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Learning, Rare Events, and Recurrent Market Crashes in Frictionless Economies Without Intrinsic Uncertainty

by

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Abstract

In this paper I consider a dynamically complete market model without intrinsic uncertainty. The only uncertainty is modelled by sunspots. Agents' beliefs are heterogeneous, but eventually become homogeneous in the sense that agents' beliefs are identical in the limit. I show that if some states of nature occur rarely then arbitrarily large market crashes may occur infinitely often. This result contrasts with Cass and Shell's (83) results which show that when beliefs are homogeneous, in complete markets without intrinsic uncertainty, sunspots do not matter.

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1. Introduction

There has been considerable amount of interest concerning the economic implications of speculative trade generated by differences in beliefs. For example, Harris and Raviv (93) argue that speculative trading accounts for a major part of stock market operations and they show how differences in opinion may help explain empirical regularities concerning prices and volume of trade. This view is reinforced by Kandel and Pearson (95), who show that the empirical evidence of volume of trade and stock returns around public announcements is inconsistent with the hypothesis that agents interpret this information identically. Kurz (97) and Timmermann (93) show how agents' disagreements about future probabilities may help explain the predictability of excess returns and the excess volatility of stock prices. Morris (96), based on a model developed by Harrison and Kreps (78), shows how a heterogeneous beliefs may help explain the pricing of initial public offerings. Nyarko (91) obtains cycles in a model in which a monopolist sets prices according to incorrect beliefs.[1] However, it has been conjectured that speculative trade should eventually disappear if agents learn over time i.e., if agents' posterior beliefs converge to the true distribution. In this paper, I present an example in which speculative trade remains relevant, although there is no asymmetric information and agents' posterior beliefs eventually become identical (and correct).

A key feature of the example is that some events occur with vanishing probability. It is clearly difficult to ascertain the probabilities of rare events and, consequently, it is natural to expect that agents will disagree about these probabilities. For example, stock market crashes (and other relevant economic phenomena) do not happen often. It is likely that most agents believe that the probability of a stock market crash is small, but some agents believe that this probability is smaller than do others. Thinking a market crash more likely, pessimistic agents may be more inclined to buy bonds, while optimistic agents may be more inclined to buy stock. The main theoretical question is whether heterogeneity of beliefs allows for the existence, in equilibrium, of a rare phenomenon, such as a market crash, which otherwise would be impossible if all agents held identical beliefs. Moreover, can this phenomenon persist even if agents' beliefs eventually become identical?

Assume that markets are frictionless and dynamically complete, and that there

[1] See Morris (95) for an excellent overview of some of these issues.

is no intrinsic uncertainty. The only uncertainty is sunspots. Cass and Shell (83) have shown that if agents have homogeneous beliefs then sunspots do not matter.[2] In particular, extrinsic uncertainty can not generate a market crash regardless of whether or not some sunspots occur infrequently. These results are of great theoretical importance because they identify the environments in which extrinsic uncertainty matters.[3] However, the assumption that agents have rational expectations is extreme. A weaker and more natural requirement might be that agents have prior beliefs which are not exactly correct. Then, after observing enough data, agents' predictions would converge to the true distribution.[4] In this paper, it is shown that if the conditional probability of a sunspot vanishes over time then sunspots may continue matter.

Consider a standard dynamic asset pricing model in which there exists a long-lived tree, as in Lucas' (78) model, and a risk-free asset. There are two states of nature h and l. The probability of state l vanishes over time slowly enough so that l occurs infinitely often. The tree gives the same fruits in both states. Therefore, the states h and l may be interpreted as sunspots. The number of assets (two) is the same as the number of states (two) and markets are dynamically complete. Two long-lived agents maximize an expected discounted logarithmic utility function according to their beliefs and discount factor. Both agents believe that state l has vanishing probability. Hence, there is convergence to rational expectations in the standard sense that, in the limit, both agents' beliefs are correct. However, agents' beliefs are different because they believe that the probability of state l vanishes at different rates.

An $\varepsilon-$market crash occurs infinitely often if the gross return on a share of a tree is smaller than ε infinitely often. The central result in this paper is that, for every $\varepsilon > 0$, there exist discount factors for both agents such that, in equilibrium, an $\varepsilon-$market crash occurs infinitely often. Hence, sunspots matter.

The intuition behind the result is as follows: Assume that the propensity to

[2]A related result was obtained by Milgrom and Stokey (81), who show that under the common prior assumption and complete markets, no speculative trade is possible.

[3]There exists a considerable literature showing that if there are frictions in the economy then market crashes (interpreted as a discontinuity in equilibrium prices) may occur. These papers usually consider a repeated static economy with asymmetric information. A price discontinuity may arise because of differences in information revealed by prices. See, for example, Gennotte and Leland (90), Jacklin, Kleidon and Pfleiderer (92), Romer (93), Caplin and Leahy (94), and Madrigal and Scheinkman (97).

[4]See Woodford (90) for a learning model of sunspots

save of some agents is greater than that of others. For example, assume that some agents are more patient than others. The price of a share will be high if the wealth is concentrated in the hands of more patient agents because they value the tree the most. The price of a share will be low if the wealth is concentrated in the hands of less patient agents because they value the tree the least. If agents' beliefs are heterogeneous there will be speculative trade in the economy. Therefore, share prices will fluctuate as the wealth becomes more or less concentrated in the hands of agents with different discount factors. I assume that the more patient agent believes that the rare state, l, has lower probability than does the less patient agent. If there is a market crash associated with state l, then patient agents will be more inclined to buy stocks than impatient agents because they assign smaller probability to market crashes. Therefore, when state l occurs and the market crashes, the wealth becomes more concentrated in the hands of the impatient agent. This is consistent with a drop in share prices because impatient agents value the tree the least.

