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The Speed of Rational Learning

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Abstract

A stage game is played infinitely many times. After observing the outcomes of the game, players revise their beliefs about opponents' strategies. I show the general conditions under which players' predictions become accurate *fast*.

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1. Introduction

The objective of this paper is to show the general conditions under which players' predictions become accurate fast in infinitely repeated games.

In each period, players' choose an action, and update their beliefs about the future evolution of the play according to the past history of outcomes. Assuming that, after observing enough data, players' posterior beliefs become accurate, it is important to know what determines the speed of the convergence process. However, obtaining rates of convergence is difficult, and, in game theoretical models, the literature on this issue is small. Jordan [92] obtained exponential rate of convergence to Nash equilibrium for a class of myopic learning processes. Vives [93] showed interesting examples of slow convergence (rate $t^{-\frac{1}{6}}$) and fast convergence (rate $t^{-\frac{1}{2}}$) in an economic model.

The question of how fast agents' predictions become accurate is relevant even if players do not fully optimize given their beliefs. In applications, it might be interesting to assume that agents take almost best-responses, or myopic best responses, or best responses subject to "complexity constraints," etc. I assume that players choose a strategy according to an unspecified criteria. If players fully optimize given their beliefs then fast convergence to an equilibrium can be obtained as a corollary of some of the results presented in this paper.

There are two standard notions of convergence. One is merging and the other is weak merging. Players' posterior beliefs merge with chosen strategies if players' prediction about all future events eventually become accurate. Players' posterior beliefs weakly merge with chosen strategies if eventually players' predictions about all future events become accurate, except, possibly, distant-future events.

Players' posterior beliefs merge with chosen strategies if and only if chosen strategies are absolute continuous with respect to players' beliefs (see Blackwell-Dubins [62], and Kalai and Lehrer [93a] and [94]). Absolute continuity requires that if an event occurs with positive probability then all players assign positive probability to this event. If players' posterior beliefs merge with chosen strategies and players' take best responses according to their beliefs, then the play is eventually close to a Nash equilibrium play (see Kalai and Lehrer [93a] and [93b]). However, convergence to Nash equilibrium also obtains if players' posterior beliefs and optimal strategies weakly merge, but do not necessarily merge. Hence, convergence to Nash equilibrium can be obtained under conditions which are weaker than absolute continuity (see Lehrer and Smorodinsky [97] and Sandroni [97]). In this paper, I show that absolute continuity implies fast converge of agents' beliefs and true strategies.

Players' beliefs and chosen strategies weakly merge fast, with density one, if players' beliefs and chosen strategies weakly merge faster than $t^{-0.5}$ in a subsequence of periods which has density one. A subsequence of periods have density one if the relative frequency of the periods in this subsequence converges to one as time progresses.

The central result of this paper is as follows: If players' posterior beliefs merge with chosen strategies then players' beliefs and chosen strategies weakly merge fast, with density one. Hence, under absolute continuity, players' beliefs and chosen strategies weakly merge fast, with density one. However, this result is obtained under some restrictions on players' beliefs and chosen strategies which are imposed to obtain mathematical tractability.

The result above is sharp. First, I show an example such that players' posterior beliefs merge with chosen strategies, but players' beliefs and chosen strategies do not weakly merge faster than $t^{-0.5}$ in a subsequence of density zero. Hence, in general, even under absolute continuity, it is possible that, very rarely, players will make mistakes in their predictions which are not consistent with fast convergence. Moreover, I show another example such that players' posterior beliefs merge with chosen strategies, but players' beliefs and chosen strategies do not weakly merge faster than t^{-v}, $v > 0.5$, with density one. Thus, the rate of convergence $t^{-0.5}$ is sharp.

A natural question is the converse of the result described above. That is, does fast weak merging imply merging? I show an example in which players' beliefs and chosen strategies weakly merge at rate t^{-1}, and consequently faster than $t^{-0.5}$, but players' beliefs and chosen strategies do not merge. However, I show that if players' beliefs and chosen strategies weakly merge faster than $t^{-(1+\varepsilon)}$, $\varepsilon > 0$, then players' beliefs and chosen strategies merge.

2. The Model

There exist n players. Each player $i \in \{1, ..., n\}$ has a finite set $\sum_i$ of possible actions. Let $\triangle(\sum_i)$ be the set of probability distributions on $\sum_i$. Let $\sum = \prod_{i=1}^{n} \sum_i$ be the set of action combinations. Let $\sum^t$ be the set of histories of length $t+1, 0 \leqslant$

$t \leq \infty$. Define $H = \bigcup_t \sum^t$ as the set of finite histories.

A cylinder with base on $h \in \sum^t$ is the set $C(h) = \{w \in \sum^\infty / w = (h, ...)\}$ of all infinite histories such that the $t+1$ initial elements coincide with h. Let

$$\Im_0 \subset ...\Im_t \subset ... \subset \Im,$$

be a filtration where $\Im_0$ is the trivial σ-algebra, $\Im_t$ is the $\sigma-$algebra generated by the cylinders with base on $\sum^t$, and $\Im$ is the σ-algebra generated by the algebra of finite histories $\Im^0 \equiv \bigcup_{t \geq 0} \Im_t$.

