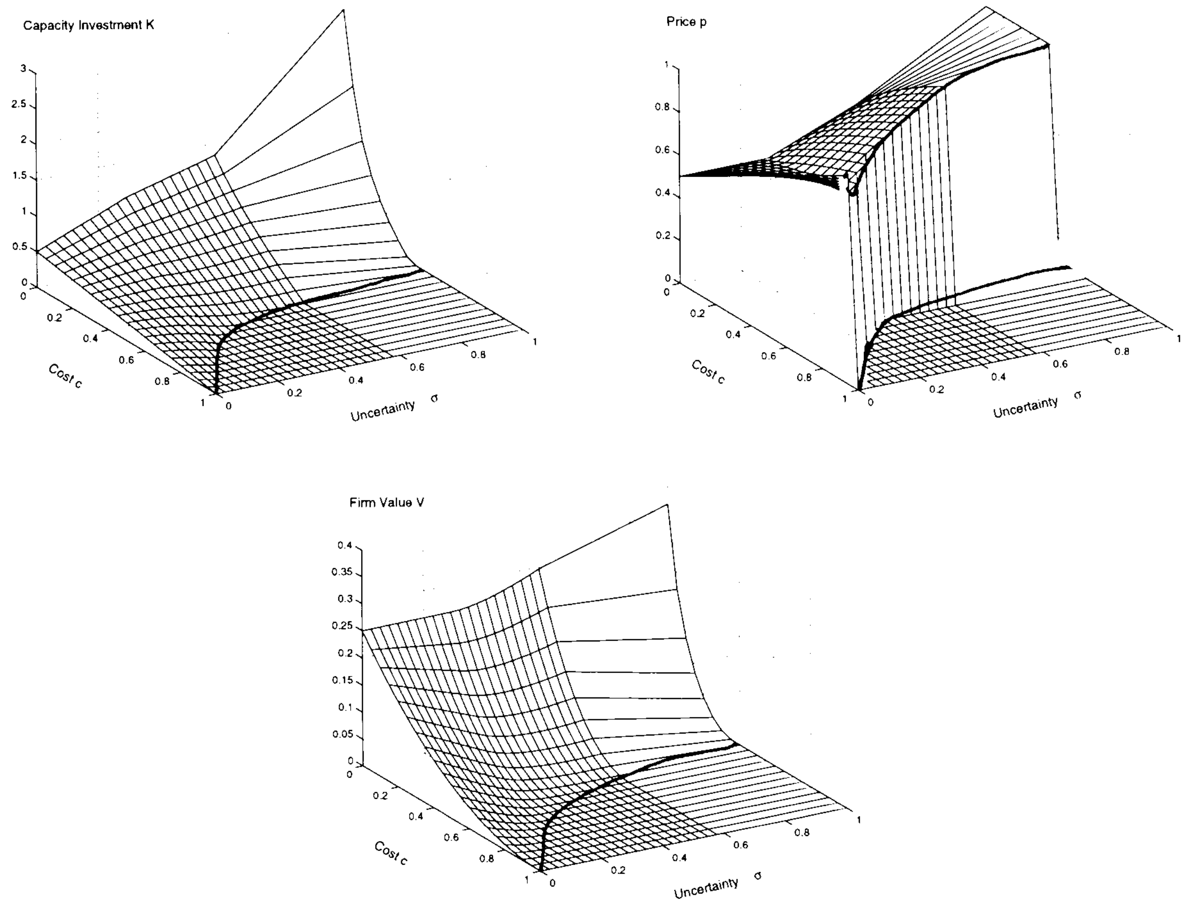

Figure 3: The optimal monopoly capacity K (top left), price p (top right) and firm value V (bottom) as a function of the marginal investment cost c and uncertainty σ. The cost threshold $\bar{c}(\sigma)$ is in bold.

3.1 Capacity-Constrained Quantity Setting

Let us first consider the optimal capacity-constrained quantity decision. The sample-path revenue functions are, similar as before, unimodal concave-convex and we will assume as before that f is such that $\mathrm{E}\pi_M^q(q, K)$ is also unimodal concave-convex. If there is ample capacity, the optimal capacity-constrained monopoly quantity $q_M(K)$ is the unique unconstrained maximum q^* of $\mathrm{E}\pi_M^q(q, K) = \int_q^\infty (\varepsilon - q)\, q f(\varepsilon) d\varepsilon$. Otherwise, if $K < q^*$, $q_M(K) = K$.

Proposition 3 *The optimal capacity-constrained monopoly quantity is $q_M(K) = \min(q^*, K)$, where q^* is the unique solution to*

$$\int_{q^*}^{\infty} (\varepsilon - 2q^*) f(\varepsilon) d\varepsilon = 0. \tag{19}$$

If the firm has excess capacity, the optimal quantity is independent of K:

$$\forall K > q_M(K) : \frac{\partial}{\partial K} q_M(K) = 0. \tag{20}$$

Our deterministic base case is extremely simple. Indeed, when the monopolist sets quantity $q \leq K$, the market price is $p = 1 - q$, with revenues $\pi = q(1 - q)$ which are indeed unimodal concave-convex and thus the optimal capacity-constrained quantity is

$$q_M(K) = \min(\frac{1}{2}, K). \tag{21}$$

For example, for exponential uncertainty we have that $\mathrm{E}\pi^q_M(q,K) = \int_q^\infty q\,(\varepsilon - q)\,e^{-\varepsilon}d\varepsilon = qe^{-q}$, and the optimal capacity-constrained quantity is $q_M(K) = \min(q^*, K)$, where

$$\int_{q^*}^{\infty} (\varepsilon - 2q^*)\,e^{-\varepsilon}d\varepsilon = (1 - q^*)\,e^{-q^*} = 0 \Leftrightarrow q^* = 1, \tag{22}$$

and associated quantity-optimized expected revenue function

$$\mathrm{E}\pi^q_M(q_M(K), K) = \begin{cases} Ke^{-K} & \text{if} \quad K \leq 1, \\ e^{-1} & \text{if} \quad 1 < K. \end{cases} \tag{23}$$

Clearly, the optimal quantity depends on uncertainty, but only if the level of uncertainty is sufficiently high. Indeed, if ε has a distribution with low variability, uncertainty "averages out." More precisely:

Corollary 2 *If ε is bounded from below with probability one by $\underline{\varepsilon} \geq \frac{1}{2}$, we have that $P(\Omega_0(q)) = 0$ for all quantities $q \leq \underline{\varepsilon}$ and the capacity-constrained quantity $q_M(K)$ and the expected revenue function are independent of variability and equal to the deterministic solutions: $\forall q \leq \min(K, \underline{\varepsilon})$*

$$q_M(K) = \min(\tfrac{1}{2}, K) \leq \underline{\varepsilon} \text{ and } \mathrm{E}\pi^q_M(q, K) = q(1 - q). \tag{24}$$

This result will have far-reaching consequences on the investment strategy as will be discussed in the next section. If ε is uniformly distributed over the interval $[1 - \sqrt{3}\sigma, 1 + \sqrt{3}\sigma]$, the corollary with $\underline{\varepsilon} = 1 - \sqrt{3}\sigma$ requires

$$\sigma \leq \tfrac{1}{\sqrt{3}} \max\left(\tfrac{1}{2}, 1 - K\right).$$

Thus, moderate levels of uncertainty ($\sigma \leq \frac{1}{2\sqrt{3}}$) do not impact the optimal quantity-decision. There exists a dual result for capacity-constrained pricing:

Proposition 4 (Duality) *The optimal capacity-constrained monopoly price is $p_M(K) = \max(p^*. p(K))$, where p^* is the unique solution to*

$$\int_{p^*}^{\infty} (\varepsilon - 2p^*)f(\varepsilon)d\varepsilon = 0, \tag{25}$$

and $p(K)$ solves (8).

Corollary 3 *If ε is bounded from above with probability one by $\bar{\varepsilon}$, then $P(\Omega_2(p)) = 0$ for all prices p such that $p + K \geq \bar{\varepsilon}$ and the capacity-constrained price is $p_M = p^*$, which is independent of capacity. If, in addition, ε is bounded from below with probability one by $\underline{\varepsilon} \geq \frac{1}{2}$, then $P(\Omega_0(p)) = 0$ and the capacity-constrained price $p_M(K)$ and the expected revenue function are independent of variability and equal to the deterministic solutions: $\forall p \leq \underline{\varepsilon} \leq \bar{\varepsilon} \leq p + K$:*

$$p_M(K) = \tfrac{1}{2} \text{ and } \mathrm{E}\pi^p_M(p, K) = p(1 - p). \tag{26}$$

The dual of $q_M(K) = \min(q^*, K)$ is $p_M(K) = \max(p^*, p(K))$, where $q^* = p^*$. In addition, the actual capacity-constrained price and quantity are related via the deterministic demand curve ($p = 1 - q$) *only if* variability is low and there is sufficient capacity: if ε has finite support $[\underline{\varepsilon}, \bar{\varepsilon}]$ with $\frac{1}{2} \leq \underline{\varepsilon} \leq \bar{\varepsilon} \leq \frac{1}{2} + K$, then $p_M(K) = q_M(K) = \frac{1}{2}$ independent of the level of moderate variability. (This provides a second explanation for the flat zones of $p_M(K)$ in Figure 3.) Because their expected revenue functions differ, price-setting and quantity-setting in general yield different investment results. From the duality result, one could hope that in the special case of low variability, both would yield identical investment outcomes, but the next section will show that this is not true either.

3.2 Optimal Capacity & Quantity Strategy for a Monopolist

For the optimal capacity investment decision under quantity-setting, any excess capacity level $K > q_M(K)$ is suboptimal because it does not satisfy necessary optimality conditions: $\frac{\partial}{\partial K}V(K) = -c < 0$. Therefore, a quantity-setting monopolist will *never* invest in excess capacity and always produce up to its capacity. (While this is obvious for a monopolist, this need not be true under competition. For example, we will show in section 5 that one may strategically choose capacity larger than production ($K > q(K)$) to deter entry.)

Proposition 5 *The optimal monopoly quantity-capacity strategy is unique: If $c \geq 1$, $q_M(K) = K = 0$, otherwise $q_M(K) = K$ where K satisfies:*

$$\int_K^{\infty} (\varepsilon - 2K) f(\varepsilon) d\varepsilon = c, \tag{27}$$

and the optimal investment level and firm value are decreasing in c

$$\frac{\partial K}{\partial c} < 0, \frac{\partial V}{\partial c} = -K < 0 \text{ and } V = K^2 P(\varepsilon \geq K) \leq \tfrac{1}{2} K(1-c). \tag{28}$$

The optimality equation (27) yields $1 - 2K(1 - F(K)) \geq c$ (recall, F is the distribution of ε) and comparing it with (19) shows that q^* is the optimal investment level if capacity is costless.

Corollary 4 *The optimal quantity-capacity level under uncertainty is never lower than under certainty, and never higher than q^* :*

$$K^{det} = \frac{1-c}{2} \leq K^{det} + \tfrac{1}{2} K^{det} F(K^{det}) \leq K^{det} + \tfrac{1}{2} K F(K) \leq K \leq q^* = K|_{c=0}. \tag{29}$$

Corollary 5 *If ε is bounded from below with probability one by $\underline{\varepsilon}$ and capacity cost is not too inexpensive $c \geq 1 - 2\underline{\varepsilon}$, the optimal investment level, the expected price and the expected revenue function are independent of variability and equal to the deterministic solutions:*

$$q = K = \frac{1-c}{2}, \mathrm{E}p = \frac{1+c}{2} \text{ and } V_M^q = \frac{(1-c)^2}{4}, \tag{30}$$

while the optimal firm value has standard deviation $\sigma_V = K\sigma_\varepsilon$.

If ε is uniformly distributed over the interval $[1 - \sqrt{3}\sigma, 1 + \sqrt{3}\sigma]$, the corollary with $\underline{\varepsilon} = 1 - \sqrt{3}\sigma$ requires

$$c + 1 \geq 2\sqrt{3}\sigma.$$

Thus, moderate levels of uncertainty ($\sigma \leq \frac{1}{2\sqrt{3}}$) *never* impact the optimal monopoly capacity-quantity decision. Capacity-quantity decisions remain insensitive to high levels of uncertainty ($\sigma > \frac{1}{2\sqrt{3}}$) provided capacity is not too expensive. This insensitivity result never holds for optimal capacity-price decisions. Earlier we showed that capacity-constrained pricing is insensitive to variability *if* variability is low and there is sufficient capacity. The latter requires investment costs that are trivially low ($c = 0$). Indeed, if $p_M(K) = \frac{1}{2}$, expected revenues $\mathrm{E}\pi_M^p(p, K) = p(1-p)$ are independent of $K \geq \frac{1}{2} + \bar{\varepsilon}$, which can never be an optimal investment level at positive cost c (because it would not satisfy necessary optimality conditions: $\frac{\partial}{\partial K} V(K) = -c < 0$).

