Discussion Paper No. 1187

EXPLAINING POSITIONAL VOTING PARADOXES II; THE GENERAL CASE

by

Donald G. Saari
Department of Mathematics
Northwestern University

April 1997

EXPLAINING POSITIONAL VOTING PARADOXES II; THE GENERAL CASE

DONALD G. SAARI

ABSTRACT. A theory is developed to explain all possible positional voting paradoxes coming from a single but arbitrarily chosen profile. This includes all pairwise voting cycles, all conflict between Borda and Condorcet winners and rankings, all disagreement in outcomes among positional procedures, and all discrepancies among rankings for any positional procedure as candidates are dropped or added. The theory explains why each of the possible paradoxes occurs while describing how to construct illustrating profiles. It is shown how to use this approach to derive properties of other procedures based on positional voting methods. The three candidate results of the companion paper [19] are extended to an arbitrary number of candidates.

1. INTRODUCTION

In this and a companion paper (Saari [19]), I explain all possible positional voting paradoxes that can be generated by a single but arbitrarily chosen profile. Positional procedures are the commonly used methods where points are assigned to candidates according to how each voter positions them on the ballot. Familiar choices are the plurality vote where a single point is assigned to a voter's top-ranked candidate and zero to all others, and the Borda Count (BC) where $n-1, n-2, \ldots, n-n = 0$ points are assigned, respectively, to a voter's first, second, ..., nth ranked candidate. This paper extends the (Saari [19]) results from three to any number of candidates.

Among the problems caused by considering more candidates are to further complicate traditional choice theory issues and to introduce new concerns. For instance, extra candidates exasperate problems with cyclic and nontransitive pairwise outcomes, the conflict between the BC and Condorcet winners, and differences in positional election outcomes. New issues include understanding why the same sincere voters' societal rankings of different subsets of candidates can differ so radically. As an illustration, paralleling the pairwise voting cycles are cycles of the three-candidate positional rankings. Explanations for all of this behavior, and for anything else that can occur, follow from the approach introduced here.

1.1. **Problems of the BC.** One longstanding choice theory theme emphasizes the BC's notoriety to radically change the societal ranking when candidates are added or dropped. Brams [3] nicely captures this with an example where the BC ranking is $C \succ B \succ A$, but once X joins the race it becomes $A \succ B \succ C \succ X$. Brams reflects a widely held belief with his comment that the BC behavior allowing A to vault to top place "when 'irrelevant' candidate X is introduced [illustrates] the extreme sensitivity of the Borda count to apparently irrelevant alternatives."

The actual situation is much worse. To create far more perplexing BC behavior, consider the nested subsets of candidates $\{c_1, c_2, \ldots, c_n\}$, $\{c_1, c_2, \ldots, c_{n-1}\}$, ..., $\{c_1, c_2\}$ obtained by

This research was supported by an NSF grant and my Arthur and Gladys Pancoe Professorship. Earlier versions were written while visiting CREME, Universite de Caen, in May of 1995 and of 1996; my thanks to my host M. Salles.

dropping one candidate at each stage. Choose a ranking for each subset; these rankings can be selected randomly or even in a perverse manner to create a particularly disturbing example. As proved in (Saari [11]), there are voters' preferences where their sincere BC ranking of each set is the chosen one. This, for instance, ensures a profile with the ten-candidate BC ranking $c_2 \succ c_3 \succ c_4 \succ \cdots \succ c_{10} \succ c_1$ even though the profile's BC rankings of the other eight subsets agree with $c_1 \succ c_2 \succ c_3 \succ \cdots \succ c_9$. The rankings of the subsets strongly support c_1 as the candidate of choice, but once the "essentially irrelevant candidate" c_{10} is admitted, c_1 falls to the bottom. Brams' troubling example, then, only hints at the actual level of perversity admitted by the BC.

This [11] result also extends other aspects of the BC's so-called notoriety that were identified by Arrow and Raynaud [2]. Arrow and Raynaud formalized appealing "sequential axioms" which, essentially, impose conditions on the relative ranking of certain candidates when candidates are added or dropped. For instance, instead of insisting that the same two candidates always are top-ranked, we might merely require their relative ranking to remain fixed as other candidates are dropped from competition. As both Arrow and Raynaud and my [11] result prove, the BC fails to satisfy their conditions.

To derive stronger assertions, relax the Arrow-Raynaud conditions so that the relative ranking does not change "too much." No matter what definition is used to quantify "too much," it must impose some restriction. Consequently, the [11] assertion proves that the BC cannot satisfy even these more forgiving axioms.

1.2. **Explanations.** These seemingly devastating conclusions appear to indict the BC. But before rendering judgement, we must understand whether these difficulties are peculiar to the BC, or suffered by all positional procedures. If the latter, why? Which procedures are more prone to these problems? Is this random appearing behavior restricted to nested subsets of candidates, or is it a general phenomenon? What causes these behaviors? As a radical counter suggestion, is this behavior sufficiently natural to signal that procedures may be flawed if they always avoid these seemingly chaotic outcomes?

These and a host of other questions reflecting central concerns of choice theory finally can be resolved with the approach developed here; many of the surprising answers counter accepted beliefs. As described in [19], the approach divides profiles[1] into component parts. Each component captures a particular election behavior. By describing a profile in terms of its components, it now is possible to determine all of its paradoxical or positive behavior. Central to this division is what I call the *basic profile* component; this is where all positional procedures and the pairwise vote *share the same election ranking and (normalized) election tally* over all subsets of candidates. In other words, no conflict of any kind occurs on the basic portion.

The profile portions orthogonal to the space of basic profiles create what I call "profile noise." These are the profile components where, arguably, the outcome should be a complete tie. Yet, subtle peculiarities of positional procedures deny this natural conclusion. So, by understanding how different procedures behave on different components of profile noise, we can explain all possible paradoxical behavior and create illustrating examples. Of particular interest, we now can determine the subtle costs and tradeoffs involved in using any procedure based on pairwise and positional outcomes.

Although these introductory comments paint a dreary picture where no procedure is reliable, this is not the case. A fundamental conclusion of this paper is that *the BC outcome for all n candidates is the unique ranking to be trusted.* In light of my recitation of disturbing BC

[1] A profile specifies the preferences of each voter.

behavior, this surprising assertion requires explanation. Support for this claim comes from the fact developed here that the BC outcome for n candidates is strictly determined by the basic profile; this is where the rankings of all procedures over all subsets of candidates agree. Consequently, all differences in the election rankings of other procedures and other subsets of candidates are caused by forms of profile noise which do not affect the BC ranking of all n candidates.

This assertion mandates that a particular type of profile noise generates the behavior of [11] where the BC rankings can change with the subsets of candidates. This is correct, and this noise requires the election rankings and tallies of *all* positional methods over all subsets to agree with that of the BC. Thus, rather than being specific to the BC, *all procedures* suffer this same disturbing behavior with the same profiles. Moreover, the theory explains that this paradoxical behavior occurs because *this particular profile noise emasculates the crucial assumption that voters have transitive preferences.* Namely, the contrary BC rankings of the subsets of candidates manifest the dismissal of information about individual rationality. Notice how this analysis surprisingly contradicts Brams' generally accepted argument. It is the BC ranking of four candidates which should be trusted because (as shown later) the conflicting BC ranking obtained by dropping X manifests a weakening of the assumption of individual rationality.

The BC rankings are immune to all remaining types of profile noise. But as this noise alters the outcomes for non-BC positional methods, all remaining positional procedures admit more kinds of paradoxical difficulties than the BC. In fact, rather than being viewed as "notorious," the surprising conclusion is that only the BC provides stability and consistency in outcomes. As illustrated in Sect. 6.3 with the Arrow-Raynaud procedure [2], an unexpected corollary is that this BC variation of rankings of subsets is sufficiently natural to cause worry about choice procedures which do *not* exhibit similar variations in rankings as candidates are added or dropped. (The theory identifies the source of the problem.) This assertion not only contradicts widely accepted previous beliefs, but even contradicts a central research objective.

To extend results from [19], I use this profile decomposition to analyze criteria such the Condorcet Principle for any number of candidates. Although the Condorcet Principle, "which asserts that a candidate that has a simple majority over every other candidate should be the social choice, has been accepted almost without question by a number of writers" (page 79, Fishburn and Gehrlein [5]), we discover that it is highly flawed. This surprising conclusion relies upon the fact that the pairwise vote dismisses valuable information concerning the individual rationality of voters. But this dismissal of a crucial assumption forces the pairwise and Condorcet outcomes to be suspect. (A closely related argument explains all cycles and other non-transitive pairwise rankings.) If the measuring stick of the Condorcet Principle is warped, how can we trust its measurements? In other words, if a procedure fails the Condorcet Principle, is the procedure, the pairwise conclusion, or both flawed?

2. Preliminaries

The notation is essentially the same as in my earlier papers. List, in any manner, the $n!$ possible transitive ways to strictly rank the $n \geq 3$ candidates $\{c_1, c_2, \dots, c_n\}$. A *profile* specifies the number of voters of the jth type, $j = 1, \dots, n!$ The n candidates define $2^n - (n+1)$ subsets of two or more candidates; list them in some manner as $S_1, S_2, \dots, S_{2^n-(n+1)}$ where $|S_j|$ denotes the number of candidates in S_j. To represent the S_j election tally as a vector in a $|S_j|$-dimensional space, assign each axis to a candidate from S_j in increasing order of the subscripts. So, for $S_j = \{c_2, c_4, c_5\}$, the $(4, 23, 13)$ *vector tally* defines the ranking $c_4 \succ c_5 \succ c_2$ with the 23:13:4 tally.

Denote a positional voting method for S_j by *voting vector* $\mathbf{w}^{S_j} = (w_1, w_2, \dots, w_{|S_j|} = 0)$ where $w_1 = 1$, $w_j \geq w_{j+1} \geq 0$ for $j = 1, \dots, |S_j| - 1$. This normalized choice (of $w_1 = 1$ and $w_{|S_j|} = 0$) simplifies the comparisons of procedures and results. Converting a voting vector into its normalized form is trivial; e.g., if a four-candidate subset is tallied with $(7, 3, 2, 0)$, the normalized version is $(\frac{7}{7}, \frac{3}{7}, \frac{2}{7}, 0)$. Similarly, if the plurality method is used with $\{c_1, c_3, c_4\}$ and the BC with $\{c_2, c_3, c_4, c_5\}$, then the normalized voting vectors are, respective, $(1, 0, 0)$ and $(1, \frac{2}{3}, \frac{1}{3}, 0)$. In tallying ballots with $\mathbf{w}^{S_j}$, w_j points are assigned to a voter's jth ranked candidate, $j = 1, \dots, |S_j|$, and the candidates are ranked according to the sum of assigned points.

The *system voting vector*, $\mathbf{W}^n = (\mathbf{w}^{S_1}, \dots, \mathbf{w}^{S_{2^n-(n+1)}})$ specifies that $\mathbf{w}^{S_j}$ is used to tally the S_j election, $j = 1, \dots, 2^n - (n+1)$. Let $F(\mathbf{p}, \mathbf{W}^n)$ and $\tilde{F}(\mathbf{p}, \mathbf{W}^n)$ be, respectively, the lists of election tallies and election rankings over all subsets of candidates defined by profile $\mathbf{p}$ with system vector $\mathbf{W}^n$. To illustrate with $S_1 = \{c_1, c_2\}$, $S_2 = \{c_1, c_3\}$, $S_3 = \{c_2, c_3\}$, $S_4 = \{c_1, c_2, c_3\}$, the system vector $\mathbf{W}^3 = [(1, 0), (1, 0), (1, 0), (1, 0, 0)]$ requires the pairs to be tallied with the majority rule (the voting vectors (1,0)) and the triplet with the plurality vote (voting vector (1,0,0)). The fifty-voter profile $\mathbf{p}$ where three voters have preferences $c_1 \succ c_2 \succ c_3$, 24 voters have $c_3 \succ c_1 \succ c_2$, and 23 voters have $c_2 \succ c_1 \succ c_3$ defines the election rankings

$$\tilde{F}(\mathbf{p}, \mathbf{W}^3) = [c_1 \succ c_2,\ c_1 \succ c_3,\ c_2 \succ c_3,\ c_3 \succ c_2 \succ c_1] \tag{2.1}$$

Notice the conflict between the pairwise and plurality rankings. All of these inconsistencies are characterized and explained. Also, I show which $\mathbf{W}^n$ choices minimize inconsistencies.

2.1. **Words.** The numbers and kinds of admissible election inconsistencies — paradoxes — are staggering. To illustrate, Eq. 2.1 lists the election ranking for each subset of candidates coming from the specified profile — I call such a listing a *word.* Different profiles can define different plurality words so the number and kinds of different words measures the complexity and randomness of a procedure. It turns out that the plurality procedure admits 351 different words for $n = 3$ candidates and over a *billion* different words (1,041,048,450) for $n = 4$ candidates. This means there are over a billion different ways to list rankings for the six pairs, four triplets, and the set of all four candidates, and each listing is the sincere plurality election outcome for some profile. These numbers overwhelm any naive belief that the election rankings of the pairs and triplets must agree with the election ranking of the four candidates. (If this naive wish were true, only 50 plurality words could occur. Of these, $4! = 24$ have no ties, the rest have at least one tie vote.) The following assertion about all subsets of candidates (not just the nested sets considered in [11]) demonstrates the severity of the problem.

Theorem 1. *(Saari* [16]*) For $n \geq 3$ candidates, suppose all subsets of candidates are tallied with the plurality method. For each subset, choose a ranking. As these rankings can be selected randomly, there need not be a relationship among any of them. There exists a profile so that the voters' sincere plurality ranking of each subset of candidates is the selected one.*[2]

According to this theorem, there is a profile where its plurality rankings of all subsets with an even number of candidates match $c_1 \succ c_2 \succ c_3 \succ \cdots \succ c_n$, but its plurality rankings of subsets with an odd number of these candidates reverse the ranking. A more disturbing conclusion is that we could use a random number generator to select the ranking of each subset, and (according to Thm. 1) there is a profile where the voters' sincere plurality election ranking

[2] To use this result to compute the number of four candidate plurality words, notice that there are three ways to rank a pair (including ties), thirteen ways to rank a triplet, and 50 ways to rank four candidates. Thus, the total number of words is $3^6 \times (13)^4 \times 50$. Similarly, the number of plurality words – election paradoxes – for five candidates escalates over ten million billion fold to $3^{10} \times (13)^{10} \times (50)^4 \times 630 = 3.21 \times 10^{25}$.

of each set agrees with the randomly generated rankings. This is not a ringing endorsement for our standard tool of democracy.

2.2. **Other procedures.** This electoral nightmare caused by the plurality method is shared by most methods. The next result uses the fact that $\mathbf{W}^n$ is a vector in a $\nu(n) = 2^{n-1}(n-4)+n+2$ dimensional Euclidean space. (The derivation of $\nu(n)$ is in Sect. 10.) In $R^{\nu(n)}$, an *algebraic set* is a lower dimensional subset representing the zeros of a particular collection of algebraic equations.

Theorem 2. *(Saari, [16]) With the exception of an algebraic set $\alpha^n \subset R^{\nu(n)}$, all other system vectors in $R^{\nu(n)}$ have the same property as described for the plurality vote in Theorem 1.*

Only the highly exceptional $\mathbf{W}^n$ tallying procedures in α^n offer any consistency in outcomes with election relationships. The α^n entries and election relationships are described.

3. Division of voting vectors

As it will be shown, all paradoxes manifest the ways positional procedures treat different kinds of profile noise. In a natural manner each subspace of profile noise defines an associated subspace of positional procedures; these are the procedures which react to this particular kind of noise. Thus, accompanying the decomposition of profiles is a dual decomposition of positional methods. To develop this duality, I exploit the linearity of $F(\mathbf{p}, \mathbf{W}^n)$ in both variables. This technical description is fundamental for the profile decomposition which starts in the next section.

To illustrate F's linearity with respect to the voting vector, suppose a four-candidate election is tallied with both $(5,2,1,0)$ and $(2,1,0,0)$. Because $(9,4,1,0) = 2(2,1,0,0)+(5,2,1,0)$, the $(9,4,1,0)$ election tally is the same as adding twice each candidate's tally from the second election to her tally from the first one. In normalized form, the computation is $(1,\frac{4}{9},\frac{1}{9},0) = \frac{5}{9}(1,\frac{2}{5},\frac{1}{5},0)+\frac{4}{9}(1,\frac{1}{2},0,0)$. Implications of the horizontal decomposition follow.

3.1. **Horizontal decomposition.** A n-candidate voting vector is a convex combination of the $(n-1)$ voting vectors $\{\mathbf{v}^n_j\}_{j=1}^{n-1}$ where $\mathbf{v}^n_j$'s first j components are ones and the rest are zeros. (So, $\mathbf{v}^n_j$ represents those n candidate elections where we vote for j candidates.) Clearly, the convex hull defined by $\{\mathbf{v}^n_j\}_{j=1}^{n-1}$ includes all possible (normalized) voting vectors for n candidates. Denote this *n-candidate pyramid of voting vectors* by

$$\mathcal{P}^n = \{\, \mathbf{w}^n = \sum_{j=1}^{n-1} \lambda_j \mathbf{v}^n_j \mid \lambda_j \geq 0, \sum_{j=1}^{n-1} \lambda_j = 1\}. \tag{3.1}$$

The $\mathcal{P}^n$ dimension of $(n-2)$ reflects the $n-2$ weights needed to define a n-candidate voting vector. The BC voting vector $\mathbf{b}^n = \frac{1}{n-1}\sum_{j=1}^{n-1}\mathbf{v}^n_j$ is at the $\mathcal{P}^n$ barycenter.

To illustrate with the ten-voter profile

Number	Preference	Number	Preference
1	$A \succ B \succ C \succ D$	1	$D \succ A \succ B \succ A$
2	$B \succ A \succ C \succ D$	3	$D \succ C \succ B \succ A$
3	$C \succ A \succ B \succ D$		

(3.2)

the $\mathbf{v}^4_1 = (1,0,0,0)$, $\mathbf{v}^4_2 = (1,1,0,0)$, and $\mathbf{v}^4_3 = (1,1,1,0)$ respective tallies are $(1,2,3,4)$, $(7,3,6,4)$, and $(7,10,9,4)$. A $\mathcal{P}^4$ voting vector is expressed as $\sum_{j=1}^{3}\lambda_j\mathbf{v}^4_j$, so its election tally for the profile is $\lambda_1(1,2,3,4)+\lambda_2(7,3,6,4)+\lambda_3(7,10,9,4)$. All election tallies are in the triangle (called the *procedure hull*, Saari [15]) with vertices defined by the three $\{\mathbf{v}^4_j\}_{j=1}^{3}$ election tallies.