Each player $i \in \{1, 2, ..., n\}$ has a payoff function $u_i : \sum \rightarrow \Re$; a chosen behavior strategy $f_i : H \rightarrow \triangle(\sum_i)$ which describes how player i randomizes among possible actions conditional on every possible history; and a belief about opponents' strategies $f^i = (f_1^i, ..., f_n^i)$. The chosen strategy profile is defined by $f = (f_1, ..., f_n)$. Each player knows his own strategy, i.e., $f_i = f_i^i$.

Given a strategy profile g, there exists a probability measure μ_g (see Kalai and Lehrer [93a] for details) that represents the probability distribution over play paths generated by g. Given a finite history $h \in H$, the induced strategy profile g_h is defined by $g_h(\acute{h}) = g(h, \acute{h})$ for any $\acute{h} \in H$.

3. Main Concepts and Results

Definition 1. *After observing history $h \in H$, the difference, in the sup-norm, between player i's posterior beliefs and chosen strategies is given by*

$$\left\| f_h - f_h^i \right\| = \sup_{A \in \Im} \left| \mu_{f_h}(A) - \mu_{f_h^i}(A) \right| .$$

If $\|f_h - f_h^i\| \leq \varepsilon$ then, after observing history $h \in H$, the absolute difference between the probability assigned by player i to any event and the true probability of the event is smaller than ε.

Definition 2. *Fix a natural number l. After observing history $h \in H$, the difference, in the d_l-metric, between player i's beliefs and the chosen strategies is given by*

$$d_l(f_h,\ f_h^i) = \sup_{A \in \Im_j,\ 0 \leq j \leq l} \left| \mu_{f_h}(A) - \mu_{f_h^i}(A) \right| .$$

If $d_l(f_h, f_h^i) \leq \varepsilon$ then, after observing history $h \in H$, the absolute difference between probability assigned by player i to an event, within l periods, and the true probability of this event is smaller than ε.

Definition 3. *Player i's beliefs and chosen strategies merge if there exists a set $\Omega \in \Im$ such that $\mu_f(\Omega) = 1$, and for every path $w \in \Omega$, $w = (w(t), ...)$, $\left\| f_{w(t)} - f^i_{w(t)} \right\|$ goes to zero as t goes to infinity.*

That is, player i's beliefs and chosen strategies merge if, given enough data, player i's posterior beliefs become close, in the sup norm, to the true probability distribution.

Definition 4. *Player i's beliefs and chosen strategies weakly merge if for every natural number l, there exists a set $\Omega \in \Im$ such that $\mu_f(\Omega) = 1$, and for every path $w \in \Omega$, $w = (w(t), ...)$, $d_l(f_{w(t)}, f^i_{w(t)})$ goes to zero as t goes to infinity.*

That is, player i's beliefs and chosen strategies weakly merge if, given enough data, player i's beliefs and the true probability measure assign similar probabilities for all measurable events, except possibly the ones that may only be observed in the distant future. Next, I define absolute continuity and local absolute continuity:

Definition 5. *Chosen strategies f are locally absolutely continuous with respect to player i's beliefs f^i if for every $A \in \Im^0$, $\mu_{f^i}(A) = 0$ implies $\mu_f(A) = 0$.*

The assumption of local absolute continuity requires that if any finite-time event occurs with positive probability, then player i assigns positive probability to this event. Local absolute continuity is a relatively mild restriction. Under local equivalence, players can unambiguously update their beliefs by Bayes' rule. However, the mere fact that players are able to revise their beliefs in a Bayesian fashion does not necessarily imply that players predictions will eventually become accurate. For example, consider the case of a player flipping a fair coin many times. Suppose the player believes that the probability of heads is 0.3. Local absolute continuity is satisfied because the player does not assign probability zero to any finite-time event. However, the player's posterior beliefs are never close to the truth.

Hereafter, I assume that chosen strategies are locally absolutely continuous with respect to all players' beliefs, unless otherwise indicated.

Definition 6. *Actually chosen strategies f are absolutely continuous with respect to player i's beliefs f^i if, for every $A \in \Im$, $\mu_{f^i}(A) = 0$ implies $\mu_f(A) = 0$.*

The assumption of absolute continuity requires that if any event occurs with positive probability, then player i assigns positive probability to this event.

Proposition 1, below, relates absolute continuity and merging.

Proposition 1. *Player i's beliefs and chosen strategies merge if and only if chosen strategies are absolutely continuous with respect to player i's beliefs.*

Proof - For the assertion, see Blackwell and Dubins [62], and Kalai and Lehrer [93]. For the converse, see Kalai and Lehrer [94].

Proposition 1 shows necessary and sufficient compatibility conditions on players' prior beliefs and the truth which ensure that players' posterior beliefs merge with chosen strategies.

Proposition 2, below, provides an alternative characterization of merging which will be useful to prove the main results of this paper. Moreover, proposition 2 is useful to determine whether merging occurs in specific examples.

Given $w \in \Sigma^\infty, w = (w(t), ...)$, $w(t) = (w(t-1), a)$, $a \in \Sigma$, let z_t^i be $\Im_t$ measurable functions defined by:

$$\begin{aligned} z_t^i(w) &= 1 \text{ if } \mu_f(C(w(t))) = 0; \text{ and} \\ z_t^i(w) &= \log\left(\frac{\mu_{f^i_{w(t-1)}}(C(w(t)))}{\mu_{f_{w(t-1)}}(C(w(t)))}\right) \text{ if } \mu_f(C(w(t))) > 0. \end{aligned}$$

That is, z_t^i is logarithm of the ratio of player i's beliefs to chosen strategies over next period's events. Let e_t^i be $E\{z_t^i / \Im_{t-1}\}$, and let v_t^i be $Var\{z_t^i / \Im_{t-1}\}$, where E and Var are expectation and variance operators associated with the true probability measure μ_f, respectively.