After the crash there may *not* necessarily be a quick share price rebound, although no bad information about the tree has been revealed by the crash nor has such information been generated by the realization of a state of nature (because those states are sunspots).[5] The patient agents would like to buy more shares then they did before the crash, but they may not precisely because a significant part of their wealth was lost during the crash. It may take time for share prices to return to the same level as before the crash and it may take time for the patient agents to recover the wealth lost during the crash. This eventually happens partially because patient agents exchange current consumption for future consumption, and partially because their speculative trade generates wealth when state h occurs and, consequently, the market return is greater than interest rates. Before patient agents accumulate a significant fraction of the total wealth the state l may happen, but the market will not necessarily crash (in the sense of there being a very small return). Perhaps only a small decline in prices will be observed. A Market crash will only occur if l happens and the patient agents hold a significant portion of the total wealth. Thus, a market crash is more likely to occur after a long period in which most of the time market returns are greater than interest rates.

In this paper, agents are long-lived and they have different discount factors and different beliefs. If the wealth eventually ends up in the hands of a single

[5]Moreover, in the examples in this paper, the conditional probabilities of states of nature do not depend upon past histories nor on current outcomes. It depends only on time.

agent then prices will eventually be as if there is a representative agent in the economy. Hence, market crashes will eventually cease. So, in the examples presented in this paper, agents' discount factors and beliefs are chosen such that the wealth does not become concentrated in a single agent's hands.[6] This restriction makes these examples somewhat special, but in a very precise way. Agents' characteristics must be such that the wealth bounces back and forth between them. This restriction must always be satisfied in models where there are a few long-lived agents and the core economic issue is related to permanent interactions between two or more agents. An alternative modeling choice would be to consider an overlapping generation model because natural death would, of course, stop the transactions between the agents. Then, agents' characteristics would not have to be precisely chosen in order for meaningful interactions among different agents to continue eternally. However, introducing natural death may not add a relevant insight to the ideas presented in this example. Moreover, this modelling choice would, perhaps, make the analysis less transparent because sunspots may matter in overlapping generations models under rational expectations. Hence, the contrast between the complete irrelevance of sunspots when agents' beliefs are homogeneous and the possibility of recurrent market crashes when there is convergence to rational expectations may not be present in an overlapping generations model. This contrast shows that, in the model presented in this paper, the market crashes are associated with speculative trade.

The general conclusion of this paper is that it is possible for speculative trade, generated by a difference in opinions regarding rare events, to persist even if agents' beliefs become eventually identical. This speculative trade may induce the wealth to bounce back and forth among agents with different propensities to save. These fluctuations may, in turn, be associated with market crashes which could not exist if agents' beliefs were identical.

This paper is organized as follows: The model is presented in section 2. Convergence to rational expectation is defined in section 3. In section 4, agents' beliefs and the true probabilities are defined. The central result is presented and demonstrated in section 5. Section 6 concludes the paper.

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[6] In the example presented in this paper, agents have identical preferences over risk and identical beliefs, in the limit, but different discount factors. A surprising feature of this example is that the wealth may bounce back and forth between the agents and does not necessarily become concentrated in the hands of the most patient agent. This would not be possible if the conditional probabilities of the sunspots were bounded away from zero.

2. Frictionless Markets Without Intrinsic Uncertainty

In this section, the basic framework is described. Time is discrete and continues forever. There are two long-lived agents, a long-lived tree, a risk-free asset in zero supply, a single consumption good c, and two states of nature given by the set $\Sigma = \{h, l\}$. Markets are dynamically complete. The tree gives d units of consumption in every state of nature. Hence, there is no intrinsic uncertainty. The only uncertainty is price uncertainty modeled by sunspots.

Agents are born with shares of the tree, as in Lucas' (78) model, and receive no other endowments. Agent i's initial share is k_1^i. Hence, $k_1^1 + k_1^2 = 1$.

Let $\sum^t$, $1 \leq t \leq \infty$, be the set of all t-histories. Let $\Im_1 \subset ...\Im_t \subset ... \subset \Im$ be the filtration on $\sum^\infty$ where $\Im_t$ is the σ-algebra generated by all t-histories, and $\Im$ is the σ-algebra generated by the algebra $\Im^0 \equiv \bigcup\limits_{t \geqslant 1} \Im_t$.

At period t, agent i's consumption, share holdings, and bond holdings are c_t^i , k_t^i, and b_t^i respectively. Share prices and interest rates are p_t and i_t, respectively. The variables c_t^i, k_t^i, b_t^i, p_t, and i_t are assumed to be $\Im_t-$measurable.

The market value of agent i's assets before consumption takes place is defined by $w_t^i \equiv (p_t + d)k_t^i + b_t^i$. I will refer to w_t^i as agent i's wealth.

At period t, agent i's observed and anticipated budget constraints are

$$c_{t+j}^i + p_{t+j}k_{t+j+1}^i + \frac{b_{t+j+1}^i}{i_{t+j}} = (p_{t+j} + d_{t+j})k_{t+j}^i + b_{t+j}^i;\ c_{t+j}^i \geqslant 0;\ w_{t+j}^i \geqslant 0.$$

Markets clear at period t if

$$c_t^1 + c_t^2 = d,\ b_t^1 + b_t^2 = 0, \text{ and } k_t^1 + k_t^2 = 1.$$

Let P and P^i be probability measures on $(\Sigma^\infty, \Im)$ representing the true probability measure and agent i's belief about the histories of states of nature.

At period t, agent i's expected discounted utility function is given by

$$E^{P^i}\left\{\sum_{j=0}^{\infty}(\beta^i)^j \log(c_{t+j}^i)/\Im_t\right\},$$

where β^i is agent i's discount factor, and E^{P^i} is the expectations operator associated with agent i's belief P^i.