In addition, with quantity-setting, the cost threshold $\bar{c} = 1$ is *independent* of demand variability. For exponential uncertainty, for example, the optimal capacity investment under quantity-setting is positive for $c < \bar{c} = 1$ and solves

$$(1-K)\exp(-K) = c. \tag{31}$$

Not only is a quantity-setting monopolist willing to invest at higher costs (its threshold cost is $\bar{c} = 1$ which is larger than a price-setting monopolist with $\bar{c} = e^{-1}$), but the investment differs: it is lower at low costs (never exceeding 1) and higher at high cost. Because the cost threshold with price-setting decreases as demand becomes more variable, it follows that a quantity-setting monopolist is willing to invest at higher costs than a price-setting monopolist.

Figure 4 shows the optimal monopoly capacity investment and firm value for the quantity-capacity problem. In the zone $c + 1 \geq 2\sqrt{3}\sigma$ of low variability levels or high costs, the capacity investment level and firm value are *independent of variability* and equal to their deterministic values. For higher levels of variability, both the capacity level and firm value increase. Thus, the non-linear model structure under quantity-setting seems to induce the firm to be risk-seeking. Effectively, demand distribution truncation, analogous to that discussed in the previous section, occurs and the effective mean demand is increasing in

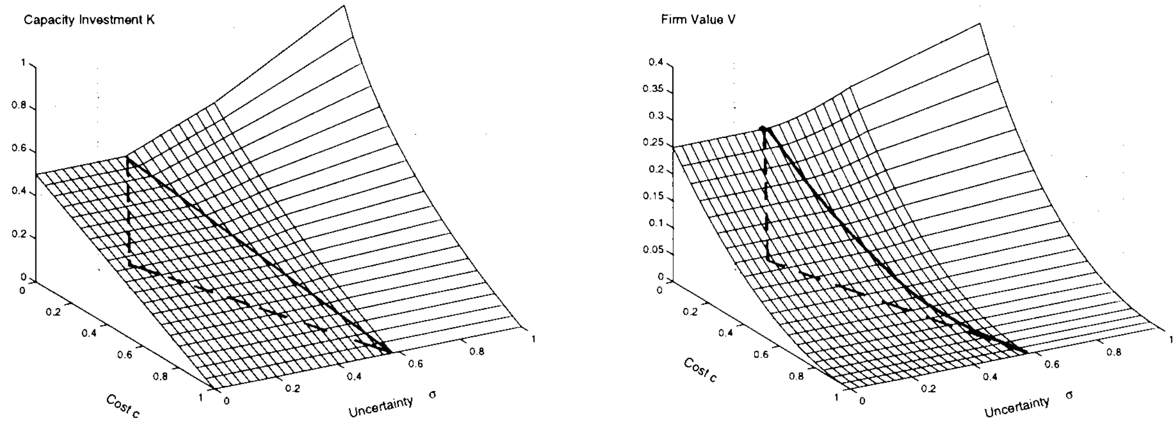

Figure 4: The optimal monopoly capacity K^q (left) and firm value V^q (right) as a function of the marginal investment cost c and uncertainty σ. The boundary of the uncertainty-insensitive zone $c + 1 \geq 2\sqrt{3}\sigma$ is in bold.

variability. Not only is more uncertainty never less profitable for a quantity-setting monopolist, quantity-setting is also more profitable than price-setting. The relative difference of the firm value under quantity-setting V^q and the value under price-setting V^p is positive and increasing in both cost c and variability σ.

Hence in the presence of uncertainty, price-setting and quantity-setting yield fundamentally different investment outcomes for a monopolist *for any positive level of variability.* Only in the deterministic limit ($\sigma \rightarrow 0$) do price- and quantity-setting give the same outcome. Moreover, the investment strategy under quantity-setting is more profitable and significantly less sensitive to variability than the price-setting strategy. (In the next section, we will show price- and quantity-setting under uncertainty are also fundamentally different for a duopoly so that the reconciliation effected by Kreps and Scheinkman [11] does not hold under uncertainty. Even stronger, it does not even hold in a monopoly under uncertainty.)

3.3 Postponement and the Expected Value of Information

For a monopolist, the sequence of the pricing decision and the capacity decision is actually irrelevant: provided both have to be made before uncertainty is resolved, it doesn't matter which one is made first because essentially no information is gained between the two decisions. (In contrast, the timing of the decisions will be important in the duopoly game in the next section.) Obviously, *postponement* of the tactical price or quantity decision until after uncertainty is resolved, can never hurt. Actually, the corresponding firm value increment is a measure of the expected *value of information* and it provides another explanation for the superiority of quantity-setting investment of its price-setting counterpart.

If the firm can postpone its quantity decision until after observing ε, then the optimal capacity-constrained quantity $q_M(K, \varepsilon)$ solves

$$\max_{q \leq K} q(\varepsilon - q), \tag{32}$$

so that

$$q_M(K, \varepsilon) = \min(\frac{\varepsilon}{2}, K) \text{ and } \mathrm{E}\pi_M = \int_0^{2K} \left(\frac{\varepsilon}{2}\right)^2 f(\varepsilon)d\varepsilon + \int_{2K}^{\infty} K(\varepsilon - K)f(\varepsilon)d\varepsilon. \tag{33}$$

Optimal investment sets marginal revenues equal to marginal costs, so that the optimal investment level under quantity-setting postponement solves

$$\int_{2K}^{\infty} (\varepsilon - 2K)f(\varepsilon)d\varepsilon = c, \tag{34}$$

with corresponding firm value

$$V = \int_0^{2K} \left(\frac{\varepsilon}{2}\right)^2 f(\varepsilon)d\varepsilon + \int_{2K}^{\infty} K^2 f(\varepsilon)d\varepsilon = \int_0^{2K} \left(\frac{\varepsilon}{2}\right)^2 f(\varepsilon)d\varepsilon + K^2\bar{F}(2K) \le K^2. \tag{35}$$

Similarly, if the firm can postpone its pricing decision until after observing ε, then the optimal capacity-constrained price $p_M(K,\varepsilon)$ solves

$$\max_{p\ge 0} p \min(\varepsilon - p, K), \tag{36}$$

so that

$$p_M(K,\varepsilon) = \max(\frac{\varepsilon}{2}, \varepsilon - K) \text{ and } \mathrm{E}\pi_M = \int_0^{2K} \left(\frac{\varepsilon}{2}\right)^2 f(\varepsilon)d\varepsilon + \int_{2K}^{\infty} K(\varepsilon - K)f(\varepsilon)d\varepsilon, \tag{37}$$

which yields the same expected revenues, and thus investment decision, as under quantity-setting!

This yields two interesting observations. First, the optimal investment under postponement exhibits uncertainty insensitivity similar to, but less pronounced than, before. If ε is bounded from below with probability one by $\underline{\varepsilon}$ and capacity cost is not too inexpensive $c \ge 1 - \underline{\varepsilon}$, the optimal investment level, the expected price and the expected revenue function under postponement are independent of variability and equal to the deterministic solutions. More importantly, in this case, postponement does not change our earlier quantity-setting investment decision, nor does it increase firm value. Thus, the value of information when setting quantities is zero (when uncertainty levels are moderate $c \ge 1 - \underline{\varepsilon}$), while it is positive when setting prices. This provides another explanation why quantity-setting investments are superior to price-setting ones.

Second, while the value of information under quantity-setting is zero when uncertainty levels are moderate, it is positive at high levels of uncertainty. Postponement then warrants higher investment levels: $K^{\text{postpone}} \ge K^q$.

3.4 Relationship to other models

What happens if we relax our model to allow for more general types of uncertainty and general demand curves:

$$p = \varepsilon_1 g(D) + \varepsilon_2 h(D), \tag{38}$$

where g and h are deterministic downward sloping functions and ε is a random (possible correlated) positive vector? If the inverse demand curve is linear in the random vector ε, the results hold and the optimal capacity investment and quantity-setting strategy under uncertainty equals the deterministic strategy $K^{(\sigma=0)}$ provided variability is moderate in the sense that $P(\varepsilon_1 g(K^{(\sigma=0)}) + \varepsilon_2 h(K^{(\sigma=0)}) \ge 0) = 1$. Indeed, the optimal investment in the deterministic problem equates marginal revenue with marginal costs: $\frac{d}{dK}K\left(g(K)\mathrm{E}\varepsilon_1 + h(K)\mathrm{E}\varepsilon_2\right)|_{K^{\sigma=0}} = c$. This equals the optimality equations under low levels of uncertainty, that is, as long as variability levels are not too high such that the market price at the supplied quantity $q = K^{\sigma=0}$ is positive with probability one (i.e., $P(\varepsilon_1 g(K^{\sigma=0}) + \varepsilon_2 h(K^{\sigma=0}) \ge 0) = 1$). In that case the effect of uncertainty in the quantity-setting model "averages out." Amihud and Mendelson [1] observed a similar property in a related monopoly production-inventory model.

It should no longer be surprising that the price-capacity decision problem which has two control variables affecting two dimensions (price and quantity via the capacity constraint) is inferior to the quantity-capacity problem which effectively controls only one dimension (quantity). The reason is clear: quantity-setting firms benefit from the market mechanism that sets a state-dependent market clearing price such that, up to medium levels of variability, no sales are ever lost due to capacity constraints nor is there ever any unused excess capacity. This perfect balance can never be achieved under price-setting: because of demand uncertainty, there are always scenarios where the announced price was either too low, leading to excess demand and lost sales, or too high, resulting in unused excess capacity.

It is surprising that quantity setting is so robust in that it is insensitive to correlated uncertainty in both slopes and intercepts of linear, or even non-linear inverse demand curves. All that is needed is linearity of the inverse demand curve in the random element ε and moderate levels of uncertainty. The quantity-setting model also bears some resemblance to an intuitively attractive max-min strategy in the sense that capacity is set such that even in the worst outcome (low ε if $p = \varepsilon - D$), revenue remains positive.

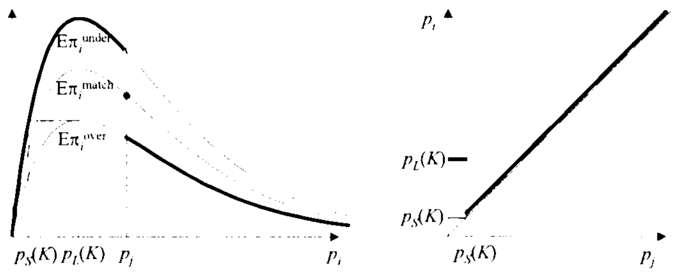

Figure 5: Firm i's expected revenues when it names a price p_i assuming firm j names p_j (left) and its optimal price-reaction curve (right).

Obviously one cannot possibly hope for similar robustness of the model in pricing decisions. Pricing under uncertainty is inherently more complicated than quantity-setting and it is sensitive to the type of uncertainty and the non-linearity of the inverse demand curve. For example, uncertainty in the slope, that is $p = 1 - \varepsilon D$, is equivalent to what is known as "multiplicative uncertainty." It is easy to verify that such type of uncertainty leads to an optimal capacity-price strategy where the price is always (i.e., for any level of cost c or variability σ) above the deterministic benchmark price, in conformance with results of Karlin and Carr [10]. In our additive model, $p = \varepsilon - D$, that is not necessarily true and the monopolist may price above, equal to, or below the deterministic benchmark depending on the level of variability (Figure 3). This is in contrast with traditional additive models such as Whitin, Mills and follow-up papers which were incomplete in the sense that they a priori restricted price to be such that there is always a positive demand at that price, i.e., $P(\Omega_0) = 0$. That restriction forces the monopoly price under uncertainty to be lower than the deterministic benchmark. Our results are also consistent with those of Li [12] who shows that if the monopolist can dynamically produce, price and hold inventory, the price will always exceed the deterministic benchmark price. In our model the same happens except at low levels of variability, and the reason may be that we assumed marginal production to be costless. If production is not costless, we expect the firm to be more cautious in its investment which does also imply a higher price.

4 Capacity Decisions under Price Competition

We now extend our model to capture the effects of competition on the investment decision. First, assume there are two independent firms. If there is no uncertainty, this is exactly the Kreps-Scheinkman model. They considered a duopoly in which firms first invest in capacity. Then, both firms observe each other's capacity levels and set a price at which they will sell a market-determined quantity constrained by their capacity level. Kreps and Scheinkman show that in equilibrium the capacity investments of both firms are low enough so that each firm is able to sell its entire potential output. Since this is identical to the equilibrium when firms compete by setting quantity, the price-setting (Bertrand) and quantity-setting (Cournot) equilibria coincide.