Definition 7. *The ratio of player i's beliefs to chosen strategies over next period's events is bounded away from zero and infinity if there exists $\delta > 0$ and $M > 0$ such that for every finite-history $h \in H$, and for every stage game outcome $a \in \Sigma$, such that $\mu_f(C(h, a)) > 0$,*

$$\delta \leqslant \frac{\mu_{f^i_h}(C(h,a))}{\mu_{f_h}(C(h,a))} \leqslant M.$$

Proposition 2. *Assume that the ratio of player i's beliefs to chosen strategies over next period's events is bounded away from zero and infinity. Then, player i's beliefs and chosen strategies merge if and only if*

$$\sum_t e_t^i > -\infty \text{ and } \sum_t v_t^i < \infty \text{ a.s. } \mu_f.$$

Proof - See Appendix.

If players' beliefs and chosen strategies merge then players' beliefs and chosen strategies weakly merge, but not conversely. Example 1, below, illustrates this point.

Example 1. *There are two players, 1 and 2. Payoffs are given by the matrix*

$$\begin{bmatrix} (1,2) & L & R \\ T & (1,0) & (0,0) \\ B & (0,0) & (1,0) \end{bmatrix}.$$

Player 2 is indifferent between the outcomes of the game and plays L in all periods. Player 1 believes that player 2 plays L with probability $1-\frac{1}{t+2}$ *at period t. Player 1 plays T in all periods. Clearly, player 1's beliefs and chosen strategies weakly merge. However, player 1's beliefs and chosen strategies do not merge because, in all periods, player 1 believes that the probability that player 2 will play L in all remaining periods is zero, and the true probability of this event is one.*

Kalai and Lehrer [94], Lehrer and Smorodinsky [97], and Sandroni [97] obtained compatibility conditions between players' beliefs and chosen strategies which are weaker than absolute continuity, but strong enough to ensure weak merging. The motivation of these authors is the notion that, as illustrated by example 1, absolute continuity is an unnecessarily strong compatibility for convergence to Nash equilibrium. On the other hand, example 2, below, suggests a connection between merging and fast weak merging.

Example 2. *Continue with the set up of example 1, but assume that player 1 believes that player 2 plays L with probability* $1-\frac{1}{(t+2)^\rho}$ *at period t. Note that if* $\rho > 0$ *then player 1's beliefs and chosen strategies weakly merge. However, merging obtains only if* $\rho > 1$ *because, in this case, player 1 assigns positive probability to the event in which player 2 plays L in all periods. On the other hand, if* $\rho \leqslant 1$ *then player 1 assigns zero probability to the event in which player 2 plays L in all periods. The higher is* ρ*, the faster is the convergence rate.*

The focus of this paper is to establish the exact connection between merging and the rate of weak merging. To achieve this objective, I make the following definitions:

Definition 8. *A sequence a_t goes to zero, with density one, if for every $\varepsilon > 0$,*

$$\frac{\sharp\{j/\ |a_j| \leq \varepsilon; j \leq t\}}{t} \underset{t\to\infty}{\longrightarrow} 1.$$

That is, a sequence a_t goes to zero with density one if $|a_t|$ is arbitrarily small with arbitrarily high frequency.

Definition 9. *Player i's beliefs and chosen strategies weakly merge, with density one, at the rate $t^{-\nu}$, if, for every natural number l, there exists a set $\Omega \in \Im$ such that $\mu_f(\Omega) = 1$, and for every path $w \in \Omega$, $w = (w(t), ...)$, $t^{\nu} d_l(f_{w(t)}, f^i_{w(t)})$ goes to zero, with density one.*

That is, player i's beliefs and chosen strategies weakly merge, with density one, at the rate $t^{-\nu}$, if the d_l-distance between player i's posterior beliefs and true strategies goes to zero, with density one, faster than $t^{-\nu}$.

Definition 10. *Player i's beliefs and chosen strategies weakly merge fast, with density one, if player i's beliefs and chosen strategies weakly merge, with density one, at the rate $t^{-0.5}$.*

The rate $t^{-0.5}$ is standard for fast convergence (see Vives [93]). For mathematical tractability, I impose the following restriction on chosen strategies:

Definition 11. *Players do not randomize using vanishingly small probabilities if there exists $\gamma > 0$ such that for every stage game outcome $a \in \Sigma$, and for every finite-history $h \in H$, such that $\mu_f(C(h)) > 0$, either $\mu_{f_h}(C(h, a) = 0$ or $\mu_{f_h}(C(h, a) > \gamma$.*

Players are typically indifferent about which probabilities to use when they decide to randomize. Players do not randomize using vanishingly small probabilities if players do not take actions with arbitrarily small (and strictly positive) probability.

If players do not randomize using vanishingly small probabilities then the ratio of player i's beliefs to chosen strategies over next period's events is bounded away from infinity.

If players do not randomize using vanishingly small probabilities, and player i's beliefs and chosen strategies weakly merge, then the ratio of player i's beliefs to chosen strategies over next period's events will eventually become close to one. However, this ratio may approach zero on paths that have zero measure (under μ_f).