The $\mathbf{b}^4$ outcome is at the barycenter (where all $\lambda_j = \frac{1}{3}$) and elementary algebra proves that this profile allows *each* candidate to win when appropriate positional voting procedures are used. Similarly, there is a ten candidate profile $\mathbf{p}$ where its *procedure hull* – the $F(\mathbf{p}, -)$ image of $\mathcal{P}^{10}$ – has over 84 million different election rankings as the choice of a tallying procedure changes (Saari [15]). This significantly extends earlier results (e.g., Fishburn [4]) limited to asserting only that there are profiles which admit two different outcomes.

3.2. **Vertical connection.** Accompanying the horizontal procedure hull connection is a vertical relationship (Saari, [17]) relating election tallies for all k-candidate subsets with certain election tallies for subsets with more candidates. To illustrate with $n = 3$ and $k = 2$, a voter with preferences $c_1 \succ c_2 \succ c_3$ votes in the following manner in the three pairwise elections.

Candidates	$\{c_1\}$	$\{c_2\}$	$\{c_3\}$
$\{c_1, c_2\}$	1	0	–
$\{c_1, c_3\}$	1	–	0
$\{c_2, c_3\}$	–	1	0
Total	2	1	0

(3.3)

The sum of votes this voter gives a candidate over all pairs equals what he assigns her in a BC election. Thus (along with neutrality and the fact that each pair is tallied with the same voting vector) a candidate's BC election tally is the sum of her two pairwise election tallies from contests against candidates in the same subset; the normalized $\mathbf{b}^3$ tally is half this.

As the Eq. 3.3 summation property extends to define the BC vector for n candidates, the normalized $\mathbf{b}^n$ outcome is the sums of pairwise outcomes divided by $(n-1)$. To illustrate, the pairwise tallies of Eq. 3.2 for $\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{B,C\}$, $\{B,D\}$, $\{C,D\}$ are, respectively 5:5, 4:6, 6:4, 4:6, 6:4, 6:4, so the BC vector tallies for all candidates are $(5+4+6, 5+4+6, 6+6+6, 4+4+4)$ with a $\mathbf{b}^4$ vector tally $(5,5,6,4)$. Similarly, the $\mathbf{b}^3$ vector tally for $\{A,B,C\}$ is $\frac{1}{2}(5+4, 5+4, 6+6)$.

An identical Eq. 3.3 summation argument shows that when the same k-candidate voting vector $\mathbf{w}^k$ is used with all k-candidate subsets, it defines voting vectors for subsets with more candidates. In this way, for instance, the plurality election outcomes over all three-candidate subsets uniquely determine the $(3,1,0,0)$ four-candidate outcome, while $(1,1,0)$ determines the $(3,3,2,0)$ election rankings. (To show this compute the number of points a voter with preferences $A \succ B \succ C \succ D$ assigns each candidate over the four three-candidate elections.) This summation approach, then, naturally associates the three-candidate plurality vector $\mathbf{v}_1^3$ with the four-candidate voting vector $\frac{1}{3}(3,1,0,0) = (1, \frac{1}{3}, 0, 0)$ and $\mathbf{v}_2^3 = (1,1,0)$ with $(1,1,\frac{2}{3},0)$. To illustrate, the plurality tallies of Eq. 3.2 for $\{A,B,C\}$, $\{A,B,D\}$, $\{A,C,D\}$, $\{B,C,D\}$ are, respectively, $(2,2,6)$, $(4,2,4)$, $(3,3,4)$, $(3,3,4)$, so the $(3,1,0,0)$ tally of $(2+4+3, 2+2+3, 6+3+3, 4+4+4)$ is obtained by adding each candidate's tallies from her three three-candidate plurality elections.

The summation approach defines a mapping

$$g_k : \mathcal{P}^{k-1} \to \mathcal{P}^k, \quad k = 3, \ldots, n. \tag{3.4}$$

which expresses a class of $(k-1)$-candidate voting vectors as

$$\mathbf{w}^k = \sum_{j=1}^{k-2} \lambda_j g_k(\mathbf{v}_j^{k-1}), \quad \sum_{j=1}^{k-2} \lambda_j = 1. \tag{3.5}$$

(Function g_k is defined over one copy of $\mathcal{P}^{k-1}$ because the same voting vector is used over all $(k-1)$-candidate subsets. Technical extensions are reported elsewhere.)

The only restriction imposed on the scalars $\{\lambda_j\}_{j=1}^{k-2}$ in Eq. 3.5 is that they define a voting vector. In turn, the election outcomes of the Eq. 3.5 voting vectors are uniquely determined by the $\{\lambda_j\}_{j=1}^{k-2}$ values and the $\{\mathbf{v}_j^{k-1}\}_{j=1}^{k-2}$ election tallies. These dependencies define valuable ordering and consistency properties.

Definition 1. *The* derived set *of voting vectors in $\mathcal{P}^k$, denoted by $\mathcal{D}^k$, consists of all voting vectors that can be expressed in the form of Eq. 3.5*

The following theorem (which is proved in Sect. 10) collects useful structural properties of the derived set $\mathcal{D}^k$ and the pyramid $\mathcal{P}^k$.

Theorem 3. *For $k \geq 3$, the* voting pyramid *$\mathcal{P}^k$ is the convex hull of the k-candidate voting vectors $\{\mathbf{v}_j^k\}_{j=1}^{k-1}$. The BC voting vector, $\mathbf{b}^k$, is at the barycenter of $\mathcal{P}^k$.*

The vectors defining the derived set *$\mathcal{D}^k$ are*

$$\mathbf{u}_j^k = g_k(\mathbf{v}_j^{k-1}) = \frac{1}{k-1}(k-1,\dots,k-1,j,0,\dots,0), \quad j = 1,\dots,k-2, \tag{3.6}$$

where the j value is in the $(j+1)$ coordinate position. $\mathcal{D}^k$ is a $(k-3)$-dimensional subspace which includes the BC voting vector $\mathbf{b}^k$. A normal vector for the $\mathcal{D}^k$ affine space in $\mathcal{P}^k$, called the deviation vector, *is*

$$\mathbf{d}^k = (0, \binom{k-1}{1}, -\binom{k-1}{2}, \dots, (-1)^{k-1}\binom{k-1}{k-2}, 0) \tag{3.7}$$

Figure 1 shows the structure of derived set within $\mathcal{P}^n$. The deviation vector points into the portion of the pyramid which does *not* contain the plurality vector.

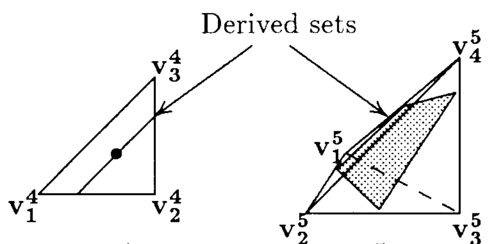


Fig. 1. Voting Pyramids $\mathcal{P}^n$

3.3. **Election relationships.** Notice how Thm. 3 relieves the previous need to individually compute the outcomes of each procedure for a given profile.

Corollary 1. *For a given n-candidate profile* $\mathbf{p}$, *all possible positional election rankings for all subsets of candidates are uniquely determined by the pairwise election outcomes and the deviation vector $\mathbf{d}^{|S_j|}$ outcomes for each S_j.*

The induction proof is immediate. First, the pairwise outcomes define all $\mathbf{b}^{|S_j|}$ outcomes. Next, all three-candidate voting vectors can be expressed as $\mathbf{w}^3 = \mathbf{b}^3 + \mu\mathbf{d}^3$, so the election outcome for a set is the $\mathbf{b}^3$ tally plus μ times the deviation $\mathbf{d}^3$ tally. This determines all $\{\mathbf{v}_j^3\}_{j=1}^2$ tallies, which, in turn, determines the $\mathbf{u}_j^4$ outcomes. These $\mathbf{u}_j^4$ tallies, along with the deviation vector $\mathbf{d}^4$ outcomes, determine all four-candidate tallies. By induction, all possible outcomes are found.

To illustrate Cor. 1 with the Eq. 3.2 profile, we already have shown how the pairwise elections determine all $\mathbf{b}^n$ tallies. The $\mathbf{d}^3 = (0,2,0)$ tallies of a three candidate subset is twice the number of times each candidate is in *second place,* while the $\mathbf{d}^4 = (0,3,-3,0)$ outcome is three times the difference between how often a candidate is in second and third place. So, the deviation vector $\mathbf{d}^n$ tallies of $\{A, B, C\}$, $\{A, B, C, D\}$ for Eq. 3.2 are, respectively, $2(5, 5, 0)$ and $3(6-0, 1-7, 3-3, 0)$.

The next step is to note from Thm. 3 that $\mathcal{D}^3 = \{\mathbf{b}^3\}$. Thus, Cor. 1 ensures that all positional outcomes are weighted sums of the $\mathbf{d}^3$ and $\mathbf{b}^3$ tallies. For instance, because

$$\mathbf{b}^3 - \frac{1}{4}\mathbf{d}^3 = (1, \frac{1}{2}, 0) - \frac{1}{4}(0, 2, 0) = (1, 0, 0), \tag{3.8}$$

the plurality tally of $\{A, B, C\}$ for Eq. 3.2 is $(4.5, 4.5, 6) - \frac{1}{4}(10, 10, 0) = (2, 2, 6)$. More generally, $\mathbf{w}^3_s = (1, s, 0) = \mathbf{b}^3 + \frac{1}{2}(s - \frac{1}{2})\mathbf{d}^3$, so the $\mathbf{w}^3_s$ outcome is $(2 + 5s, 2 + 5s, 6)$.

The same elementary computations handles the outcomes for four candidates. Because $\mathbf{v}^4_2 = (1, 1, 0, 0) = \mathbf{b}^4 + \frac{1}{9}\mathbf{d}^4 = (1, \frac{2}{3}, \frac{1}{3}, 0) + \frac{1}{9}(0, 3, -3, 0)$, the $\mathbf{v}^4_2$ outcome for Eq. 3.2 is $(5, 5, 6, 4) + \frac{1}{9}(18, -18, 0, 0) = (7, 3, 6, 4)$. All remaining four-candidate outcomes come from the computation of the $\mathbf{u}^4_j$ vertices which are determined from the three candidate tallies.

An immediate Cor. 1 consequence is that only the BC outcomes are related to the pairwise tallies – the tallies of all other procedures are distanced from the pairwise outcomes through the tallies of deviation $\mathbf{d}^k$ terms. This structure provides a new explanation for the known result that *the BC outcomes must be related to the pairwise ranking, but the rankings of any other procedure need not be related in any manner!* (The first part is due to Nanson [9]; the second part was found, in a very different manner, by Saari [16]. Sieberg [24] also noted and used this separation effect with a statistical interpretation.)

With Thm. 3, we can identify certain α^n system voting vectors which enjoy the following election relationships.

Corollary 2. *For k satisfying $2 \leq k < n$, let $\mathbf{w}^k$ be a voting vector. When all k-candidate elections are tallied with $\mathbf{w}^k$ and all $(k+1)$-candidate elections with $g_{k+1}(\mathbf{w}^k)$, the election outcomes satisfy the following relationships.*

1. *A candidate who is top-ranked in all $\mathbf{w}^k$ elections cannot be bottom-ranked in the $(k+1)$-candidate election tallied with $g_{k+1}(\mathbf{w}^k)$.*
2. *A candidate who is bottom-ranked in all $\mathbf{w}^k$ elections cannot be top-ranked in the $g_{k+1}(\mathbf{w}^k)$ election. Also, this candidate is $g_{k+1}(\mathbf{w}^k)$ strictly ranked below a candidate who always is $\mathbf{w}^k$ top-ranked.*
3. *If all $\mathbf{w}^k$ outcomes end in a complete tie, than the $g_{k+1}(\mathbf{w}^k)$ outcome also is a complete tie.*

As an illustration of Cor. 2, if a candidate wins all three candidate plurality elections, then she cannot be bottom ranked in the $g_4(\mathbf{v}^3_1) = \mathbf{u}^4_1 = \frac{1}{3}(3, 1, 0, 0)$ election. Similarly, if all three candidate antiplurality elections are tied (using $\mathbf{v}^3_2 = (1, 1, 0)$), then the four candidate election also is tied when tallied with $g_4(\mathbf{v}^3_2) = \mathbf{u}^4_2 = \frac{1}{3}(3, 3, 2, 0)$. However, the plurality, or the $\mathbf{u}^4_1$ outcomes need not be tied.

Proof. The proof of this important result, which significantly extends similar BC assertions, is trivial. A candidate who always is $\mathbf{w}^k$ top-ranked receives more than the average number of total votes cast over all k-candidate elections; consequently, she cannot be $g_{k+1}(\mathbf{w}^k)$ bottom-ranked. The proof of the second assertion is similar. The third assertion requires the same number of points to be added for each candidate. □

A natural way to avoid inconsistencies and election paradoxes, then, is to avoid deviation vector, $\mathbf{d}^k$, effects. To do so, tally all k-candidate elections with $\mathbf{w}^k$, the $(k+1)$-candidate elections with $g_{k+1}(\mathbf{w}^k)$, the $(k+2)$-candidate elections with $g_{k+2}(g_{k+1}(\mathbf{w}^k))$, ... It now follows that the fewest paradoxes along with the ultimate consistency and the largest number of election relationships require starting this string of voting vectors with the smallest value of $k = 2$. This is the Borda Count; this explains why the BC admits more consistency in election outcomes and more election relationships than any other positional procedure.

3.4. **Paradoxes and examples.** Theorem 3 and its corollaries provide unlimited opportunities to generate new paradoxes while explaining why they occur. For instance, a procedure which rewards a voter's top-ranked candidate but penalizes his bottom-ranked candidate is the five-candidate voting vector $\mathbf{v}_1^5 + \mathbf{v}_4^5 = (2,1,1,1,0)$. Its normalized form $\mathbf{w}^5 = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) = \frac{2}{3}\mathbf{u}_1^5 - \frac{1}{3}\mathbf{u}_2^5 + \frac{2}{3}\mathbf{u}_3^5$ ensures that $\mathbf{w}^5 \in \mathcal{D}^5$. Thus (from Eq. 3.6), the $\mathbf{w}^5$ outcomes are uniquely determined by the three scalars and the election tallies of the four-candidate procedures $\mathbf{v}_1^4$, $\mathbf{v}_2^4$, $\mathbf{v}_3^4$.

While the inclusion $\mathbf{w}^5 \in \mathcal{D}^5$ guarantees election relationships, the negative scalar $(-\frac{1}{3})$ ensures that not all of them are "positive." This is because the $\mathbf{v}_2^4 = (1,1,0,0)$ tallies are *subtracted* from the sum of the two other four-candidate elections. Causing further doubt about $\mathbf{w}^5$ is that Eqs. 3.5, 3.4 allow the $\mathbf{w}^5$ outcomes to be treated as the sums of each candidate's four-candidate $\frac{2}{3}\mathbf{v}_1^4 - \frac{1}{3}\mathbf{v}_2^4 + \frac{2}{3}\mathbf{v}_3^4 = (1, \frac{1}{3}, \frac{2}{3}, 0)$ tallies. But, this is *not* a voting vector because this perverse procedure assigns twice as many points to a third-place candidate as to a second ranked one. Thus one of the promised $(2,1,1,1,,0)$ relationships is to *penalize* a candidate who often is second ranked in four candidate subsets over a candidate who is consistently third-ranked! Consequently, a candidate who always is the $\mathbf{v}_2^4 = (1,1,0,0)$ winner over the four-candidate subsets by virtue of often being a voter's second choice could do poorly in the five-candidate $(2,1,1,1,0)$ election outcome.

For other $\mathbf{w}^5$ relationships, note that $g_5((1, \frac{1}{2}, \frac{1}{2}, 0)) = g_5(\frac{1}{2}(1,0,0,0) + \frac{1}{2}(1,1,1,0)) = \frac{1}{2}(\mathbf{u}_1^5 + \mathbf{u}_3^5)$ is a procedure which rewards a candidate when a voter has her top-ranked in *four-candidate* elections and penalizes her if she is bottom-ranked. To add perversity, modify this procedure by *penalizing* a candidate who does well in pairwise elections. As $\mathbf{b}^5$ is the sum of pairwise votes, a choice is $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) = \mathbf{u}_1^5 + \mathbf{u}_3^5 - \mathbf{b}^5$. But as this returns us to $\mathbf{w}^5$, another disturbing $\mathbf{w}^5 \in \alpha^5$ election relationship is that $\mathbf{w}^5$ *punishes* candidates who do well in pairwise elections. Namely, $\mathbf{w}^5$ rewards the Condorcet loser while hurting the Condorcet winner. The derivation of this surprising assertion is not restricted to $\mathbf{w}^5$; once the $\mathcal{D}^k$ dimension at least unity (so $k \geq 4$), any $\mathcal{D}^k$ voting vector which is not $\mathbf{b}^k$ can be expressed in a sum where the Borda point is subtracted off. This allows an argument resembling the one for $\mathbf{w}^5$ to be fashioned for *any* non-BC $\mathcal{D}^k$ voting vector; i.e., the election relationships promised by Thm. 3 can be negative.

Corollary 3. *For a non-BC voting vector $\mathbf{w}^k$ in $\mathcal{D}^k$, $k \geq 4$, there exists a $(k-1)$ candidate voting procedure $\mathbf{w}^{k-1}$ so that $\mathbf{w}^k$ penalizes a strong pairwise performance at the expense of rewarding positive $\mathbf{w}^{k-1}$ outcomes.*

Proof. For such a $\mathbf{w}^k \in \mathcal{D}^k$, there is a $\mathbf{w}^{k-1}$ so that $g_k(\mathbf{w}^{k-1})$ is in the interior of the line segment with vertices $\mathbf{w}^k$ and $\mathbf{b}^k$. Thus there is a $\mu \in (0,1)$ so that $\mu\mathbf{b}^k + (1-\mu)\mathbf{w}^k = g_k(\mathbf{w}^{k-1})$. The conclusion follows by solving for $\mathbf{w}^k$. □

Even more disturbing outcomes are generated by using the voting vectors off of the derived set $\mathcal{D}^n$. This is because the deviation vector $\mathbf{d}^n$ can modify the election outcomes for these positional procedures in any manner without regard for the election outcomes of the subsets. So, the outcomes of a non-BC procedure in $\mathcal{D}^n$ can be thought of as *subtracting* each candidate's pairwise tallies from the tallies of other procedures, while the $\mathbf{d}^k$ component ignores the outcomes of other procedures.

