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THE GENERAL CONCEPT OF MULTIDIMENSIONAL CONSISTENCY: SOME ALGEBRAIC ASPECTS OF THE AGGREGATION PROBLEM (*) (**)

by

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I. INTRODUCTION

1.1. Looking back at the historical development of some widely accepted models of economic theory often reveals some striking similarities. In Koopman's words: "With the help of more fundamental mathematical tools, the common logical structure of received economic theories of quite diverse origin can be brought out." [13]. One such similarity has largely gone unnoticed so far: the behavior of most economic agents is assumed to be understandable through the simple maximization hypothesis of a single objective function. The traditional model of consumer choice describes the consuming agent solely in terms of a complete preordering relation over some commodity space where he is to choose an optimal attainable commodity bundle. Similarly the producer's fundamental incentive is profit maximization, within the framework of the current technological constraints. In both cases, the rationality of these agents is, so to speak, one dimensional in nature. Indeed, it would be a mistake to ignore the fruitfulness of this assumption. Simple as it may be it still captures some essential features of the agents it purports to describe. Moreover it often serves as a good proxy for some other motivations which we may want to assign to those agents. A producer, for instance, may want to think in terms of market share besides profit; similarly a consumer's utility function may be made multi-dimensional as one way of introducing various consumption periods over time. And finally, another explanation for this approach may be found in the simple fact that the mathematical theory of multiple objective optimization is by no means as advanced as it is for a single objective function.

In this paper we shall consider an important case of the multiple objective optimization problem which was originally considered in the context of social choice theory. The model, however, is by no means restricted to this context of interpretation and, for the sake of generality, we shall briefly sketch the general aggregation problem to set the stage for our discussion.

1.2. Let us consider an economic agent having to decide on a "best" course of action to be chosen from a finite[1] set of feasible alternatives. Let

(1) $A = \{a_1, a_2, \ldots, a_i, \ldots, a_m\}$ denote this alternative set. The evaluation of the respective worths of the various alternatives (a_i) is carried out via a set of ℓ individual representation mappings:

(2) $\varphi_h : A \to S_h$ where S_h denotes the h^{th} criterion space. The ℓ criteria S_h $(h=1,a,\ldots,\ell)$ could be various agents or various viewpoints taken by a single agent. Aggregating these various objectives involves the following problem: is there a "best" way of combining the ℓ-dimensional images of the m objects (a_i) into a set of (m) one-dimensional images. Formally, we want to find some aggregation mapping σ that maps the image set of A, $\Phi(A)$ (where $\Phi(A) \subset \prod_{h=1}^{\ell} S_h$) into a one-dimensional aggregate image set 0:

(3) $\sigma : \Phi(A) \to 0$. Clearly, unless we endow the S_h's spaces and the 0 space with a certain structure - algebraic and/or

topological as the case may be - no unique solution can be offered for this problem. To choose between various solutions means that we have to agree on (1) some specified structure for the various spaces and mappings involved, and (2) some goodness of fit criterion. There exist as many "solutions" to the aggregation problem as there are answers to these two questions.

1.3. In the sequel we shall only consider a version of the problem with which social choice theorists have mostly concerned themselves: the individual representation spaces S_h are all identical viz.

(4) $S_h = \{1,2,\dots,i,\dots,m\}$ the finite set of the first m integers; the individual representation mappings φ_h are the permutation operators, i.e.

(5) $\varphi_h \in \mathscr{S}_m$ where $\mathscr{S}_m$ denotes the group of permutation operators of order m; and finally the aggregate representation mapping σ is also taken from the group $\mathscr{S}_m$. In other words, the various viewpoints (criteria, voters, etc...) define ℓ total strict orderings of the m alternatives a_i. For reasons that will become clear in the subsequent discussion, these linear orders will be called preference patterns and denoted by L_h. In this paper we wish to study some fundamental algebraic properties of this version of the aggregation problem (Section II) and then show how they provide various solutions to this problem (Section III).