It is easy to check that in examples 1 and 2, the ratio of player 1's beliefs to chosen strategies over next period's events are bounded away from zero and players do not randomize using vanishingly small probabilities.

The main result of this paper is proposition 3 stated below:

Proposition 3. *Assume that the ratio of player i's beliefs and chosen strategies over next period's events is bounded away from zero and players do not randomize using vanishingly small probabilities. If chosen strategies are absolutely continuous with respect to player i's beliefs then player i's beliefs and chosen strategies weakly merge fast, with density one.*

Proof - See Appendix.

Corollary 1. *Assume that the ratio of player i's beliefs and chosen strategies over next period's events is bounded away from zero and players do not randomize using vanishingly small probabilities. If player i's beliefs and chosen strategies merge then player i's beliefs and chosen strategies weakly merge fast, with density one.*

That is, if chosen strategies are absolutely continuous with respect to player i's beliefs (or if player i's beliefs and chosen strategies merge) then, except in some rare periods, the d_l-distance between player i's posterior beliefs and chosen strategies, at period t, is smaller than $t^{-0.5}$.

Example 3, below, shows that even if player i's beliefs and chosen strategies merge, then player i's beliefs and chosen strategies may not weakly merge at rate $t^{-0.5}$ in a subsequence of upper density zero. (The upper density of a subsequence $L \subseteq N$ is defined as $\limsup_{m\to\infty} \sharp\{L \bigcap \{1, ..., m\}/m.$) Hence, even under absolute continuity, the d_l-distance between player i's posterior beliefs and chosen strategies, at period t, may, in some rare periods, be greater than $t^{-0.5}$.

Example 3. *Continue with the set up from example 1, but assume that player 2 plays R with probability 0.5 in every period. Player 1 believes that player 2 plays R with probability $0.5(1 + 1/t^{0.25})$ in all periods t such that there exists a natural number $j \in N$, $j > 1$ such that $t = j^4$. In all other periods, player 1 believes that player 2 plays R with probability 0.5.*
The subsequence $L = \{t/ \ t = j^4, \ j \in N\}$ has upper density zero. For all paths $w = (w(t), ...)$ $\sqrt{t}d_1(f_{w(t)}, f^i_{w(t)}) = 0.5t^{0.25}$, $t \in L$. Hence, player 1's beliefs and chosen strategies do not weakly merge at rate $t^{-0.5}$ in this subsequence.
By definition, $e^1_t = 0$ if $t \neq j^4$, and $e^1_t = 0.5\log(1 - 1/j^2)$ if $t = j^4$; $v^1_t = 0$ if $t \neq j^4$, and $v^1_t = 0.5\log^2(1 - 1/j) + 0.5\log^2(1 + 1/j) - (e^1_t)^2$ if $t = j^4$. However, if $x \geqslant 0$ then $\log(1 + x) \leqslant x$; and if $x < 0$, but sufficiently close to zero, then $\log(1 - x) \geqslant (-2)x$. Hence, if j is sufficiently large then $0 \geqslant e^1_t \geqslant -1/j^2$, and $0 \leqslant v^1_t \leqslant 2.5/j^2$. By proposition 3, player 1's beliefs and chosen strategies merge.

Example 4, below, shows that even if player i's beliefs and chosen strategies merge then player i's beliefs and chosen strategies may not weakly merge at rate $t^{-v}, v > 0.5$, with density one. This example shows that the rate $t^{-0.5}$ is sharp.

Example 4. *Continue with the set up from example 1, but assume, as in example 2, that player 2 plays R with probability 0.5 in every period. Player 1 believes that player 2 plays R with probability $0.5(1 + 1/t^{\gamma}), \gamma > 0.5$, in all periods.*
If $v > \gamma > 0.5$ then, for all paths $w = (w(t), ...)$, $t^v d_1(f_{w(t)}, f^i_{w(t)})$ goes to infinity as t goes to infinity.
By definition, $e^1_t = 0.5\log(1-2/t^{2\gamma})$, and $v^1_t = 0.5\log^2(1-2/t^{\gamma}) + 0.5\log^2(1+2/t^{\gamma}) - (e^1_t)^2$. Hence, if t is sufficiently large then $0 \geqslant e^1_t \geqslant -4/t^{2\gamma}$, and $0 \leqslant v^1_t \leqslant 10/t^{2\gamma}$. By proposition 3, player 1's beliefs and chosen strategies merge.

A natural question is whether the converse of proposition 3 holds. That is, does fast weak merging imply merging? Example 2 shows that this is not necessarily true. In example 2, if $\rho = 1$, then player 1's beliefs and chosen strategies weakly merge at rate t^{-1}. Hence, player 1's beliefs and chosen strategies weakly merge fast, with density one, but player 1's beliefs and chosen strategies do not merge.

Definition 12. *Fix $\varepsilon > 0$. Player i's beliefs and chosen strategies weakly merge faster than $t^{-(1+\varepsilon)}$ if there exists a set $\Omega \in \Im$ such that $\mu_f(\Omega) = 1$, and for every path $w \in \Omega$, $w = (w(t), ...)$, $t^{-(1+\varepsilon)}d_1(f_{w(t)}, f^i_{w(t)})$ goes to zero as t goes to infinity.*

A "partial converse" to proposition 3 is given by proposition 4.