Unfortunately, the Kreps-Scheinkman result is not robust. Davidson and Deneckere [4] show that the deterministic model does not have a pure strategy equilibrium with any allocation rule (which specifies how demand is allocated to each firm when both announce identical prices) other than a fixed, pre-determined 50-50 splitting rule. Hviid [9] showed that after introducing uncertainty (as in our model presented above) no pure strategy equilibria exist for any allocation rule. Let us provide the following explanation of this subtle result. Assuming a capacity vector K, both firms must decide on a price in the pricing subgame. Assuming firm j names price p_j, firm i has three options: underprice, match or overprice. (For the exponential distribution, the resulting revenues are as shown in Figure 5.) When underpricing, firm i gets all market demand up to its capacity, while with price matching, demand is split between the two firms. Thus underpricing revenues are strictly larger than matching revenues, and firm i will either overprice (for low p_j) or underprice, but never match its rival's price. Thus, firm i's price-reaction curve is as shown in Figure 5: it is always strictly above (it may be tangent, but approaching from above) the diagonal $p_i = p_j$. The same argument shows that firm j's price-reaction curve is strictly below the diagonal. Because the reaction curves do not intersect, there is no pure pricing equilibrium. This scenario holds for all levels of positive variability.

Only for the deterministic case and a fixed 50-50 allocation rule do under-, match and overpricing revenues coincide for low prices[4] and an equilibrium on the diagonal $p_i = p_j$ follows. In conclusion, whereas price and quantity-setting yield identical monopoly results only in the deterministic case, competition reduces this possibility even further and requires a very specialized setting with pre-determined market allocations.

Mixed strategy equilibria do exist, but such equilibria do not seem reasonable predictions of an outcome of firms' strategic pricing decisions. Prices are not rigid for a sufficiently long period to support the realized equilibrium prices (a firm always has an incentive to undercut its rival when there is uncertainty). Thus, Hviid concludes that two-stage models with simultaneous strategy choices and where the pricing subgame is modeled as a one-shot game are not always a good approximation. One way to deal with this is to consider alternative formulations of a duopoly game. Hviid [8] proposes to change the timing of the decisions and he considers a sequential pricing game to study first-mover advantage. Arthur [3] introduces product differentiation which guarantees a pure strategy equilibrium. We propose to analyze different modes of competition. We will show next that under uncertainty the (simultaneous) quantity-setting duopoly does have a pure equilibrium strategy which may suggest that competition with a homogeneous product (e.g., a commodity) is better modeled by quantity-setting rather than price-setting strategies.

5 Capacity Decisions under Quantity Competition

Consider the multi-firm competitive version of our capacity and quantity-setting model. First, each firm i simultaneously invests in capacity K_i. Then, investment levels K are observed by all players and each firm i simultaneously announces the quantity q_i (constrained by its earlier investment: $q_i \leq K_i$) that it will bring to the market. Finally, uncertainty is resolved and the market mechanism determines the market clearing price $p = \varepsilon - q_+$ for the supplied market quantity $q_+ = \sum q_i$. As before, this market clearing price is zero with oversupply $q_+ > \varepsilon$.

Both firms will make their decisions so as to maximize expected profits, taking into account the other firm's likely decisions. Thus, we have a two stage non-cooperative game which is solved by working backwards: first solve the capacity-constrained quantity-setting subgame for a given capacity vector K, and then solve for the capacity decisions. Unlike under price competition, the revenue functions are now continuous in the actions (i.e., in the quantities q) and a pure strategy equilibrium for the full quantity-investment game exists. First we will consider a duopoly and then oligopoly and perfect competition.

5.1 The Capacity-Constrained Quantity-Setting Duopoly Subgame

Our question here is: given capacity vector $K = (K_1, K_2)$, what are the (subgame perfect) quantity-setting strategies for both competitors? We will show that there exists a pure strategy equilibrium by showing that the firms' reaction curves intersect.

Denote firm i's reaction function by $R_i(\cdot|K)$, where $q_i = R_i(q_j|K)$ denotes firm i's optimal quantity response when firm j chooses quantity q_j:

$$R_i(q_j|K) = \arg\max_{0 \leq q_i \leq K_i} \int_{q_i+q_j}^{\infty} (\varepsilon - q_i - q_j) q_i f(\varepsilon) d\varepsilon. \tag{39}$$

Notice that the space of interest is the rectangle $[0, K_1] \times [0, K_2]$ and that $R_i(0|K) = q_M(K_i)$, the monopoly capacity-constrained quantity which equals $\min(q_M^*, K_i)$. In the appendix it is shown that $\left|\frac{\partial R_i}{\partial q_j}\right| \leq 1$ for a large class of probability distributions so that both reaction curves intersect and a pure strategy equilibrium exists (a representative situation is shown in Figure 6). As in the monopoly case, quasi-concavity (unimodularity) of the revenue functions is a sufficient condition for the existence of a pure equilibrium.

Proposition 6 *For any capacity vector K and for a large class of probability distributions, there is a unique pure strategy equilibrium $q(K)$ in the quantities q, which is independent of K_i if firm i has excess capacity:*

$$\forall K_i > q_i(K) : \frac{\partial}{\partial K_i} q(K) = 0. \tag{40}$$

[4] Both firms are then on the linear capacity-constrained part of their revenue curves, which are similar to those in Figure 1 with $\varepsilon = \varepsilon_3$.

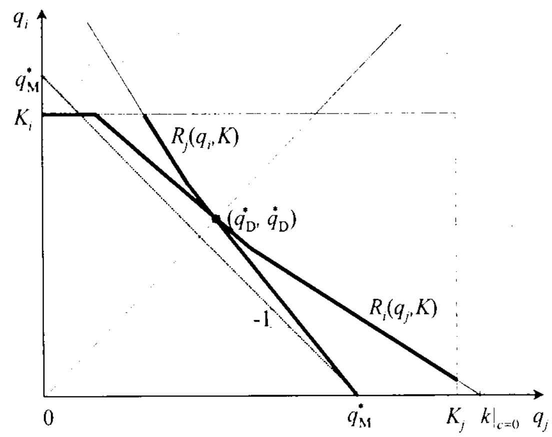

Figure 6: Reaction curves for quantity-setting when the duopolists are constrained by capacity vector K.

This generalizes the capacity-constrained quantity-setting monopoly result that $q_M = \min(q^*_M, K)$. Denote the unconstrained duopoly equilibrium by (q^*_D, q^*_D), the intersection of the reaction curves if K is arbitrarily large. If there is sufficient capacity, $K > q^*_D$, then $q(K) = q^*_D$, which is independent of the capacity vector K. Otherwise the equilibrium is on the intersection of a reaction curve with a constraint of the form $q_i = K_i$. For the deterministic example, the reaction curves are

$$q_i = R_i(q_j|K) = \min(\tfrac{1}{2}(1 - q_j), K_i). \tag{41}$$

with a unique solution (either interior at $q = (q^*_D, q^*_D) = (\frac{1}{3}, \frac{1}{3})$ if $K \geq \frac{1}{3}$, or at the boundary $q_i = K_i$ or $q_i = K_j$.) With exponential uncertainty, the reaction curves are trivial and have a unique intersection:

$$q_i = R_i(q_j|K) = \min(1, K_i). \tag{42}$$

5.2 Optimal Capacity Investment under Quantity Competition (the full duopoly game)

From Proposition 6, it follows that, similar to the monopoly case, any excess capacity level $K_i > q_i(K)$ is a suboptimal investment ($\frac{\partial}{\partial K_i} V_i = -c < 0$) provided both firms invest in which case each will produce up to its capacity: $q = K$. The capacity reaction curves become:

$$\max_{0 \leq K_i} V_i(K) = \int_{K_+}^{\infty} (\varepsilon - K_+) K_i f(\varepsilon) d\varepsilon - cK_i, \tag{43}$$

where $K_+ = \sum_i K_i$ denotes the total industry investment level. If both firms invest ($K > 0$), a similar argument as in the capacity-constrained quantity subgame shows that there is a unique intersection of the reaction curves and the equilibrium is symmetric: $K = \frac{1}{2}(K_+, K_+)$. Clearly if $\frac{\partial}{\partial K_i} V_i|_{K_i=0} = \int_{K_j}^{\infty} (\varepsilon - K_j) f(\varepsilon) d\varepsilon - c < 0$, firm i will not invest. Thus, as before, there is a maximal cost-threshold $\bar{c}$:

$$\bar{c} = \int_0^{\infty} \varepsilon f(\varepsilon) d\varepsilon = 1, \tag{44}$$

which is independent of uncertainty and above which no firm will invest. If capacity is not too expensive ($c < \bar{c}$), there exists a unique symmetric duopoly equilibrium investment, in the sense that both firms will invest.

In addition, two asymmetric equilibria with excess capacity investments are possible. Indeed, if firm j invests in excess capacity $K_j \geq k$, where $\int_k^{\infty} (\varepsilon - k) f(\varepsilon) d\varepsilon = c$, then firm i will not find it profitable to invest. Thus, any investment of k or higher deters the other firm from entering the market. However, for such entry deterrent investment strategies $K = (0, k)$ or $(k, 0)$ to be equilibria (that is, to be credible), the deterring

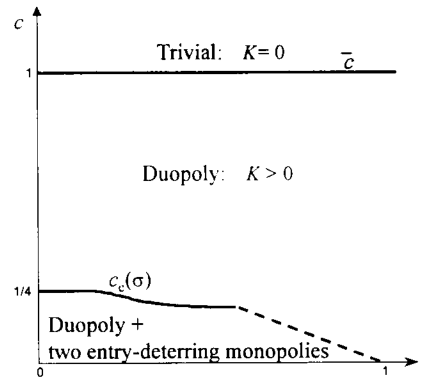

Figure 7: The different optimal duopoly strategies depend on the marginal cost c and the level of variability σ: at high cost ($c > \bar{c} = 1$) no investment is profitable, at medium cost ($\bar{c} > c > c_e$) only a true duopoly equilibrium is possible, while at low cost entry can be deterred.

firm must find this strategy at least as profitable as investing less and not deterring entry. If firm j deters entry, it behaves as a monopolist in the quantity-setting subgame and will choose a lower (the next section will explain why this is the case) output $q_j = q_M(k) < k$ resulting in positive firm value of $V_M(q_M(k), k)$. Thus, for this entry deterrence strategy to be credible, its resulting firm value $V_M(q_M(k), k)$ cannot be lower than the firm value that would obtain in the symmetric duopoly equilibrium $V_i(q = K, K)$, where K is the unique symmetric duopoly investment.

Proposition 7 *The optimal capacity-quantity duopoly strategy is unique and trivial if* $c \geq 1 : q = K = 0$. *If* $c < \varepsilon_0$, *there exists a unique interior pure strategy equilibrium such that* $q = K$ *and* $K_i = K_j = \frac{1}{2}K_+$, *and industry investment* K_+ *and value* V_+ *satisfy:*

$$\int_{K_+}^{\infty} (\varepsilon - \tfrac{3}{2}K_+) f(\varepsilon) d\varepsilon = c \text{ and } V_+ = \tfrac{1}{2}K_+^2 P(\varepsilon \geq K_+) \leq \tfrac{1}{3}K_+(1-c). \tag{45}$$

In addition, there exists a cost threshold $c_e(\sigma)$, *such that if* $c < c_e(\sigma)$, *there are two additional asymmetric entry-deterring equilibria* $K = (k, 0)$ *and* $K = (0, k)$ *where* k *satisfies:*

$$\int_{k}^{\infty} (\varepsilon - k) f(\varepsilon) d\varepsilon = c. \tag{46}$$

For example, the capacity investment reaction curves for our deterministic base case scenario are

$$K_i(K_j) = \tfrac{1}{2}(1 - K_i - c), \tag{47}$$

with unique interior "true duopoly" equilibrium (for all $c < 1$) $q_i = K_i = \frac{1}{3}(1-c)$ with $V_i = \frac{1}{9}(1-c)^2$. In addition, deterministic entry-deterrent equilibria require an investment level k such that:

$$K_j(k) = \tfrac{1}{2}(1 - k - c) = 0 \Leftrightarrow k = 1 - c. \tag{48}$$

Using (21), the corresponding entry-deterring firm value $V_M(q_M(k), k) = \frac{1}{4} - c(1-c)$ if $c < \frac{1}{2}$ and zero otherwise. Thus, for the entry-deterring investment to be credible, we must have that $V_M(q_M(k), k) > V_i$, or $c < c_e = \frac{1}{4}$. However, the existence of entry-deterrent equilibria not only depends on the investment cost c, but also on the level of uncertainty. Figure 7 shows that entry deterrence is progressively more difficult as uncertainty increases. (As before, we give explicit solutions for the uniform distribution in [18, Appendix].) With exponential uncertainty, deterring entry is impossible and $c_e = 0$. Indeed, for exponential uncertainty, the quantity-optimized revenue is

$$\mathrm{E}\pi_i(q_i(K), K) = \min(1, K_i) \exp\left(-\min(1, K_i) - \min(1, K_j)\right), \tag{49}$$

with capacity reaction curves

$$(1 - K_i)\exp\left(-K_i - \min(1, K_j)\right) = c \text{ and } K_i \leq 1. \tag{50}$$

For entry-deterrent equilibria to exist, firm i's response to an investment of $K_j = k$ must be $K_i = 0$, which yields $k = -\ln c$,where $0 \leq k \leq 1$ if $e^{-1} \leq c \leq 1$. The entry-deterring firm then acts as a monopolist and we know from (23) that the optimal output $q_M = k \leq 1$ and the firm value $V_M(k, k) = k\exp(-k) - kc = 0$. Thus, the entry-deterrent strategy is not credible because the firm can make money by investing according to the symmetric duopoly equilibrium $K = \frac{1}{2}(K_+, K_+)$ where K_+ solves for $c < \bar{c} = 1$:

$$\left(1 - \tfrac{1}{2}K_+\right)\exp\left(-K_+\right) = c. \tag{51}$$

The qualitative results from the quantity-setting monopoly section extend to a quantity-setting duopoly under uncertainty: the functional dependence on cost and uncertainty is similar to Figure 4. Also, quantity-setting investment is insensitive to demand variability but less so than in a monopoly: for low variability levels ($\frac{2}{3}(1-c) < \underline{\varepsilon}$ or, for the uniform distribution, $\frac{\sigma}{\varepsilon_0} \leq \frac{1}{3\sqrt{3}}$), the capacity investment level $K_+ = \frac{2}{3}(\varepsilon_0 - c)$ is independent of variability and equal to the deterministic capacity investment level. Kreps and Scheinkman showed that in a deterministic duopoly setting quantity-setting yields the same investment decision as price-setting. However, like in the monopoly case, this is not true if demand is uncertain. Only in the deterministic limit ($\sigma \to 0$) and pre-determined 50-50 market allocation do price- and quantity-setting give the same outcome. This structural difference in the policy structure demonstrates that a reconciliation between price-setting and quantity-setting competition cannot be achieved when there is uncertainty in demand.