II. SOME ALGEBRAIC PROPERTIES OF THE SET OF INDIVIDUAL PREFERENCE PATTERNS.

2.1. A natural order relation on the set of individual preference patterns.

As we know the ℓ individual preference patterns L_h are generated by ℓ permutation mappings $\varphi_h : A \to A$. By definition these mappings are bijective, and, algebraically they form a group $\mathcal{S}_m$.

A special type of permutation known as a transposition $t \in \mathcal{S}_m$ is defined as any permutation which interchanges only two elements in A and leaves the other (m-2) elements unchanged, i.e.:

$$(6) \begin{cases} k \neq i,\ k \neq j \Rightarrow t(k) = k \\ t(j) = i\ ;\ t(i) = j \quad \forall\ i \neq j \quad i, j, k = 1,2,\dots,m. \end{cases}$$

Without any loss of generality we can adopt some arbitrary labelling of the alternatives a_i. The natural order $L_0 = \{1,2,\dots,m\}$ of the first m intergers provides such a "reference order" (pattern). An inversion is defined as a transposition such that:

$$(7)\ i < j \iff t(i) > t(j).$$

In any preference pattern the number of inversions is unique and can be counted directly.

These notions enable us to define a natural order relation (R) over the family $\mathcal{L}$ of preference patterns L_h as follows.

Definition 1: The agreement set $V(L_h)$ associated with any preference pattern $L_h \in \mathcal{L}$ is defined as that subset of $A \times A$ such that

$$(8)\ (a_i\ a_j) \in V(L_h) \iff a_i\ L_h\ a_j \text{ whenever } a_i\ \ L_0\ a_j.$$

In other words for any individual preference pattern L_h the agreement set $V(L_h)$ singles out in the set of $\frac{m(m-1)}{2}$ ordered pairs (a_i, a_j), the subset of those whose order in L_h agrees with the reference order L_0. Those are the non-inverted pairs.

We can now define a binary relation (R) on $\mathcal{L}$ as follows:

(9) $\forall L_h, L_k \in \mathcal{L} : L_h \, R \, L_k \iff V(L_h) \supseteq V(L_k)$ (R can be read ... "precedes"...).

The proof of the following theorem is immediate.

Theorem 1: The relation R defined by (9) above is a partial order relation, i.e.

$$R \text{ is } \begin{cases} \text{(i) } \underline{\text{Reflexive}}: \forall L_h \in \mathcal{L} : L_h \, R \, L_h \\ \text{(ii) } \underline{\text{Antisymmetric}}: \forall L_h, L_k \in \mathcal{L} : L_h \, R \, L_k \text{ and } L_k \, R \, L_h \\ \qquad \Rightarrow L_h = L_k \\ \text{(iii) } \underline{\text{Transitive}}: \forall L_h, L_k, L_g \in \mathcal{L} : L_h \, R \, L_k \text{ and } L_k \, R \, L_g \\ \qquad \Rightarrow L_h \, R \, L_g. \end{cases}$$

Proof: This follows directly from the fact that $\supseteq$ defines a partial order relation on the power set of A, $\mathcal{P}(A)$. Q.E.D.

Thus, in graph-theoretic terms we can represent the family $\mathcal{L}$ of preference patterns as a finite graph $G(\mathcal{L}, \Gamma)$ where $\mathcal{L}$ denotes the set of nodes (the strict total orders L_h which we agreed to call the preference patterns) and Γ is a multi-valued mapping from $\mathcal{L}$ into itself defined by:

$$(10) \begin{cases} \forall L_h, L_{k_1}, L_{k_2}, \ldots, L_{k_s} \\ \Gamma(L_h) = \{L_{k_1}, L_{k_2}, \ldots, L_{k_s}\} \iff \\ L_h \, R \, L_{k_j} \quad \forall j \in \{1, 2, \ldots, j, \ldots, s\}. \end{cases}$$

An immediate corollary to Theorem 1 can now be stated as follows:

Corollary 1: The graph $G(\mathcal{S}_m, \Gamma)$ defined above is finite, reflexive, transitive and antisymmetric.

Proof: This follows at once from Theorem 1 and equation (10) above. Q.E.D.