Proposition 4. *Assume that the ratio of player i's beliefs and chosen strategies over next period's events is bounded away from zero and players do not randomize using vanishingly small probabilities. If player i's beliefs and chosen strategies weakly merge faster than $t^{-(1+\varepsilon)}$, for some $\varepsilon > 0$, then player i's beliefs and chosen strategies merge.*

Proof - See Appendix.

Appendix

Definition A.1 Chosen strategies f are asymptotically continuous with respect to player i's beliefs f^i if, for every sequence of sets $\{A_k\}$, $A_k \in \Im^0$,

$$\mu_{f^i}(A_k) \underset{k\to\infty}{\longrightarrow} 0 \Rightarrow \mu_f(A_k) \underset{t\to\infty}{\longrightarrow} 0.$$

Proposition A.1 Chosen strategies f are absolutely continuous with respect to player i's beliefs f^i if and only if chosen strategies f are asymptotically continuous with respect to player i's beliefs f^i.

Proof Assume that f is absolutely continuous with respect to f^i. Take a sequence $A_k \in \Im_k$ such that $\mu_{f^i}(A_k) \to 0$ as $k \to \infty$. Assume, by contradiction, that there exists a subsequence, also indexed by k, $\{A_k\}$ such that $\mu_f(A_k) \geq \varepsilon > 0$.
Take a sub-subsequence, also indexed by k, such that $\mu_{f^i}(A_k) \leq \frac{1}{k^2}$.
Let $B = \bigcap\limits_{s=1}^{\infty} B_s$ where $B_s = \bigcup\limits_{k\geq s} A_k$. Then, $\mu_{f^i}(B_s) \leq \sum\limits_{k\geq s} \frac{1}{k^2} \underset{s\to\infty}{\to} 0$.
Thus, $\mu_{f^i}(B) \leq \mu_{f^i}(B_s) \underset{s\to\infty}{\to} 0$. Hence, $\mu_{f^i}(B) = 0$ and $\mu_f(B) = 0$.
On the other hand, $A_s \subset B_s \; \forall s \in N$ and $\mu_f(A_s) \geq \varepsilon \; \forall s \in N$.
So, $\mu_f(B_s) \geq \varepsilon \; \forall s \in N$. However, $B_{s+1} \subset B_s$. So, $B_s \downarrow B$.
Thus, $\mu_f(B_s) \downarrow \mu_f(B) = 0$. This is a contradiction.
Assume that f is asymptotically continuous with respect to f^i. Suppose, by contradiction, that there exists a set $A \in \Im$ such that $\mu_{f^i}(A) = 0$ and $\mu_f(A) > \delta > 0$.
By the Caratheodory extension theorem, it is possible to define the probability measure of a set in $\Im$ as a "limit" of the probability measure of sets in $\Im^0$. That is,

by the very construction of a probability measure in the Caratheodory extension theorem,

$$\mu_{f^i}(A) = \inf\{\sum_k \mu_{f^i}(B_k)/A \subset \sum_k B_k,\ B_k \in \Im_k\}.$$

The symbol $\sum$ appears instead of $\bigcup$, indicating that we consider only countable *disjoint* sets $B_k \in \Im_k$, whose union covers A. So, for every $\epsilon > 0$, there exists sets $B(\epsilon)_k \in \Im_k$ such that $\sum_{k=1}^{\infty} \mu_{f^i}(B(\epsilon)_k) < \epsilon$ and $A \subset \sum_{k=1}^{\infty} B(\epsilon)_k$. Hence, $\delta < \sum_{k=1}^{\infty} \mu_f(B(\epsilon)_k)$. Thus, there exists $k(\epsilon)$ such that $\delta < \sum_{k=1}^{k(\epsilon)} \mu_f(B(\epsilon)_k)$. Let $D(\epsilon)$ be the set $\sum_{k=1}^{k(\epsilon)} B(\epsilon)_k$. Then,

$$\mu_{f^i}(D(\epsilon)) \leq \mu_{f^i}\left(\sum_{k=1}^{\infty} B(\epsilon)_k\right) < \epsilon. \text{ Thus, } \mu_{f^i}(D(\epsilon)) \underset{\epsilon\to 0}{\to} 0.$$

Consider the sequence $D(\frac{1}{s}) \in \Im_{k(\frac{1}{s})}$. $\mu_f\left(D(\frac{1}{s})\right) \underset{s\to\infty}{\to} 0$ because $\mu_{f^i}\left(D(\frac{1}{s})\right) \underset{s\to\infty}{\to} 0$.

But, $\mu_f\left(D(\frac{1}{s})\right) = \mu_f\left(\sum_{k=1}^{k(\frac{1}{s})} B(\frac{1}{s})^k\right) > \delta$. This is a contradiction.

q.e.d.

Given $w \in \Sigma^{\infty}, w = (w(t), ...)$, $w(t) = (w(t-1), a)$, $a \in \Sigma$, let x_t^i be $\Im_t$ measurable functions defined by:

$$\begin{aligned} x_t^i(w) &= 1 \text{ if } \mu_f(C(w(t))) = 0; \text{ and} \\ x_t^i(w) &= \frac{\mu_{f^i}(C(w(t)))}{\mu_f(C(w(t)))} \text{ if } \mu_f(C(w(t))) > 0. \end{aligned}$$

That is, x_t^i is the ratio of player i's subjective probability over $(t+1)$-histories and the true probability of this history.

Proposition A.2 Chosen strategies are absolutely continuous with respect to player i's beliefs if and only if, almost surely with respect to μ_f, x_t^i converges to a strictly positive number as t goes to infinity.