To elaborate while developing an important tool for what follows, notice that the plurality vector and the other $\{\mathbf{v}_j^k\}_{j=1}^{k-1}$ never are in $\mathcal{D}^k$. But the derived set $\mathcal{D}^k$ is only a dimension lower than the voting pyramid $\mathcal{P}^k$, so the normal vector $\mathbf{d}^k$ can be expressed as a linear combination of vectors from $\mathcal{D}^k$ and the plurality $\mathbf{v}_1^k$. (See, for instance, Eq. 3.8.) In turn, a profile's deviation $\mathbf{d}^k$ outcome needed for Cor. 1 is obtained from the same linear combination of this profile's plurality tally and the tallies of the specified $\mathbf{u}_j^k$ methods. But the $\mathbf{u}_j^k$ outcomes are determined by the outcomes of $\mathbf{d}^{k-1}$ and entries of $\mathcal{D}^{k-1}$. In turn, the deviation $\mathbf{d}^{k-1}$ outcome can be determined from the $(k-1)$ candidate plurality outcomes and those from $\mathcal{D}^{k-1}$. Tracing this induction argument to its base, we discover that *all* election tallies can be obtained from the pairwise and plurality tallies. Consequently all pathologies of election outcomes are reflected by the commonly used plurality vote.

Corollary 4. *For $n \geq 3$ candidates and a given profile, all possible election tallies for all positional methods over all subsets of candidates are linear expressions of the tallies of the $\binom{n}{2}$ pairs of candidates and the plurality tallies of all subsets with three or more candidates.*

3.5. **Positive behavior.** Corollary 4, along with Thm. 1, motivates the following assertion.

Corollary 5. *For $n \geq 3$, a necessary condition for a system vector $\mathbf{W}^n$ to be in α^n is that at least one of the voting vectors in $\mathbf{W}^n$ is in a derived set.*

This assertion specifies that the only way to ensure relationships among election rankings of a profile is through the summation process motivated by Eq. 3.3. This is not a sufficient condition. For instance, no relationships among the $\mathbf{W}^4$ rankings occur if the four-candidate election is tallied with $(3,1,0,0) \in \mathcal{D}^4$ while each of the four triplets is tallied with one of $(1,0,0)$, $(1,\frac{1}{4},0),(1,\frac{2}{3},0),(1,1,0)$. On the other hand, should even three of the triplets be tallied with the same voting vector, then some restrictions are imposed upon the rankings of the sole candidate who is in these three subsets. For instance, if this common voting vector is on the plurality side of the BC, and if this candidate is top-ranked in all two and three candidate elections, then she cannot be bottom ranked in the full four-candidate election.

As a partial summary, we have established and explained the following:

1. The Borda Count offers more consistency of election outcomes over all subsets of candidates than any other positional procedure.
2. Election relationships among rankings require using voting vectors from the various $\mathcal{D}^k$ sets.

These comments mean that *only the BC is spared a negative Cor. 3 conclusion.*

4. Profiles

As demonstrated, much more goes wrong with voting outcomes than previously suspected. This perverse behavior is captured by the following profile decomposition. As this decomposition uses the differences between profiles with the same number of voters, it allows a *negative* number of votes to have certain preferences. This, however, creates no problems in computing election outcomes.

Definition 2. *A* profile differential *is the difference between two profiles involving the same number of voters. Equivalently, a listing of the number of voters of each type is a profile differential if and only if the sum of voters is zero.*

The profile differentials are divided into classes determined by whether they have no effect on outcomes, compromise the assumption that voters have transitive preferences, exhibit various biases, etc. In the decomposition, I occasionally need the *space of normalized profiles* where, instead of specifying the number of voters of a particular type, the fraction of all voters is used. This space is identified with the $n! - 1$ dimensional simplex

$$Si(n!) = \{\mathbf{x} = (x_1, \dots, x_{n!}) \in R^{n!} \mid \sum_{j=1}^{n!} x_j = 1,\, x_j \geq 0\} \tag{4.1}$$

4.1. **Universal Kernel.** The first part of my characterization is to describe those profiles with no effect upon pairwise or positional election rankings — all elections end in a complete tie. An obvious example is the profile, $\mathbf{K}^n$, with one voter for each of the $n!$ types. But, there are many more such profiles. This follows from Thm. 2 which requires $F(-, \mathbf{W}^n)$ to have maximum rank for almost all $\mathbf{W}^n$ choices, so the linearity of F with respect to profiles (and the dimension of F's image space) ensures that the $Si(n!)$ kernel of $F(-, \mathbf{W}^n)$ is a $n! - 2^{n-1}(n-2) - 2$ dimensional subspace. By definition, the profiles in the kernel cannot affect the $\mathbf{W}^n$ election ranking of any subset. Our analysis would be hindered if this kernel changed with $\mathbf{W}^n$. It does not; Cor. 4 requires the kernel of the plurality and pairwise outcomes to be in the kernel of all other procedures.

Theorem 4. *For $n \geq 3$, there exists a $n! - 2^{n-1}(n-2) - 2$ dimensional subspace $\mathcal{UK}^n$ of the profile space $Si(n!)$, called* the universal kernel, *so that if $\mathbf{p} \in \mathcal{UK}^n$, then its word for all choices of $\mathbf{W}^n$ is a complete tie for each subset of candidates.*

For $n = 3$, $\mathcal{UK}^3$ has dimension zero; it is the barycentric point $\frac{1}{6}\mathbf{K}^3$. However, the $\mathcal{UK}^n$ dimension grows rapidly; for $n = 4$ it is 6, and for $n = 5$ it is 70. In fact, a simple computation proves that the kernel constitutes the largest portion of profiles once $n \geq 5$ (i.e., its dimension is over half that of the profile space $Si(n!)$) and that the ratio of the $\mathcal{UK}^n$ and $Si(n!)$ dimensions rapidly approaches unity as $n \to \infty$. (So, with enough candidates, most of profile space is consumed by profiles with no effect on pairwise or positional outcomes.) While it is important to characterize $\mathcal{UK}^n$, these dimensions prove for $n \geq 5$ that the analysis would dominate the discussion. Thus, I provide a complete description of $\mathcal{UK}^n$ for $n = 4$ and a nearly complete one for $n \geq 5$.

4.2. **Kernel profiles.** To create $\mathcal{UK}^n$ entries, take the difference between two profiles with identical tallies for each subset of candidates. To illustrate, start with a profile differential where 1 voter has the ranking $(C \succ D) \succ (A \succ B)$ and -1 have $(C \succ D) \succ (B \succ A)$. The cancelling effect of the -1 term forces all plurality tallies in the four candidate subset and the four triplets to be zero; only the $A \succ B$ pairwise outcome (with tally 1:-1) avoids a complete zero tally. According to Cor. 4, these tallies determine all possible positional tallies over all subsets of candidates.

The same argument with identical tallies occurs by replacing the $(C \succ D)$ portion of both rankings with $(D \succ C)$. The difference between these profiles defines the $\mathcal{UK}^4$ profile differential

Number	Ranking	Number	Ranking
1	$C \succ D \succ A \succ B$	-1	$C \succ D \succ B \succ A$
-1	$D \succ C \succ A \succ B$	1	$D \succ C \succ B \succ A$

(4.2)

By changing the identity of the candidates in each pair, we obtain $\binom{4}{2} = 6$ versions of Eq. 4.2.

To extend this argument to $n \geq 4$ candidates, partition the n candidates into two sets G_1, G_2, where each G_i has at least two candidates. Let r_i be a strict ranking of the G_i candidates, $i = 1, 2$. Let $\sigma(r)$ be a ranking which permutes the ranking of r, and let $\rho(r)$ be the special permutation which reverses the ranking r. (So, $\rho(A \succ B \succ C \succ D) = D \succ C \succ B \succ A$; one σ choice is $\sigma(A \succ B \succ C \succ D) = D \succ A \succ B \succ C$.)

Choose non-identity permutations σ_j for the candidates in G_j, $j = 1, 2$. The plurality and pairwise tally of any subset for a candidate in G_1 is zero with the profile differential where 1 voter has the preference $r_1 \succ r_2$ and -1 have $r_1 \succ \sigma_2(r_2)$. The plurality and pairwise tallies for a G_2 candidate depend upon whether a subset has any G_1 candidates and the choice of σ_2. Whatever these tallies, identical pairwise and plurality tallies arise with the profile differential where 1 voter has $\sigma_1(r_1) \succ r_2$ and -1 have $\sigma_1(r_1) \succ \sigma_2(r_2)$. According to Cor. 4, the difference between these profile differentials has a zero tally for all candidates in all subsets. The difference, the *symmetry changing profile differential*,

Number	Ranking	Number	Ranking
1	$r_1 \succ r_2$	-1	$r_1 \succ \sigma(r_2)$
1	$\sigma_1(r_1) \succ \sigma_2(r_2)$	-1	$\sigma_1(r_1) \succ r_2$

(4.3)

is in $\mathcal{UK}^n$. The special case (and only choice for $n = 4$) where $\sigma_j = \rho$ is called *the double reversal profile differential.*

Theorem 5. *A basis for $\mathcal{UK}^4$ is given by the six double reversal profile differentials. Thus, all kernel vectors are weighted sums of double reversal profile differentials and $\mathbf{K}^4$.*

For $n \geq 5$, all symmetry changing profile differentials are in $\mathcal{UK}^n$.

Proof. It remains to prove that the six vectors described for $n = 4$ are linearly independent. As each entry of each vector involves a $Si(4!)$ component not in any other vector, the conclusion is immediate. □

The more general assertion for $n \geq 5$ requires showing that the symmetry changing profile differentials span $\mathcal{UK}^n$. This is not overly difficult for $n = 5$, but the combinatorics become messy for $n \geq 6$. Of more importance, because the $\mathcal{UK}^n$ profiles have no effect on election outcomes for any subset, the smaller dimensional orthogonal subspace, *the space of effective profiles $\mathcal{EP}^n$*, totally determines all election outcomes of all subsets of candidates for all positional procedures. The $\mathcal{EP}^n$ dimension of $2^{n-1}(n-2) - 1$ agrees with the sum of dimensions of the $2^n - (n+1)$ subspaces of normalized outcomes. The huge $\mathcal{UK}^n$ dimension, however, is mischief in waiting; when other procedures recognize these profiles, they generate paradoxes.[3]

5. Representation tetrahedrons and simplices

As a convenient way to display profiles, I use simplices with vertices equal distance from each other. (This equilateral simplex is in R^{n-1}. For $n = 3$, it is the equilateral triangle of [19].) Associate each of the $n \geq 2$ candidates $\{c_1, c_2, \dots, c_n\}$ with a vertex, and assign a ranking to a simplex point according to its distance from each vertex where "closer is better." The resulting division of the simplex into "ranking regions" is the "representation simplex."

The $n = 4$ simplex is an equilateral tetrahedron where each of its four faces is an equilateral triangle. When this "representation tetrahedron" is "opened" by cutting down along the edges from the D vertex, we obtain the figure depicted in Fig. 2. Each face is defined by three candidates, so the missing candidate corresponds to the vertex in the tetrahedron vertex that is most distant from the face; she is bottom ranked. Thus the Fig. 2 ranking region with a "•" in the B-C-D face corresponds to $C \succ B \succ D \succ A$.

Represent a profile by listing the number of voters with each ranking in the appropriate ranking region. To compute election tallies, notice that A is top-ranked in those regions with

[3] The new paradoxes occur because the new procedure has a non-tied outcome on the profile portion where all pairwise and positional methods are completely tied.

A as a vertex. Thus, the sum of terms in the lightly shaded region of Fig. 2 determine A's four-candidate plurality ranking. When only candidates $\{A, B, C\}$ are considered, A also receives the votes from voters who have A second-ranked and D top-ranked, so A's tally is augmented by the values of the two more heavily shaded regions. (A similar description holds for A's tally for the other two three-candidate subsets.) This change in tallying an outcome already explains why rankings differ as candidates are added or dropped.

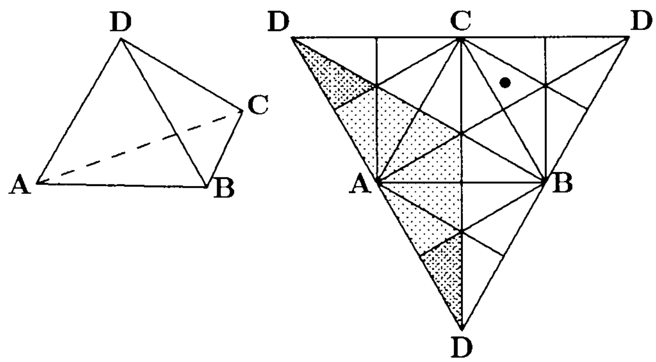
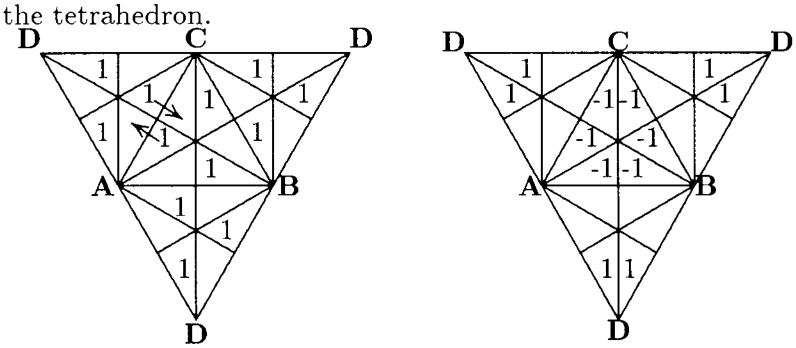


Fig. 2. Representation tetrahedron

In the $\{A, B\}$ pairwise election, A's tally is the sum of numbers to the left of the middle $A \sim B$ line; B's tally is the sum of numbers to the right. This line connects the D vertex at the bottom of the figure to the C vertex in the middle of the top line. A similar description holds for all other pairwise elections which do not involve D. In the $\{A, D\}$ pairwise election, however, the $A \sim D$ line is not straight because of how the tetrahedron is opened. Here, A's tally is the sum of points in the square where two edges are the $A \sim D$ lines in the $\{A, B, D\}$ and the $\{A, C, D\}$ faces. The last edge connects vertices B and C; it separates two faces of the tetrahedron.



a. Kernel Profile **b.** D-Basic profile

Fig. 3. A kernel and a basic profile

Applying these counts to the profile in Fig. 3-a, we find that all plurality and pairwise outcomes are zero; it is a $\mathcal{UK}^4$ profile. The arrows show that the double reversal profile differential (a voter for each preference $(A \succ C) \succ (D \succ B)$, $(C \succ A) \succ (B \succ D)$ and -1 voters for each of $(A \succ C) \succ (B \succ D)$, $(C \succ A) \succ (D \succ B)$) has the geometric interpretation of symmetrically moving voters across an edge to an adjacent face. All double reversal profiles admit this geometric description. Indeed, each of the six edges defines a double reversal profile moving preferences among the four adjacent ranking regions. By use of this geometric representation, it is easy to show that by adding certain double reversal differentials vectors to $\mathbf{K}^4$, we obtain twice the profile of Fig. 3a. To further illustrate this geometric tallying approach, the rankings of all subsets for the profile differential of Fig. 3b are compatible with $D \succ A \sim B \sim C$.

6. Pairwise and BC outcomes

The first $\mathcal{EP}^n$ *effective profiles* that I describe are those that determine all pairwise and all BC outcomes. The first of two parts is the *basic profile* where the tallies of all procedures agree. The second is the *Condorcet portion*; this profile subspace is responsible for all cyclic and non-transitive pairwise outcomes as well as the conflicting BC outcomes over different subsets of candidates. The Condorcet portion, then, has critical importance because it causes all conflict between the pairwise and BC rankings, as well as all paradoxes and difficulties of all other procedures based on pairwise rankings including those introduced by Copeland [20], [8], Kemeny [7], [21], and Arrow and Raynaud [2].

6.1. **Basic profiles.** The definition of the basic profiles and a statement indicating their fundamental importance follows. The proof is in Sect. 10.

Theorem 6. *A n-candidate* basic profile differential *for candidate c_j, denoted by $\mathbf{B}^n_{c_j}$, has a voter for each ranking where c_j is top-ranked and -1 voters for each ranking where c_j is bottom-ranked. For each subset of candidates, the $\mathbf{B}^n_{c_j}$ tallies of all normalized positional voting procedures agree. In particular, for $k \geq 2$, if c_j is in a k-candidate subset of candidates, she receives $(n-1)!$ points and each of the other candidates receives $-\frac{(n-1)!}{k-1}$ points. If c_j is not in a k-candidate subset, then all candidates receive zero points.*

The basic profile differentials satisfy the equation

$$\sum_{j=1}^{n} \mathbf{B}^n_{c_j} = \mathbf{0} \tag{6.1}$$

The $\mathbf{B}^4_D$ basic profile is displayed in Fig. 3-b. A word of caution; the n basic profiles define a $(n-1)$ dimensional linear subspace of profiles. As true with any (portion of a) vector space, this convex set admits a variety of choices of spanning vectors; different choices involve different tradeoffs. My choice of the basic profiles emphasizes simplicity of form and efficiency of use in understanding election properties. The cost of using profiles that are not orthogonal, as guaranteed by Eq. 6.1, is to slightly complicate finding the basic profile components for a specified profile.

To appreciate the power of the basic profiles, recall how much of choice theory has been devoted toward understanding why procedures have different conclusions. As all procedures totally agree on the basic profile for all subsets of candidates, no such difficulty occurs here. Consequently the basic profile finally captures the long sought after state of rationality for choice theory.