5.3 Extension to Oligopoly and Perfect Quantity Competition

Our results directly generalize to an oligopoly of n firms and to perfect quantity competition (the case when $n \to \infty$). Indeed there is a quantity-capacity strategy where all n firms invest: if $c \geq \varepsilon_0$, $q = K = 0$, and if $c < \varepsilon_0$, an interior pure strategy equilibrium is such that all firms produce up to capacity ($q = K$), and all firms invest in the same capacity level $K_1 = \ldots = K_i = \ldots = K_n = \frac{1}{n}K_+^{(n\text{ firms})}$, where the industry capacity level $K_+^{(n\text{ firms})}$ and firm value $V_+^{(n\text{ firms})}$ satisfies:

$$\int_{K_+^{(n\text{ firms})}}^{\infty}\left(\varepsilon - \tfrac{n+1}{n}K_+^{(n\text{ firms})}\right)f(\varepsilon)d\varepsilon = c \text{ and } V_+^{(n\text{ firms})} = \tfrac{1}{n}K_+^2 P(\varepsilon \geq K_+) \leq \tfrac{1}{n+1}K_+(1-c). \tag{52}$$

As above, n entry-deterring equilibria $K_i = k$ and $K_{j\neq i} = 0$ are possible if costs are sufficiently low. (In addition, coalitions of subsets of firms may form and lead to additional equilibria, but we will not investigate this complication in this article.)

The oligopoly extension yields two interesting insights. First, comparing (52) with (46), shows that the industry investment $K_+^{(n\text{ firms})}$ is increasing in the industry size n while industry firm values are decreasing:

$$\begin{aligned} K_M &\leq K_+^{(\text{duopoly})} \leq K_+^{(n\text{ firms})} \leq K_+^{(n+1\text{ firms})} \leq K_+^{(\text{perfect competition})} = k, \\ V_M &\geq V_+^{(\text{duopoly})} \geq V_+^{(n\text{ firms})} \geq V_+^{(n+1\text{ firms})} \geq V_+^{(\text{perfect competition})} = 0 < V_{\text{deterrence}}. \end{aligned}$$

Not only is this in line with economic intuition, it also provides a nice interpretation of the entry-deterring investment k, which is independent of the industry size n and is given by (46). This investment level *equals* the industry investment that would obtain with perfect quantity competition. Thus, a monopolist investing in k has a credible threat to produce $q = k$ at which point no incoming firm could make any money. It also shows that under entry-deterrence $q_M(k) < k$, because a monopolist will bring a lower output to the market than the total industry output under perfect competition. This allows a positive value $V_{\text{deterrence}}$, while perfect competition erodes any producer surplus and has $V_+^{(\text{perfect competition})} = 0$.

Second, our result that quantity-setting firms under low uncertainty invest exactly like deterministic (quantity- or price-setting) firms remains valid, but is more subdued, in an oligopoly with n firms. Indeed, the optimal industry investment equals the deterministic investment

$$K_+^{(n\text{ firms})} = \tfrac{n}{n+1}(1-c), \tag{53}$$

if $\frac{n}{n+1}(1-c) <\underline{\varepsilon}$ or, for the uniform distribution, $c > \frac{1}{n}(\sqrt{3}\sigma(n+1)-1)$. Oligopoly firms at low levels of relative uncertainty ($\sigma \leq \frac{1}{(n+1)\sqrt{3}}$) invest exactly like deterministic firms *regardless* of the capacity cost c. However, as competition intensity n rises, uncertainty becomes more important because the insensitivity zone shrinks, but never disappears: insensitivity to uncertainty remains at higher levels of uncertainty *and* in perfect quantity-competition, provided capacity costs are high ($1 > c > \sqrt{3}\sigma$).

6 Conclusion

In this article we developed a framework for understanding how capacity decisions are made in an industry that faces uncertain demand. Under uncertainty, these decisions depend crucially on the mode of competition. Even in the case of a monopoly, the investment differs when the firm sets quantities instead of prices. We show that the appropriate response to increased demand variability need not always be an increase in price. Nor is it always true that increased variability will lead to lower expected profitability. More importantly, optimal firm value of a quantity setting monopoly is never lower than its price-setting counterpart. We also find that for a monopoly that sets prices when uncertainty is modeled as "additive," the well known result of Mills [14] that the optimal price is below the corresponding deterministic price need not hold. Indeed, the optimal price can be higher than the deterministic benchmark because, if the optimal price is sufficiently high, there is a positive probability that there is zero demand ($\Omega_0 \neq \emptyset$). In contrast, a quantity-setting monopolist is rather insensitive to uncertainty.

Investment followed by capacity-constrained price-setting or quantity-setting provides the firm not only with an additional lever to affect profitability, but also with a strategic weapon to deter entry. We show that deliberate excess-capacity investments can credibly deter entry when investment costs are low, yet they are more difficult as variability increases. Incorporating competitive interactions complements existing capacity models in operations management, which typically assume single decisions makers (monopolists). We show that our quantity-setting results extend to oligopolies of arbitrary size, including the limiting case of perfect competition. Insensitivity to uncertainty remains (in moderated form) which suggests that when firms compete by setting output, uncertainty need not influence the investment decision. Therefore, quantity competition provides a more robust and profitable investment policy. Analogous results when firms compete by setting prices cannot be obtained under uncertainty because a pricing equilibrium in pure strategies does not exist. This may suggest that competition with a homogeneous product (*e.g.*, a commodity) is better modeled by quantity-setting than price-setting. Thus, in order to have realistic models of capacity investment under price-competition, some product or decision timing differentiation must exist and our model may be extended by introducing a substitution parameter in the demand curves. Another possibly fruitful opportunity for future investigation is the extension to dynamic competitive models.

Finally, the article proposes a few hypotheses that are empirically testable: prices may fall when uncertainty increases in non-volatile markets (which have low levels of uncertainty). Quantity-setting firms are willing to invest at higher costs than their price-setting counterparts. And, if investment costs are low, quantity-setting monopolists operate with excess capacity to deter entry.

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References

[1] Y. Amihud and H. Mendelson. Multiperiod sales-production decisions under uncertainty. *Journal of Economic Dynamics and Control*, 5:249–265, 1983.

[2] Y. Amihud and H. Mendelson. Price smoothing and inventory. *Review of Economic Studies*, pages 87–98, 1983.

[3] D. S. Arthur. *Optimal Capacities with Uncertainties and applications to the Truckload Sector of the Motor Carrier Industry.* PhD thesis, Field of Economics, Northwestern University, Evanston, Illinois, 1997.

[4] C. Davidson and R. Deneckere. Long-run competition in capacity, short-run competition in price, and the cournot model. *Rand Journal of Economics*, 17:404–415, 1986.

[5] A. Dixit. A model of duopoly suggesting a theory of entry barriers. *Bell Journal of Economics*, 10:20–32, 1979.

[6] A. Federgruen and A. Heching. Combined pricing and inventory control under uncertainty. *Operations Research*, 1997. forthcoming.

[7] G. Gallego and G. J. van Ryzin. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, 40(8):999–1020, 1994.

[8] M. Hviid. Sequential capacity and price choices in a duopoly model with demand uncertainty. *Journal of Economics*, 51(2):121–144, 1990.

[9] M. Hviid. Capacity constrained duopolies, uncertain demand and non-existence of pure strategy equilibria. *European Journal of Political Economy*, 7:183–190, 1991.

[10] S. Karlin and C. R. Carr. Prices and optimal inventory policy. In K. J. Arrow, S. Karlin, and H. Scarf, editors, *Studies in Applied Probability and Management Science*, pages 159–172. Stanford University Press, Stanford, 1962.

[11] D. Kreps and J. Scheinkman. Quantity precommitment and bertrand competition yield cournot outcomes. *Bell Journal of Economics*, 14:326–337, 1983.

[12] L. Li. A stochastic theory of the firm. *Math. of OR*, 13(3):447–466, 1988.

[13] S. A. Lippman and K. F. McCardle. The competitive newsboy. *Operations Research*, 45:54–65, 1997.

[14] E. S. Mills. Uncertainty and price theory. *Quarterly Journal of Economics*, 73:116–130, 1959.

[15] M. Parlar. Game theoretic analysis of the substitutable product inventory problem with random demand. *Naval Research Logistics*, 35:397–405, 1988.

[16] N. L. Petruzzi and M. Dada. Pricing and the newsvendor model: A review with extensions. Technical report, Center for the Management of Manufacturing Enterprises, Purdue University, August 1996.

[17] J. A. Van Mieghem. Investment and subcontracting under uncertainty. *submitted*, pages 1–20, March 1997.

[18] J. A. Van Mieghem and M. Dada. Capacity investment under demand uncertainty: Price vs. quantity competition. Technical report, Center for Mathematical Studies in Economics and Management Science, Northwestern University, 1997. Available at http://www.kellogg.nwu.edu/research/math.

[19] T. M. Whitin. Inventory control and price theory. *Management Science*, 2:61–68, 1955.

Appendix to:
Capacity Investment under Demand Uncertainty: Price vs. Quantity Competition

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Here we first summarize the explicit solutions to our pricing-capacity and quantity-capacity models for both the monopoly and duopoly when uncertainty is uniformly distributed. Then follow the proofs.

1 Explicit Solutions for the Uniform Distribution

1.1 Monopoly Price-Capacity Decisions

Lemma 1 *If ε is uniformly distributed with mean ε_0 and standard deviation σ, then the optimal capacity-constrained monopoly price $p_M(K)$ for a monopolist depends on the coefficient of variation $\frac{\sigma}{\varepsilon_0}$ as follows:*

1. *If $0 \leq \frac{\sigma}{\varepsilon_0} \leq \frac{1}{3\sqrt{3}}$ (low variability), then*

$$p_M(K) = \begin{cases} \frac{1}{3}(\varepsilon_0 - \sqrt{3}\sigma - K)\left(2 + \sqrt{1 + \frac{12\sqrt{3}\sigma K}{(\varepsilon_0 - \sqrt{3}\sigma - K)^2}}\right) & \text{if } K \leq \frac{1}{2}\varepsilon_0 + \sqrt{3}\sigma, \\ \frac{1}{2}\varepsilon_0 & \text{otherwise.} \end{cases}$$

2. *If $\frac{1}{3\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{2\sqrt{3}}$ (medium variability)*

$$p_M(K) = \begin{cases} \frac{1}{2}\varepsilon_0 + \frac{\sqrt{3}}{2}\sigma - \frac{1}{4}K & \text{if } K \leq 2\left(3\sqrt{3}\sigma - \varepsilon_0\right), \\ \frac{1}{3}(\varepsilon_0 - \sqrt{3}\sigma - K)\left(2 + \sqrt{1 + \frac{12\sqrt{3}\sigma K}{(\varepsilon_0 - \sqrt{3}\sigma - K)^2}}\right) & \text{if } 2\left(3\sqrt{3}\sigma - \varepsilon_0\right) < K \leq \frac{1}{2}\varepsilon_0 + \sqrt{3}\sigma, \\ \frac{1}{2}\varepsilon_0 & \text{otherwise.} \end{cases}$$