Proof - $\Rightarrow$) Assume, by contradiction, that f is not absolutely continuous with respect to f^i. By proposition $A.1$, there exist $\varepsilon > 0$ and a sequence $\{A_n\} \in \Im^0$, such that

$$\mu_f(A_n) > \varepsilon \text{ and } \mu_{f^i}(A_n) \underset{n\to\infty}{\longrightarrow} 0$$

Let $B_{m,\delta} \in \Im$, $m \in N$, be defined by

$$B_{m,\delta} = \left\{ w \in \Sigma^\infty,\ x_t(w) \geq \frac{\delta}{2}\ \forall t \geq m \right\}.$$

By assumption, x_t converges to a strictly positive random variable x (almost surely with respect to μ_f). Hence, there exists $\bar{\delta} > 0$ small enough and $\bar{m} \in N$ large enough such that

$$\mu_f\left(B_{\bar{m},\bar{\delta}}\right) \geq 1 - \frac{\varepsilon}{2}.$$

Thus, any set such that the intersection with $B_{\bar{m},\bar{\delta}}$ is empty has μ_f-measure smaller than $\frac{\varepsilon}{2}$. Any set $A_n \in \Im^0$ is a finite disjoint union of cylinders that belongs to $\bigcup\limits_{j\geq\bar{m}} \Im^j$. Let $D_n \in \Im^0$ be A_n minus all its cylinders that do not intersect $B_{\bar{m},\bar{\delta}}$. Note that the *union* of all these cylinders has μ_f-measure smaller than $\frac{\varepsilon}{2}$. Thus,

$$\mu_f(D_n) \geq \frac{\varepsilon}{2} \text{ and } \mu_{f^i}(D_n) \underset{n\to\infty}{\longrightarrow} 0.$$

By definition, D_n is a finite sum of disjoint cylinders $\{C_k\} \in \Im^0$ such that

$$\mu_{f^i}(C_k) \geq \frac{\bar{\delta}}{2}\mu_f(C_k).$$

Therefore,

$$\mu_{f^i}(D_n) \geq \frac{\bar{\delta}}{2}\mu_f(D_n).$$

A contradiction.

$\Leftarrow$)Assume that f is absolutely continuous with respect to f^i. By the Radon-Nikodym theorem there exists a random variable y^i (the Radon-Nikodym derivative) such that $\mu_f = \int y^i \partial\mu_{f^i}$ and y^i is strictly positive (almost surely with respect to μ_{f^i}). However, under absolute continuity, $y_t^i \equiv 1/x_t^i$ converges to y^i almost surely with respect to μ_{f^i}, (see Kalai and Lehrer [94], proposition 1). Hence, x_t^i converges to $x^i \equiv 1/y^i$ almost surely with respect to μ_f.

q.e.d.

Proof of Proposition 2 - Assume that player i's beliefs and chosen strategies merge. By proposition 1, chosen strategies are absolutely continuous with respect to player i's beliefs.

By definition, $\sum_{j=1}^{t} z_j^i = \log x_t^i$. Hence, by proposition $A.2$, $\sum_{j=1}^{\infty} z_j^i > -\infty$ $a.s.$ μ_f.

Define $\hat{z}_t^i = z_t^i - e_t^i$. It is known that $e_t^i \leq 0$, (see Lehrer and Smorodinsky [96], lemma 2). Thus, $\liminf \sum_{j=1}^{t} \hat{z}_j^i > -\infty$ $a.s.$ μ_f. By definition, $E\{(\hat{z}_t^i)/\Im_{t-1}\} = 0$. By assumption, $|\hat{z}_t^i|$ is uniformly bounded. That is, there is an $T > 0$ such that $|\hat{z}_t^i| \leq T$. Assume, by contradiction, that $\sum_{t=0}^{\infty} v_t^i = \infty$ on a set of paths to which μ_f assigns strictly positive probability. By Freedman [75], corollary 4.5, part (a), $\liminf \sum_{j=1}^{t} \hat{z}_j^i = -\infty$ in a set to which μ_f assigns strictly positive probability. A contradiction. Thus, $\sum_{t=0}^{\infty} v_t^i < \infty$ $a.s.$ μ_f. Hence, by Freedman [75], corollary 4.5, part (b), $\left|\sum_{j=1}^{\infty} \hat{z}_j^i\right| < \infty$ $a.s.$ μ_f. Therefore, $\sum_{j=1}^{\infty} e_j^i > -\infty$ $a.s.$ μ_f.

On the other hand, assume that $\sum_{t=0}^{\infty} v_t^i < \infty$ and $\sum_{j=1}^{\infty} e_j^i > -\infty$ $a.s.$ μ_f. Then, by Freedman [75], corollary 4.5, part (b), $\left|\sum_{j=1}^{\infty} \hat{z}_j^i\right| < \infty$ $a.s.$ μ_f. Thus, $\sum_{j=1}^{\infty} z_j^i > -\infty$ $a.s.$ μ_f.By proposition $A.2$, chosen strategies are absolutely continuous with respect to player i's beliefs. By proposition 1, player i's beliefs and chosen strategies merge.

q.e.d.

Lemma A.1 If player i's beliefs and chosen strategies merge then

$$\sum_{t=0}^{\infty} E\{\left(z_t^i\right)^2 / \Im_{t-1}\} < \infty \ a.s. \ \mu_f.$$

Proof - By proposition 2, $\sum_{t=0}^{\infty} v_t^i < \infty$ and $\sum_{j=1}^{\infty} e_j^i > -\infty$ $a.s.$ μ_f.