Going beyond providing compatible ordinal rankings, the tallies of the basic profile satisfy a strong cardinal transitivity that holds both in the horizontal (over subsets of the same size) and vertical (over subsets of different size) directions. To explain, the ultimate transitivity is attained if the tallies share the additive properties of points $x_1, x_2, \dots, x_k$ on the line given by the equality

$$(x_1 - x_2) + (x_2 - x_3) + \cdots + (x_{k-1} - x_k) = x_1 - x_k. \tag{6.2}$$

Not only is the powerful Eq. 6.2 cardinality condition normally missing from election tallies, but even the much weaker ordinal ranking conditions need not be satisfied. However, the basic profile tallies of all positional procedures satisfy an Eq. 6.2 condition.

Corollary 6. *For a basic profile* $\sum_{j=1}^{n} a_j \mathbf{B}^n_{c_j}$, *if both* c_i *and* c_j *are in the same subset of* k*-candidates, then the difference between their common positional tally is*

$$\tau^k(c_i, c_j) = (a_i - a_j)[(n-1)! - \frac{(n-1)!}{(k-1)!}]. \tag{6.3}$$

In particular, the election tallies from any *combination of* k *candidate subsets satisfy the equality*

$$\tau^k(c_1, c_2) + \tau^k(c_2, c_3) + \dots \tau^k(c_{j-1}, c_j) = \tau^k(c_1, c_j) \tag{6.4}$$

Proof. This is an immediate consequence of Thm. 6. □

Because Cor. 6 holds for *any* positional method, we can choose the relative tallies of pairs from different subsets involving different positional methods. To illustrate with

$$9\mathbf{B}^5_{c_1} + 7\mathbf{B}^5_{c_2} + 2\mathbf{B}^5_{c_3} + \mathbf{B}^5_{c_4},$$

the $\{c_1, c_3\}$ relative plurality tally from $\{c_1, c_2, c_3\}$ plus the $\{c_3, c_2\}$ relative antiplurality $\mathbf{v}^3_2$ tally from $\{c_2, c_3, c_5\}$ determines, say, the $\{c_1, c_2\}$ relative BC tally for $\{c_1, c_2, c_4\}$. A slight modification of Eq. 6.3 even tells how the relative basic profile tallies of a pair of *any* k-candidate subset uniquely determines the two candidates' relative tally for *all subsets and all positional procedures.* For instance, it is easy to find from Eq. 6.3 the appropriate multiples μ_4, μ_5 so that μ_4 times the $\{c_1, c_2\}$ relative plurality tally from $\{c_1, c_2, c_3, c_5\}$ plus the μ_5 multiple of the $\{c_2, c_4\}$ relative BC tally from all five candidates equals the $\{c_1, c_4\}$ relative $(1, \frac{6}{7}, 0)$ tally from the $\{c_1, c_4, c_5\}$ election.[4]

6.2. **Condorcet profiles.** Not all profiles have the desirable Eq. 6.4 property. For instance, with the one person profile $A \succ B \succ C \succ D$, we have $\tau^2(A, B) = \tau^2(A, C) = \tau^2(B, C)$, so $\tau^2(A, B) + \tau^2(B, C) \neq \tau^2(A, C)$. This fact implies that there are profile differentials other than the basic ones which influence the pairwise outcomes. Indeed, when Thm. 1, which asserts that all possible pairwise rankings can occur, is expressed in a geometric setting (e.g., see Saari [14]) the pairwise outcomes span a space of dimension $\binom{n}{2}$. Consequently the profiles supporting all pairwise outcomes span an $\binom{n}{2}$ dimensional subspace of $Si(n!)$. Thus we still need $\binom{n}{2} - (n-1) = \binom{n-1}{2}$ additional spanning profile vectors to handle all pairwise outcomes. The importance of these new profile components is that they define the crucial profile noise which completely explains all possible pairwise election problems including those of the Condorcet winner, agendas, any procedure using pairwise rankings such as Copeland's method, Kemeny's rule, and the Arrow-Raynaud procedure.

The building blocks for these profile differentials are the *Condorcet* n*-tuples.* To construct one, attach to a fixed background a disk that rotates about its center. Equally spaced along its circular boundary place the ranking numbers $1, 2, \dots, n$. To represent a ranking r of the candidates, place each candidate's name on the fixed background next to the appropriate ranking number. Rotate the disk in a fixed direction until the number 1 points to the next candidate; the numbers define a second ranking. Continue this process until n rankings are defined. Notice how a Condorcet n-tuple is uniquely defined by any of these n rankings. (The reader who knows group theory will recognize this as the Z_n orbit of r.)

It is arguable that the election outcome for a Condorcet n-tuple should be a complete tie because each candidate is ranked in each position precisely once. But this natural outcome does not hold for the pairwise vote. Instead the outcome of each pair depends upon their relative positions in r. For instance, if c_1 is ranked immediately above c_2 in r, then one of the

[4]For instance, $\mu_4 = [4! - 4!/2!]/[4! - 4!/3!]$.

rankings from the tuple has c_2 and c_1, respectively, top and bottom ranked. In the remaining $n-1$ rankings c_1 is ranked above c_2. Thus, c_1 beats c_2 with a $n-1:1$ tally. Using the same argument, if some ranking of the n-tuple has, say, c_3 ranked $s \leq \frac{n}{2}$ candidates above c_1, then c_3 beats c_1 in the pairwise vote with a $n-s:s$ tally. The symmetry of the Condorcet n-tuple ensures that if c_i is ranked s candidates above a specified candidate in some term of the Condorcet n-tuple, then she is ranked s candidates below another candidate in another Condorcet term. This proves the following.

Proposition 1. *In a Condorcet n-tuple defined by r, the sum of each candidate's tallies over the $n-1$ pairwise elections all agree. If c_i is ranked s candidates above c_j in a ranking of the n-tuple, then the $\{c_i, c_j\}$ election tally is $n-s:s$.*

The following profile differential uses the reversal ranking $\rho(r)$ from Thm. 5.

Definition 3. *The* Condorcet profile differential defined by r *is where there is one voter for each ranking in the Condorcet n-tuple defined by r and -1 voters for each ranking in the Condorcet n-tuple defined by $\rho(r)$.*

The next statement (proof in Sect. 10) describes the role of the Condorcet profile differentials in pairwise voting.

Theorem 7. *The Condorcet profile differential defined by a ranking r of the $n \geq 4$ candidates satisfies the following:*

1. *If c_i is ranked s candidates above c_j in any ranking of the Condorcet n-tuple defined by r, then the $\{c_i, c_j\}$ election tally of the Condorcet profile differential is $n-2s:2s-n$. Consequently, the sum of a candidate's pairwise tallies over all possible opponents is zero.*
2. *If $\mathbf{p}$ is a profile differential orthogonal to the space spanned by the basic and the Condorcet profile differentials, then all pairwise tallies of $\mathbf{p}$ are zero. Consequently, all admissible pairwise election tallies coming from rational voters are obtained with the weighted sum of kernel vectors, basic vectors, and Condorcet profile differentials; all remaining profile components have no effect upon pairwise or BC outcomes.*
3. *For any n-candidate positional method, the tally of the Condorcet profile differential assigns zero to all candidates.*
4. *The Condorcet profile differential is orthogonal to all basic profiles and it is orthogonal to all double reversal profile differentials.*
5. *The Condorcet profile differentials span a space of dimension $\frac{1}{2}(n-1)!$*
6. *When a Condorcet profile differential is restricted to a k-candidate subset for an odd integer value of k, or for $k=n$, the deviation $\mathbf{d}^k$ tally is zero. However, if $k<n$ has an even integer value, then the deviation $\mathbf{d}^k$ tally need not be zero.*

With respect to our goal to simplify the analysis of voting procedures, the good news (part 2) is that all concerns about pairwise voting can be completely analyzed with the basic and Condorcet profile differentials; the other profile differentials are superfluous for this analysis as they have no effect on the outcome. Moreover, we learn from part 4 that the effects of the Condorcet profile differentials on the pairwise outcomes differs significantly from that of the basic profile; this difference is that (part 1) the Condorcet portion creates a cyclic effect.

More good news (part 3) is that the Condorcet profile differentials does not effect the positional tallies of all n candidates. Consequently, the only linkage between pairwise rankings and the positional procedures ranking of all n candidates comes from the basic profile portion. Because the basic portion uniquely determines the BC outcome (of all n candidates) and because the outcomes of all other procedures are altered by deviation $\mathbf{d}^k$ effects, we find further support for the integrity of the BC outcome.

Part 6 identifies a problem that arises with $n \geq 5$ candidates. It asserts that the Condorcet profile differentials can have a non-zero deviation vector tally. This difficulty, where the profile differential influences more than just the pairwise tallies, is resolved in Sect. 7. For a preview why this occurs with $k = 4$ and $n = 5$, consider the Condorcet profile differential defined by $r = A \succ B \succ C \succ D \succ E$. With the Condorcet five-tuple defined by r, the absence of E in the subset $\{A, B, C, D\}$ places B in second place twice and third place once, while in $\rho(r)$, she is in second place once and third place twice to give B a positive $\mathbf{d}^4$ tally. Notice, however, that part 6 does not assert that the tally *must* be non-zero. An example with a zero tally is where $r = c_1 \succ c_2 \succ \ldots \succ c_8$ and the subset is $\{c_2, c_4, c_6, c_8\}$.[5]

The Condorcet profile differential defines all sorts of cycles. To illustrate with $r = c_1 \succ c_2 \succ \cdots \succ c_9$, accompanying the cycle where $c_1 \succ c_2, c_2 \succ c_3, \ldots, c_8 \succ c_9, c_9 \succ c_1$ is the cycle involving every other candidate where $c_1 \succ c_3, c_3 \succ c_5, c_5 \succ c_7, c_7 \succ c_9, c_9 \succ c_2, c_2 \succ c_4, c_4 \succ c_6, c_6 \succ c_8, c_8 \succ c_1$, three cycles obtained by considering every third candidate to obtain $c_1 \succ c_4, c_4 \succ c_7, c_7 \succ c_1$ with similar cycles starting with c_2 and with c_3. The final cycle has every fourth candidate to derive $c_1 \succ c_5, c_5 \succ c_9, c_9 \succ c_3, \ldots$. Notice that the cycles obtained by skipping a larger number of candidates have a closer (common) election tally. As a way to describe the general behavior, let the subscript for c_j represent j if $j \leq n$, or the remainder obtained when n is divided into j for $j > n$.

Corollary 7. *The Condorcet profile differential defined by $r = c_1 \succ c_2 \succ \cdots \succ c_n$ defines the following cycles. The primary cycle is where $c_j \succ c_{j+1}$, $j = 1, \ldots, n$. The sth level cycle, $1 \leq s < \frac{n}{2}$, is where $c_j \succ c_{j+s}$, $j = 1, \ldots, n$. If $\frac{n}{s}$ is the integer α, then there are s different s level cycles containing α candidates. If $\frac{n}{s}$ is not an integer, then there is a unique sth level cycle that involves all candidates.*[6]

By using these cycles as building blocks, a wide variety of behavior emerges. For instance, the above nine candidate example also admits $c_1 \succ c_3 \succ c_4 \succ c_8 \succ c_1$. For this cycle, however, the tally between successive terms varies while it remains fixed with the values from Thm. 7 for the sth level cycles.

6.3. **Three and four candidates.** Part 5 tells us that the subspace dimension of Condorcet profile differentials is $\frac{1}{2}(n-1)!$ while we only need a subspace of dimension $\binom{n-1}{2}$. These dimensions agree for $n = 3, 4$, so nothing further needs to be done. But once $n \geq 5$, the inequality $\frac{1}{2}(n-1)! > \binom{n}{2}$ suggests that the Condorcet profile differentials have effects other than influencing pairwise tallies. As indicated, they do; this is described separately.

So, for three and four candidates, we have completely identified all profiles differentials with any impact on the pairwise election outcomes. For three candidates, the dimension of the space of Condorcet differential is $\frac{1}{2}(3-1)! = 1$, so this space is spanned by the Condorcet profile differential $\mathbf{C}^3$ defined by $A \succ B \succ C$. All three candidate pairwise concerns are completely determined by this Condorcet profile differential and the three basic profiles; this is described in (Saari [19]).

The four candidate Condorcet subspace dimension is $\frac{1}{2}(4-1)! = 3$. To find a defining basis start with any ranking of the four candidates, say $r_1 = A \succ B \succ C \succ D$, and compute the

[5] The reader comfortable with algebraic group theory will recognize that this result is due to a subgroup structure of the Z_8 orbit. This observation generalizes and leads to assertions such as "if n is not a multiple of k, then the deviation vector tally is not zero."

[6] Again, the reader familiar with abstract algebra will find here, and in most results, statements that follow immediately from group theory.

associated Condorcet profile differential. Call it $\mathbf{C}^4_{r_1}$. Next, choose a ranking which differs from the eight rankings that appear in the r_1 profile differential, say $r_2 = A \succ C \succ B \succ D$, and compute its Condorcet profile differential $\mathbf{C}^4_{r_2}$. Sixteen of the 24 possible rankings are used, so the last Condorcet profile differential, $\mathbf{C}^4_{r_3}$, is determined by a remaining ranking, say $r_3 = A \succ B \succ D \succ C$.

Corollary 8. *The three Condorcet profile differentials* $\{\mathbf{C}^4_{r_j}\}_{j=1}^3$ *are mutually orthogonal.*

The proof of this assertion follows from the construction which places each ranking into a unique Condorcet $\mathbf{C}^4_{r_j}$ term. Contrast this assertion with the non-orthogonality property of basic profiles. As shown later, this technical difference makes it easier to find the Condorcet components of a specified profile.

6.4. **Arrow-Raynaud procedure.** To illustrate the importance of this Condorcet subspace I use it to analyze the Arrow-Raynaud [2] procedure for multicriterion decision making and pairwise voting. This procedure uses the *outranking matrix* $A = [a_{ij}]$ where a_{ij} is c_i's pairwise vote in a $\{c_i, c_j\}$ contest. A ranking of the candidates is obtained with their *primal algorithm* where they first identify the maximum in each row. (The maximum in the ith row identifies c_i's largest pairwise election outcome; the column identifies her competitor.) The candidate associated with the smallest of these values is designated as bottom-ranked. Next, delete the row and column of this candidate, and repeat the process with the reduced outranking matrix to identify the candidate second from the bottom.[7] Continue until all candidates are ranked.

To illustrate, applying this procedure to the matrix

$$\begin{pmatrix} - & 40 & 62 & 48 \\ 36 & - & 76 & 62 \\ 14 & 0 & - & 40 \\ 28 & 14 & 36 & - \end{pmatrix} \tag{6.5}$$

defines the ranking $A \succ B \succ C \succ D$. Because the sum of the entries in row j defines c_j's BC score, we find that the Arrow-Raynaud ranking conflicts with the BC ranking of $B \succ A \succ D \succ C$.

To analyze the Arrow-Raynaud procedure, let $a_{ij} = a(B)_{ij} + a(r_1)_{ij} + a(r_2)_{ij} + a(r_3)_{ij}$ be given, respectively, by the basic, $\mathbf{C}^4_{r_1}$, $\mathbf{C}^4_{r_2}$, and $\mathbf{C}^4_{r_3}$ portions of a profile. According to Thm. 7, only these profile portions have any effect upon the pairwise outcomes. So, profile $\sum_{j=1}^4 a_j \mathbf{B}^4_j + \sum_{j=1}^3 \gamma_j \mathbf{C}^4_{r_j}$ defines matrix $A = A_B + \sum_{j=1}^3 \gamma_j A_{r_j}$ where the A_B term is the outreach matrix defined by the basic profile and the other three matrices are defined by the indicated $\mathbf{C}^4_{r_j}$. For instance, with coefficients $a_2 = 5, a_1 = 4, a_4 = 1, a_3 = 0$

$$A_B = 6\begin{pmatrix} - & a_1 - a_2 & a_1 - a_3 & a_1 - a_4 \\ a_2 - a_1 & - & a_2 - a_3 & a_2 - a_4 \\ a_3 - a_1 & a_3 - a_2 & - & a_3 - a_4 \\ a_4 - a_1 & a_4 - a_2 & a_4 - a_3 & - \end{pmatrix} = \begin{pmatrix} - & -6 & 24 & 18 \\ 6 & - & 30 & 24 \\ -24 & -30 & - & -6 \\ -18 & -24 & 6 & - \end{pmatrix} \tag{6.6}$$

It is clear from Eq. 6.6 that A_B ranking completely agrees with the ranking of the a_j scalars. In turn (Cor. 6) this requires the A_B ranking $B \succ A \succ D \succ C$ to agree with that of all positional methods over all subsets of candidates. (For instance, the plurality and BC ranking of this profile for $\{B, C, D\}$ must be $B \succ D \succ C$.) To modify the profile to change the outcome, notice that the direction of the cycle attached to a Condorcet portion is determined by the defining r ranking. So, to have a new tally favoring A over B and C over D, let

[7] In case of ties, candidates are selected randomly.

$r_1 = (A \succ B) \succ (C \succ D)$. Adding a $\gamma_1 \mathbf{C}^4_{r_1}$ component to the basic profile changes the outreach matrix to

$$\begin{pmatrix} - & -6+2\gamma_1 & 24 & 18-2\gamma_1 \\ 6-2\gamma_1 & - & 30+2\gamma_1 & 24 \\ -24 & -30-2\gamma_1 & - & -6+2\gamma_1 \\ -18+2\gamma_1 & -24 & 6-2\gamma_1 & - \end{pmatrix} \tag{6.7}$$

where $\gamma_1 = 4$ creates the new outcome $A \succ B \succ C \succ D$. By adding 38 to each entry to eliminate the negative signs (i.e., by adding an appropriate multiple of $\mathbf{K}^4$ to the profile), the resulting 76 voter profile changes Matrix 6.7 into the initial Matrix 6.5. In other words, the reason the Arrow-Raynaud ranking of Matrix 6.5 differs from the BC ranking is that the Arrow-Raynaud method must reflect any bias introduced by the Condorcet portion. According to Thm. 7, this Condorcet profile portion has no effect upon the BC ranking of all four candidates; the BC ranking only reflects the profile's basic portion. The following summarizes the general situation; the proof is immediate from the profile decomposition.

Theorem 8. *On basic vectors, the BC and Arrow-Raynaud rankings always agree. Not only are there profiles where the two rankings disagree, but all such examples are caused and completely explained by how the Arrow-Raynaud procedure treats the Condorcet portion of a profile.*

As the example and theorem demonstrate, the Condorcet portion can significantly alter a procedure's ranking. Arrow and Raynaud inadvertently underscore this important point in their book [2] when they contrast their method with competing procedures that also rely upon pairwise rankings or tallies. By use of Thm. 7, it is easy to prove that all of these procedures agree on the basic portion of the profile. Consequently, any and all differences among them are due to how each procedure treats the Condorcet portion. For this reason, we must expect, and it is the case, that all illustrating profiles in this section of their book [2] exhibit a strong Condorcet component.[8] In turn, this means that the only difference among these procedures is their treatment of the Condorcet portion of a profile — a portion that should have a neutral outcome. It also means that all of thes procedures admit a conclusion of the Thm. 8 type.