3. *If $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \leq \frac{1}{\sqrt{3}}$ (high variability):*

$$p_M(K) = \begin{cases} \frac{1}{2}\varepsilon_0 + \frac{\sqrt{3}}{2}\sigma - \frac{1}{4}K & \text{if } K \leq \frac{2}{3}\left(\varepsilon_0 + \sqrt{3}\sigma\right), \\ \frac{1}{3}\varepsilon_0 + \frac{\sqrt{3}}{3}\sigma & \text{otherwise.} \end{cases}$$

Lemma 2 *If ε is uniformly distributed with mean ε_0 and standard deviation σ, then the optimal price-capacity strategy for a monopolist depends on the coefficient of variation $\frac{\sigma}{\varepsilon_0}$ and the investment cost c as follows:*

1. *If $0 \leq \frac{\sigma}{\varepsilon_0} \leq \frac{1}{3\sqrt{3}}$ (low variability): if $c \leq \bar{c}_1 = \varepsilon_0 - \sqrt{3}\sigma$, then*

$$p = \text{the unique root of } 4p^3 - 2(\varepsilon_0 + c)p^2 + 2\sqrt{3}\sigma c^2 \text{ with } \frac{\varepsilon_0}{2} \leq p \leq \varepsilon_0 - \sqrt{3}\sigma, \quad (1)$$

$$K = \varepsilon_0 - p + \sqrt{3}\sigma\left(1 - \frac{2c}{p}\right). \quad (2)$$

If $c > \varepsilon_0 - \sqrt{3}\sigma$, *then* $K = 0$ *and* p *is arbitrary.*

2. *If* $\frac{1}{3\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \le \frac{1}{2\sqrt{3}}$ *(medium variability): if* $c \le c_2 = \frac{\varepsilon_0^2 - 3\sqrt{3}\sigma\varepsilon_0 + 6\sigma^2}{\sqrt{3}\sigma}$, *then* K *and* p *are determined by (1)-(2). If* $c_2 < c < \bar{c}_2 = \frac{(\varepsilon_0 + \sqrt{3}\sigma)^2}{8\sqrt{3}\sigma}$, *then*

$$p = \frac{1}{6}\left(\varepsilon_0 + \sqrt{3}\sigma + \sqrt{\left(\varepsilon_0 + \sqrt{3}\sigma\right)^2 + 24\sqrt{3}\sigma c}\right), \tag{3}$$

$$K = 2\left(\varepsilon_0 + \sqrt{3}\sigma - 2p\right). \tag{4}$$

If $c \ge \bar{c}_2$, *then* $K = 0$ *and* p *is arbitrary.*

3. *If* $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \le \frac{1}{\sqrt{3}}$ *(high variability): if* $c \le \bar{c}_2$, *then* K *and* p *are determined by (3)-(4). If* $c \ge \bar{c}_2$, *then* $K = 0$ *and* p *is arbitrary.*

1.2 Monopoly Quantity-Capacity Decisions

Lemma 3 *If* ε *is uniformly distributed with mean* ε_0 *and standard deviation* σ, *then the optimal capacity-constrained monopoly quantity* $q_M(K)$ *for a monopolist depends on the coefficient of variation* $\frac{\sigma}{\varepsilon_0}$ *as follows:*

1. *If* $0 \le \frac{\sigma}{\varepsilon_0} \le \frac{1}{2\sqrt{3}}$ *(low & medium variability), then*

$$q_M(K) = \min\left(\frac{\varepsilon_0}{2}, K\right),$$

2. *If* $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \le \frac{1}{3\sqrt{3}}$ *(high variability)*

$$q_M(K) = \min\left(\frac{\varepsilon_0 + \sqrt{3}\sigma}{3}, K\right).$$

Lemma 4 *If* ε *is uniformly distributed with mean* ε_0 *and standard deviation* σ, *then the optimal quantity-capacity strategy for a monopolist depends on the coefficient of variation* $\frac{\sigma}{\varepsilon_0}$ *and the investment cost* c *as follows: if* $c \ge \varepsilon_0$, *then* $K = 0$, *otherwise:*

1. *If* $\frac{\sigma}{\varepsilon_0} \le \frac{1}{2\sqrt{3}}$ *(low & medium variability), then*

$$K = \frac{1}{2}(\varepsilon_0 - c). \tag{5}$$

2. *If* $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \le \frac{1}{\sqrt{3}}$ *(high variability): if* $c < 2\sqrt{3}\sigma - \varepsilon_0$, *then*

$$K = \frac{1}{3}\left(2\left(\varepsilon_0 + \sqrt{3}\sigma\right) - \sqrt{\left(\varepsilon_0 + \sqrt{3}\sigma\right)^2 + 12\sqrt{3}c\sigma}\right), \tag{6}$$

otherwise $K = \frac{1}{2}(\varepsilon_0 - c)$.

1.3 Duopoly Quantity-Capacity Decisions

Lemma 5 *If* ε *is uniformly distributed with mean* ε_0 *and and standard deviation* σ, *then the optimal quantity-capacity strategy for a duopolist depends on the coefficient of variation* $\frac{\sigma}{\varepsilon_0}$ *and the investment cost* c *as follows: If* $c \ge \bar{c} = \varepsilon_0$, *then* $q = K = 0$, *otherwise* $q = K$ *where*

1. *If* $0 \le \frac{\sigma}{\varepsilon_0} \le \frac{1}{3\sqrt{3}}$ *(low variability):*

$$K_i = K_j = \frac{1}{3}(\varepsilon_0 - c).$$

2. If $\frac{1}{3\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \le \frac{1}{\sqrt{3}}$ *(medium & high variability): if* $c \le \bar{c}_1 = \frac{-\varepsilon_0+3\sqrt{3}\sigma}{2}$*, then*

$$K_i = K_j = \frac{3}{8}\left(\varepsilon_0 + \sqrt{3}\sigma\right) - \frac{1}{8}\sqrt{\left(\varepsilon_0 + \sqrt{3}\sigma\right)^2 + 32c\sqrt{3}\sigma}. \tag{7}$$

Otherwise, $K_i = K_j = \frac{1}{3}(\varepsilon_0 - c)$..

In addition, if $c < c_e \le \frac{\varepsilon_0}{4}$*, two entry-deterring equilibria* $K = (k, 0)$ *and* $K = (0, k)$ *exists, with* $q = (q_M(k), 0)$ *and* $(0, q_M(k))$ *respectively, where*

1. If $0 \le \frac{\sigma}{\varepsilon_0} \le \frac{1}{2\sqrt{3}}$ *(low & medium variability):*

$$k = \begin{cases} \varepsilon_0 + \sqrt{3}\sigma - 2\sqrt{c\sigma\sqrt{3}} & \text{if} \quad c < \sqrt{3}\sigma, \\ \varepsilon_0 - c & \text{if} \quad \sqrt{3}\sigma < c \le c_e. \end{cases} \tag{8}$$

where $c_e = \frac{\varepsilon_0}{4}$ *if* $\frac{\sigma}{\varepsilon_0} \le \frac{1}{4\sqrt{3}}$ *and* $c_e < \frac{\varepsilon_0}{4}$ *elsewhere.*

2. If $\frac{1}{2\sqrt{3}} < \frac{\sigma}{\varepsilon_0} \le \frac{1}{\sqrt{3}}$ *(high variability):*

$$k = \varepsilon_0 + \sqrt{3}\sigma - 2\sqrt{c\sigma\sqrt{3}} \quad \text{if} \quad 0 < c < c_e < \frac{1}{9\sqrt{3}\sigma}(\varepsilon_0 + \sqrt{3}\sigma)^2 < \frac{\varepsilon_0}{4}. \tag{9}$$

2 Proofs

2.1 Proof of Proposition 1

$\mathrm{E}\pi^p_M(p, K)$ is twice differentiable in p and

$$\frac{\partial V_M}{\partial p} = \int_p^{p+K} (\varepsilon - 2p) f(\varepsilon) d\varepsilon + \int_{p+K}^{\infty} K f(\varepsilon) d\varepsilon, \tag{10}$$

$$\frac{\partial^2 V_M}{\partial p^2} = pf(p) + 2F(p) - pf(p+K) - 2F(p+K). \tag{11}$$

We have assumed that the measure P is such that $\mathrm{E}\pi_M(p, K)$ is unimodal so that the first order condition is sufficient.

If $K > 0$, $\frac{\partial V_M}{\partial p} > 0$ if $p = 0$ so that $p(K)$ must be strictly positive and the first order condition is sufficient for the interior maximum. Implicit differentiation yields

$$\frac{\partial^2 V_M}{\partial p^2}\frac{dp_M}{dK} = -\frac{\partial^2 V_M}{\partial K \partial p} = -\left(1 - F(p_M + K) - p_M f(p_M + K)\right). \tag{12}$$

The following sample path argument shows that $\frac{dp_M}{dK} \le 0$. Assume that $p_M(K)$ is unique and consider the three representative sample paths of $\pi_M(p, K, \varepsilon)$ (refer to Figure 1 in the paper). If K increases to $K + dK$, the revenue maximizing price for each sample path either remains the same (for low and medium values of $\varepsilon : \varepsilon < 2K$) or decreases (for high values of $\varepsilon > 2K$). Therefore, the unique maximum $p_M(K)$ of $\mathrm{E}\pi_M(p, K)$ (which is a convex superposition of the sample paths $\pi_M(p, K, \varepsilon)$) cannot increase when K increases to $K + dK$. ■

2.2 Proof of Proposition 2

The objective function

$$V_M = \int_p^{p+K} p(\varepsilon - p) f(\varepsilon) d\varepsilon + \int_{p+K}^{\infty} pK f(\varepsilon) d\varepsilon - cK \tag{13}$$

is differentiable with, in addition to (10), the necessary first order condition:

$$\frac{\partial V_M}{\partial K} = \int_{p+K}^{\infty} pf(\varepsilon) d\varepsilon - c = pP(\varepsilon \ge p + K) - c \tag{14}$$

For a boundary solution $p = 0, K > 0$ to be optimal we would need $c = 0$. Boundary solutions $p \geq 0, K = 0$ yield $V_M = 0$. Thus, any non-trivial solution is an interior solution of the first-order conditions which were assumed to be sufficient.

Implicit differentiation of $\frac{\partial V_M}{\partial p} = 0$ and $\frac{\partial V_M}{\partial K} = 0$ yields

$$\frac{\partial K}{\partial c} = \frac{\frac{\partial^2 V_M}{\partial p^2}}{\frac{\partial^2 V_M}{\partial K^2}\frac{\partial^2 V_M}{\partial p^2} - \left(\frac{\partial^2 V_M}{\partial p \partial K}\right)^2} \text{ and } \frac{\partial p}{\partial c} = -\frac{\frac{\partial^2 V_M}{\partial p \partial K}}{\frac{\partial^2 V_M}{\partial p^2}}. \tag{15}$$

Because there is an interior maximum, V is concave at that point (p, K) and thus $\frac{\partial K}{\partial c} < 0$. Also, if $\frac{\partial^2 V_M}{\partial p \partial K} > 0$, then $\frac{\partial p}{\partial c} > 0$. ■

2.3 Proof of Proposition 3

Proof. We have that

$$q_M(K) = \arg \max_{0 \leq q \leq K} \mathrm{E}\pi_M(q, K) = \int_{q \leq K}^{\infty} (\varepsilon - q) q f(\varepsilon) d\varepsilon. \tag{16}$$

Given our assumption that f is such that $\mathrm{E}\pi_M(q, K)$ is unimodal concave-convex, the first order equations are sufficient. The unconstrained maximum q^* must satisfy $\int_{q^*}^{\infty} (\varepsilon - 2q^*) f(\varepsilon) d\varepsilon = 0$. If $q^* < K$, then $q_M(K) = q^*$, otherwise $\mathrm{E}\pi_M(q, K)$ is increasing over $[0, K]$ so that $q_M(K) = K$. ■

2.4 Proof of Proposition 4 (duality)

Proof. First note that for arbitrary large capacity K, clearly $p^* = p(K)$. Now, invoking Proposition 1 yields that $p(K)$ increases as K decreases, so that in general $p(K) \geq p^*$. ■

2.5 Proof of Proposition 5

Proof. Any choice $K > q^*$, implies that $\frac{d}{dK}\mathrm{E}\pi_M(q^*, K) = 0$ and thus $\frac{d}{dK}V_M(q^*, K) = -c$, which cannot be optimal at positive cost. Thus, it must be that $K = q^*$ and necessary optimality equation is $\int_K^{\infty} (\varepsilon - 2K) f(\varepsilon) d\varepsilon = c$, which has only has a solution if $c < \bar{c}$, where $\bar{c} = \frac{d}{dK}\mathrm{E}\pi_M(K, K)|_{K=0} = \int_K^{\infty} (\varepsilon - 2K) f(\varepsilon) d\varepsilon|_{K=0} = \mathrm{E}\varepsilon = \varepsilon_0$. ■

2.6 Proof of Proposition 6

Proof. First assume that K_i is large such that

$$\max_{0 \leq q_i \leq K_i} \int_{q_i + q_j}^{\infty} (\varepsilon - q_i - q_j) q_i f(\varepsilon) d\varepsilon \tag{17}$$

has an interior optimum $q_i(q_j)$ which satisfies

$$\int_{q_i(q_j)+q_j}^{\infty} (\varepsilon - 2q_i(q_j) - q_j) f(\varepsilon) d\varepsilon = 0 \text{ and } q_i f(q_+) - 2\bar{F}(q_+) < 0, \tag{18}$$

where $1 - F(x) = \bar{F}(x) \geq 0$. It follows directly that the unconstrained intersection of the reaction curves, if it exists, is symmetric:

$$\int_{q_+}^{\infty} (\varepsilon - q_+) f(\varepsilon) d\varepsilon = q_i \overset{q_i = q_D^* = \frac{1}{2}q_+}{\Leftrightarrow} \int_{2q_D^*}^{\infty} (\varepsilon - 3q_D^*) f(\varepsilon) d\varepsilon = 0.$$

Clearly, from Proposition 3 we know that $q_i(q_j = 0) = q_M(K_i) = q_M^*$ if K_i is large ($K_i \geq q_M^*$). The implicit function theorem yields

$$\frac{\partial q_i(q_j)}{\partial q_j} = -\frac{\bar{F}(q_+) - q_i f(q_+)}{2\bar{F}(q_+) - q_i f(q_+)}, \tag{19}$$

so that if $f(q_+)/\bar{F}(q_+) \leq q_i^{-1}$, we have that $-1 \leq \frac{\partial q_i(q_j)}{\partial q_j} \leq 0$ with $q_i(0) = q_M^*$. If K_j is also large with interior optimum, it's reaction curve is decreasing with slope ≤ -1 and it intersects the $q_i = 0$ axis at q_M^*. Thus, the two reaction curves have exactly one intersection and that equilibrium is symmetric and is denoted by (q_D^*, q_D^*) where $q_D^* \leq q_M^*$.