However, $(-1)e_j^i > 0$. Thus, $\sum_{j=1}^{\infty} \left(e_j^i\right)^2 < \infty$ $a.s.$ μ_f. By definition, $v_t^i =$

$E\{(z_t^i)^2/\Im_{t-1}\} - (e_t^i)^2$. Hence, $\sum_{t=0}^{\infty} E\{(z_t^i)^2/\Im_{t-1}\} < \infty$ $a.s.$ μ_f.

q.e.d.

Given a natural number $k \geqslant 1$, let $z_{k,t}^i$ be $\Im_{t+k}-$measurable functions defined by:

$$z_{k,t}^i(w) = 1 \text{ if } \mu_f(C(w(t+k))) = 0; \text{and}$$
$$z_{k,t}^i(w) = \log\left(\frac{\mu_{f^i_{w(t)}}(C(w(t+k)))}{\mu_{f_{w(t)}}(C(w(t+k)))}\right) \text{ if } \mu_f(C(w(t+k))) > 0,$$

where $w \in \Sigma^\infty$, $w = (w(t+k), ...)$, $w(t+k) = (w(t), ...)$.

Lemma A.2 If player i's beliefs and chosen strategies merge then for every natural number k,

$$\sum_{t=0}^{\infty} E\{(z_{k,t}^i)^2/\Im_t\} < \infty \ a.s. \ \mu_f.$$

The proof of lemma $A.2$ is completely analogous to the arguments given in proposition 2 and lemma $A.1$. Therefore, this proof is omitted.

Lemma A.3 Consider a sequence $\{c_n, n \geqslant 0\}$ such that $c_n > 0$ and $\sum_{n \geqslant 0} c_n < \infty$. Then, nc_n goes to zero with density one.

Proof - Assume, by contradiction, that there exists $\varepsilon > 0$ and $\delta > 0$ and a subsequence $n(k)$ such that, for every $k > 0$,

$$\sharp\{j/\ jc_j \geqslant \varepsilon, 1 \leqslant j \leqslant n(k)\} \geqslant \delta n(k).$$

Let $A_{n(k)}$ be the set $\{j/\ jc_j \geqslant \varepsilon, 1 \leq j \leq n(k)\}$. Then, $\sharp\{A_{n(k)}\} \geqslant \delta n(k)$. Consider a sub subsequence, $k(r)$, such that $n(k(r+1)) > \frac{2}{\delta}n(k(r))$. By definition,

$$\sum_{j \in A_{n(k(r+1))},\ j \geqslant n(k(r))}^{n(k(r+1))} c_j \geqslant \varepsilon \sum_{j \in A_{n(k(r+1))},\ j \geqslant n(k(r))}^{n(k(r+1))} \frac{1}{j}.$$

However, $\sharp\{A_{n(k(r+1))}\} \geqslant \delta n(k(r+1))$. Hence,

$$\sharp\{j/\ j \in A_{n(k(r+1))},\ j \geqslant n(k(r))\} \geqslant \delta n(k(r+1)) - n(k(r)) > \frac{\delta}{2} n(k(r+1).$$

Thus,

$$\sum_{j \in A_{n(k(r+1))},\ j \geqslant n(k(r))}^{n(k(r+1))} \frac{1}{j} \geqslant \frac{\sharp\{j/\ j \in A_{n(k(r+1))},\ j \geqslant n(k(r))\}}{n(k(r+1)} \geqslant \frac{\delta}{2}.$$

Therefore,

$$\sum_{j \in A_{n(k(r+1))},\ j \geqslant n(k(r))}^{n(k(r+1))} c_j \geqslant \varepsilon \frac{\delta}{2} \Longrightarrow \sum_{j}^{\infty} c_j = \infty.$$

A contradiction.

q.e.d.

Lemma A.4 If $s \in (0,1]$ and $r \in (0,1]$ then $\left|\log(\frac{s}{r})\right| \geqslant |s-r|$.

Proof - Consider the function $f(x) = log(x) - x$. Then, $\frac{\partial}{\partial x} f(x) = \frac{1}{x} - 1$. Hence, $\frac{\partial}{\partial x} f(x) \geqslant 0$ if $x \in (0,1]$. So, $log(s) - s \geqslant log(r) - r$ if $s \geqslant r$ and $log(s) - s \leqslant log(r) - r$ if $s \leqslant r$. Thus, $\left|\log(\frac{s}{r})\right| = |\log(s) - \log(r)| \geqslant |s-r|$.

q.e.d.