To review the central concern, the natural outcome for the Condorcet portion is a tie vote, but the pairwise vote twists the conclusion with its cyclic effect. To remove this bias to obtain a more reliable conclusion, we could either eliminate the Condorcet portion from a profile, or, more pragmatically, use a procedure which ignores the Condorcet portion. According to Thm. 7, the sole procedure which does the latter is the BC. To illustrate with Eq. 6.7, recall that the BC tally for c_j is the sum of the entries in row j and notice how this summation cancels the Condorcet portion leaving only the effects of the basic profile.

6.5. **Loss of individual rationality.** Any critique of the Arrow-Raynaud procedure, the Condorcet winner, or any method using pairwise tallies requires providing an interpretation for the effects of the Condorcet profile differentials on the pairwise tallies. As the argument extends the one used to describe what occurs for $n = 3$ (in Saari [19]), I offer only a brief description using $n = 4$. All arguments immediately extend to all $n \geq 3$.

By construction, in a Condorcet n-tuple no candidate has an advantage over another; each is in first, second, ..., last place exactly once. With neutrality and anonymity arguments,

[8] To determine the Condorcet component in a $\mathbf{C}^4_r$ direction, add the number of voters with preferences from the Condorcet four-cycle defined by r and the number with preferences from the Condorcet cycle defined by $\rho(r)$. The difference between these sums reflects the strength of the $\mathbf{C}^4_r$ component in the profile. So, the profile does not have a $\mathbf{C}^4_r$ component if and only if the sums agree. This computation uses the orthogonality of the Condorcet profile differentials as promised by Cor. 8. The same simple approach does not extend to non-orthogonal profile differentials such as, for instance, basic profiles.

this suggests that the natural outcome for the n-tuple is a complete tie; this is true for any n-candidate positional procedure (Thm. 7, Part 3). Therefore, to interpret this profile differential, we must explain why pairwise cycles replace the natural conclusion.

The argument involves decomposing each ranking from a Condorcet n-tuple into its pairwise parts as in Table 6.8. Each row is assigned to the voter identified by the ranking in the left hand column; the other row entries are the associated binary rankings.

Ranking	$\{A,B\}$	$\{B,C\}$	$\{C,D\}$	$\{A,D\}$
$A \succ B \succ C \succ D$	$A \succ B$	$B \succ C$	$C \succ D$	$A \succ D$
$B \succ C \succ D \succ A$	$B \succ A$	$B \succ C$	$C \succ D$	$D \succ A$
$C \succ D \succ A \succ B$	$A \succ B$	$C \succ B$	$C \succ D$	$D \succ A$
$D \succ A \succ B \succ C$	$A \succ B$	$B \succ C$	$D \succ C$	$D \succ A$

(6.8)

By satisfying anonymity, the pairwise vote cannot determine how each voter voted; it cannot determine whether, say, the voter the first row voted $C \succ B$ or $B \succ C$.[9] Consequently the pairwise vote cannot distinguish the Condorcet four-tuple of individually transitive preferences from a profile constructed by permuting the entries of each column in *any desired manner.* But most of these permutations define profiles of voters with irrational preferences. One choice, for instance, has three voters with the cyclic preferences $\mathcal{A} = \{A \succ B, B \succ C, C \succ D, D \succ A\}$ while the last voter has the reversed cyclic preferences $\{B \succ A, C \succ B, D \succ C, A \succ D\}$. As three voters have one (cyclic) belief while the last has the exact opposite opinion, this profile constitutes a single issue comparison where the "fair" outcome is $\mathcal{A}$ by a 3:1 vote. This is the pairwise tally.[10]

The point of this example is that in computing a $\{c_j, c_k\}$ pairwise outcome, the procedure ignores all information about how the voters rank other candidates. But by ignoring how rational voters sequence pairs in a transitive manner, the pairwise vote dismisses all information corroborating the individual rationality of voters. Indeed, this feature is precisely why the pairwise vote can be used with equal ease with rational or irrational voters; it is designed to fairly (as determined by the majority rule) service either society. This dual service creates no difficulties with a sufficiently homogeneous rational society. But once a rational society is sufficiently heterogeneous, as totally captured by the Condorcet profile differentials (Thm. 7), the pairwise vote cannot distinguish between whether the voters are rational or irrational. Namely, *the effect of the Condorcet profile differentials on the pairwise vote is to drop the crucial assumption of individual rationality.*

To summarize, the pairwise votes are completely determined by the basic and the Condorcet profile differentials. The basic profile retains the rationality of voters for all procedures; the Condorcet portion explicitly drops the individual rationality assumption for the pairwise vote. The stronger the Condorcet portion (relative to the basic part) of a profile, the more the pairwise outcomes reflects at least a partial loss of the assumption of individual transitivity.

6.6. **"Reasonable" procedures.** This observation about the role of the Condorcet portion explains all flaws of all procedures using pairwise rankings. If the procedure does not cancel the Condorcet profile differential, then the outcome exhibits a bias — the portion of the outcomes from the Condorcet portion reflects a loss of individually transitive preferences.

[9] Anonymity requires the procedure either to be incapable of determining this information, or to be equivalent to a procedure that cannot.

[10] Applying this argument to a Condorcet n-tuple generates a profile where $n-1$ voters have one cyclic ranking and the last has the opposite belief.

Using this observation, it now is easy to construct examples illustrating all possible cycles, or where the outcomes of a procedure fail to reflect the voters' true views. As in the description of the Arrow-Raynaud procedure, the analysis reduces to simple algebra. To capture this in a formal statement, we introduce the following terms.

Definition 4. *A procedure where its ranking of the candidates is determined by the outcomes of pairwise comparisons is said to be* reasonable *if the ranking for a basic profile always agrees with the ranking of the pairs. The procedure is said to be* monotonic *if the top-ranked candidate with profile* $\mathbf{p}$*, say* c_i*, remains top-ranked with profile* $\mathbf{p} + \mathbf{p}_1$ *where all voters in* $\mathbf{p}_1$ *have* c_i *top-ranked.*

On the basic profiles, all positional and pairwise outcomes agree. Therefore, a procedure should be immediately suspected if it gives a contrary outcome on this space of agreement. Indeed, all pairwise procedures that I know about are "reasonable." This includes agendas, tournements, the Arrow-Raynaud method, the Copeland method, the Kemeny method, the Condorcet ranking, and so forth. However, it also is known that these procedures can have different outcomes. The reason for this difference is specified in the following theorem.

It also is well known that many of these procedures, such as an agenda, are *not* monotonic. The problem with a non-monotonic procedure is that a candidate can lose the election because more of her supporters arrived to vote. Consequently, this issue of monotonicity is a much studied topic. But as described in (Saari [13, 14]) and as illustrated in a special case in (Merlin and Saari [8]) this is just one of several "multiple profile issues," which includes strategic voting, etc., which can be analyzed in essentially the same manner. Therefore, the second part of the following theorem indicates the role this profile decomposition plays in the analysis of all of these concerns. (A more complete discussion requires the development of a related tool, so it will appear elsewhere.)

Theorem 9. *For* $n \geq 3$ *candidates, if the ranking of a reasonable procedure disagrees with the BC ranking, then the difference is completely due to the procedure's treatment of the Condorcet portion of the profile. Consequently, the difference is because the procedure allows a partial loss of the assumption of individual rationality of the voters. Similarly, any difference in ranking between any two reasonable procedures is completely due to how they treat the Condorcet portion of a profile.*

All reasonable procedures are monotonic on the space of basic profiles. Consequently, if a reasonable procedure is not monotonic, it is because the procedure admits a partial loss of the assumption of the individual rationality of the voters.

Proof. The proof is simple; the pairwise outcomes are based solely on the basic and the Condorcet portions of a profile. As the BC and the reasonable procedures agree on the basic portion, all disagreement comes from the Condorcet portion. The BC ignores this portion. So, if a reasonable procedure does not agree with the BC, its outcome must be modified by this portion of the profile. The remainder of the assertion follows from the analysis of the Condorcet portion.

To prove the monotonicity assertion, notice that the basic profiles define a vector space; that is, the sum of two basic profiles is again a basic profile. If $\mathbf{p}$ and $\mathbf{p}_1$ are basic profiles satisfying the above conditions, then the $\mathbf{B}^n_{c_i}$ coefficient for both profiles is the largest. In turn, this coefficient also is the largest for the $\mathbf{p} + \mathbf{p}_1$. Combining this observation with the definition for "reasonable," it follows that all reasonable procedures are monotonic on basic profiles. Therefore, if a reasonable procedure is not monotonic, it is strictly due to how it treats the Condorcet portion of a profile. The assertion now follows. □

Because the Condorcet ranking of candidates is a reasonable procedure, it is subject the the conclusions of this theorem. This means that rather than serving as a standard for choice theory, the Condorcet winner is highly suspect. To show this, just add a significantly strong Condorcet portion to ensure that the Condorcet winner is, say, the same candidate who is ranked second to the bottom with the basic profile. Because the BC outcome ignores the Condorcet portion, the BC and Condorcet rankings are in serious conflict. But, as the analysis proves, it is the Condorcet ranking which is troubling because its difference is cause by a profile differential which compromises the assumption that voters are rational. Indeed, the three candidate examples of (Saari [19]) can be extended to create situations with almost any degree of perversity as long as the Condorcet winner is not BC bottom ranked. In all cases, the culprit for these differences is the Condorcet profile differential. Because of the importance of the Condorcet Principle, these comments are repeated in a formal statement.

Theorem 10. *Any disagreement between the BC and Condorcet winners, or between the way any candidate is ranked, is due to the fact that the Condorcet approach does not ignore the Condorcet portion of a profile. Therefore, the Condorcet outcome is influenced by the partial loss of the assumption of the individual rationality of the voters. Conversely, profiles can be constructed to illustrate any difference between the BC and Condorcet rankings by use of basic and Condorcet profile differentials.*

6.7. **Explanation of the BC problems.** The BC is partially immune to these criticisms because with *transitive preferences*, the BC tallies are equivalent to summing each candidate's pairwise tallies over all opponents. As this summation cancels the Condorcet's cyclic effect over all n candidates, it immunizes the BC outcome for n-candidates from this loss of individual transitivity.

This comforting assertion does not extend to the BC ranking for $k < n$ candidates because the cancellation fails. This is most easily seen with $k = 2, n = 3$ where the cancellation of the $A \succ B$ ranking from the Condorcet triplet fails because it requires the tallies from the $B \succ C, C \succ A$ rankings. These terms are unavailable because they involve an excluded candidate C. An identical explanation holds for all k where $3 \leq k < n$. (From a mathematical perspective and as a computation readily discloses, when the Condorcet n-candidate profile differential is restricted to a k-candidate subset, it is *not* orthogonal to the k-candidate basic profiles. This requires a portion of the Condorcet differential to influence the k-candidate basic profile, so it alters the BC outcome for this set.) Consequently the Condorcet profile differentials *must affect the BC tallies and rankings for k-candidate subsets.* This important fact completely explains the source of all changes in the BC rankings as candidates are added or dropped (including those from (Saari [11]). It also identifies new BC election relationships based on the fact that the terms needed for a cancellation are in other k-candidate subsets.

Theorem 11. *Let k satisfy $2 < k < n$. For each candidate, the sum of her BC tallies from a n-candidate Condorcet profile differential over all k candidate subsets is zero.*

According to Thm. 11, all distortions among k-candidate BC tallies are caused by the noise of the Condorcet profile differentials. My earlier observation that this noise causes the pairwise vote to lose the assumption of individual rationality extends to the BC elections of k-candidate sets. Thus, again, the Condorcet profile differential seriously erodes a basic assumption from choice theory.

To explain this important effect, I apply to the setting of $k = 3$ subsets with $n = 4$ candidates an argument almost identical to the one used for pairs. (This argument extends in a natural manner to all n and k values.) In Table 6.9, the left ranking in a row identifies the voter from the Condorcet four-tuple; the remaining entries are the rankings for the triplets.

Ranking	$\{A,B,C\}$	$\{B,CD\}$	$\{A,C,D\}$	$\{A,B,D\}$
$A \succ B \succ C \succ D$	$A \succ B \succ C$	$B \succ C \succ D$	$A \succ C \succ D$	$A \succ B \succ D$
$B \succ C \succ D \succ A$	$B \succ C \succ A$	$B \succ C \succ D$	$C \succ D \succ A$	$B \succ D \succ A$
$C \succ D \succ A \succ B$	$C \succ A \succ B$	$C \succ D \succ B$	$C \succ D \succ A$	$D \succ A \succ B$
$D \succ A \succ B \succ C$	$A \succ B \succ C$	$D \succ B \succ C$	$D \succ A \succ C$	$D \succ A \succ C$

(6.9)

By mimicking the arguments associated with Eq. 6.8, it follows that when a positional procedure ranks a three-candidate subset, it ignores all information about how the voters rank other subsets of candidates. For instance, when a procedure ranks all four triplets (or, more generally, all $\binom{n}{k}$ subsets of k candidates), anonymity precludes the procedure from determining how a voter ranks the candidates from different subsets. But by severing these connections, the procedure drops information about the individual rationality of voters. In particular, with the Condorcet profile of Table 6.9, the BC, or any other positional procedure cannot distinguish the original Condorcet profile from any profile constructed by permuting the entries of each column in any desired manner.

Most permutations of the entries of Table 6.9 (and its natural extension to any $k < n$) define settings where the voters have only k-fold transitivity. Namely, the procedures cannot distinguish between rational voters, or those voters who can rank subsets of three candidates in a transitive manner but cannot connect the triplets into a four candidate transitive ranking. One permutation of Table 6.9, for instance, defines the following table where each row lists a particular voter's preferences.

$A \succ B \succ C$	$B \succ C \succ D$	$C \succ D \succ A$	$D \succ A \succ B$
$A \succ B \succ C$	$B \succ C \succ D$	$C \succ D \succ A$	$D \succ A \succ B$
$B \succ C \succ A$	$C \succ D \succ B$	$D \succ A \succ C$	$A \succ B \succ D$
$C \succ A \succ B$	$D \succ B \succ C$	$A \succ C \succ D$	$B \succ D \succ A$

(6.10)

The imaginary voter of each row has a transitive ranking for each triplet, but the triplets are not compatible with any four-candidate transitive ranking. So, transitivity going from the level of three candidates to four is lost. This is particularly demonstrated by, say, the voter of row one whose rankings of triplets defines a cycle. But because the BC, or any other procedure, cannot distinguish the rational voter from these partially rational ones, it follows that a level of individual rationality is dismissed by using the BC on k-candidate subsets. Notice, the Condorcet and basic terms are the only ones effecting BC outcomes, so these comments completely explain all BC paradoxes including those from (Saari [11]). Although the same argument extends to all positional methods, these other methods remain subject to deviation effects of $\mathbf{d}^j$ which further distort the outcomes.

Proposition 2. *All k-candidate BC rankings which do not completely agree with the n-candidate BC ranking are caused by the profile's Condorcet differential portions. These profile differentials admit the interpretation that they drop portions of the assumption about the voters' individual rationality.*

To understand how the Condorcet profile differential alters the BC outcomes, notice from Table 6.10 that each column has a Condorcet triplet and an additional ranking. The Condorcet triplet defines a tie outcome with all positional methods, so the preference of the remaining ranking uniquely determines the BC outcome. This ranking, which duplicates another ranking from the column, is due to the non-cancellation of certain terms of the Condorcet n-tuple. Also notice how the repeated outcomes create a cycle over the three-candidate subsets. The same phenomenon extends to all k, $2 < k < n$. By exploiting this observation, it now is easy to create profiles demonstrating a variety of different paradoxes. For instance, by adding appropriate multiples of the Condorcet terms to a basic profile, the basic profile determines the four-candidate BC outcome, while the Condorcet portion twists the pairwise and three-candidate BC rankings.

These comments about the BC can be illustrated with Brams example [3] from the introductory section. His seven voter profile has 3 voters with the preferences $C \succ B \succ A \succ X$, two with $B \succ A \succ X \succ C$ and two with $A \succ X \succ C \succ B$. The BC ranking for all four candidates is $A \succ B \succ C \succ X$, but when X is dropped it becomes $C \succ B \succ A$. According to Prop. 2, this occurs only if the profile has a strong Condorcet element to alter the ranking for the triplet by weakening the assumption that voters are individually rational. This feature is easy to see because the preferences nearly complete the Condorcet four-tuple generated by $C \succ B \succ A \succ X$. (Only the ranking $X \succ C \succ B \succ A$ is missing.) This strong Condorcet portion significantly undermines the assumption of individual rationality of the voters. Indeed, it turns out that the basic and Condorcet portions are

$$\frac{1}{24}\{[7\mathbf{B}^4_A + 6\mathbf{B}^4_B + 5\mathbf{B}^4_C] - 21\mathbf{C}^4_{A\succ B\succ C\succ X}\}.$$

The bracketed basic term defines the natural ranking of $A \succ B \succ C \succ X$ while the dominant Condorcet term distorts the BC ranking of the triplet and weakens the assumption of individual rationality.

6.8. **Geometry of the BC and Copeland Method.** This discussion allows for a convenient geometric description of the BC in terms of pairwise outcomes. To do so, we deal with the normalized tally of pair $\{c_i, c_j\}$ by defining

$$x_{i,j} = \tau^2(c_i, c_j)/v \tag{6.11}$$

where v is the total number of voters. Notice that $x_{i,j} = -x_{j,i}$. The normalization requires $-1 \le x_{i,j} \le 1$ where -1, 0, 1 mean, respectively, that c_i does not receive a single vote, is tied, wins with a unanimous vote when compared with c_j. To create a geometric representation, assign each pair an axis from a $\binom{n}{2}$ dimensional space. The relevant portion of $R^{\binom{n}{2}}$ is the *orthogonal cube* defined by the product of all the $-1 \le x_{i,j} \le 1$ conditions.

