A PROOF OF CALIBRATION VIA
BLACKWELL'S APPROACHABILITY THEOREM

by
Dean P. Foster*
Department of Statistics
Wharton School, The University of Pennsylvania

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* Foster is also a visiting professor in the Center for Mathematical Studies in Economics and Management Science at Northwestern University, Winter 1997.
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Abstract

Over the past few years many proofs of calibration have been presented (Foster and Vohra (1991, 1997), Hart (1995), Fudenberg and Levine (1995), Hart and Mas-Colell (1996)). Does the literature really need one more? Probably not, but this algorithm for being calibrated is particularly simple and doesn't require a matrix inversion. Further the proof follows directly from Blackwell's approachability theorem. For these reasons it might be useful in the classroom.

*This work was done while I was visiting the Center for Mathematical Studies in Economics and Management Science, Northwestern University. Permanent affiliation: Dept. of Statistics, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104. E-mail: foster@bellspark.wharton.upenn.edu.
Thanks to Sergiu Hart who provided the proof of the only result in the paper. The algorithm is a modification of the original algorithm in Foster and Vohra (1991).
Suppose at time $t$ a forecast $f_t$ is made which takes on the value of the midpoint of each of the intervals $[0, 1/m)$, $[1/m, 2/m)$, ..., $[(m-1)/m, 1]$, namely, $\frac{2i-1}{2m}$ for $i$ equals 1 to $m$. Let $A_t^i$ the vector of indicators as to which forecast is actually made:

$$A_t^i = \begin{cases} 1 & \text{if } f_t = \frac{2i-1}{2m} \\ 0 & \text{otherwise} \end{cases}$$

Let $X_t$ be the outcome at time $t$. We can now define the empirical frequency $\rho_t^i$ as:

$$\rho_t^i = \frac{\sum_{t=1}^{T} X_t A_t^i}{\sum_{t=1}^{T} A_t^i}$$

Hopefully, $\rho_t^i$ lies in the interval $[\frac{i-1}{m}, \frac{i}{m}]$. If so, the forecast is approximately calibrated. If not, I will measure how far outside the interval it is by two distances: $\overline{d}_t^i$ and $\overline{e}_t^i$ (for deficit and excess) which are defined as:

$$\overline{d}_t^i = \frac{1}{T} \sum_{i=1}^{T} (\frac{i-1}{m} - X_t) A_t^i = [\frac{i-1}{m} - \rho_t^i] A_t^i$$

$$\overline{e}_t^i = \frac{1}{T} \sum_{i=1}^{T} (X_t - \frac{i}{m}) A_t^i = [\rho_t^i - \frac{i}{m}] A_t^i$$

where $\overline{A}_t = \sum A_t^i/T$. I will show that the following forecasting rule will drive both of these distances to zero:

1. If there exist an $i^*$ such that $\overline{e}_{T}^{i^*} \leq 0$ and $\overline{d}_{T}^{i^*} \leq 0$, then forecast $\frac{2i^*-1}{2m}$.

2. Otherwise, find an $i^*$ such that $\overline{d}_{T}^{i^*} > 0$ and $\overline{e}_{T}^{i^*-1} < 0$ then randomly forecast either $\frac{2i^*-1}{2m}$ or $\frac{2i^*+1}{2m}$ with probabilities:

$$P \left( f_{T+1} = \frac{2i^*-1}{2m} \right) = 1 - P \left( f_{T+1} = \frac{2i^*+1}{2m} \right) = \frac{\overline{d}_{T}^{i^*}}{\overline{d}_{T}^{i^*} + \overline{e}_{T}^{i^*-1}}$$
It is clear that an $i^*$ can be found in step 2, since $i = 1$ always under forecasts and $i = m$ always over-forecasts.

The L-1 calibration score:

$$ C_{1, T} \equiv \sum_{i=0}^{m} |\rho_i - \frac{2i+1}{m} | \bar{A}_T = \frac{1}{2m} + \sum_{i=1}^{m} \max(\bar{d}_T, \bar{c}_T) $$

so showing that all the $\bar{c}_T$ and $\bar{d}_T$ converge to zero, implies that $C_{1, T}$ converges to $\frac{1}{2m}$.

**Theorem 1 (Foster and Vohra)** For all $\epsilon > 0$, there exists a forecasting method which is calibrated in the sense that $C_{1, T} < \epsilon$ if $T$ is sufficiently large. In particular the above algorithm will achieve this goal if $m \geq \frac{1}{\epsilon}$.