If K_i is small ($K_i < q_M^*$), the response function $q_i(q_j)$ is constant at $q_i = K_i$ for small q_j. After a certain value of q_j, the optimum of (17) becomes the interior point $q_i(q_j)$ from before. Thus, if $K \leq (q_D^*, q_D^*)$, the unique equilibrium is $q = K$ and if $K \geq (q_D^*, q_D^*)$, the unique equilibrium remains (q_D^*, q_D^*). Thus, the only remaining case is that $K_i < q_D^*$, while $K_j > q_D^*$ (or its symmetric counterpart). Let q_c denote the unique intersection of firm j's (unrestricted) reaction curve $q_j(q_i)$ with $q_i = K_i$: $q_c = q_j(K_i)$. It directly follows that $q_D^* \leq q_c \leq q_M^*$. Now: if $K_j \in (q_D^*, q_c]$, the unique equilibrium is $q = K$; otherwise if $K_j > q_c$, the unique equilibrium is $q = (K_i, q_c)$. This also shows that if a firm has excess capacity ($\forall K_i >$ the unique equilibrium $q_i(K)$) we have that $\frac{\partial}{\partial K_i} q(K) = 0$.

In conclusion: there is a unique pure strategy equilibrium which cannot be larger than (q_M^*, q_M^*), and thus, because in that case $q_i(q_j)$ and $q_j(q_i) \leq q_M^*$, a sufficient condition is that

$$\forall x, y \in [0, q_M^*] : \frac{f(x+y)}{\bar{F}(x+y)} \leq \frac{1}{x}. \tag{20}$$

(The argument can be relaxed by requiring $f(q_+)/\bar{F}(q_+) < \frac{3}{2} q_i^{-1}$ so that $-1 \leq \frac{\partial q_i(q_j)}{\partial q_j} < 1$.) ■

2.7 Proof of Proposition 7

Proof. Any choice $K_i > q_i(K)$, implies that $\frac{d}{dK_i} \mathrm{E}\pi_i(q(K), K) = 0$ and thus $\frac{d}{dK_i} V_i(q(K), K) = -c$, which cannot be optimal at positive cost. Thus, it must be that $K = q(K)$ and the capacity reaction curves become:

$$\max_{0 \leq K_i} V_i(K) = \int_{K_+}^{\infty} (\varepsilon - K_+) K_i f(\varepsilon) d\varepsilon - cK_i. \tag{21}$$

Clearly if $\frac{\partial}{\partial K_i} V_i|_{K_i=0} = \int_{K_j}^{\infty} (\varepsilon - K_j) f(\varepsilon) d\varepsilon - c < 0$, firm i will not invest. Thus, if $\int_0^{\infty} \varepsilon f(\varepsilon) d\varepsilon = \varepsilon_0 \leq c$, no firm will invest and a unique trivial equilibrium follows: $q = K = 0$. If $c < \varepsilon_0$, firm $i's$ reaction curves becomes:

$$K_i \bar{F}(K_+) = \int_{K_+}^{\infty} (\varepsilon - K_+) f(\varepsilon) d\varepsilon - c \text{ and } K_i f(K_+) - 2\bar{F}(K_+) < 0. \tag{22}$$

A similar argument as in the preceeding proof shows that there is a unique intersection of the reaction curves if $\forall x, y \in [0, q^*] : \frac{f(x+y)}{\bar{F}(x+y)} \leq \frac{1}{x}$. Moreover, this intersection is symmetric: $K_i = K_j = \frac{1}{2} K_+$ and because $\frac{f(x+y)}{\bar{F}(x+y)} \leq \frac{1}{x} < 2\frac{1}{x}$ the sufficient optimality condition is satisfied. ■

2.8 Lemma 1: Capacity-constrained monopoly pricing under uniform uncertainty

Depending on the location of (p, K) relative to the domain $[a, b]$ of ε we distinguish the six possible cases for $\mathrm{E}\pi_M(K, p)$. For ease of notation let $a = \varepsilon_0 - \sqrt{3}\sigma, b = \varepsilon_0 + \sqrt{3}\sigma$ and $\mu = (b-a)^{-1}$ if $b > a$.

Case 1: $a < p < p + K < b$: We have

$$\mathrm{E}\pi_M(p, K) = p\left(\frac{K}{2}\right) K\mu + pK(b - p - K)\mu, \tag{23}$$

$$\mathrm{E}\pi_M(p, K) = \frac{1}{2} pK\mu(-2p - K + 2b). \tag{24}$$

Capacity constrained monopoly pricing has sufficient interior optimality condition:

$$\frac{\partial V_M}{\partial p} = \frac{1}{2} K\mu(-K - 4p + 2b) = 0 \text{ and } \frac{\partial^2 V_M}{\partial p^2} = -2K\mu < 0. \tag{25}$$

Thus the capacity-constrained monopoly price is

$$p_M(K) = \frac{2b - K}{4}, \tag{26}$$

with corresponding revenue function

$$\mathrm{E}\pi_M(p_M(K), K) = \frac{\mu}{16} K (2b - K)^2 . \tag{27}$$

This case is optimal for any K such that $a < p < p + K < b$ or:

$$a < \frac{b}{2} - \frac{K}{4} \text{ and } \frac{3K}{2} < b \Leftrightarrow K < \min(2(b-2a), \frac{2}{3}b) \tag{28}$$

This requires $b > 2a$ (medium and high variability). Note that $2(b-2a) < \frac{2}{3}b$ iff $b < 3a$ (medium variability).

Case 2: $p < a < p + K < b$: We have

$$\begin{aligned} \mathrm{E}\pi_M &= p(a - p + \frac{p+K-a}{2}) (p + K - a) \mu + pK(b - p - K)\mu, & (29) \\ \mathrm{E}\pi_M(p, K) &= -\frac{1}{2} p\mu \left(p^2 - 2ap + 2pK - 2Kb + a^2 + K^2\right). & (30) \end{aligned}$$

The sufficient conditions for an optimal (p, K) interior in $\{(p, K) : 0 < p < a < p + K < b\}$ are

$$\begin{aligned} \frac{\partial \mathrm{E}\pi_M}{\partial p} &= -\frac{1}{2}\mu \left(3p^2 - 4ap + 4pK - 2Kb + a^2 + K^2\right) = 0, \\ \frac{\partial^2 \mathrm{E}\pi_M}{\partial p^2} &= -\mu (3p - 2a + 2K) < 0. \end{aligned}$$

Capacity constrained monopoly pricing has necessary interior optimality condition $\frac{\partial V_M}{\partial p} = 0$ or

$$p_M(K) = \frac{2}{3}(a - K) + \frac{1}{3}\sqrt{\left((a-K)^2 + 6K(b-a)\right)}, \tag{31}$$

(the other root is not a maximum) with corresponding revenue function

$$\mathrm{E}\pi_M(p_M(K), K) = \frac{\mu}{27}\left[(a-K)\left(18(b-a)K - (a-K)^2\right) + \left(\sqrt{(a-K)^2 + 6K(b-a)}\right)^3\right]. \tag{32}$$

This case is optimal for any level K such that $p < a < p + K < b$ or:

$$\begin{aligned} \frac{2}{3}(a-K) + \frac{1}{3}\sqrt{a^2 - 8aK + K^2 + 6Kb} &< a & (33) \\ \Leftrightarrow \sqrt{a^2 - 8aK + K^2 + 6Kb} &< (a + 2K) & (34) \\ \Leftrightarrow a^2 - 8aK + K^2 + 6Kb - (a+2K)^2 &< 0 & (35) \\ \Leftrightarrow a^2 - 8aK + K^2 + 6Kb - (a+2K)^2 &< 0 & (36) \\ \Leftrightarrow 2(b - 2a) &< K, & (37) \end{aligned}$$

and

$$\begin{aligned} a &< \frac{1}{3}(2a + K) + \frac{1}{3}\sqrt{(a-K)^2 + 6K(b-a)} < b & (38) \\ \Leftrightarrow &\quad (a - K) < \sqrt{(a-K)^2 + 6K(b-a)} < 3b - (2a + K) & (39) \\ \Leftrightarrow &\quad a^2 - 8aK + K^2 + 6Kb < (3b - (2a+K))^2 \text{ and } 0 < 3b - (2a + K) & (40) \\ \Leftrightarrow &\quad 4K < 3b - a \text{ and } K < 3b - 2a. & (41) \end{aligned}$$

Thus we have

$$2(b - 2a) < K < \min\left(3b - 2a, \frac{3b - a}{4}\right), \tag{42}$$

and because $3b - 2a > \frac{3b-a}{4}$ and we must have $2(b-2a) < \frac{3b-a}{4} \Leftrightarrow b < 3a$. This, this case requires

$$b < 3a \text{ (low \& medium variability) and } 2(b-2a) < K < \frac{3b-a}{4}.$$

Case 3: $p + K < a$: We have $E\pi_M = pK$. Thus the capacity-constrained monopoly price is

$$p_M(K) = a - K. \tag{43}$$

with corresponding revenue function

$$E\pi_M(p_M(K), K) = (a-K)K.$$

This case is suboptimal as it is a boundary solution. ($p + K = a - (a-c)/2 < a \Leftrightarrow c < a$)

Case 4: $p < a < b < p + K$: In this case (??) becomes $E\pi_M = p(a - p + \frac{b-a}{2}) = p(\frac{a+b}{2} - p)$ with optimal response:

$$p_M(K) = \frac{a+b}{4}, \tag{44}$$

with corresponding revenue function

$$E\pi_M(p_M(K), K) = \left(\frac{a+b}{4}\right)^2.$$

This case is optimal for any level K such that $p < a < b < p + K$ or

$$b < 3a \text{ and } \frac{3b-a}{4} < K. \tag{45}$$

Case 5: $a < p < b < p + K$: We have $E\pi_M = \frac{\mu}{2}p(b-p)^2$, with optimal response:

$$p_M(K) = \frac{b}{3}, \tag{46}$$

and corresponding revenue function

$$E\pi_M(p_M(K), K) = \frac{2}{27}\mu b^3.$$

This case is optimal for any level K such that $a < p < b < p + K$ or

$$3a < b \text{ and } \frac{2b}{3} < K. \tag{47}$$

Case 6: $a < b < p < p + K$: We have $E\pi_M = 0$, so that any price $p > b$ cannot be optimal for a positive K. ■

2.9 Lemma 2: Monopoly capacity investment with price-setting (uniform uncertainty)

We build on the results of lemma 1. There are three scenarios, depending on the level of variability. In all scenarios, $E\pi_M(K, p_M(K))$ is strictly concave increasing in K up to a (scenario-dependent) level after which it becomes constant. Let $\bar{c} = \frac{d}{dK}E\pi_M(K, p_M(K))|_{K=0}$. Thus, for any positive cost $c < \bar{c}$, there is a unique optimal capacity level K. Now, depending upton the scenario, we may have to break up the interval $(0, \bar{c})$ to derive the optimal K. For the low variability scenario we have that only case 2 (from lemma 1) is needed and

$$\begin{aligned} & \frac{d}{dK}E\pi_M(K, p_M(K)) \\ = \; & \frac{\mu}{9}\left[K^2 - 12Kb + 10aK - 5a^2 + 6ab - \sqrt{(a^2 - 8aK + K^2 + 6Kb)}\,(4a - K - 3b)\right]. \end{aligned}$$

It is easily verified that $\bar{c}_{M,\text{low}} = \frac{d}{dK}\text{E}\pi_M(K, p_M(K))|_{K=0} = a$ and $\frac{d}{dK}\text{E}\pi_M(K, p_M(K))|_{K=\frac{3b-a}{4}} = 0$.