In equilibrium, agents maximize expected discounted utility subject to the budget constraints, and markets clear in every period.

3. Learning

In this section, convergence to rational expectations is defined. In a rational expectations equilibrium, agents' beliefs are identical to the true probability measure. That is, in a rational expectations equilibrium, $P^i = P$. Hence, agents' beliefs are exactly correct. In this case, it follows from Cass and Shell (83) that equilibrium allocations may vary with time but not with states of nature. Therefore, equilibrium allocations are deterministic. In this example, if both agents have rational expectations, and identical discount factors, then share prices would be constant. But, if agents have different discount factors then share prices increase deterministically since the agent with the highest discount factor accumulates all wealth.

A weaker requirement than rational expectations is that agents have prior beliefs which are not exactly correct, but, after observing enough data, their predictions converge to the true distribution. In this case, convergence to rational expectations obtains. To measure the distance between the true probability and agents' beliefs we use the standard metric associated with the weak topology. The distance, d, between two probability measures Q and $\tilde{Q}$ is given by

$$d(Q, \tilde{Q}) = \sum_{k=1}^{\infty} 2^{-k} \left(\sup_{A \in \Im_k} \left| Q(A) - \tilde{Q}(A) \right| \right).$$

According to the distance d, two probability measures are close if they assign similar probabilities to all events except, possibly, those in the distant future.

Given a $t-$history $s_t \in \Sigma^t$, let P_{s_t} and $P^i_{s_t}$ be the true posterior probabilities and agent i's posterior belief, respectively.

Definition 1. *Convergence to rational expectations occurs if there exists a set Ω such that $P(\Omega) = 1$, and for every $s \in \Omega$, $s = (s_t, ..)$,*

$$d(P^i_{s_t}, P_{s_t}) \underset{t \to \infty}{\longrightarrow} 0.$$

That is, convergence to rational expectations occurs if both agents' posterior beliefs are eventually arbitrarily close to the true distribution.[7]

[7] In a game-theoretic framework, Lehrer and Smorodinsky (97) and Sandroni (97a) showed that if players eventually make accurate short-run predictions then the actual play is close to a Nash equilibrium play. A similar result could be obtained in this model.

4. Rare Events

In this section, true probabilities and agents' beliefs are specified. Let $\delta > 0$ be a parameter of the model. Let the function y be defined by

$$y(t) = \frac{\delta + \ln\left(\frac{\ln t - t^{-\delta}}{\ln t - 1}\right)}{\delta \ln t + \ln\left(\frac{\ln t - t^{-\delta}}{\ln t - 1}\right)} \ln t.$$[8]

It is easy to check that $\lim_{t \to \infty} y(t) = 1$. Moreover, if the period t is greater or equal to 3 then $\frac{y(t)}{\ln(t)} \in (0, 1)$; $\frac{1}{\ln t} \in (0, 1)$; and $\frac{1}{t^{\delta} \ln t} \in (0, 1)$.

The true probability of the state of nature l, conditional on all information available at period $t - 1$, $t \geqslant 3$, is $\frac{y(t)}{\ln(t)}$.

Agent 1 believes that the probability of the state of nature l, conditional on all information available at period $t - 1$, $t \geqslant 3$, is $\frac{1}{\ln(t)}$.

Agent 2 believes that the probability of the state of nature l, conditional on all information available at period $t - 1$, $t \geqslant 3$, is $\frac{1}{t^{\delta} \ln t}$.

Agents' beliefs and the true probability of the state of nature l at period 1 are not relevant and, therefore, are not defined.

Clearly, the state of nature l is the rare event. Convergence to rational expectations occurs because the state l has vanishing probability and both agents believe that state l has a vanishing probability. However, agent 2 believes the probability of state l vanishes faster than it actually does.

The probability of the state of nature l converges to zero slowly. By the Borel-Cantelli lemma, l will occur infinitely often. Market crashes will be associated with the rare event l. That is, the market will crash only when l occurs. However, in the limit economy, when agents have rational expectations, the state of nature l has probability zero. Hence, when there are rare events, there is no logical contradiction between the fact that a certain phenomenon cannot occur under rational expectations and the possibility that the same phenomenon will occur infinitely often during the learning process of convergence to rational expectations.

[8] The results would be the same if $y(t)$ were simply defined as 1. However, the complicated expression for $y(t)$ will actually simplify some calculations.

5. Recurrent Market Crashes

The main result is described in this section. The gross return of a share of the tree is $\Phi_t \equiv \frac{p_t + d}{p_{t-1}}$. I refer to Φ_t as the market return.

Definition 2. *Given a path $s \in \Sigma^\infty$, an $\varepsilon-$market crash occurs at period t if $\Phi_t(s) \leqslant \varepsilon$. An $\varepsilon-$market crash occurs infinitely often if there exists a set $\Omega \in \Im$ such that $P(\Omega) = 1$, and for every $s \in \Omega$, an $\varepsilon-$market crash occurs at infinitely many periods.*

In equilibrium, a 0-market crash cannot occur. Even if the tree becomes worthless, i.e., $p_t = 0$, the market return would still be $\frac{d}{p_{t-1}} > 0$. If an $\varepsilon-$market crash occurs then the gross return of the share is smaller then ε. In this case, the value of a share and its dividends is the fraction ε of the value of a share during the last period.

The interpretation of a market crash is that there is an extremely low market return. I will not specify precisely how low the market return must be to qualify as a market crash. Instead, I show that, for all $\varepsilon > 0$, an $\varepsilon-$market crash may arise as an equilibrium phenomenon.

Both agents are assumed to hold log utility functions. The true probability measure and agents' beliefs are fully specified, given the parameter δ. Therefore, the economy is defined given δ and the discount factors β^1 and β^2. The main objective of this section is to prove that there exists an economy such that, in equilibrium, the market will crash infinitely often.