As an illustration if $\mathcal{L} \equiv \mathcal{S}_3$, i.e., $\mathcal{L}$ describes the set of all permutation of a three element set $L_0 = \{a,b,c\}$ and if we denote:

$$(11) \begin{cases} L_0 = (a\ b\ c) & ; \quad L_4 = (c\ b\ a) \\ L_2 = (b\ a\ c) & ; \quad L_5 = (c\ a\ b) \\ L_3 = (b\ c\ a) & ; \quad L_6 = (a\ c\ b) \end{cases}$$

we have:

$$(12) \begin{cases} \Gamma(L_0) = \{L_2,\ldots,L_6\} \\ \Gamma(L_2) = \{L_3,L_4\} \\ \Gamma(L_3) = \{L_4\} \\ \Gamma(L_6) = \{L_4,L_5\} \\ \Gamma(L_5) = \{L_4\} \end{cases} \qquad \text{(as shown on Figure 1 below)}$$

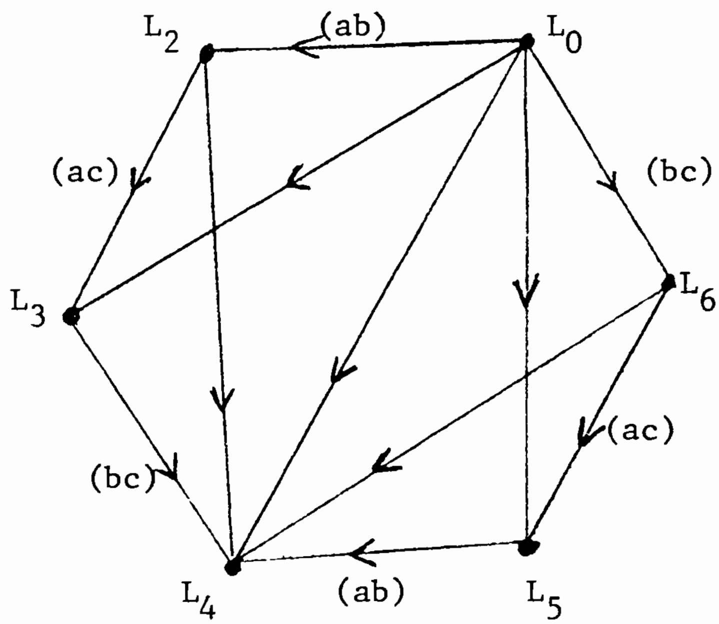

Figure 1: $G(\mathscr{S}_3, \Gamma)$

(Note: For clarity we have omitted the loops at each node)

From a geometric standpoint, as we can see from this figure, the set $\mathscr{S}_3$ can be viewed as a regular hexagon. More generally it can be stated:

Theorem 2:[2] The set $\mathscr{S}_m$ forms a convex polyhedron.

Proof: This fact follows, at once, from the well-known Birkhoff-Von Neumann theorem which states that the set of bistochastic matrices[3] of order m is a convex polyhedron embedded in an m^2-dimensional space, and whose profile is identical with the permutation matrices $\varphi_h \in \mathscr{S}_m$. Q.E.D.

2.2. The general concept of multidimensional consistency

2.2.1. If we now consider the graph $G(\mathscr{S}_3, \Gamma)$ as depicted on Figure 1 above, we notice that, even though the order relation R on $\mathcal{S}$ is only partial, some proper subsets of $\mathcal{S}$ are totally ordered. Such proper subsets are normally referred to as "chains."

Definition 2: A proper subset $\mathcal{L} \subset \mathcal{S}_m$ forms a maximal chain if and only if (1) the patterns $L_h \in \mathcal{L}$ can be linearly ordered e.g., $L_1 \; R \; L_2 \; \ldots \; R \; L_h \; \ldots \; R \; L_t$ and (2) $t = \frac{m(m-1)}{2}$.

Put another way a maximal chain (1) forms a Hamiltonian path on the partial subgraph generated by $\mathcal{L}$ and (2) the length of this path is equal to $t = C_m^2$. We will now show how these concepts naturally lead to the notion of multidimensional consistency.