Lemma A.5 Let $\{a_i, i = 1, ..., T\}$ and $\{b_i, i = 1, ..., T\}$ be two finite sequences of real numbers such that $a_i \geqslant 0$, $b_i \geqslant 0$, $\sum_{i=1}^{T} a_i = 1$, and $\sum_{i=1}^{T} b_i = 1$. Assume that $b_i \geqslant \gamma > 0$ whenever $b_i > 0$. Then,

$$\max_{i=1,..,T} |a_i - b_i| \leqslant \frac{1}{\gamma} \sum_{i=1}^{T} b_i \left|a_i - b_i\right|.$$

Proof - Let j be such that $|a_j - b_j| \geqslant |a_i - b_i|$, $i = 1, ..., T$. If $b_j > 0$ then $b_j \geqslant \gamma$ and so, $|a_j - b_j| \leqslant \frac{b_j}{\gamma} |a_j - b_j| \leqslant \frac{1}{\gamma} \sum_{i=1}^{T} b_i |a_i - b_i|$. Consider the case $b_j = 0$. By assumption, $\sum_{i=1}^{T} a_i = \sum_{i=1}^{T} b_i = 1$. Then,

$$a_j = a_j - b_j = \sum_{b_i > 0, i \neq j} (b_i - a_i) + \sum_{b_i = 0, i \neq j} (b_i - a_i) = \sum_{b_i > 0, i \neq j} (b_i - a_i) - \sum_{b_i = 0, i \neq j} a_i.$$

Hence,

$$|a_j - b_j| = a_j \leqslant a_j + \sum_{b_i=0, i\neq j} a_i = \sum_{b_i>0, i\neq j} (b_i - a_i) \leqslant \sum_{i=1}^{T} \frac{b_i}{\gamma} |a_i - b_i| .$$

q.e.d.

Proof of Proposition 3 - By lemma $A.2$, for every natural number k,

$$\sum_{t=0}^{\infty} E\{\left(z_{k,t}^i\right)^2 / \Im_t\} < \infty \ a.s.\ \mu_f.$$

By lemma $A.3$, almost surely with respect to μ_f, $tE\{\left(z_{k,t}^i\right)^2 / \Im_t\}$ converges to zero, with density one. Thus, almost surely with respect to μ_f, $t^{0.5}E\{\left|z_{k,t}^i\right| / \Im_t\}$ converges to zero, with density one.

Given a natural number $k \geqslant 1$, let $y_{k,t}^i$, be $\Im_{t+k}$-measurable functions defined by:

$$y_{k,t}^i(w) = \left|\mu_{f_{w(t)}^i}(C(w(t+k))) - \mu_{f_{w(t)}}(C(w(t+k)))\right|$$

where $w \in \Sigma^\infty$, $w = (w(t+k), ...)$, $w(t+k) = (w(t), ...)$. By lemma $A.4$,

$$E\{y_{k,t}^i / \Im_t\} \leqslant E\{\left|z_{k,t}^i\right| / \Im_t\}.$$

Therefore, almost surely with respect to μ_f, $t^{0.5}E\{y_{k,t}^i / \Im_t\}$ converges to zero, with density one. By lemma $A.5$,

$$d_l(f_{w(t)}^i, f_{w(t)}) \leqslant \max_{k=1,...,l} \frac{1}{\gamma} E\{y_{k,t}^i / \Im_t\}.$$

Hence, there exists a set $\Omega \in \Im$ such that $\mu_f(\Omega) = 1$, and for every path $w \in \Omega$, $w = (w(t), ...)$, $t^{0.5}d_l(f_{w(t)}, f_{w(t)}^i)$ goes to zero, with density one.

q.e.d.

Lemma A.6 - Fix $\varepsilon > 0$. If $n^{(1+\varepsilon)} |c_n|$ goes to zero then $\left|\sum_{n\geqslant 0} c_n\right| < \infty$.

Proof - For n large enough, $|c_n| \leqslant \frac{1}{n^{(1+\varepsilon)}}$. So, $\sum\limits_{n\geqslant 0} |c_n| < \infty$. However, for every natural number k, $\left|\sum\limits_{n=1}^{k} c_n\right| \leqslant \sum\limits_{n=1}^{k} |c_n|$.

q.e.d.

Lemma A.7 - If $\frac{s}{r} \geqslant \delta > 0$ and $r \geqslant \gamma > 0$ then $\left|\log \frac{s}{r}\right| \leqslant \frac{1}{\gamma\delta} |s - r|$.

Proof - By assumption, $s \geqslant \gamma\delta$ and $r \geqslant \gamma\delta$. By the mean value theorem, there exists $\xi \geqslant \gamma\delta$ such that $\log s - \log r = \frac{1}{\xi}(s - r)$. Hence, $\left|\log \frac{s}{r}\right| \leqslant \frac{1}{\gamma\delta} |s - r|$.

q.e.d.

Proof of Proposition 4 - Let $y^i_{1,t}$ and $z^i_{1,t}$ be defined as in the proof of proposition 3. By lemma $A.7$, for every $w \in \Sigma^\infty$, $w = (w(t), ...)$, if $\mu_f(C(w(t))) > 0$ then $\left|z^i_{1,t}(w)\right| \leqslant \frac{1}{\gamma\delta} y^i_{1,t}(w)$. By definition, $y^i_{1,t}(w) \leqslant d_1(f_{w(t)}, f^i_{w(t)})$. By assumption, and lemma $A.6$, there exists a set $\Omega \in \Im$, such that $\mu_f(\Omega) = 1$, and if $w \in \Omega$ then $\sum\limits_{j=0}^{\infty} d_1(f_{w(t)}, f^i_{w(t)}) < \infty$. Hence, almost surely with respect to μ_f, $\sum\limits_{j=0}^{\infty} \left|z^i_{1,j}\right| < \infty$. However, $\sum\limits_{j=0}^{t-1} z^i_{1,j} = \log x^i_t$. So, almost surely with respect to μ_f, x^i_t converges to a strictly positive number. By proposition $A.2$, chosen strategies are absolutely continuous with respect to player i's beliefs. By proposition 1, player i's beliefs and chosen strategies merge.

q.e.d.

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