Proposition 1. *For every $\varepsilon > 0$, there exists an economy $(\bar{\delta}, \bar{\beta}^1, \bar{\beta}^2)$ such that, in equilibrium, an $\varepsilon-$market crash occurs infinitely often.*

Proposition 1 will be demonstrated as follows: First, the equilibrium market return will be calculated as a function of agents' discount factors and agents' relative wealth. Second, it will be shown that under certain restrictions on the parameters of the model, the wealth will bounce back and forth between the two agents. Then, it will be shown that some fluctuations in agents' relative wealth are be associated with market crashes. Moreover, these fluctuations in agents' relative wealth recur.

Let $\delta_t^i \equiv 1 - \frac{c_t^i}{w_t^i}$ be agent i's saving ratio. It is known that if agents have log utility then their optimal savings ratio is identical to their discount factor. Lemma 1, below, shows this result in the model presented in this paper.

Lemma 1. *The optimal savings ratio of agent i is $\delta_t^{*i} = \beta^i$.*

Proof - See Appendix.

Lemma 1 simplifies many calculations. This simplification is the primary reason why it is assumed that both agents have log utility functions. The central result should extend to other utility functions, but explicit expressions relating the market return and agents' relative wealth would be more difficult to obtain.

Let $w_t \equiv w_t^1 + w_t^2$ be the aggregate wealth. Let $\varpi_t^i \equiv \frac{w_t^i}{w_t}$ be agent i's fraction of the aggregate wealth. Let $\nu_t^2 \equiv \frac{w_t^2}{w_t^1}$ be agent 2's relative wealth.

By definition, $\varpi_t^1 = \frac{1}{1+\nu_t^2}$ and $\varpi_t^2 = \frac{\nu_t^2}{1+\nu_t^2}$. In equilibrium, $w_t = p_t + d$, and $c_t^i = (1-\beta^i)w_t^i$. Moreover,

$$(1-\beta^1)\varpi_t^1 + (1-\beta^2)\varpi_t^2 = \frac{d}{w_t} \Rightarrow p_t + d = \frac{d}{(1-\beta^1)\varpi_t^1 + (1-\beta^2)\varpi_t^2}.$$

Hence,

$$p_t + d = \frac{d(1+\nu_t^2)}{(1-\beta^1) + (1-\beta^2)\nu_t^2} \text{ and } p_t = \frac{d(\beta^1 + \beta^2\nu_t^2)}{(1-\beta^1) + (1-\beta^2)\nu_t^2}.$$

Therefore, the market return can be computed as a function of the agents' discount factor and the ratio of agent 2's wealth relative to agents 1's wealth. That is,

$$\Phi_{t+1} = \frac{(1-\beta^1) + (1-\beta^2)\nu_t^2}{(\beta^1 + \beta^2\nu_t^2)} \frac{(1+\nu_{t+1}^2)}{(1-\beta^1) + (1-\beta^2)\nu_{t+1}^2}.$$

For future reference, let us call the expression above equation 1. Let $\eta_{t+1}^2 \equiv \frac{\nu_{t+1}^2}{\nu_t^2}$ be the ratio of agent 2's relative wealth in consecutive periods. With some elementary calculus, it is possible to show that lemma 2, below, follows from equation 1.

Lemma 2. *For every $\varepsilon > 0$, there exist $(\bar{\beta}^2, \bar{\beta}^1)$, $\bar{\beta}^2 > \bar{\beta}^1$, and strictly positive numbers $\bar{\tau}$, $\bar{\varphi}$, and $\bar{\eta}$ such that if $\bar{\varphi} \leqslant \nu_t^2 \leqslant \bar{\tau}$ and $\eta_{t+1}^2 \leqslant \bar{\eta}$, then $\Phi_{t+1} \leqslant \varepsilon$. Moreover, if $\nu_t^2 \geqslant \bar{\tau}$ and $\nu_{t+1}^2 \leqslant \bar{\varphi}$ then $\Phi_{t+1} \leqslant \varepsilon$.*

Proof - See Appendix.

Lemma 2 shows that there exist discount factors such that the market will crash in two situations: first, if the relative wealth of agent 2 is within certain limits and becomes smaller during the next period; and second, if the relative wealth of agent 2 is starts above a certain level and during the next period falls below another level. In both cases, the market crash is associated with a wealth transfer from agent 2 to agent 1. This makes intuitive sense. Assume that agent 1 is a representative agent in this economy. Then, in equilibrium, the price of a share would be $p_t = d\frac{\beta^1}{1-\beta^1}$. On the other hand, if agent 2 is a representative agent in the economy then, in equilibrium, the price of a share would be $p_t = d\frac{\beta^2}{1-\beta^2}$. So, lower share prices are associated with lower relative wealth in the hands of more patient agents.

Clearly, agents' relative wealth is fundamentally important in determining market returns. Lemma 3, below, shows that if agent 2 is more patient than agent 1, then there exists a true probability measure and agents' beliefs, given by a parameter δ, such that the wealth will bounce back and forth between the agents.

Lemma 3. *If $\beta^2 > \beta^1$ then there exists a parameter δ and a set $\Omega \in \Im$ such that $P(\Omega) = 1$ and for every $s \in \Omega$, $\limsup_{t\to\infty} \nu_t^2(s) = \infty$ and $\liminf_{t\to\infty} \nu_t^2(s) = 0$.*

Proof - See Appendix.

Lemma 3 is quite surprising. The two agents have the same utility function and their beliefs are similar in the sense that both agents' posterior beliefs are eventually arbitrarily close to the true probability distribution. However, their discount factors may be completely different and the wealth may still bounce back and forth between them. This could not occur if there were no rare events. For instance, it follows from Sandroni (97b) that if convergence to rational expectations occurs and the conditional probability of states of nature next period are bounded away from zero, then the aggregate wealth will eventually be concentrated in the hands of the most patient agent even if the difference between the discount factors is small.