Ever since Arrow's theorem was presented a number of authors, starting with Arrow himself, and Black, mentioned the possibility of reaching an aggregate transtive order from individual orders whenever these individual preference patterns present a certain "similarity." Black's single-peaked preference patterns are the most well-known example of such a restricted set of individual orders that allow transitive preference aggregation through majority voting. But this is a very restrictive and special case of a much more general phenomenon which we shall call "multidimensional consistency." The following illustration will help motivate the introduction of this concept.

2.2.2. Illustration: Consider a family $\mathcal{L}$ of 5 preference patterns (i.e., total strict orders) over a four-element alternative set $A = \{a,b,c,d\}$. Let $\mathcal{L} = \{L_0, L_1, L_2, L_3, L_4\}$, e.g.

$$(13)\begin{cases} L_0 = (a\ b\ c\ d) & ; \quad L_3 = (b\ d\ a\ c) \\ L_1 = (b\ a\ c\ d) & ; \quad L_4 = (b\ d\ c\ a) \\ L_2 = (b\ a\ d\ c) & \end{cases}$$

It is readily seen that this family $\mathcal{L}$ displays a linear order for the ordering relation R defined previously. Or, equivalently, in terms of the associated partial subgraph of $G(\mathcal{L}_4, \Gamma)$ this subgraph is linear since we have

(14) $L_0 \; R \; L_1 \; R \; L_2 \; R \; L_3 \; R \; L_4$

i.e. <=>

$$(15)\begin{cases} \Gamma(L_0) = \{L_1, \ldots, L_4\} \\ \Gamma(L_1) = \{L_2, \ldots, L_4\} \\ \vdots \qquad\qquad \vdots \\ \Gamma(L_4) = \{\emptyset\} \end{cases}$$

In this case we say that the family of preference patterns $\mathcal{L} = \{L_h | h=1,\ldots,4\}$ displays the property of multidimensional consistency.

Definition 3: A family $\mathcal{L} = \{L_h | h \in H\}$ - where H denotes some index set - is said to possess the property of multidimensional consistency if and only if there exists some permutation mapping f on H such that

(16) $f(h) < f(h')$ whenever $L_h \; R \quad L_{h'}$, $\forall \; h, h' \in H$

In other words the family $\mathcal{L}$ is said to be multidimensionally consistent whenever we can relabel the individual patterns L_h that they form a chain for the order relation R defined previously (see equation 9). When this chain is maximal we say that the family $\mathcal{L}$ in question displays maximal multidimensional consistency.

2.2.3. Example: Coombs orderings: the notion of multidimensional consistency we have just defined derives its name from a very special case first pointed out by Coombs [9]. In attitude measurement theory Coombs noticed that often enough when a group of individuals rank a set of stimuli many of the potential orderings are conspicuously absent. It seems as if there exists some underlying structure - linear or otherwise - on the set of stimuli A (alternatives), and this structure is implicitly respected by the various individuals (view-points, criteria, etc.). This experimental evidence confirms what seems quite logical at first viz. the fact that in any problem of multiple criteria decision-making the alternative set most often displays some generic characteristic, some homogeneity which is not explicitly recorded in the mere labeling of the alternatives $\{a_1, \ldots, a_m\}$. In short, some information loss often occurs when we start indexing the alternatives without considering their very nature. For instance, in Coombs experiment let us suppose that the family $\mathcal{L}$ consist of only seven orderings, viz.

$$(17)\begin{cases} L_0 = (a\ b\ c\ d) \quad ; \quad L_4 = (c\ b\ d\ a) \\ L_1 = (b\ a\ c\ d) \quad ; \quad L_5 = (c\ d\ b\ a) \\ L_2 = (b\ c\ a\ d) \quad ; \quad L_6 = (d\ c\ b\ a) \\ L_3 = (c\ b\ a\ d) \end{cases}$$

A possible rationalization of this clustering phenomenon is that the alternative set displays "an underlying unidimensional continuum" sometimes known as a joint quantitative scale. It is easily shown that if the underlying order is L_0 (the reference order) and the distance from a to b is less than the distance from c to d, varying the location of the most preferred point along this linear scale generate this family of "Coombs patterns." This family is but one very special example of the general property of maximal multidimensional consistency. Whenever a finite family $\mathcal{L}$ of preference patterns $L_h \in \mathcal{L}$ $(h=1,2,\ldots,\ell)$ possesses the property of multidimensional consistency it is clear that the solution of the aggregation problem is greatly simplified as we shall show in Section 3 below. It is also clear that the class of all families of total orders which are multidimensionally consistent, is very large.