To illustrate, the unanimity profile $c_1 \succ c_2 \succ \cdots \succ c_n$ requires $x_{i,j} = 1$ if $i < j$, so it defines a vertex of the orthogonal cube. More generally, the $n!$ unanimity profiles define $n!$ of the $2^{\binom{n}{2}}$ vertices of the cube. The remaining (and dominant number for $n \ge 4$) vertices cannot be election outcomes as this would require all voters to have nontransitive rankings. Indeed, the set of all possible pairwise outcomes is given by the rational points (where all components are fractions) in the convex hull defined by the unanimity profiles. (For details and motivation, see Saari [14].) I call this set the representation cube.

A natural coordinate system for the representation (and orthogonal) cube comes from the fact that all pairwise election outcomes $\mathbf{q}_n$ are due to the basic and Condorcet profile differentials. The outcomes from the basic profiles must satisfy the desired properties of Cor. 6 and Eq. 6.3; this set of points spans what I call the *transitivity plane* of $R^{\binom{n}{2}}$. Each Condorcet profile differential defines an associated *Condorcet direction* of the orthogonal cube and of $R^{\binom{n}{2}}$. The connections among them are specified next.

Theorem 12. *The transitivity plane passes through the center point* $\mathbf{0}$ *of the orthogonal cube and the point where* c_i *unanimously beats each of the other candidates, and all other pairwise elections end in a complete tie;* $i = 1, \ldots, n$. *Each Condorcet direction is orthogonal to the transitivity plane. For* $n = 3, 4$, *the Condorcet directions are orthogonal to each other; this is* *not true for $n \geq 4$. All points in the orthogonal or representation cubes can be represented as the vector sum of points in the transitivity plane and Condorcet directions.*

Proof. The assertion about the spanning vectors for the transitivity plane follows by observing that these points are the election outcomes for the basic profiles. The assertion that these vectors span $R^{\binom{n}{2}}$, or the orthogonal, or representation cubes, follows from the fact that the representation cube contains an open subset of $R^{\binom{n}{2}}$ and that all outcomes in the cube can be obtained from the basic and Condorcet profile differentials.

To prove that each Condorcet direction is orthogonal to the transitivity plane, it suffices to compute the scalar product of one of them with the c_j-basic profile outcomes. But (a multiple of) the c_j-basic profile outcome has the components $x_{j,s} = 1$ for all $s \neq j$ (c_j wins unanimously over all candidates) and zero for all others (the remaining outcomes are ties causing $x_{i,s} = 0$). Thus, the scalar product is equivalent to the sum of c_j's pairwise election tallies over all possible opponents. According to Thm. 7, this is zero.

The proof that the Condorcet directions are orthogonal for $n = 3, 4$, but not so for $n \geq 5$ follows from a direction computation using the pairwise values given in Thm. 7. □

A consequence of this structure is to provide a convenient geometric representation for the BC and for Copeland's Method (CM) (see Saari and Merlin [20].) The CM is where instead of dealing with pairwise tallies, candidate c_i receives $1, 0, -1$ points if, respectively, c_i wins, ties, or loses to c_j. Her CM score is the sum of points received in each comparison. This means that if $\mathbf{q}_n$ is the actual representation cube point representing the tallies of all pairwise elections, then the corresponding CM point is $\mathbf{q}_{CM}$ where each $x_{i,j} \neq 0$ is replaced with the nearest of $1, -1$, otherwise it keeps the zero value. Thus, unless there is a tie outcome, $\mathbf{q}_{CM}$ is a vertex of the orthogonal cube.

Theorem 13. *If $\mathbf{q}_n$ represents the pairwise tallies in the representation cube, then the corresponding BC outcome is given by the ranking associated with the unique point in the transitivity plane which is closest to $\mathbf{q}_n$. The CM outcome is given by the unique point in the transitivity plane which is closest to $\mathbf{q}_{CM}$.*

Proof. Because $\mathbf{q}_n$ can be expressed as $\mathbf{q}_n = \mathbf{q}_T + \mathbf{q}_{Con}$ where $\mathbf{q}_T$ and $\mathbf{q}_{Con}$ are, respectively, the transitivity plane and orthogonal component, it suffices (from our derived properties of the basic profiles) to show that the BC returns a zero value for $\mathbf{q}_{Con}$. According to Thm. 12, this term is given by the Condorcet directions. As each candidate's BC tally for $\mathbf{q}_{Con}$ comes from summing her tallies in the pairwise elections over all opponents, the conclusion now follows from Thm, 7. Similarly, $\mathbf{q}_{CM}$ has a similar decomposition, and each candidate's tally also depends on her sum of points over all opponents, so the same argument applies. □

7. More candidates; more procedures

The problem with $n \geq 5$ candidates is that the Condorcet profile differentials span a space with dimension larger than the needed $\binom{n-1}{2}$. This suggests that these differentials influence more than just pairwise and BC outcomes. This is the case; Thm. 7, part 6 asserts that for certain even values of k, the Condorcet-profile differential is *not* orthogonal to the deviation vector $\mathbf{d}^k$. Consequently these Condorcet terms go beyond influencing the BC conclusion to change the plurality, antiplurality, and other positional outcomes. To be specific with $n = 8$, a Condorcet profile differential *is* orthogonal to $\mathbf{d}^3$ for each triplet (Thm. 7), so this profile has no additional impact on positional three-candidate outcomes beyond the description given by Table 6.10. For four candidate subsets, however, the Condorcet differential is *not* orthogonal to the deviation vectors $\mathbf{d}^4$. So, this profile differential not only changes the BC four-candidate tallies, but further alters the plurality outcomes. In turn, these changes influence outcomes of procedures in the derived five candidate subsets $\mathcal{D}^5$. This five candidate distortion is advanced to the six candidate outcomes; and an added distortion comes from the fact that the Condorcet differential is not orthogonal to the $\mathbf{d}^6$ terms.

My next goal, then, is to separate from the Condorcet profile differentials those effects which influence the pairwise ranking from those which influence the non-BC positional rankings. A natural resolution is to find an orthogonal basis using standard approaches (such as the Gram-Schmidt orthonormalization process). For instance, since we do not want the Condorcet portion to have any component in the deviation vector directions, subtract profile components in this direction from the Condorcet differential. We could also eliminate all portions of the Condorcet profile differentials that are in the kernel direction, but this is not necessary from a practical perspective (because the kernel portion does not effect any positional or pairwise ranking). Indeed, after computing the resulting differentials, I found that they are unnecessarily difficult to explain and to use when constructing examples of profiles. Therefore, the following definition captures the portions of a Condorcet profile differential that effects *only* pairwise votes and some terms in the kernel, but has no further effect upon positional procedures.

Definition 5. *For candidates c_i and c_j, the $c_i \succ c_j$ Condorcet profile differential is defined by using all Condorcet profile differentials determined by rankings r where the top and second ranked candidates are, respectively, c_i and c_j.*

To illustrate, the $A \succ B$ Condorcet profile differential for four candidates combines the two Condorcet profile differentials defined by $A \succ B \succ C \succ D$ and $A \succ B \succ D \succ C$. The following theorem asserts the $c_i \succ c_j$ Condorcet profile differentials does not influence the remaining election outcomes. The tallies associated with the the new differentials are easier to use because they emphasize two candidates rather than several.

Theorem 14. *Assume there are $n \geq 4$ candidates.*

1. *Each $c_i \succ c_j$ Condorcet profile differential is orthogonal to the basic vectors as well as to all $\mathbf{d}^k$ vectors for each subset of three or more candidates. As such, the set of all basic vectors and all $c_i \succ c_j$ Condorcet vectors uniquely determine all pairwise and all BC outcomes. For each subset of k candidates, a positional outcome based on these profiles agrees with the BC outcome.*
2. *A $c_i \succ c_j$ Condorcet profile differential is orthogonal to all double reversal profile differentials. There exist, however, symmetry changing profile differentials from $\mathcal{UK}^n$ which are not orthogonal to the $c_i \succ c_j$ Condorcet profile differential.*
3. *For the $c_i \succ c_j$ Condorcet profile differential, c_i beats c_j in a pairwise election with the $(n-2)(n-2)! : -(n-2)(n-2)!$ tally. However, c_j beats and c_i loses to all other candidates with a $(n-2)! : -(n-2)!$ tally. The pairwise outcome for any other pair of candidates is a tie where each candidate receives zero votes.*
4. *For a k-candidate subset, $2 < k < n$, the tally of a $c_i \succ c_j$ Condorcet profile differential is the same for all positional methods. If both c_i and c_j are in the set, then c_i receives $(n-k)(n-2)!$ points, c_j receives the negative of this, and all other candidates receive zero points. If c_i is in the set, but c_j is not, then c_i receives $-(k-1)(n-2)!$ points while each other candidate receives $(n-2)!$ points. If c_j is in the set when c_i is not, then c_j receives $(k-1)(n-2)!$ points and each other candidate receives $-(n-2)!$ points. For all other sets, all candidates receive zero points.*

The proof of this theorem is in Sect. 10. The use of the theorem to create examples, and an explanation of the meaning of these profile differentials follows the lead of the discussion on Condorcet profile differentials. After all, the $c_i \succ c_j$ Condorcet profile differential is just a sum of $(n-2)!$ of the simpler Condorcet profile differentials.

While the $c_i \succ c_j$ Condorcet profile differential is cumbersome, it offers two advantages. The first reflects the purpose of defining it from the Condorcet profile differentials; the profile does not introduce added profile noise for positional voting outcomes. (This is part 4.) The second advantage (part 2) is that the relative tallies for only a limited number of pairs of candidates are affected. To illustrate, in the introductory section I claimed there is a profile where the BC ranking is $c_2 \succ c_3 \succ \cdots \succ c_{10} \succ c_1$ even though the rankings of $\{c_1, c_2\}, \ldots, \{c_1, c_2, \ldots, c_9\}$ reflect the $c_1 \succ c_2 \succ \cdots \succ c_9$ ranking. It now is clear how to construct a supporting profile. Namely, start with a basic profile which gives the indicated ten-candidate BC outcome. Next, add appropriate multiples of $c_1 \succ c_j$ Condorcet profile differentials, $j = 2, 3, \ldots, 10$ so that the desired outcome for the different subsets occurs. Incidentally, all profiles with this same behavior do so because of their $c_{\succ} c_j$ Condorcet components.

8. The deviation profiles

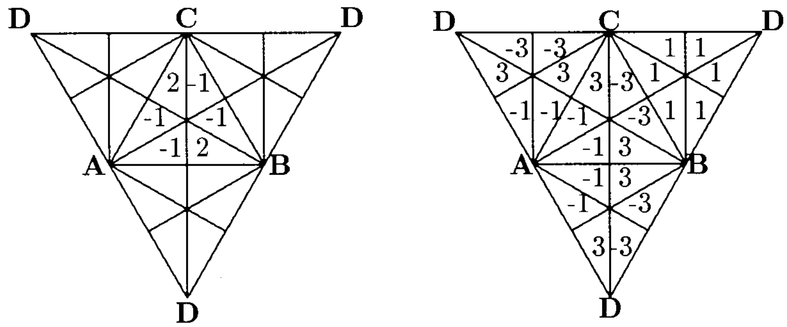
Now that we understand the source of all pairwise and BC outcomes, it remains to find the profile differentials which cause the outcomes of positional methods to deviate from the BC and the pairwise conclusions. The initial form of these profiles is easy to determine by using the deviation vectors. However we encounter a problem similar to the one that arose for Condorcet profile differentials; these profile differentials also influence the positional rankings of larger subsets of candidates. Again, the tradeoff is to use the profile differentials in the relatively simple manner given below, or, to ensure orthogonality, modified them via standard procedures to remove all extraneous influences. Because the approach is well understood, I present and describe properties of the "raw profile differentials." The reader interested in applying these differentials has two options; either carry out the orthogonal process or adjust the rankings of the different subsets. I illustrate both approaches with $n = 4$ candidates.

Definition 6. *The c_i* deviation profile differential for the set of all $n \geq 3$ candidates, $\mathbf{D}^n_{c_i}$ *is defined in the following manner. For each possible ranking where c_j is sth ranked, $s = 1, \ldots, n$, assign $(-1)^s\binom{n-1}{s-1}$ voters.*

For a subset $\mathcal{S}$ of k candidates, $3 \leq k < n$, the c_i deviated profile differential $\mathbf{D}^k_{c_i,\mathcal{S}}$ *is where the $\mathbf{D}^k_{c_i}$ profile is augmented $(n-k)!$ times by adding all possible rankings of the candidates not in $\mathcal{S}$ as bottom ranked.*

It is important to remember that the profile differentials determine the decomposition of profile space. It would be rare, indeed, for a particular profile to consist completely of these profile differentials rather than only portions of them. As examples, Fig. 4 shows the $\mathbf{D}^3_{A,\mathcal{S}}$ for $\mathcal{S} = \{A, B, C\}$ and $\mathbf{D}^4_A$. So, the unanimity profile $A \succ B \succ C \succ D$ has a profile decomposition with components in the $\mathbf{D}^3_{A,\{A,B,C\}}$ and $\mathbf{D}^4_A$ directions.

Some of the particularly important properties of these deviation profiles differentials are introduced in the next theorem.



a. $\mathbf{D}^3_{A.\mathcal{S}}$ b. $\mathbf{D}^4_A$

Fig. 4. Deviation profiles

Theorem 15. *Assume there are $n \geq 3$ candidates and that $\mathcal{S}$ is a subset of k candidates where $3 \leq k \leq n$. The following statements concern the tally of $\mathbf{D}^k_{c_i.\mathcal{S}}$, $c_i \in \mathcal{S}$.*

1. *All pairwise tallies end in a complete tie where each candidate receives zero points.*
2. *If there is at least one candidate in $\mathcal{S}$ that is not in $\mathcal{S}'$, then all $\mathcal{S}'$ positional tallies of $\mathbf{D}^k_{c_i.\mathcal{S}}$ are zero.*
3. *If $\mathbf{w}^k$ is a voting vector from the derived set $\mathcal{D}^k \subset \mathcal{S}$, then the $\mathbf{w}^k$ tally of $\mathbf{D}^k_{c_i.\mathcal{S}}$ is a complete tie where all candidates receive zero points.*
4. *Let $\mathcal{S}$ is a proper subset of the set of candidates $\mathcal{S}'$ where the deviation vector for $\mathcal{S}'$ is $\mathbf{d}'$. The $\mathbf{d}'$ tally of $\mathbf{D}^k_{c_i.\mathcal{S}}$ has a positive tally for c_i, and an equal tally for all remaining candidates in $\mathcal{S}$. The sum of the tallies equals zero. If $\mathcal{S}'$ has one more candidate than $\mathcal{S}$, then this candidate has a zero tally. In general, each candidate in $\mathcal{S}' - \mathcal{S}$ has the same tally.*
5. *The $\mathcal{S}$ plurality tally for c_i has her bottom ranked with $-(n-k)!(k-1)!$ votes. The tally for each of the remaining $(k-1)$ candidates is $(n-k)!(k-2)!$ votes.*
6. *If $\mathbf{d}^k$ is the deviation vector for $\mathcal{S}$, then the $\mathbf{d}^k$ tally of $\mathbf{D}^k_{c_i.\mathcal{S}}$ has c_i top ranked and all remaining candidates tied for bottom. As the sum of the votes equals zero, a bottom ranked candidate's tally is the $\frac{-1}{k-1}$ multiple of c_i's tally.*
7. *The deviation profile differentials are not orthogonal; instead they satisfy the relationship*

$$\sum_{c_j \in \mathcal{S}} \mathbf{D}^k_{c_j.\mathcal{S}} = \mathbf{0}. \tag{8.1}$$

It is easy to argue that the natural election outcome for $\mathbf{D}^k_{c_i.\mathcal{S}}$ should be a completely tied vote. As a way to see this, start with a two person profile where one preference is given by r and the other with the exact opposite preferences $\rho(r)$. Here, it is arguable, that the outcome should be a complete tie; this is similar to where a husband and wife justify not voting because their preferences cancel. Indeed, it is a simple exercise to show that this special profile has a completely tied pairwise vote and (hence) a completely tied BC outcome for each subset of candidates.

To use this intuition to analyze $\mathbf{D}^n_{c_i}$ for an odd integer n, notice that $\mathbf{D}^n_{c_i}$ is the sum of two-person profiles of this completely conflicting type. Indeed, if r is one of the rankings where c_i is jth ranked, then $\rho(r)$ is one of the rankings where c_i is $(n-j)$th ranked. This one-to-one relationship and the equal number of voters for each setting completes the proof of this assertion.

A closely related argument that $\mathbf{D}^k_{c_i.\mathcal{S}}$ should lead to a complete tie comes from parts 1, 2 of Thm. 15; they assert that $\mathbf{D}^k_{c_i.\mathcal{S}}$ has no influence on the pairs, on the rankings of any subset, or on the rankings of any other subset with the same number of candidates. For instance, with ten candidates, $\mathbf{D}^{10}_{c_i}$ has a completely tied tie vote for all sets of pairs, triplets, ... , sets of nine candidates, and then, in stark contradiction, it suddenly ranks c_i as the top-ranked candidate for the set of all ten-candidates. Finding a justification for such a conclusion is not obvious

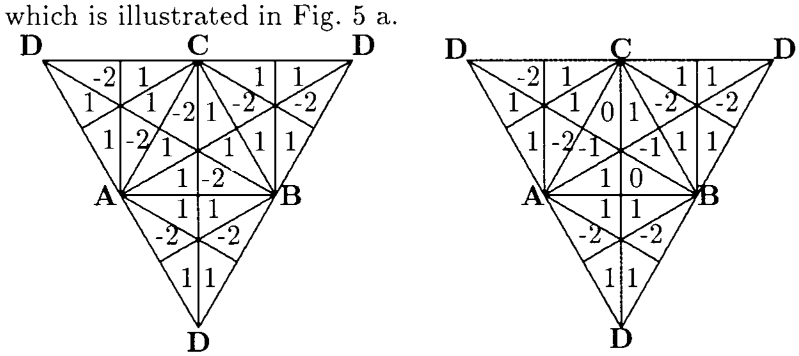
9. Applications

To indicate implications, I use Thm. 15 in three ways. The first illustrates how to create profiles with certain desired behavior. The second briefly indicates how to analyze voting procedures. The third underscores the actual complexity of standard profiles.