The dual result of lemma 3 is also true. Given δ, there exist β^1 and β^2 such that the wealth will bounce back and forth between the two agents. The relationship that agents' beliefs and discount factors must satisfy is $\ln \beta^2 = \ln \beta^1 + \delta$. If this is not satisfied, then the wealth will end up in the hands of a single agent and share prices will be as if there were a representative agent in the economy. In this case, of course, share prices converge to a certain level and market crashes will not occur after a certain period.

Lemma 4. *For every path $s \in \Sigma^\infty$, $s = (s_{t-1}, l, ...)$, $\lim\limits_{t \to \infty} \eta_t^2(s) = 0$. Moreover, if $\beta^2 > \beta^1$ then for every path $s \in \Sigma^\infty$, $s = (s_{t-1}, h, ...)$, $\eta_t^2(s) \geqslant 1$.*

Proof - See Appendix.

Lemma 4 shows that agent 2's relative wealth will decrease if l happens and increase if h happens. This is natural because agent 2 assigns a higher probability to h than does agent 1. Moreover, lemma 4 shows that agent 2's relative wealth will be much smaller than before if l happens. This result, per se, does not imply that once l happens most of the wealth will go to agent 1, because before l happened the wealth could have been very concentrated in agent 2's hands.

The proof of proposition 1 is now as follows: Fix $\varepsilon > 0$. Let $\bar{\beta}^2$, $\bar{\beta}^1$, $\bar{\tau}$, $\bar{\varphi}$, and $\bar{\eta}$ be defined as in lemma 2. By lemma 3, there exists $\bar{\delta}$ such that $\limsup\limits_{t \to \infty} \nu_t^2 = \infty$ and $\liminf\limits_{t \to \infty} \nu_t^2 = 0$. By lemma 4, there exists a time $\bar{t}$ such that if $t \geqslant \bar{t}$ and l occurs at period t then η_t^2 is smaller than $\bar{\eta}$. By lemma 3, after period $\bar{t}$, agent 2's relative wealth will cross $\bar{\varphi}$ downwards infinitely often. That is, after period $\bar{t}$, ν_t^2 will be higher than $\bar{\varphi}$ and ν_{t+1}^2 will be smaller than $\bar{\varphi}$ infinitely often. If $\nu_t^2 \geqslant \bar{\tau}$ then, by lemma 2, an $\varepsilon-$market crash will occur at period $t+1$. If $\nu_t^2 \leqslant \bar{\tau}$ then l must have occurred at $t+1$, otherwise, by lemma 4, ν_t^2 would have risen. Then, η_{t+1}^2 is smaller than $\bar{\eta}$. Hence, by lemma 2, an $\varepsilon-$market crash will occur at period $t+1$. Therefore, an $\varepsilon-$market crash occurs infinitely often.

6. Conclusion

In this paper, I consider a dynamically complete market model without intrinsic uncertainty. Speculative trade, generated by differences in opinion regarding the probability of rare events, may result in wealth bouncing back and forth between agents with different propensities to save. These fluctuations of relative wealth may, in turn, be associated with market crashes which could not exist if agents' beliefs were identical. These market crashes may not cease to occur even if agents' beliefs become eventually identical.

7. Appendix

Proof of Lemma 1 - Consider period 0. The proof in all other periods is analogous. The fraction of agent i's savings allocated in tree shares and in the risk-free asset are defined by

$$a_t^i = (\alpha_{1,t}^i, 1 - \alpha_{1,t}^i) = \left(\frac{p_t k_{t+1}^i}{p_t k_{t+1}^i + \frac{b_{t+1}^i}{i_t}}, \frac{\frac{b_{t+1}^i}{i_t}}{p_t k_{t+1}^i + \frac{b_{t+1}^i}{i_t}} \right).$$

Let r_{t+1} be (Φ_{t+1}, i_t). By definition, agent i's budget constraint can also be written as

$$w_{t+1}^i = w_t^i \delta_t^i (a_t^i r_t),\ c_t^i = (1 - \delta_t^i) w_t^i,$$

$$c_t^i \geq 0,\ 0 \leq \delta_t^i \leq 1, t \geqslant 0.$$

Note that

$$w_t^i = w_0^i \prod_{k=0}^{t-1} \delta_k^i (a_k^i r_{k+1}),$$

and

$$\log(c_0^i) + E^{P^i} \left\{ \sum_{t=1}^{\infty} \left(\beta^i\right)^t \log(c_t^i) / \Im_0 \right\} =$$

$$\log\left(1 - \delta_0^i\right) + \log\left(w_0^i\right) + E^{P^i} \left\{ \sum_{t=1}^{\infty} \left(\beta^i\right)^t \log\left((1 - \delta_t^i) w_t^i\right) / \Im_0 \right\} =$$

$$\log\left(1 - \delta_0^i\right) + \log\left(w_0^i\right) + E^{P^i} \left\{ \sum_{t=1}^{\infty} \left(\beta^i\right)^t \log\left((1 - \delta_t^i) w_0^i \prod_{k=0}^{t-1} \delta_k^i (a_k^i r_{k+1})\right) / \Im_0 \right\} =$$

$$\log\left(1 - \delta_0^i\right) + \sum_{t=1}^{\infty} \left(\beta^i\right)^t E^{P^i} \left\{ \log(a_0^i r_1) / \Im_0 \right\} + \sum_{t=1}^{\infty} \left(\beta^i\right)^t \log(\delta_0^i)$$

$+$ extra terms that do not depend on δ_0^i and a_0^i.