In graph-theoretic terms, the notion of a Hamiltonian path affords a very simple characterization theorem for the members of this class.

Theorem 3: Let $\mathcal{C}$ denote the class of all families of preference patterns that possess the property of multidimensional consistency. An arbitrary family $\mathcal{L}_\alpha$ of preference patterns belongs to $\mathcal{C}$ if and only if we can find a hamiltonian path μ on $G(\mathcal{J}_m,\Gamma)$ or G' some partial subgraph of G such that $\mu = (L_1,L_2,\ldots,L_j,\ldots,L_n)$ with $L_j \in \mathcal{L}_\alpha \ \forall\ j=1,\ldots,n$.

Proof: By definition a hamiltonian path μ in a graph G goes through each node L_j once and only once.

(i) Let $\mathcal{L}_\alpha$ possess the property of multidimensional consistency. Then $\exists$ f a permutation operator on $J = \{1,2,\ldots,j,\ldots,n\} \ni L_{f(1)}\ R\ L_{f(2)}\ R \ldots R\ L_{f(j)}\ R \ldots R\ L_{f(n)}$ which is a hamiltonian path of length n on the partial subgraph $G'(\mathcal{J}_{\mathcal{L}_\alpha},\Gamma_{\mathcal{L}_\alpha})$.

(ii) Let $\mu = (L_1, L_2, \ldots, L_j, \ldots, L_n)$ be a hamiltonian path on some graph $G(\mathcal{S}_{\mathcal{L}_\alpha}, \Gamma_{\mathcal{L}_\alpha})$. Then from the definition of the mapping $\Gamma_{\mathcal{L}_\alpha}$ we can use the identity permutation i on the index set $J = \{1,2,\ldots,j,\ldots,n)$ to obtain

$$(L_1 \; R \; L_2, \ldots, R \; L_j, \ldots, R \; L_n)$$

as required. Q.E.D.

The problem of devising an efficient enumerative algorithm for all such multidimensionally consistent families $\mathcal{L}_\alpha \in \mathcal{C}$ will be discussed elsewhere. Here, it suffices to note that for a three element alternative set this class $\mathcal{C}$ has 21 members and, more generally, this number is an increasing function of (m!). Thus from the mere size of this class is seems most likely that the property of multidimensional consistency will be verified for a large number of families of total orderings.

Moreoever it can also be shown that any family $\mathcal{L}$ of total orders on a finite set A, can always be represented as the <u>finite union</u> of multidimensionally consistent subfamilies of total orderings.

<u>Theorem 4</u>: Let $\mathcal{L} = \{L_h | h \in H\}$ be a finite family of total orders on a finite set A. Then there always exists a class $\mathcal{J}$ of sub-families $\mathcal{J} = \{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_j, \ldots, \mathcal{L}_n\}$ such that

(i) $\forall\; j=1,2,\ldots,n \qquad \mathcal{L}_j$ is multidimensionally consistent

(ii) $\bigcup_{j=1}^{n} \mathcal{L}_j = \mathcal{L}$.

Proof: Several cases are to be considered.

(i) If $\mathcal{L}$ is itself multidimensionally consistent, the theorem holds.

(ii) If $\mathcal{L}$ is not multidimensionally consistent take an arbitrary $L_h \in \mathcal{L}$ and search for $L_{h'} \in \mathcal{L} - \{L_h\}$ such that $L_h \; R \; L_{h'}$. (Note: to simplify this search procedure it is advisable to determine the size of the agreement sets $V(L_h)$ for all patterns L_h and start with the patterns for which $|V(L_h)|$ is maximum.) Since R is a partial order on $\mathcal{L}_m \supset \mathcal{L}$ two cases are possible:

Case 1: $\nexists \; L_{h'} \in \mathcal{L} \ni L_h \; R \; L_{h'}$

In this case set $L_h = \mathcal{L}_1$ and repeat the above procedure on the reduced set $\mathcal{L} - \{\mathcal{L}_1\}$.