9.1. **Constructing profiles.** Suppose we want to create a four-candidate profile with a complete tie for all pairwise, all BC, and all four-candidate positional rankings. The plurality rankings of the triplets, however, are to form a cycle where, say, $A \succ B \sim C$, $B \succ C \sim D$, $C \succ D \sim A$, $D \succ A \sim B$. According to Thm. 15, part 5, the construction starts with

$$\mathbf{p}' = -\mathbf{D}^3_{A.\{A.B.C\}} - \mathbf{D}^3_{B.\{B.C.D\}} - \mathbf{D}^3_{C.\{C.D.A\}} - \mathbf{D}^3_{D.\{D.A.B\}} \tag{9.1}$$

which is illustrated in Fig. 5 a.



a. Example profile **b.** $C \succ A \succ B$ ranking

Fig. 5. Profile examples

According to parts 1, 2 of Thm. 15, profile $\mathbf{p}'$ has no impact upon the pairwise or BC votes; all end in complete ties with zero tallies. Part 2 also asserts that the $\mathbf{D}^3_{A.\mathcal{S}}$ portion for each triplet $\mathcal{S}$ has no effect upon the outcome for any other triplet. Therefore, the plurality triplet outcomes are as desired. Part 4, on the other hand, warns that $\mathbf{p}'$ might change the positional outcomes of the larger set of four candidates. The extreme symmetry of this particular example, however, creates a cancellation leaving the four-candidate plurality ranking in a complete tie. All of this can be verified from Fig. 5 by using the tallying procedures introduced in Sect. 5.

(To convert $\mathbf{p}'$ into a profile, add an appropriate $\mathcal{UK}^4$ profile to make all terms non-negative. As one choice, notice how the indicated double reversal of Fig. 3a (given by the arrows) changes the original $\mathcal{UK}^4$ profile to another one with unity in the same ranking regions of $\{A, B, C\}$ and $\{A, C, D\}$ with negative entries in Fig. 5a. Adding similar double reversal profiles to the $\{A, B, D\}$, $\{B, C, D\}$ regions creates a twelve voter $\mathcal{UK}^4$ profile with unity in all regions with a Fig. 5a negative entry. By doubling each entry and adding the result to the profile differential of Fig. 5a, we obtain a 24 voter profile with the desired properties.)

To indicate how to handle situations where a four-candidate cancellation does not occur, change the $\{A, B, C\}$ ranking of the Fig. 5a example to $C \succ A \succ B$. This occurs by replacing the $-\mathbf{D}^3_{A.\{A.B.C\}}$ portion of profile $\mathbf{p}'$ with $\frac{1}{3}[-\mathbf{D}^3_{C.\{A.B.C\}} + \mathbf{D}^3_{A.\{A.B.C\}}]$. The new profile is in

Fig. 5b where the difference are in the $\{A, B, C\}$ triangle. This choice, however, changes the four candidate plurality ranking from a complete tie to $C \succ A \sim B \succ D$ with tally 2:0:0:-2. To obtain the desired four-candidate complete tie, just add $\frac{1}{4}[\mathbf{D}^4_C - \mathbf{D}^4_A]$. As we know from Thm. 15, this profile differential does not effect the pairwise or triplet rankings, but it returns the four-candidate ranking to a complete tie.

An alternative approach is to replace the $\mathbf{D}^3_{c_j,\mathcal{S}}$ profiles with a profile that only influences $\mathcal{S}$ rankings. To see how to do this, notice from Fig. 4a that the $\mathbf{D}^3_{A,\{A,B,C\}}$ profile has the four candidate plurality ranking of $B \sim C \succ D \succ A$ with the tally $1:1:0:-2$. So, to remove this influence while retaining the properties of $\mathbf{D}^3_{A,\{A,B,C\}}$, add $\frac{1}{4}[-\mathbf{D}^4_A + \frac{1}{2}[\mathbf{D}^4_B + \mathbf{D}^4_C]$ to cancel the four-candidate effects. To remove fractions from the resulting profile has fractions, use $4\mathbf{D}^3_{A,\{A,B,C\}} - \mathbf{D}^4_A + \frac{1}{2}[\mathbf{D}^4_B + \mathbf{D}^4_C]$. This profile differential, given in Fig. 6, has the desired properties of Thm. 15 plus the added property that it does not change the four candidate outcomes. (This can be shown by a direct computation of the tallies.) The choices for other sets and candidates follows from symmetry.

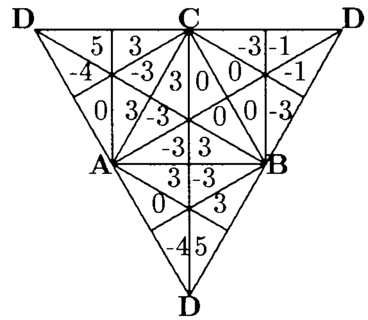


Fig. 6. Profile $4\mathbf{D}^3_{A,ABC} - \mathbf{D}^4_A + \frac{1}{2}[\mathbf{D}^4_B + \mathbf{D}^4_C]$

The tally for $\mathbf{D}^3_{A,\{A,B,C\}}$ satisfies the conditions of Part 4 of Thm. 15, and it dictates the appropriate adjustment term. However with a five candidate setting, a further adjustment term is needed to capture the added Part 4 assertion that $\mathbf{D}^3_{A,\{A,B,C\}}$ also affects the deviation vector for five candidates. For this reason, when the number of candidates is not specified, if it is large, and/or if this number is intended to be vague, I prefer using the deviation profiles as defined. However, for specified $n \leq 5$ values, it seems easier to modify the deviation vectors so they have no terms upon other subsets. This easy modification follows the pattern of Fig. 6.

9.2. **Runoffs.** The way Thm. 15 is used to analyze other procedures mimics the use of the basic and Condorcet profiles to analyze procedures based on binary rankings. (See Sect. 6.4.) Namely, with the basic profile the outcomes over all subsets of candidates agree. Here there is no disagreement; any reasonable procedure behaves as we might hope. Conflict, then, is generated only by adding profile noise. As we now know how to alter the rankings of any subset of candidates, we know how to create examples illustrating all possible flaws of all procedures.

To illustrate with a five-candidate plurality runoff, start with a basic profile with the universal ranking $A \succ B \succ C \succ D \succ E$. Suppose we want E to be the runoff winner where the order of candidates being eliminated is A, B, C, D, but all subsets of candidates not being voted upon have a ranking consistent with the basic profile ranking. (This demonstrates flaws of the procedure.) If the basic profile has all binaries with a stronger victor margin than in $\{D, E\}$ election, then appropriate multiple of the $E \succ D$ Condorcet profile differential changes only the $\{D, E\}$ outcome to have E beating D. Adding appropriate multiples of $\mathbf{D}^3_{C.\{C.D.E\}}$ and $\mathbf{D}^3_{D.\{C.D.E\}}$ ensure the $E \succ D \succ C$ plurality outcome, while all other triplets retain the basic profile ranking.

The rest of the induction step is clear. First, add appropriate four candidate deviation profile differentials to return the four candidate outcomes to the basic profile rankings, and then add appropriate profile differentials to convert the $\{B, C, D, E\}$ outcome into $E \succ D \succ C \succ B$. The same is done for the five-candidate election.

A closely related construction, but using only Condorcet profile differentials can identify flaws of the Nanson method. This is a runoff where at each stage the BC bottom ranked candidate is dropped. While the outcome is the Condorcet winner, when one exists, it now is easy to create examples with explanations showing that this is the wrong candidate. To illustrate, the development of Chap. 5 from [14] prove that Nanson's method is not monotonic. If, however, only basic profiles are being used, the Nanson's method is monotonic. This flaw, then, is caused by the Condorcet portion of a profile. In turn, this raises doubts about Nanson's approach.

9.3. **Unanimity.** To conclude, notice that the number of voters needed to support a profile $\mathbf{p}$ is not correlated with its apparent complexity when $\mathbf{p}$ is described in terms of its profile components. To illustrate, the following four-candidate profile differential has components in the basic, the Condorcet, deviation profiles in each of the four triplet directions and even deviation profiles for the set of all four candidates.

$$\mathbf{p} = [3\mathbf{B}^4_A + 2\mathbf{B}^4_B + \mathbf{B}^4_C] + 3\mathbf{C}^4_{A\succ B\succ C\succ D} + \\ 4[\mathbf{D}^3_{B.\{A.B.C\}} + \mathbf{D}^3_{C.\{B.C.D\}} + \mathbf{D}^3_{C.\{C.D.A\}} + \mathbf{D}^3_{B.\{A.B.D\}}] - [\mathbf{D}^4_B + 2\mathbf{D}^4_C] \qquad (9.2)$$

While $\mathbf{p}$ appears to promises all sorts of complications, it is merely the unanimity profile differential for $A \succ B \succ C \succ D$. (Elsewhere I describe how to start with a profile and then compute its component parts.) It remains to add the appropriate kernel term to have a positive number of voters for the $A \succ B \succ C \succ D$ ranking and zero for all others.

This Eq. 9.2 representation proves that even the unanimity profile has interesting properties. To illustrate, the first bracketed term of Eq. 9.2 is the important basic profile; it specifies that the natural ranking for this profile is, indeed, $A \succ B \succ C \succ D$. The Condorcet term, the second term which introduces a twist to the pairwise rankings, explains why the pairwise tallies fail to reflect A's preferred status. (Recall, the pairwise tallies for $\{A, B\}$ and $\{A, C\}$ agree, but the tally for the basic portion gives A a higher tally in the second election.) The next bracket of four terms is the portion of the profile which changes the three candidate plurality outcomes. These terms cause the plurality ranking to be $A \succ B \sim C$ rather than the natural $A \succ B \succ C$ for $\{A, B, C\}$. To explain, the BC outcome remains compatible with expectations, but the $\mathbf{D}^3_{B.\{A.B,C\}}$ portion of the unanimity profile reduces B's basic tally to obtain the distorted plurality outcome. A similar explanation holds for the four-candidate elections. Here, the plurality vote changes the basic profile ranking of $A \succ B \succ C \succ D$ to $A \succ B \sim C \sim D$ because of the profile's component of deviation profile noise that appears in the last bracketed term.

9.4. **Summary.** In summary, we now can explain all differences in outcomes coming from any positional procedures and/or from methods based on pairwise and positional outcomes. The analysis starts with the basic profiles where there is agreement. Any disagreement in pairwise outcomes can be completely explained by the Condorcet portion of a profile. Any disagreement in the positional rankings of any other subset is based on deviation profile affects.

All of these effects have interpretations. An important theme throughout this analysis is that an important reason election outcomes can appear to be irrational is that the procedures negate, at different levels, the assumption of individual transitivity. Here there are different levels; the first comes from the Condorcet terms. However, even the deviation terms create a similar problem. A way to see this is to apply the earlier arguments developed for the Condorcet terms to the deviation profile terms for $(k-1)$ candidate subsets. Because the sets are separate and the deviation profiles have no effect on the other subsets, there is no assumption of rationality. When appropriate procedures are used at the next level of k candidates, however, these these irrationality effects are canceled in a manner consistent with the BC. Namely, the $g_k(\mathbf{w}^{k-1})$ outcomes preserve a higher level of rationality than the $k\ \mathbf{w}^{k-1}$ results. This means that comflicts are caused by the weakening of the assumption of voter rationality along with the deviation vector and profile affects.

Of equal interest, the profile decomposition allows us to construct examples to illustrate any admissible behavior. This statement holds for all methods based on positional and pairwise outcomes.

10. Proofs

Proof of Thm. 2. All that remains to be proved in Thm. 2 is the assertion that $\mathbf{W}^n$ is a vector in a $\nu(n) = 2^{n-1}(n-4)+n+2$ dimensional Euclidean space. To derive this $\nu(n)$ value, notice that a j-candidate voting vector has j components where the first is unity and the last is zero. This leaves $j-2$ weights free to be chosen, so the number of free variables in $\mathbf{W}^n$ is $\nu(n) = \sum_{j=2}^{n} \binom{n}{j}(j-2)$. By differentiating the binomial expression

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j \tag{10.1}$$

we obtain

$$n(1+x)^{n-1} = \sum_{j=1}^{n} j \binom{n}{j} x^{j-1}. \tag{10.2}$$

After setting $x=1$ in Eqs. 10.1, 10.2 and using some algebra, the expression for $\nu(n)$ follows. □

Proof of Thm. 3. To prove that the $\mathbf{u}_j^k$ vectors have the desired form, it suffices to consider how many points a voter with preferences $c_1 \succ c_2 \succ \cdots \succ c_k$ will cast for each candidate when all $(k-1)$ candidate elections are tallied with $\mathbf{v}_j^{k-1}$. In any of the subsets where candidate c_i, $i \le j$, is included, c_i receives one point. To count the number of these subsets, notice that each $(k-1)$-candidate subset can be characterized in terms of the missing candidate; each candidate is missing from precisely one set. As this requires c_i to be in all but one subset (that is, she is in precisely $k-1$ sets), she receives $k-1$ points.

Candidate c_{j+1} receives a point only if one of the top j ranked candidates is not present. As each candidate is absent from precisely one subset, this means that c_{j+1} receives a point in j subsets. This means that c_{j+1} receives j points.

It remains to consider a candidate c_i where $i > j+1$. For c_i to receive any points, at least two candidates ranked above her must be missing from a $(k-1)$ candidate subset. But this never can occur, so c_i receives zero points. This proves that the $\mathbf{u}_j^k$ vectors have the indicated form.

To prove that $\mathbf{d}^k$ is a normal vector for $\mathcal{D}^k$, it suffices to prove that $\mathbf{d}^k$ is orthogonal to $\mathbf{u}_j^k - \mathbf{b}^k$. (This is because these vectors span the space $\mathcal{D}^k$.) To simplify the form, recall that $\mathbf{B}^k = (k-1)\mathbf{b}^k$, so an equivalent problem is to show that

$$\mathbf{d}^k \cdot (\mathbf{u}_j^k - \mathbf{b}^k) = \mathbf{d}^k \cdot ((k-1)\mathbf{u}_j^k - \mathbf{B}^k) = 0. \tag{10.3}$$

By use of the equality $\binom{n}{j} = \binom{n}{n-j}$, we have that

$$\mathbf{d}^k \cdot \mathbf{B}^k = \sum_{j=1}^{k-2} (-1)^j \binom{k-1}{j} j.$$

By comparing this expression with the summation obtained from Eq. 10.2 when $n = k-1$ and $x = -1$, it follows that

$$\mathbf{B}^k \cdot \mathbf{d}^k = \sum_{j=1}^{k-2} (-1)^j j \binom{k-1}{j} = (k-1)(1-1)^{k-2} + (k-1) = k-1 \tag{10.4}$$

It remains to prove that $(k-1)\mathbf{u}_j^k \cdot \mathbf{d}^k = k-1$.

The proof that $(k-1)\mathbf{u}_j^k \cdot \mathbf{d}^k = k-1$ uses induction. Only the second coordinate of both $(k-1)\mathbf{u}_1^k$ and $\mathbf{d}^k$ are nonzero; they are, respectively 1 and $k-1$. It now follows trivially that $(k-1)\mathbf{u}_1^k \cdot \mathbf{d}^k = k-1$.

Using the induction hypothesis, assume that $(k-1)\mathbf{u}_j^k \cdot \mathbf{d}^k = k-1$ for $j < s$. We need to establish that $(k-1)\mathbf{u}_s^k \cdot \mathbf{d}^k = k-1$, or that $[(k-1)\mathbf{u}_s^k - (k-1)\mathbf{u}_{s-1}^k] \cdot \mathbf{d}^k = 0$.

The vector in the bracket has non-zero values only in the sth and $(s+1)$th position; they are, respectively, $k-s$ and s. In turn, this means that the scalar product becomes (up to a sign) $(k-s)\binom{k-1}{k-s} - s\binom{k-1}{k-(s+1)}$. Using the binomial expressions, this becomes

$$\begin{aligned} &(k-s)\frac{(k-1)!}{(k-s)!((k-1)-(k-s))!} - s\frac{(k-1)!}{(k-1-s)!s!} \\ &= (k-1)![\frac{1}{(k-s-1)!(s-1)!} - \frac{1}{(k-s-1)!(s-1)!}] = 0. \end{aligned} \tag{10.5}$$

This completes the induction proof. □

Proof of Cor. 4. An outline for the simple induction proof is given. The pairwise tallies determine the BC tallies for all subsets of candidates. For all triplets, the BC and plurality tally determines all positional tallies. These tallies determine the tallies of the derived set for four candidates. To obtain all four-candidate tallies, the plurality tally determines the required deviation $\mathbf{d}^4$ tally (Cor. 1) for the four-candidate subsets. The obvious induction argument completes the proof. □

Proof of Thm. 4. The linearity of the tallying procedure ensures there is a kernel. The fact the universal kernel, determined by the pairwise and plurality votes, is contained in the kernel of all procedures is a direct consequence of Cor. 4. All that remains is to find the dimension of the universal kernel. Again from the linearity of the tallying procedure, this value is the difference between the dimension of the normalized space of profiles $(n!-1)$ and the dimension of the normalized space of vote tallies.

The normalized space of vote tallies is where instead of election tallies, we compute the fraction of the total vote received by each candidate. Thus a k candidate election has $(k-1)$ degrees of freedom. This means that the dimension of all pairwise elections is $\binom{n}{2}(2-1)$, of the triplets is $\binom{n}{3}(3-1), \ldots$. The total dimension is $\sum_{j=2}^{n}(j-1)\binom{n}{j} = \sum_{j=2}^{n} j\binom{n}{j} - \sum_{j=2}^{n}\binom{n}{j}$..

It follows from Thm. 1 that this is the dimension of the image space for the plurality vote. To find the value of this summation and to show that the dimension of the kernel is as specified in the theorem is a straightforward computation that uses Eqs. 10.1, 10.2 in a manner similar to the derivation of the $\nu(n)$ value. □

Proof of Thm. 6. In a plurality (or pairwise) tally of $\mathbf{B}^n_{c_i}$ for any subset containing c_i, she receives one point for each ranking where she is top-ranked. There are $(n-1)!$ of them, so this gives her tally. All remaining candidates are treated symmetrically, so each receives the same point total over the subset. Also, $\mathbf{B}^n_{c_i}$ is a profile differential, so the some of each candidate's tally must be zero. This means that in a k-candidate subset with c_i, each of the other candidates receives $-\frac{(n-1)!}{k-1}$ votes. In a set where c_i is not a candidate, each candidate receives zero votes.