Thus,

$$a_0^{*i} = \arg\max E^{P^i}\left\{\log\left(a_0^i r_1\right)/\Im_0\right\} \text{ and } \delta_0^{*i} = \arg\max \log\left(1-\delta_0^i\right) + \frac{\beta^i}{1-\beta^i}\log(\delta_0^i)$$

The first order conditions of the problem above imply that

$$\frac{1}{1-\delta_0^{*i}} = \frac{\beta}{1-\beta}\frac{1}{\delta_0^{*i}} \Rightarrow \delta_0^{*i} = \beta^i \text{ and } E^{P^i}\left\{\frac{\Phi_1 - i_0}{a_0^{*i} r_1}/\Im_0\right\} = 0.$$

q.e.d.

Consider the function $f(x) = \frac{b+cx}{d+ex}$. The derivative of f is $\frac{\partial f}{\partial x}(x) = \frac{dc-eb}{(d+ex)^2}$. Hence, f is a non-decreasing function of x if $dc \geqslant eb$ and f is a non-increasing function of x if $dc \leqslant eb$. For future reference, lets call this result lemma $A.1$.

Proof of lemma 2 Fix $\varepsilon > 0$. Let $\bar{\varphi}$ and $\bar{\tau}$ be defined by

$$1+\bar{\varphi} = \frac{4}{\varepsilon} \text{ and } 1+\bar{\tau} = \frac{16}{\varepsilon^2}.$$

Consider the functions $h_1(a) = \frac{1+\frac{\bar{\tau}}{a}}{1+\bar{\varphi}}$ and $h_2(a) = \frac{1+\frac{\bar{\tau}}{a}}{1+\bar{\tau}}\frac{1+\bar{\varphi}}{1+\frac{\bar{\varphi}}{a}}$. Clearly, $h_1(a)$ and $h_2(a)$ converges to $\frac{\varepsilon}{4}$ as a goes to infinity. Let $\bar{a} > 1$ be large enough such that

$$\frac{1+\frac{\bar{\tau}}{\bar{a}}}{1+\bar{\varphi}} \leqslant \frac{\varepsilon}{3} \text{ and } \frac{1+\frac{\bar{\tau}}{\bar{a}}}{1+\bar{\tau}}\frac{1+\bar{\varphi}}{1+\frac{\bar{\varphi}}{\bar{a}}} \leqslant \frac{\varepsilon}{3}.$$

Consider the functions $g(\beta^1,\beta^2) = \frac{1+\frac{\bar{\tau}}{a}}{\beta^1+\beta^2\bar{\varphi}}$ and $k(\beta^1,\beta^2) = \frac{1+\frac{\bar{\tau}}{a}}{\beta^1+\beta^2\bar{\tau}}\frac{1+\bar{\varphi}}{1+\frac{\bar{\varphi}}{\bar{a}}}$. Clearly,

$$\lim_{\substack{\beta^1\to 1,\ \beta^2\to 1\\ (1-\beta^1)=\bar{a}(1-\beta^2)}} (\beta^1,\beta^2) = \frac{1+\frac{\bar{\tau}}{\bar{a}}}{1+\bar{\varphi}} \leqslant \frac{\varepsilon}{3} \text{ and } \lim_{\substack{\beta^1\to 1,\ \beta^2\to 1\\ (1-\beta^1)=\bar{a}(1-\beta^2)}} k(\beta^1,\beta^2) = \frac{1+\frac{\bar{\tau}}{\bar{a}}}{1+\bar{\tau}}\frac{1+\bar{\varphi}}{1+\frac{\bar{\varphi}}{\bar{a}}} \leqslant \frac{\varepsilon}{3}.$$

Let $\bar{\beta}^2$ and $\bar{\beta}^1$ be such that

$$(1-\bar{\beta}^1) = \bar{a}(1-\bar{\beta}^2);\ \frac{1+\frac{\bar{\tau}}{\bar{a}}}{\bar{\beta}^1+\bar{\beta}^2\bar{\varphi}} \leqslant \frac{\varepsilon}{2};\ \text{and } \frac{1+\frac{\bar{\tau}}{\bar{a}}}{\bar{\beta}^1+\bar{\beta}^2\bar{\tau}}\frac{1+\bar{\varphi}}{1+\frac{\bar{\varphi}}{\bar{a}}} \leqslant \frac{\varepsilon}{2}.$$

Consider the function $u(\eta)=\frac{\bar{a}+\bar{\tau}}{\beta^1+\beta^2\bar{\varphi}}\frac{1+\bar{\tau}\eta}{\bar{a}+\bar{\varphi}\eta}$. Clearly, $u(\eta)$ converges to $\frac{1+\frac{\bar{\tau}}{\bar{a}}}{\beta^1+\beta^2\bar{\varphi}}\leqslant\frac{\varepsilon}{2}$ as η goes to infinity. Let $\bar{\eta}$ be large enough such that

$$\frac{\bar{a}+\bar{\tau}}{\beta^1+\beta^2\bar{\varphi}}\frac{1+\bar{\tau}\bar{\eta}}{\bar{a}+\bar{\varphi}\bar{\eta}}\leqslant\varepsilon.$$

By the definition of η_{t+1}^2, and equation 1,

$$\Phi_{t+1}=\frac{(1-\bar{\beta}^1)+(1-\bar{\beta}^2)\nu_t^2}{(\bar{\beta}^1+\bar{\beta}^2\nu_t^2)}\frac{(1+\nu_t^2\eta_{t+1}^2)}{(1-\bar{\beta}^1)+(1-\bar{\beta}^2)\nu_t^2\eta_{t+1}^2}.$$

However, $(1-\bar{\beta}^1)=\bar{a}(1-\bar{\beta}^2)$. Then,

$$\Phi_{t+1}=\frac{\bar{a}+\nu_t^2}{(\bar{\beta}^1+\bar{\beta}^2\nu_t^2)}\frac{(1+\nu_t^2\eta_{t+1}^2)}{\left(\bar{a}+\nu_t^2\eta_{t+1}^2\right)}.$$

Assume that $\bar{\varphi}\leqslant\nu_t^2\leqslant\bar{\tau}$, and $\eta_{t+1}^2\leqslant\bar{\eta}$.