Case 2: $\exists \; L_{h'}, L_{h''} \ldots \in \mathcal{L} \ni L_h \; R \; L_{h'} \; R \; L_{h''}$

Then set $\mathcal{L}_1 = \{L_h, L_{h'}, L_{h''}, \ldots,\}$ and repeat the above procedure by choosing some $L_k \in \mathcal{L} - \{\mathcal{L}_1\}$ and comparing L_k to $\mathcal{L}$.

After a finite number of steps this algorithm will terminate having exhausted the finite family $\mathcal{L}$. And also, it will always be the case that this finite union of subfamilies $\mathcal{L}_1 \; \mathcal{L}_2, \ldots, \mathcal{L}_n$ will necessarily cover $\mathcal{L}$ as required by (2).

(Note: This finite covering of $\mathcal{L}$ does not need to be a partition since we may, for instance, have two chains with the reference order as their common intersection.) Q.E.D.

From Theorem 4 above it follows that the concept of multidimensionally consistent preference patterns is always applicable either directly or via this decomposition mechanism. In the limit, however, we could conceivably encounter a family $\mathcal{L}$ which, upon decomposition, yields a union of one-element subfamilies of the form

(18) $\mathcal{L}_h = \{L_h\} \quad \forall\, h \in H$

In this extreme case when the consistency check fails for all patterns in $\mathcal{L}$, it is only natural to ask: what alternative course of action should we follow?

Several answers to this question are possible: (i) on the one hand we can try to see whether or not the set of all total orderings $\mathcal{S}_m$ partially ordered by R has any other algebraic structure, this route will not be followed here but it suffices to note that we have been able to show that this set forms a symmetric (finite) lattice ([7]); this result and the various aggregation methods that it leads to will be discussed elsewhere. (ii) On the other hand we may wish to strengthen the partial order relation R on $G(\mathcal{S}_m, \Gamma)$ by finding one or several total orders T that are compatible with R. This route will now be very briefly outlined.[4]

2.3. Linear extension and dimension of the partial order R.

2.3.1. Definitions

Definition 4: A total order T is said to be a linear extension of the partial ordering R if and only if

$$\forall\, L_h, L_{h'} \in \mathcal{S}_m \quad L_h \, R \, L_{h'} \Rightarrow L_h \, T \, L_{h'}$$

The following lemma can readily be established.

Lemma 1: Let $\mathcal{T} = \{T_j \mid j=1,\ldots,n\}$ be a collection of total orders on any set $\mathcal{S}$. Let $R = \bigcap_{j=1}^{n} T_j$. Then R is a partial order.

Proof: Trivial.

Definition 5: A collection $\mathcal{T} = \{T_j | j=1,\ldots,n\}$ of total orders is said to realize (span) a partial order R if and only if

$$\bigcap_{j=1}^{n} T_j = R$$

Definition 6: The dimension d(R) of a partial order R is the smallest cardinal number of total orders T_j that realize R.

Definition 7: A basis $\mathcal{T}$ of R is any collection of total orders such that

(i) $\bigcap_{j=1}^{n} T_j = R$ (i.e. $\mathcal{T}$ is a spanning set)

(ii) $|\mathcal{T}| = d(R)$ (and $\mathcal{T}$ is minimal)

We can now proceed to determine the class $\mathcal{T}$ of total orders that are compatible with the partial order R defined on $\mathcal{S}_m$. First of all we should prove the existence of such a non-empty class $\mathcal{T}$.

Lemma 2: The class $\mathcal{T}$ compatible with R is non-empty.

Proof: This corollary follows directly from Szpilrajn theorem.[5] Q.E.D.