By use of the summation process defining the $\mathbf{b}^k$ outcome, it follows that c_1 in a k candidate subset receives $\frac{1}{k-1}(k-1)((n-1)!)$ voters. (The fraction is to normalize the BC outcome, the $(k-1)$ term is the number of pairwise elections.) If c_1 is not in a set, then the BC tally for all candidates is zero. Thus the BC and plurality outcomes for all subsets is as stated. According to Cor. 4, the normalized tally for all procedures agree.

To prove the summation assertion, notice that in the $(n-1)!$ terms where c_i is top-ranked, c_j is bottom-ranked in precisely $\frac{(n-1)!}{n-1} = (n-2)!$ of them, $j \neq i$. Similarly, in the $(n-1)!$ terms where c_i is bottom ranked, c_j is top-ranked in precisely $(n-2)!$ of them, $j \neq i$. The conclusion now follows with a simple computation. □

Proof of Thm. 7. Part 1. This is a simple computation.

Part 2. Of the several ways to prove this assertion, an easy one involves the dimension of the set of deviation profiles. The difference between the dimension of $\mathcal{P}^n$ and the dimension of the deviation profiles is $\binom{n}{2}$. Because each of these deviation profiles has a zero tally for any binary election, they have no component in the binary vote direction. Thus, the space orthogonal to these vectors includes the kernel, the basic, and the $c_i \succ c_j$ Condorcet profile differentials. In turn, if $\mathbf{p}$ is orthogonal to the basic and Condorcet profile differentials, it must be in the space of deviation profiles. This completes the proof.

Part 3. The total number of points in a positional method $\mathbf{w}^n = (1, w_2, \dots, w_{n-1}, 0$ is $\sum_{j=1}^n w_j$. Because each candidate is ranked first, second, ... , last precisely once in the Condorcet n-tuple defined by r, each candidate receives $\sum_{j=1}^n w_j$ points. The only change in this argument for a candidate in the $\rho(r)$ portion is that each candidate receives $-\sum_{j=1}^n w_j$ points. This completes the proof.

Part 4. Let $\mathbf{C}^n_r$ be the Condorcet profile differential defined by r. To show that $\mathbf{C}^n_r$ is orthogonal to an arbitrarily chosen basic profile $\mathbf{B}^n_{c_j}$, notice that the only terms they have in common is when c_j is top and bottom ranked in the Condorcet n-tuple defined by r and the one defined by $\rho(r)$. In the r-Condorcet n-tuple, these two rankings have the same number of voters; in the basic profile, one term has a positive number of voters and the other has a negative number of these voters. Thus, these terms cancel. The same argument holds for the $\rho(r)$ portion.

To prove the statement about the double-reversal profiles, let the kernel profile be where there is one voter for each of the $r_1 \succ r_2$ and $\rho(r_1) \succ \rho(r_2)$ preferences and -1 voters for each of the $r_1 \succ \rho(r_2)$ and $\rho(r_1) \succ r_2$ rankings. Now, either $\mathbf{C}^n_r$ has no preferences in common (which means they are orthogonal) with this double-reversal profile, or there is at least one preference shared by both profiles. Assume without loss of generality that one of the common preferences is $r = r_1 \succ r_2$. This means that $\rho(r) = \rho(r_1 \succ r_2) = \rho(r_2) \succ \rho(r_1)$. Because the number of voters with these two preferences agree in the double reversal profile, but differ by sign in $\mathbf{C}^n_r$, the scalar product of these terms cancel. It is easy to show that if $r = r_1 \succ r_2$ is one of the $\mathbf{C}^n_r$ rankings, then the rankings $r_1 \succ \rho(r_2)$ and $\rho(r_1) \succ r_2$ are not in $\mathbf{C}^n_r$. This completes the proof.

Part 5. The proof uses the fact that the Condorcet profile differentials partition the set of preferences. This partitioning occurs because, by construction, if preferences r_1 and r_2 appear a Condorcet profile differential, then (up to sign of the number of voters with each preference), the profile differentials are the same. This proves that the sets (the orbits) are disjoint. That they fill the space is an immediate corollary of the fact that each ranking defines a Condorcet profile differential. The dimension statement now follows from the fact that there are $n!$ preferences and each Condorcet profile differential has $2n$ of them, so there are $\frac{1}{2}(n-1)!$ sets of these profile differentials which do not have any preferences in common.

Part 6. First consider $k = n$ and the deviation vector $\mathbf{d}^n$. In the Condorcet n-tuple defined by r, each candidate is in each position the same number of times, so each candidate's $\mathbf{d}^n$ tally is the sum of the $\mathbf{d}^n$ components. The same argument applies to the Condorcet n-tuple defined by $\rho(r)$ except there are a negative number of voters. Consequently, the two sums cancel.

To handle the setting where k is an odd integer, $2 < k < n$, let c_i be a candidate in the k-candidate subset $\mathcal{S}$. For each ranking r_1 from $\mathbf{C}^n_r$ the companion ranking $\rho(r_1)$ in this differential has a negative number of the voters. If c_i is jth ranked in $\mathcal{S}$ with r_1, then she is $(k-j)$th ranked in $\mathcal{S}$ with $\rho(r_1)$. But, the jth and $(k-j)$th coefficient of $\mathbf{d}^k$ are the same. This means that the difference in sign for the number of voters forces a cancellation in the $\mathbf{d}^k$ tally.

The proof for the setting where k is an even integer $2 < k < n$ only involves creating an example. This is done following the theorem. □

Proof of Thm. 14. Part 1. Because each Condorcet profile differential is orthogonal to each basic profile, the $c_i \succ c_j$ profile differential (which is the sum of Condorcet profile differentials) also is orthogonal to the basic profiles. Similarly, because Thm. 7 asserts that the Condorcet profile differentials have a zero tally with the deviation vectors $\mathbf{d}^k$ for odd values of k, the same conclusion holds for the $c_i \succ c_j$ Condorcet profile differentials.

To prove the assertion about the $A \succ B$ Condorcet profile differential for k even, let $\mathcal{S}$ be a k-candidate subset. If A and B are not in $\mathcal{S}$, then because we are dealing with a profile differential where candidates other than $\{A, B\}$ are treated symmetrically, my standard argument shows that all candidates receive a zero $\mathbf{d}^k$ tally. If both candidates are in $\mathcal{S}$, then to determine the A tally, we need to determine how often A is jth ranked in $\mathcal{S}$. In the $A \succ B$ Condorcet profile differentials, the only condition on these rankings is that A is ranked immediately above B; all such rankings are included. The important point is that the number of rankings with this property in a $A \succ B$ Condorcet profile differential is the same for all j satisfying the relevant values of $2 \le j \le k-1$. (It is not necessary to determine when A is top or bottom ranked in $\mathcal{S}$ because the corresponding $\mathbf{d}^k$ coefficient is zero.)

To see this, notice that once the slots for the $\mathcal{S}$ candidates within the n possible positions are determined, we need to compute the number of ways to rank the $\mathcal{S}$ candidates. Of these, there are $\binom{k-2}{j-1}$ ways to select which candidates are ranked above A, and each choice can be ranked in $(j-1)!$ ways. The number of ways to rank the candidates below B is $(k-2-(j-1))!$. The product, which gives the total number of such rankings, is $(k-2)!$. It remains to determine how many ways to select the rankings for k positions within n positions where two slots are together in the jth and $(j+1)$th positions (to accommodate the adjacent A, B ranking. Using standard combinatoric approaches, where the adjacent rankings are treated as one unit, this is $\binom{n-1}{k-1}$. Therefore the total number of rankings where A is in jth position is $\binom{n-1}{k-1}(k-2)!$. Of more value than the actual number is that it does not depend upon j. Consequently, A is ranked in jth position as often as in $(k-j)$th position. But, as these $\mathbf{d}^k$ coefficients differ only in sign, the terms cancel. A similar argument holds for the $\rho(r)$ portion of each profile, and for B. Therefore the A and B tallies with the deviation vector $\mathbf{d}^k$ are zero. By using my standard symmetry argument, the same assertion extends to all candidates.

If only one of A or B is in $\mathcal{S}$, then the only minor changes in the above argument show that the candidate is ranked in jth place as often as she is ranked $(k-j)$th place. This gives the same argument to prove that the $\mathbf{d}^k$ tally is zero.

Part 2. It is shown in Thm. 7 that a Condorcet profile differential is orthogonal to a double-reversal profile, so the same assertion holds for the $c_i \succ c_j$ profile differentials. It suffices to provide an example to prove that there are symmetry changing profile differentials that are not orthogonal to a $A \succ B$ Condorcet profile differential. One such example is where $r = (C \succ D \succ A) \succ (B \succ E)$ where the other three rankings come from $\sigma_1(C \succ D \succ A) = C \succ A \succ D$. In this setting, only one of the four rankings in the symmetry changing profile is in the $A \succ B$ profile differential, so orthogonality is impossible.

Part 3. This is a simple computation involving the tallies from Thm. 7.

Part 4. This is a direct consequence of parts 1, 2 and the computations from Thm. 7. Alternatively, because these computations are not difficult for the normalized borda and the plurality vote, these computations provide an alternative proof for part 1 when k is even. □

Proof of Thm. 15. Part 1. We compute the $\{c_i, c_j\}$ pairwise tally. If $c_i \in \mathcal{S}$ but c_j is not, then $c_i \succ c_j$ in each ranking in $\mathbf{D}^n_{c_i,\mathcal{S}}$. As the number of rankings which has c_i in sth spot is $(k-1)!(n-k)!$, the total number of votes c_i wins in the pairwise elections is $(k-1)!(n-k)!\sum_{s=1}^{k}(-1)^s\binom{k-1}{s-1}$. According to Eq. 10.1 where $x=-1$, this summation equals zero.

Now suppose both $c_i, c_j \in \mathcal{S}$. Among all of the $\mathbf{D}^n_{c_i,\mathcal{S}}$ rankings which has c_i in sth position, $(k-s)(k-2)!(n-k)!$ have c_j ranked lower. Therefore, the total number of points earned by c_i in the pairwise vote is a $(k-2)!(n-k)!$ multiple of

$$\sum_{s=1}^{k}(-1)^s(k-s)\binom{k-1}{s-1} = \sum_{s=1}^{k}(-1)^s(k-s)\binom{k-1}{k-s}. \tag{10.6}$$

According to Eq. 10.2 where $x=-1$, this sum is zero. This completes the proof of this part.

Part 2. According to Cor. 4 and part 1, it suffices to show that the plurality votes for all candidates in $\mathcal{S}'$ is a tie. If no candidates from $\mathcal{S}'$ are in $\mathcal{S}$, then each candidate is treated symmetrically. This means that because $\mathbf{D}^n_{c_i,\mathcal{S}}$ is a profile differential, the sum of the votes equals zero and each candidate receives the same vote. Thus, each candidate has a zero plurality tally.

Now suppose there is at least one candidate in $\mathcal{S}$ that is not in $\mathcal{S}'$ and that $\mathcal{S}\cap\mathcal{S}' \neq \varnothing$. The plurality tally of any $\mathcal{S}'$ candidate not in $\mathcal{S}$ is, trivially, zero. If c_i is in $\mathcal{S}'\cap\mathcal{S}$, then her plurality tally is determined by the number of rankings where she is top ranked in $\mathcal{S}$ or where candidates in $\mathcal{S}$ but not in $\mathcal{S}'$ are ranked above her. Suppose there are $s \geq 1$ candidates of this type. This means c_1 is in top-place in $(n-k)!(k-1)!$ rankings, second place in $(n-k)!\binom{s}{1}1!(k-2)!$ rankings, third place in $(n-k)!\binom{s}{2}2!(k-3)!$ rankings, … , $(n-k)!\binom{s}{j}j!(k-j-1)!$ rankings in jth place. The plurality vote is determined by the number of voters with each of these rankings, so the tally is $-(n-k)!$ times the value

$$\sum_{j=0}^{s}(-1)^j\binom{s}{j}j!(k-j-1)!\binom{k-1}{j} = \sum_{j=0}^{s}(-1)^j\binom{s}{j}j!(k-j-1)!\frac{(k-1)!}{j!(k-j-1)!}$$
$$= (k-1)!\sum_{j=0}^{s}(-1)^j\binom{s}{j} = (1-1)^s = 0. \quad (10.7)$$

Again, the symmetry for the other voters in $\mathcal{S}\cap\mathcal{S}'$ ensures that their plurality vote is zero. A similar argument holds for $\mathcal{S}\cap\mathcal{S}' \neq \varnothing$ where c_i is not in this set.

Part 3. To prove this assertion, it suffices to prove that all $\mathbf{u}_j^k$ tallies of $\mathbf{D}^n_{c_i,\mathcal{S}}$ in $\mathcal{S}$ are zero. Because there are only two non-zero terms in $(k-1)\mathbf{u}_j^k$, its tally is determined by the number of ways the candidates can be ranked to have c_i is in first and in second place. Because this number is the same (it is $(k-1)!(n-k)!$), we only need to multiply the number of voters times the assigned points. All other candidates from $\mathcal{S}$ are treated symmetrically. This means that the tally is $(n-k)!(k-1)![((k-1)(-1)+\binom{k-1}{1}(1)] = 0$. As $\mathbf{D}^n_{c_i,\mathcal{S}}$ is a profile differential, the sum of the total vote is zero, and each of the other candidates receives the same tally. Thus, their tally also is zero.

With the induction hypothesis, assume that the c_i tally with $(k-1)\mathbf{u}_j^k$ is zero for $j<s$. We now must show that the tally is zero for $(k-1)\mathbf{u}_s^k$. But this computation is the same as that given in Eq. 10.5. (This reflects the duality of the construction.) The same symmetry argument shows that the tally for the other candidates also is zero.

Part 4. To see that c_i has a positive tally with $\mathbf{d}'$, notice that the sign of the coefficients of $\mathbf{d}'$ and the number of voters for each of the admissible c_i rankings agree. Hence, c_i receives a positive vote. All other candidates are treated symmetrically within the groups $\mathcal{S}-\{c_i\}$ and $\mathcal{S}'-\mathcal{S}$, so, within these groups, they receive the same tally. Suppose there are β more candidates in $\mathcal{S}'$ then in $\mathcal{S}$. The tally for each candidate in $\mathcal{S}'-\mathcal{S}$ is the sum of the last β coefficients of $\mathbf{d}'$. So, if $\beta=1$, then this sum is zero. For $\beta>1$, the sign of this total depends upon the parity of $|\mathcal{S}|$ and of β. Because $\mathbf{D}^n_{c_i,\mathcal{S}}$ is a profile differential, the sum of the total number of points is zero.

Part 5. The $\mathcal{S}$ plurality tally for c_i is determined by the number of times she is top-ranked in $\mathbf{D}^n_{c_i,\mathcal{S}}$. But this is $(k-1)!(n-k)!$. As (-1) voters are assigned to each ranking, the tally is as stated. Each of the other $\mathcal{S}$ candidates is treated symmetrically, so each receives the same plurality tally. But $\mathbf{D}^n_{c_i,\mathcal{S}}$ is a profile differential, so the sum of the votes equals zero. Therefore, each of these other candidates receives a $(n-k)!(k-2)!$ plurality tally.

Part 6. By use of the symmetry argument in the proof of part 5, it suffices to prove that c_i is $\mathbf{d}^k$ top-ranked in $\mathcal{S}$ with the $\mathbf{D}^n_{c_i,\mathcal{S}}$ profile. But the signs of the jth coefficient of $\mathbf{d}^k$ and the number of voters when c_i is jth ranked agree. Therefore, c_i's tally is $(k-1)!(n-k)!\sum_{j=2}^{k-1}[\binom{k-1}{j}]^2$.

Part 7. This is a direct computation. □

References

[1] Arrow. K.J. *Social Choice and Individual Values* 2nd. ed.. Wiley. New York. 1963.

[2] Arrow. K.J.. and H. Raynaud. *Social choice and Multicriterion Decision-Making*. MIT Press. 1986.

[3] Brams. S.. *Voting Procedures*. in *Handbook of Game Theory. Vol. 2*. ed. by Aumann. R. and S. Hart. North-Holland. 1994.

[4] Fishburn. P. *Inverted orders for monotone scoring rules*. Discrete Applied Math. **3** (1981). 27-36.

[5] Fishburn. P. and W. Gehrlein.

[6] Gehrlein, W. and P. Fishburn
[7] Kemeny, J., *Mathematics without numbers*, Daedalaus **88** (1959), 571-591.
[8] Merlin, V., and D. G. Saari, *Copeland Method II: Manipulation, monotonicity, and paradoxes*, *Jour Economic Theory*, **72** (1997), 148-172.
[9] Nanson, E.J., *Methods of election*, Trans Proc Roy Society Victoria **18** (1882), 197-240.
[10] Saari, D. G., *A chaotic exploration of aggregation paradoxes*, SIAM Review, **37** (1995), 37-52.
[11] ______, *The ultimate of chaos resulting from weighted voting systems*, Adv. Applied Math **5** (1984), 286-308.
[12] ______ *Mathematical complexity of simple economics*, Notices Am Math Soc., **42** (1995), 222-230
[13] ______ *Geometry of Voting*, Springer-Verlag, 1994
[14] ______ *Basic Geometry of Voting*, Springer-Verlag, 1995
[15] ______ *Millions of election outcomes from a single profile*, Social Choice & Welfare **9** (1992), 277-306.
[16] ______ *A dictionary for voting paradoxes*, JET **48** (1989), 443-475.
[17] ______ Symmetry extensions of neutrality I; *Social Choice and Welfare*, **9** (1992), 307-336.
[18] ______ *Connecting and resolving Arrow's and Sen's Theorems*, to appear in Social Choice & Welfare
[19] ______ *Explaining positional voting paradoxes I: The simple case*, NU Center for Math Econ Discussion paper # 1179 (Jan. 1997)
[20] Saari, D. G., and V. Merlin, *The Copeland Method I: Relationships and the dictionary*, **8** (1996), 51-76.
[21] ______ *A geometric examination of Kemeny's rule*, NU Preprint; April 1997.
[22] Saari, D. G., and M. Tataru, *The likelihood of dubious election outcomes*, to appear
[23] Schofield, N. (ed.), *Collective decision-making: Social choice and political economy*, Kluwer, 1996
[24] Sieberg, K., *Voting analyzed with probability theory*, NYU preprint, March, 1994.

Department of Mathematics, Northwestern University, Evanston, Illinois 60208-2730
E-mail address: d_saari@math.nwu.edu