Note that if $\varepsilon\leqslant 4$ then $\bar{\varphi}\leqslant\bar{\tau}$. Moreover, by definition, $\bar{a}\geqslant 1$. Hence, $\bar{a}\bar{\tau}\geqslant\bar{\varphi}$. By lemma $A.1$, $\frac{1+\bar{\tau}\eta}{\bar{a}+\bar{\varphi}\eta}$ is an increasing function of η. Therefore, if $\eta_{t+1}^2\leqslant\bar{\eta}$ then $\frac{(1+\nu_t^2\eta_{t+1}^2)}{\bar{a}+\nu_t^2\eta_{t+1}^2}$ is smaller than $\frac{1+\bar{\tau}\bar{\eta}}{a+\bar{\varphi}\bar{\eta}}$. Thus, by the definition of $\bar{\eta}$,

$$\Phi_{t+1}=\frac{\bar{a}+\nu_t^2}{(\bar{\beta}^1+\bar{\beta}^2\nu_t^2)}\frac{(1+\nu_t^2\eta_{t+1}^2)}{\left(\bar{a}+\nu_t^2\eta_{t+1}^2\right)}\leqslant\frac{\bar{a}+\bar{\tau}}{(\bar{\beta}^1+\bar{\beta}^2\bar{\varphi})}\frac{1+\bar{\tau}\bar{\eta}}{\bar{a}+\bar{\varphi}\bar{\eta}}\leqslant\varepsilon.$$

Assume that $\nu_t^2\geqslant\bar{\tau}$ and $\nu_{t+1}^2\leqslant\bar{\varphi}$.

By lemma $A.1$, $\frac{(1+\nu_t^2\eta_{t+1}^2)}{\left(\bar{a}+\nu_t^2\eta_{t+1}^2\right)}$ increases with $\nu_t^2\eta_{t+1}^2=\nu_{t+1}^2$. Hence, $\frac{(1+\nu_t^2\eta_{t+1}^2)}{\left(\bar{a}+\nu_t^2\eta_{t+1}^2\right)}$ is smaller than $\frac{1+\bar{\varphi}}{\bar{a}+\bar{\varphi}}$.

By lemma $A.1$, $\frac{\bar{a}+\nu_t^2}{(\bar{\beta}^1+\bar{\beta}^2\nu_t^2)}$ decreases with ν_t^2. Hence, $\frac{\bar{a}+\nu_t^2}{(\bar{\beta}^1+\bar{\beta}^2\nu_t^2)}$ is smaller than $\frac{\bar{a}+\bar{\tau}}{(\bar{\beta}^1+\bar{\beta}^2\bar{\tau})}$. Thus, by the definition of $\bar{\beta}^2$ and $\bar{\beta}^1$,

$$\Phi_{t+1}=\frac{\bar{a}+\nu_t^2}{(\bar{\beta}^1+\bar{\beta}^2\nu_t^2)}\frac{(1+\nu_t^2\eta_{t+1}^2)}{\left(\bar{a}+\nu_t^2\eta_{t+1}^2\right)}\leqslant\frac{\bar{a}+\bar{\tau}}{(\bar{\beta}^1+\bar{\beta}^2\bar{\tau})}\frac{1+\bar{\varphi}}{\bar{a}+\bar{\varphi}}\leqslant\frac{\varepsilon}{2}\leqslant\varepsilon.$$

q.e.d.

Let x_t be a $\Im_t-$measurable random variable defined by

$$x_t(s) = \ln\left(\frac{P^2_{s_{t-1}}(C(s_t))}{P^1_{s_{t-1}}(C(s_t))}\right) + \delta.$$

where $s = (s_t, ...) \in \Sigma^\infty$, $s_t \in \Sigma^t$, $s_t = (s_{t-1}, a)$, $s_{t-1} \in \Sigma^{t-1}$, $a \in \Sigma$.

By the definition of agents beliefs P^1 and P^2, $x_t = -\delta \ln t + \delta$ if l occurs at period t, and $x_t = \ln\left(\frac{\ln t - t^{-\delta}}{\ln t - 1}\right) + \delta$ if h occurs at period t. Hence, by the definition of $y(t)$ and the true probability measure P, if $t \geqslant 3$ then

$$E^P\left\{x_t/\Im_{t-1}\right\} = -\delta y(t) + (1 - \frac{y(t)}{\ln t})\ln\left(\frac{\ln t - t^{-\delta}}{\ln t - 1}\right) + \delta = 0.$$

Let $Var^P\left\{x_t/\Im_{t-1}\right\}$ be the conditional variances of x_t according to the probability P. By definition,

$$Var^P\left\{x_t/\Im_{t-1}\right\} = \delta^2 y(t)\ln t + (1 - \frac{y(t)}{\ln t})\left(\ln\left(\frac{\ln t - t^{-\delta}}{\ln t - 1}\right)\right)^2 - \delta^2.$$

Hence, $Var^P\left\{x_t/\Im_{t-1}\right\} \geqslant \delta^2(\ln t - 1)$. Therefore,

$$\sum_{t=3}^{\infty} Var^P\left\{x_t/\Im_{t-1}\right\} = \infty.$$

Let T_m be $\sum\limits_{t=3}^{m} Var^P\left\{x_t/\Im_{t-1}\right\}$. Then, $T_m \geqslant \sum\limits_{t=3}^{\infty} \delta^2(\ln t - 1)$. Moreover, $\ln t$ goes to infinity as t goes to infinity. Hence, there exists $\bar{m}$ such that if $m \geqslant \bar{m}$ then $T_m \geqslant m$.