Consider this as a game between a statistician and nature. The statistician picks the forecast $f_t$ and nature picks the data sequence $X_t$. The statistician's goal is to force all of the $\bar{c}_i$ and $\bar{d}_i$ to be negative (or at least approach this in the limit). Nature's goal is to keep the statistician from doing this. This set up is a game of "approachability" which was studied by Blackwell. He found a necessary and sufficient condition for a set to be approachable.

**Theorem 2 (Blackwell 1956)** Let $L_{ij}$ be a vector valued payoff taking values in $\mathbb{R}^n$, where the statistician picks an $i$ from $\mathcal{I}$ at round $i$ and nature picks a strategy $j$ from $\mathcal{J}$ at time $t$. Let $G$ be a convex subset of $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$ and let $c \in G$ be the closest point in $G$ to the point $a$. Then $G$ is approachable by the statistician if for all such $a$, there exist a weight vector $w_i$ such that for all $j \in \mathcal{J}$.

$$ (\sum_{i \in \mathcal{I}} w_i L_{ij} - c)'(a - c) \leq 0. \tag{1} $$

To prove Theorem 1, we need to translate the calibration game into a Blackwell approachability game. The set of strategies for the statistician, $\mathcal{I}$.
is the set of the \( m \) different forecasts. The set of strategies for nature, \( \mathcal{J} \), is the set \( \{0, 1\} \). Define:

\[
\epsilon_X^i = (X - \frac{1}{m}) A^i
\]
\[
d_X^i = (\frac{i-1}{m} - X) A^i
\]

The vector loss is the vector of all the \((d^i, \epsilon^i)\)'s. In other words, it is a point in \( R^{2m} \). The goal set \( G \subset R^{2m} \) is \( G = \{ x \in R^{2m} | (\forall k)x_k \leq 0 \} \). Let \( \bar{c}_T = \frac{1}{T} \sum_{t=1}^{T} c_{X_t} \) and \( \bar{d}_T = \frac{1}{T} \sum_{t=1}^{T} d_{X_t} \). The \((d_X, \epsilon_X)_i\) will be our \( L_i \) in the Blackwell game, and \((\bar{d}_T, \bar{c}_T)\) will be the point \( c \). The closest point in \( G \) to the current average \( a = (\bar{d}_T, \bar{c}_T)_{i \in I} \) is

\[
c = (\bar{d}_T^{-}, \bar{c}_T^{-})_{i \in I}
\]

where we have defined the positive and negative parts as \( x^+ = \max(0, x) \) and \( x^- = \min(0, x) \). The weight vector \( w \) is the vector of probability of forecasting \( i/k \).

**Proof:** (Hart 1996) Now to check equation (1). Writing it in terms of \( d^i \)'s and \( \epsilon^i \)'s equation (1) is:

\[
\sum_{i=1}^{m} \left( (w^i d^i - (\bar{d}^i)^-) (\bar{d}_T - (\bar{d}^i)^-) + (w^i \epsilon^i - (\bar{c}^i)^-) (\bar{c}_T - (\bar{c}^i)^-) \right) \leq 0
\]

from \( x - x^- = x^+ \) equation (1) is equivalent to

\[
\sum_{i=1}^{m} \left( (w^i d^i - (\bar{d}^i)^-) (\bar{d}_T)^+ + (w^i \epsilon^i - (\bar{c}^i)^-) (\bar{c}_T)^+ \right) \leq 0
\]

Since, \( (x^-)(x^+) = 0 \), it is sufficient to show:

\[
\sum_{i=1}^{m} w^i (\epsilon^i (\bar{c}^i)^+ + d^i (\bar{d}^i)^+) \leq 0.
\]
If step 1. of the algorithm is used the weight vector is just $w' = 1$ if $i^*$ is the forecast chosen and zero otherwise. So $w' \neq 0$ only when both $(d')^+$ and $(\bar{r}')^+$ are zero, so the entire sum is zero.

If step 2. is used, the non-zero terms are $w'^*$ and $w'^*-1$. But, $(\bar{r}')^+$ is zero and $(\bar{d}'^*-1)^+$ is zero. So, it is sufficient to show:

$$w'^* d'^*(\bar{d}')^+ + w'^*-1 \bar{r}'^*-1(\bar{r}'^*-1)^+ \leq 0$$

But, $d'^* = -\bar{r}'^*-1$, so it is sufficient to show:

$$w'^* (\bar{d}')^+ - w'^*-1(\bar{r}'^*-1)^+ \leq 0$$

But, this follows (with equality) from the definition of our probabilities.  

References