For the medium variability scenario we have that for $K < 2(b-2a)$ case 1 is valid:

$$\frac{d}{dK}\text{E}\pi_M(K, p_M(K)) = \frac{1}{16}\mu(2b-K)(2b-3K),$$

so that $\bar{c}_{M,\text{medium}} = \frac{1}{4}\mu b^2$. At $K = 2(b-2a)$ case 2 becomes valid, at that point the derivative of $\text{E}\pi_M(K, p_M(K))$ is $\mu a(3a-b)$. Thus, for the medium variability scenario, we have that there is a unique optimal capacity level K, which is determined by case 2 if $0 < c < \mu a(3a-b)$, and by case 1 if $\mu a(3a-b) \leq c < \bar{c}_{M,\text{medium}} = \frac{1}{4}\mu b^2$.

Finally, for the high variability scenario we have that only case 1 applies so that we have $\bar{c}_{M,\text{high}} = \bar{c}_{M,\text{medium}} = \frac{1}{4}\mu b^2$.

Thus, to get the explicit expressions for the optimal K, we only need to consider cases 1 and 2:

Case 1: $a < p < p+K < b$: We have that

$$V_M(K, p_M(K)) = \frac{\mu}{16}K(2b-K)^2 - cK, \tag{48}$$

The optimal capacity level satisfies

$$\frac{\partial}{\partial K}V_M(K, p_M(K)) = 0 \Leftrightarrow \mu(2b-K)(2b-3K) = 16c \tag{49}$$

with unique positive solution $K = \frac{2}{3}\left(2b + \sqrt{b^2 + 12c/\mu}\right)$ and thus $p = \frac{1}{6}\left(b + \sqrt{b^2 + 12c/\mu}\right)$ and corresponding objective value:

$$\begin{aligned} V_i^*(c) &= \mu p(b-2p)(-2p-2b+4p+2b) - c2(b-2p), & (50)\\ &= 2\left(p^2\mu - c\right)(b-2p). & (51) \end{aligned}$$

Conditions $a < p, p+K = 2b-3p < b$ and $K = 2(b-2p) \geq 0$ require $\max(a, \frac{b}{3}) < p < \frac{b}{2}$. Thus: either $\frac{b}{2} > a > \frac{b}{3}$ (medium variability) and

$$a < \frac{1}{6}\left(b + \sqrt{b^2 + 12c/\mu}\right) \leq \frac{b}{2} \Leftrightarrow c_2 = a\mu(3a-b) < c \leq \bar{c}_{\text{medium}} = \frac{1}{4}\mu b^2,$$

or $\frac{b}{3} > a$ (high variability) and

$$\frac{b}{3} < \frac{1}{6\mu}\left(\mu b + \sqrt{(\mu^2 b^2 + 12c\mu)}\right) \leq \frac{b}{2} \Leftrightarrow 0 < c \leq \bar{c}_{\text{high}} = \frac{1}{4}\mu b^2.$$

Case 2: $p < a < p+K < b$: The optimality equation for this case $\frac{\partial}{\partial K}\text{E}\pi_M(K, p_M(K)) = c$ is a third-order polynomial with a unique solution satisfying $p < a < p+K < b$ and

$$4K^3 + (2c + a - 11b)K^2 + 2\left(2a^2 + 3b^2 + 10ca - 12cb - ab\right)K + (a-c)\left(a^2 - 9ca - 3ab + 9cb\right) = 0.$$

However, an easier (but equivalent) expression is obtained by swiching the order of optimization (which is allowed, given the uniqueness of the optimum in the zone $p < a < p+K < b$). We have that

$$V_M = -\frac{1}{2}p\mu\left(p^2 - 2ap + 2pK - 2Kb + a^2 + K^2\right) - cK. \tag{52}$$

The necessary conditions for an optimal (p, K) interior in $\{(p, K) : 0 < p < a < p+K < b\}$ are

$$\begin{aligned} \frac{\partial V_M}{\partial p} &= -\frac{1}{2}\mu\left(3p^2 - 4ap + 4pK - 2Kb + a^2 + K^2\right) = 0, \\ \frac{\partial V_M}{\partial K} &= -p\mu(p - b + K) - c = 0. \end{aligned}$$

The optimal capacity level is most easily solved for algebraically by first solving for K as a function of p using $\frac{\partial V}{\partial K} = 0$ such that $K = \frac{-p^2\mu + p\mu b - c}{p\mu}$. Then substitute into $\frac{\partial V}{\partial p} = 0$ and solve for p. The solution p is the unique root of $f(x) = -4\mu x^3 + \mu(a + b + 2c)x^2 - c^2$ in the interval $[0, a]$. [Necessary and sufficient condition for $f(x)$ to have a single root in $[0, a]$ is that $c^2 - 2\mu a^2 c + 4\mu a^3 - \mu(a+b)a^2 = (c - \alpha_1)(c - \alpha_2) \leq 0$. Because $\alpha_1 + \alpha_2 < 0$, we have a positive root α iff $\alpha_1\alpha_2 = 4\mu a^3 - \mu(a+b)a^2 > 0 \Leftrightarrow 3a > b$. One can verify that this single positive root is $\alpha = c_2$ so that there is a valid price for this case if $3a > b$ and $c < c_2$. Because $c_2 > a$ when $b < 2a$, the condition $c < a$ when $b < 2a$ guarantees the uniqueness of the root.] ■

2.10 Lemma 3: Capacity-constrained monopoly quantity setting

We have two cases:

Case 1: $K < a$: Because $q \leq K$, the revenue function becomes

$$\mathrm{E}\pi_M = \mathrm{E}(\varepsilon - q)q = (\varepsilon_0 - q)q,$$

so that the capacity-constrained monopoly quantity is $q_M(K) = \min\left(\frac{\varepsilon_0}{2}, K\right)$. (Note that if $b > 3a$ (high variability) we have that $\frac{\varepsilon_0}{2} > a$, so that $q_M(K) = K$.)

Case 2: $K \geq a$: If $q = \frac{\varepsilon_0}{2} < a \Leftrightarrow b < 3a$ (low & medium variability), the solution above holds. Otherwise $a \leq q \leq b$, (clearly, one will never set $q > b$), and the revenue function becomes

$$\mathrm{E}\pi_M = \mathrm{E}\left[(\varepsilon - q)q | a \leq q \leq b\right] P(a \leq q \leq b) = \int_q^b (\varepsilon - q)q\mu d\varepsilon = \frac{1}{2}q\mu\left(b - q\right)^2, \tag{53}$$

with a unique maximum at $q = \frac{b}{3}$ so that $q_M(K) = \min\left(\frac{b}{3}, K\right)$.

Conclusion:

1. If $b < 3a$ (low & medium variability):

$$q_M(K) = \min\left(\frac{\varepsilon_0}{2}, K\right), \tag{54}$$

2. If $b > 3a$ (high variability):

$$q_M(K) = \min\left(\frac{b}{3}, K\right). \tag{55}$$

2.11 Lemma 4: Monopoly capacity investment with quantity-setting (uniform uncertainty)

We have two cases:

Case 1: $b < 3a$ (low & medium variability). The value function becomes

$$V(K) = \begin{cases} K\left(\varepsilon_0 - K\right) - cK & \text{if} \quad K \leq \frac{\varepsilon_0}{2}, \\ \left(\frac{\varepsilon_0}{2}\right)^2 - cK & \text{if} \quad K > \frac{\varepsilon_0}{2}. \end{cases} \tag{56}$$

Clearly, $K > \frac{\varepsilon_0}{2}$ is suboptimal. Thus, $K \leq \frac{\varepsilon_0}{2}$, and the optimal capacity investment is

$$K = \frac{1}{2}\left(\varepsilon_0 - c\right) \Rightarrow V = \left(\frac{\varepsilon_0 - c}{2}\right)^2 \text{ for low \& medium variability,} \tag{57}$$

with conditions: $c < \varepsilon_0$ and $\frac{1}{2}\left(\varepsilon_0 - c\right) < \frac{\varepsilon_0}{2}$ (which is clearly satisfied with low&medium variability).

Case 2: $b > 3a$ (high variability). The value function becomes

$$V(K) = \begin{cases} K\left(\varepsilon_0 - K\right) - cK & \text{if} \quad K \leq a, \\ \frac{1}{2}K\mu\left(b - K\right)^2 - cK & \text{if} \quad a < K \leq \frac{b}{3}, \\ \frac{2}{27}b^3\mu - cK & \text{if} \quad K > \frac{b}{3}. \end{cases} \tag{58}$$

Clearly, $K > \frac{b}{3}$ is suboptimal. Depending on the cost c, V can have a maximum in the zone $[0, a]$ or in $[a, b/3]$. A maximum in $[0, a]$ is as in case 1 with conditions $c < \varepsilon_0$ and $K = \frac{1}{2}\left(\varepsilon_0 - c\right) < a \Leftrightarrow \frac{a+b}{2} - 2a = \frac{b-3a}{2} < c$. If $c < \frac{b-3a}{2}$, then the optimal K is in between a and $b/3$ and maximizes

$$V = \frac{1}{2}K\mu\left(b - K\right)^2 - cK, \tag{59}$$

with unique maximum:

$$K = \frac{1}{3}\left(2b - \sqrt{b^2 + 6c/\mu}\right), \tag{60}$$

with conditions $0 < a < \frac{1}{3}\left(2b - \sqrt{b^2 + 6c/\mu}\right) < b \Leftrightarrow 3a - 2b < -\sqrt{b^2 + 6c/\mu} < b$ and because we must have high variability this reduces to $c < \frac{1}{2}\left(b - 3a\right) = 2\sqrt{3}\sigma - \varepsilon_0$. ■

2.12 Lemma 5: Optimal duopoly capacity investment under quantity-setting with uniform uncertainty

2.12.1 The quantity-setting subgame

We now consider the case of two firms; initially each firm i and j sets capacity at K_i and K_j for a total industry capacity of $K_+ = K_i + K_j$. Given the capacity choices in stage 1, each firm sets production quantity $q_i \leqslant K_i$ and $q_j \leqslant K_i$ for a total industry production of $q_+ = q_i + q_j$. Hence, the expected profit for firm i is:

$$\mathrm{E}\pi_i = \int_{q_+}^{\infty} (\varepsilon - q_+)\, q_i f(\varepsilon) d\varepsilon.$$

Case 1: $q_+ < a$. The revenue function becomes $\mathrm{E}\pi_i = \mathrm{E}(\varepsilon - q_+)\, q_i = (\varepsilon_0 - q_+)q_i$ so that firm i's quantity reaction curve is

$$q_i(q_j|K_i) = \min\left(\frac{\varepsilon_0 - q_j}{2}, K_i\right) \text{ if } q_+ < a \Leftrightarrow \min\left(\frac{a+b+2q_j}{4}, K_i + q_j\right) < a.$$

An interior solution $\frac{\varepsilon_0 - q_j}{2}$ requires $\frac{\varepsilon_0}{2} = \frac{a+b}{4} < a \Leftrightarrow b < 3a$ (low & medium variability).

Case 2: $q_+ \geq a$. Clearly, one will never set $q_+ > b$, so that the revenue function becomes

$$\mathrm{E}\pi_i = \int_{q_i+q_j}^{b} (\varepsilon - q_i - q_j) q_i \mu d\varepsilon = \frac{1}{2} q_i \mu \left(b - q_i - q_j\right)^2,$$

with a unique maximum at $q_i = \frac{1}{3}(b - q_j)$ so that

$$q_i(q_j|K_i) = \min\left(\frac{1}{3}(b - q_j), K_i\right) \text{ if } q_+ \geq a \Leftrightarrow \min\left(\frac{1}{3}(b + 2q_j), K_i + q_j\right) \geq a. \tag{61}$$

Thus, firm i's reaction curve $q_i(q_j|K)$ is piecewise linear. If capacity K_i is sufficiently large, the reaction curve is strictly decreasing in q_j whenever $q_i > 0$ with decreasing slopes > -1 as shown in Figure 8. If capacity is low, the reaction curve is the same, but restricted to the rectangle $[0, K_i] \times [0, K_j]$. In any case, as discussed in the proof of Proposition 5, there is a unique equilibrium at the intersection of both firm's reaction curves.