2.3.2. An example

The problems of (1) enumerating all the total orders $T_j \in \mathcal{T}$ and (2) finding one (or several) basis for R will not be discussed here in all its generality. We shall restrict our discussion to the simple case where A is a three-element set. Then R is defined on $\mathcal{S}_3$ as shown on Figure 1 above. It can be proven that the class $\mathcal{T}$ has six elements viz:

$$(19)\begin{cases} T_1 = L_0 > L_6 > L_5 > L_2 > L_3 > L_4 \\ T_2 = L_0 > L_6 > L_2 > L_5 > L_4 > L_4 \\ T_3 = L_0 > L_2 > L_6 > L_5 > L_3 > L_4 \\ T_4 = L_0 > L_2 > L_6 > L_3 > L_5 > L_4 \\ T_5 = L_0 > L_2 > L_3 > L_6 > L_5 > L_4 \\ T_6 = L_0 > L_6 > L_2 > L_3 > L_5 > L_4 \end{cases}$$

One can easily verify that the subclass $\{T_1\ T_5\}$ forms a basis for R. Thus in this case R is of dimension 2.[6] Thus, we can see that even if the family $\mathcal{L}$ is not multidimensionally consistent we can always extend R to find a linear ordering on $\mathcal{S}_m$ and hence on $\mathcal{L} \subset \mathcal{S}_m$, as in the case of multidimensional consistency. The following section will now use these results to provide some reasonable aggregation mappings on $\mathcal{L}$.

III. SOME AGGREGATION PROCEDURES

3.1. Case #1: The family $\mathcal{L}$ is multidimensionally consistent

As we said earlier, the advantage afforded by the property of multidimensional consistency is that it provides a natural total order on the preference patterns in $\mathcal{L}$. The simplest way to aggregate such patterns is then to expolit this complete ordering.

The median pattern can be defined as that pattern which leaves 50% of the elements in $\mathcal{L}$ behind; and, as we know, it minimizes the sum of the distances between itself and all the patterns in $\mathcal{L}$, when the distance is measured by the absolute value. This median solution possesses an interesting property viz. it is also the majority voting solution in the social choice context. This result can be established quite simply.

Lemma 3: Let $\mathcal{L}$ be a finite family of multidimensionally consistent patterns. Let L_{med} be the median pattern in $\mathcal{L}$. Then L_{med} is also the simple majority voting solution when all pairs of alternatives in $A \times A$ are voted upon sequentially.

Proof: Let $V(\overline{L}_{med})$ denote the agreement set of all patterns L_h such that $L_h \; R \; L_{med} \;\forall\; L_h \in \mathcal{L}$.

By definition of L_{med} we have:

$$(a_i, a_j) \in V(\overline{L}_{med}) \iff |\overline{L}_{med}| \geq \begin{cases} \frac{\ell}{2} & \text{for } \ell \text{ even} \\ \frac{\ell+1}{2} & \text{for } \ell \text{ odd} \end{cases}$$

i.e. for all such pairs a_i has a simple majority over a_j.

Symmetrically for all other pairs in $A \times A$, a_j has a simple majority over a_i. Q.E.D.

Goodman pointed out a similar result [11].

Clearly, other solutions could be obtained for a "best" aggregate pattern if we were to use other distance functions over $\mathcal{L}$. On the other hand if a given family $\mathcal{L}$ does not meet the multidimensional consistency criterion, we must devise alternative aggregation procedures as explained below.

3.2. Case #2: In cases where we can no longer rely on the existence of a single total ordering over $\mathcal{L}$, the following methods can be used.

3.2.1. Using the totals orderings in the basis of R provides one feasible option. However, several preliminary questions must first be settled: is the basis unique and, even if it is unique, how do we go about choosing a single total ordering in the basis? On purely bayesian grounds one may adopt a completely randomized strategy to pick such a total order T*. If this is done we are then confronted with a situation analogous to that of multidimensional consistency since the patterns in $\mathcal{L}$ can be uniquely located along the linear scale T*, - a linear extension of R. In effect this amounts to introducing some kind of shadow consistency on $\mathcal{L}$ - such consistency having been reconstructed ab initio on strictly logical grounds. If we follow this route we are thus led back to the previous setting where $\mathcal{L}$ was linearly ordered; all the solutions to this case now become applicable (e.g. the median or the mean solution, etc....). If it appears that a randomized choice of some basic linear extension T* of R is unwarranted in a given situation - for instance when a sensitivity analysis has shown that the final solution was highly dependent upon the initial choice of some basic T* - we can still aggregate the patterns in $\mathcal{L}$.