It is easy to check, by L'Hopital theorem, that the ratio $\frac{m^{0.25}}{\delta\ln(m+1)+\delta}$ goes to infinity as m goes to infinity. Moreover, $\ln\left(\frac{\ln(m+1)\ -(m+1)^{-\delta}}{\ln(m+1)\ -1}\right) + \delta$ goes to δ as m goes to infinity. Hence, the ratio $\frac{m^{0.25}}{\ln\left(\frac{\ln(m+1)\ -(m+1)^{-\delta}}{\ln(m+1)\ -1}\right)+\delta}$ goes to infinity as m goes to infinity. Therefore, there exists a positive constant ζ such that

$$\zeta\left(T_m\right)^{0.25} \geqslant \max\left\{\ln\left(\frac{\ln(m+1)\ -\ (m+1)^{-\delta}}{\ln(m+1)\ -1}\right) + \delta,\ \delta\ln(m+1) + \delta\right\}$$

for every $m \geqslant 3$.

Let $\epsilon(b) \equiv \zeta b^{-0.25}$. Let $\phi(b) \equiv b^{0.5}$. Clearly, $\epsilon(b) \equiv \zeta b^{-0.25}$ tends to zero as b goes to infinity and $\epsilon(b)\phi(b)$ is a non-decreasing function of b. By construction,

$$\left|x^2_{m+1}\right| \leqslant \epsilon(T_m)\phi(T_m).$$

Let S_m be $\sum_{t=3}^{m} x_t$. By Freedman (75), proposition 2.6,

$$\limsup_{m\to\infty} S_m = \infty \text{ and } \liminf_{m\to\infty} S_m = -\infty \ a.s.\ P.$$

For future reference, lets call this last result lemma $A.2$.

Proof of Lemmas 3 and 4 Assume that $\ln\beta^2 = \ln\beta^1 + \delta$.

Agents' first order conditions imply that, in equilibrium, for every path $s = (s_m, ...) \in \Sigma^\infty$, $s_m \in \Sigma^m$,

$$\frac{(\beta^2)^{m-1} P^2(C(s_m))(c^2_m(s))^{-1}}{(\beta^1)^{m-1} P^1(C(s_m))(c^1_m(s))^{-1}} = \frac{\lambda^2}{\lambda^1},$$

where λ^i is agent i's Lagrange multiplier.

By lemma 1,

$$\frac{(\beta^2)^{m-1} P^2(C(s_m))}{(\beta^1)^{m-1} P^1(C(s_m))} = \nu^2_m(s)\frac{\lambda^2}{\lambda^1}\frac{1-\beta^2}{1-\beta^1}.$$

Taking logs on both sides,

$$(m-1)\ln\frac{\beta^2}{\beta^1} + \ln\frac{P^2(C(s_m))}{P^1(C(s_m))} = \ln\nu^2_m(s) + \ln\frac{\lambda^2}{\lambda^1}\frac{1-\beta^2}{1-\beta^1}.$$

By assumption,

$$\ln\nu^2_m(s) = m\delta + \ln\frac{P^2(C(s_m))}{P^1(C(s_m))} - \ln\frac{\lambda^2}{\lambda^1}\frac{1-\beta^2}{1-\beta^1}$$

By Bayes' rule,

$$\frac{P^2(C(s_m))}{P^1(C(s_m))} = \frac{\prod_{t=2}^{m} P^2_{s_{t-1}}(C(s_t))}{\prod_{t=2}^{m} P^1_{s_{t-1}}(C(s_t))}$$

where $s_m = (s_t, ...)$.

Hence,

$$\ln \nu_m^2(s) = \sum_{t=3}^{m} x_t + \ln \frac{P_{s_1}^2(C(s_2))}{P_{s_1}^1(C(s_2))} + \delta - \ln \frac{\lambda^2}{\lambda^1} \frac{1-\beta^2}{1-\beta^1}$$

By lemma $A.2$,

$$\limsup_{m\to\infty} \nu_m^2 = \infty \text{ and } \liminf \nu_m^2 = 0 \ a.s. \ P.$$

The equation above proves lemma 3. In order to prove lemma 4, consider again the equality

$$\frac{(\beta^2)^{m-1} P^2(C(s_m))}{(\beta^1)^{m-1} P^1(C(s_m))} = \nu_m^2(s) \frac{\lambda^2}{\lambda^1} \frac{1-\beta^2}{1-\beta^1}.$$

Then,

$$\frac{\beta^2 P_{s_m}^2(C(s_{m+1}))}{\beta^1 P_{s_m}^1(C(s_{m+1}))} = \frac{\nu_{m+1}^2(s)}{\nu_m^2(s)} = \eta_{m+1}(s).$$

If the state of nature l at period $m+1$ occurs then, by definition,

$$\frac{P_{s_m}^2(C(s_{m+1}))}{P_{s_m}^1(C(s_{m+1}))} = t^{-\delta}.$$

Hence, $\eta_{t+1}(s)$ is arbitrarily small if the state of nature l at period $m+1$ and m is large enough. Moreover, $\eta_{m+1}(s) = \frac{\beta^2 P_{s_m}^2(C(s_{m+1}))}{\beta^1 P_{s_m}^1(C(s_{m+1}))}$ and, by the definition of agents' beliefs, $\frac{P_{s_m}^2(C(s_{m+1}))}{P_{s_m}^1(C(s_{m+1}))}$ is greater than 1 if h happens at period $t+1$.

q.e.d.

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