Assuming capacity is sufficiently large, the equilibrium is $q = (q_D^*, q_D^*)$ depends on the level of variability:

$$q_D^* = \begin{cases} \frac{a+b}{6} & \text{if} \quad \frac{1}{2}(b-a) \leq \frac{1}{2}(3a-b) \Leftrightarrow b < 2a \text{ (low variability)}, \\ \frac{b}{4} & \text{otherwise} \quad \text{(medium \& high variability)}. \end{cases} \tag{62}$$

In general, denoting the unrestricted reaction curves by $q_i^U(\cdot)$, the equilibrium is

$$q(K) = \begin{cases} (K_i, K_j) & \text{if } \ 0 \leq K_i, K_j < q_D^*, \\ \left(\min\left(K_i, q_i^U(K_j)\right), K_j\right) & \text{if } \ 0 \leq K_j < q_D^* \leq K_i, \\ \left(K_i, \min\left(K_j, q_j^U(K_i)\right)\right) & \text{if } \ 0 \leq K_i < q_D^* \leq K_j, \\ (q_D^*, q_D^*) & \text{if } \ q_D^* \leq K_i, K_j. \end{cases} \tag{63}$$

2.12.2 The Capacity Stage (Full Game)

First focus on the interior equlibrium. We know from Proposition 6 that in equilibrium $q = K$ and $K = \frac{1}{2}(K_+, K_+)$ where

$$\int_{K_+}^{\infty} \left(\varepsilon - \frac{3}{2} K_+\right) f(\varepsilon) d\varepsilon = c \tag{64}$$

for $c < \varepsilon_0$, and $q = K = 0$ for $c \geq \varepsilon_0$.

Case 1:$0 \leq K_+ \leq a$: $\varepsilon_0 - \frac{3}{2}K_+ = c \Leftrightarrow$

$$K_+ = \frac{2}{3}(\varepsilon_0 - c) \text{ and } V_+ = \mathrm{E}(\varepsilon - K_+)\, K_+ - cK_+ = \frac{2}{9}\left(c - \varepsilon_0\right)^2, \tag{65}$$

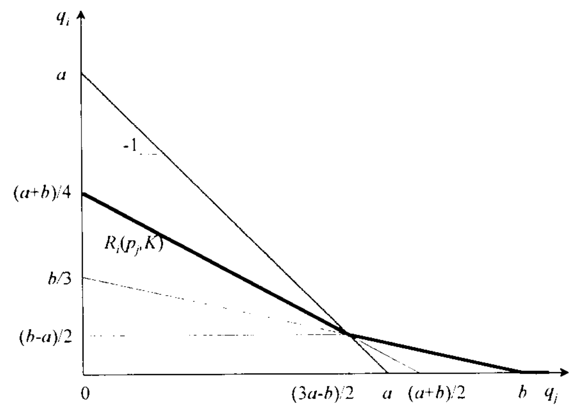

Figure 8: Quantity-reaction curve for firm i for uniform uncertainty (assuming $K_i > \frac{1}{4}(a+b)$).

with condition:

$$\frac{2}{3}(\varepsilon_0 - c) \leq a \Leftrightarrow c \geq \varepsilon_0 - \frac{3}{2}a. \tag{66}$$

With low variability ($b < 2a$), $\varepsilon_0 - \frac{3}{2}a < 0$, so that this holds for all $c < \varepsilon_0$. For medium and high variability this case requires $c \geq \varepsilon_0 - \frac{3}{2}a = \frac{b-2a}{2}$.

Case 2: $a < K_+$:

$$V_i = \int_{K_+}^{b} (\varepsilon - K_i - K_j) K_i \mu d\varepsilon - cK_i, \tag{67}$$

$$\frac{\partial}{\partial K_i} V_i = \int_{K_i + K_j}^{b} (\varepsilon - 2K_i - K_j) \mu \, d\varepsilon - c = 0, \tag{68}$$

$$\frac{\partial^2}{\partial K_i^2} V_i = \mu (3K_i + 2K_j - 2b) < 0. \tag{69}$$

Thus, with $K_i = K_j = \frac{1}{2}K_+$:

$$\int_{K_+}^{b} \left(\varepsilon - \frac{3}{2}K_+\right) \mu d\varepsilon = \frac{1}{2}\mu (b - K_+)(b - 2K_+) = c \text{ and } \mu \left(\frac{5}{2}K_+ - 2b\right) < 0 \text{ and } K_+ > a, \tag{70}$$

$$\Leftrightarrow \quad a < K_+ = \frac{1}{4}\left(3b \pm \sqrt{b^2 + 16c/\mu}\right) < \frac{4}{5}b, \tag{71}$$

so that only the negative root remains and

$$V_+(c) = \int_{K_+}^{b} (\varepsilon - K_+) K_+ \mu d\varepsilon - cK_+ = \frac{1}{2}K_+ \mu (b - K_+)^2 - cK, \tag{72}$$

$$= \frac{1}{8}\left(3b - \sqrt{b^2 + 16c/\mu}\right) \mu \left(b - \frac{1}{4}\left(3b - \sqrt{b^2 + 16c/\mu}\right)\right)^2 - c\frac{1}{4}\left(3b - \sqrt{b^2 + 16c/\mu}\right), \tag{73}$$

$$= \frac{1}{128}\left(3b - \sqrt{b^2 + 16c/\mu}\right)\left[\mu \left(b + \sqrt{b^2 + 16c/\mu}\right)^2 - 32c\right], \tag{74}$$

with conditions:

$$a < \frac{1}{4}\left(3b - \sqrt{b^2 + 16c/\mu}\right) < \frac{4}{5}b, \tag{75}$$

$$4a - 3b < -\sqrt{b^2 + 16c/\mu} < \frac{1}{5}b. \tag{76}$$

Thus, we must have $b > \frac{4}{3}a$ and

$$b^2 + 16c/\mu < (3b - 4a)^2 \Leftrightarrow c < \frac{1}{2}(b - 2a). \tag{77}$$

This requires $b > 2a$ (medium & high variability). ■

2.12.3 Entry-deterring capacity investment

Let us now investigate the deterrent equilibria $K = (k, 0)$ and $K = (0, k)$ where

$$\int_k^\infty (\varepsilon - k) f(\varepsilon) d\varepsilon = c. \tag{78}$$

Case 1: $k < a$: Then $(\varepsilon_0 - k) = c$, or

$$0 < k = \varepsilon_0 - c < a \Leftrightarrow \frac{b-a}{2} < c < \frac{b+a}{2}. \tag{79}$$

Case 2: $k \geq a$: Clearly, $k < b$ so that

$$\int_k^b (\varepsilon - k)\mu d\varepsilon = c \text{ and } k \geq a, \tag{80}$$

$$\Leftrightarrow k = b - \sqrt{2c/\mu} \geq a, \tag{81}$$

$$\Leftrightarrow k = b - \sqrt{2c/\mu} \text{ and } c < \frac{b-a}{2}. \tag{82}$$

To calculate the corresponding firm value, we must calculate the capacity-constrained monopoly quantity which is most easily done by differentiating between levels of variability:

1. Low & Medium variability ($a < b < 3a$):

$$q_M(K) = \min\left(\frac{\varepsilon_0}{2}, K\right), \tag{83}$$

so that

$$q_M(k) = \begin{cases} \frac{\varepsilon_0}{2} & \text{if } 0 < c < \frac{b+a}{4}, \\ k & \text{if } \frac{b+a}{4} < c < \frac{b+a}{2}, \end{cases} \tag{84}$$

with corresponding firm value

$$V_{\text{deterring}} = \begin{cases} \text{E}(\varepsilon - \frac{\varepsilon_0}{2})\frac{\varepsilon_0}{2} - ck = \frac{\varepsilon_0^2}{4} - c\left(b - \sqrt{2\frac{c}{\mu}}\right) & \text{if } 0 < c < \frac{b-a}{2}, \\ \text{E}(\varepsilon - \frac{\varepsilon_0}{2})\frac{\varepsilon_0}{2} - ck = \frac{\varepsilon_0^2}{4} - c(\varepsilon_0 - c) & \text{if } \frac{b-a}{2} < c < \frac{b+a}{4}, \\ \text{E}(\varepsilon - k)k - ck = (\varepsilon_0 - k - c)k = 0 & \text{if } \frac{b+a}{4} < c < \frac{b+a}{2}. \end{cases} \tag{85}$$

2. High variability ($3a < b$):

$$q_M(K) = \min\left(\frac{b}{3}, K\right), \tag{86}$$

and $k = b - \sqrt{2c/\mu} > \frac{b}{3} \Leftrightarrow \frac{2b^2}{9}\mu > c$ (notice that $\frac{b-a}{2} > \frac{2b^2}{9}\mu$ with high variability), so that

$$q_M(k) = \begin{cases} \frac{b}{3} & \text{if } 0 < c < \frac{2b^2}{9}\mu, \\ k & \text{if } \frac{2b^2}{9}\mu < c < \frac{b+a}{2}, \end{cases} \tag{87}$$

with corresponding firm value

$$V_{\text{deterring}} = \begin{cases} \int_{\frac{b}{3}}^b (\varepsilon - \frac{b}{3})\frac{b}{3}\mu d\varepsilon - ck = \frac{2}{27}\mu b^3 - c\left(b - \sqrt{2\frac{c}{\mu}}\right) & \text{if } 0 < c < \frac{2b^2}{9}\mu, \\ \int_k^b (\varepsilon - k)k\mu d\varepsilon - ck = \left(b - \sqrt{2c/\mu}\right)\left[\frac{1}{2}\mu\left(b - \left(b - \sqrt{2c/\mu}\right)\right)^2 - c\right] = 0 & \text{if } \frac{2b^2}{9}\mu < c < \frac{b-a}{2}, \\ \text{E}(\varepsilon - k)k - ck = (\varepsilon_0 - k - c)k = 0 & \text{if } \frac{b-a}{2} < c < \frac{b+a}{2}. \end{cases}$$

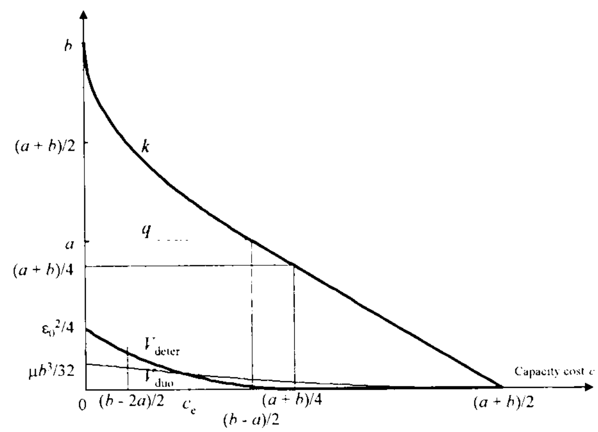

Figure 9: The entry-deterring capacity investment k, output q and firm value$V$ compared with the duopoly firm value.

Thus, the solutions are shown for medium variability in Figure 9. Finally, for the entry-deterring strategy to be an equilibrium and be credible, it's firm value must not be less than the firm value that would obtain if the duopolist invests in the "true" duopoly interior equilibrium investment.

Clearly, for very low variability

$$V_{\text{deter}} > V_i^{\text{duo}} \quad \Leftrightarrow \quad \frac{\varepsilon_0^2}{4} - c(\varepsilon_0 - c) > \frac{1}{9}(c - \varepsilon_0)^2, \tag{88}$$

$$\Leftrightarrow \quad (\varepsilon_0 - 4c)\left(\varepsilon_0 - \frac{8}{5}c\right) > 0, \tag{89}$$

$$\Leftrightarrow \quad c < c_e = \frac{1}{4}\varepsilon_0, \tag{90}$$

which requires $\frac{b-a}{2} < \frac{1}{4}\varepsilon_0 \Leftrightarrow b < \frac{5}{3}a$. For low & medium variability ($\frac{5}{3}a < b < 2a$), $V_{\text{deter}} > V_i^{\text{duo}} \Leftrightarrow c < \frac{1}{4}\varepsilon_0$ and

$$\begin{cases} \frac{\varepsilon_0^2}{4} - c\left(b - \sqrt{2\frac{c}{\mu}}\right) - \frac{1}{9}(c - \varepsilon_0)^2 > 0 & \text{if} \quad c > \frac{b-2a}{2}, \\ \frac{\varepsilon_0^2}{4} - c\left(b - \sqrt{2\frac{c}{\mu}}\right) - \frac{1}{256}\left(3b - \sqrt{b^2 + 16c/\mu}\right)\left[\mu\left(b + \sqrt{b^2 + 16c/\mu}\right)^2 - 32c\right] > 0 & \text{if} \quad c < \frac{b-2a}{2}. \end{cases}$$

Finally, for high variability we know that $c_e < \frac{1}{4}\frac{a+b}{2} < \frac{b-2a}{2}$ so that $V_{\text{deter}} > 0 \Leftrightarrow c < \frac{1}{2}(b - 2a)$ and

$$\frac{2}{27}\mu b^3 - c\left(b - \sqrt{2\frac{c}{\mu}}\right) - \frac{1}{256}\left(3b - \sqrt{b^2 + 16c/\mu}\right)\left[\mu\left(b + \sqrt{b^2 + 16c/\mu}\right)^2 - 32c\right] > 0.$$

■