3.2.2. Another possible solution would consist in directly defining a reasonable metric on the set $\mathcal{J}_m$ of all total orders of the set A. One such metric will now be discussed.

A natural metric $d(L_h,L_k)$ on $\mathcal{J}_m$ can be defined as the <u>minimal number</u> of transpositions necessary to transform L_h into L_k.

<u>Definition 4</u>: The integer valued function $d(L_h,L_k)$ is defined as:

(20) $d(L_h,L_k) = |T|$

where $|T|$ denotes the cardinal number of the set of operators:

(21) $T = \{t_1,t_2,\ldots,t_g\}$ (all t's are transposition mappings)

and

(22) $t_g\{t_{g-1}[\ldots t_1(L_h)]\} = L_k$

To show that this function $d(L_h,L_k)$ does define a metric on $\mathcal{J}_m$ we should first note that there exists an equivalent representation of d in terms of boolean variables. Specifically if we represent each preference pattern L_h by an $\frac{m(m-1)}{2}$ dimensional boolean vector (i.e. a boolean variable for each pair $[a_i,a_j]$), equation (20) becomes:

(20') $d(L_h,L_k) = |L_h-L_k|$

i.e. the absolute value of the difference between the two boolean vectors L_h and L_k. For instance, if

$L_0 = (a,b,c) = (1,1,1)$

and $L_2 = (b,a,c) = (0,1,1)$

$d(L_0,L_2) = 1$

which reflects the fact that the transposition (a b) → (b a) transforms L_0 into L_1.

Lemma 3: The function $d(L_k,L_h)$ defined by (20) is a metric on $\mathscr{S}_m$.

Proof: This follows from the fact that (20) and (20′) are equivalent and (20′) is the definition of the well known Hamming metric over boolean vectors. Q.E.D.

We can now use this metric to aggregate an arbitrary family $\mathscr{L}$ of patterns by choosing a "generalized median" pattern L_{med} such that the linear form

(23) $\sum_{L_h \in \mathscr{L}} d(L_h,L_{med})$ is minimized

This solution possesses a number of interesting properties: first, it will always lead to a transitive aggregate pattern in the set $\mathscr{S}_m$; second, it can be given another theoretically attractive interpretation viz. the generalized median can be viewed as the center of gravity of the cluster $\mathscr{L}$ in $\mathscr{S}_m$[7]; and last, if $\mathscr{L}$ happens to be multidimensionally consistent it leads to the same result as the median solution (see 3.1.).

3.3. Concluding comments

The foregoing analysis has suggested that there often exists a certain underlying cohesion among multiple criteria rankings of a set of alternatives. Such cohesive aspects can best be expressed algebraically as we have just illustrated. A general discussion of some related algebraic notions would extend far beyond the confines of this paper (see [7]). At any rate, it is now clear that this way of approaching the aggregation problem provides valuable insights into various aggregation methods.

Footnotes

1. The assumption of finiteness is not as restrictive as it may seem since we can interpret the alternative set as a finite set of neighborhoods over a compact continuum.

2. This result has been obtained independently and in various contexts by several authors ([12],[14],[7]).

3. Bistochastic matrices arise from the definition of a probability measure on the set of permutation matrices in $\mathcal{S}_m$.

4. The problem we are about to discuss is actually quite involved and a comprehensive discussion will be provided separately. (see [8-a]).

5. Theorem (Szpilrajn, [15]): Every partial order R possesses a linear extension T. Moreover, if L_1 and L_2 are any two non-comparable elements of R, there exists an extension T_1 with $L_1 < L_2$ and an extension T_2 with $L_2 < L_1$.

6. For a general study of the dimension of R in $\mathcal{S}_m$ (for $m \neq 3$) and its interpretation, see [8-a].

7. For a cluster-theoretic analysis of the aggregation problem see [8-b].

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