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MARKOV ADDITIVE PROCESSES AND SEMI-REGENERATION*

by

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1. INTRODUCTION

Our aim is to give an introduction to Markov additive processes in an informal setting, and discuss their more important properties and applications.

Some of the motivation for studying Markov additive processes comes from modelling real-life cumulative processes whose probability laws depend on a randomly changing environment. The following are two such direct applications.

(1.1) EXAMPLE. Let X_t be the velocity of a vehicle at time t, and let Y_t denote the amount of fuel consumed by that vehicle during the time interval $[0, t]$. Suppose (X_t) is a Markov process. The rate of fuel con-

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sumption depends, in addition to velocity and acceleration, on such things as temperature, humidity, etc. Therefore, Y will not be a simple additive functional of X in general. Instead, it seems that the increment $Y_{t+s} - Y_t$ would have the probability law of a process with independent increments with the parameters of that law depending on X_u, $t \leq u \leq t+s$. Then, the pair (X, Y) is a Markov additive process.

(1.2) EXAMPLE. This problem comes up in optical communications in which photon counters are used as detectors. The photon counting process is a non-stationary Poisson process. The intensity function is the sum of two functions: one is the deterministic component carrying the message, the other is the stochastic component caused by the presence of heat, turbulence, etc. The problem is to estimate the deterministic component from the counting data. For instance, if the randomness is due only to heat, one may reasonably take X_t to denote the amount of heat present at t, and then, the intensity of emission due to heat will be $|X_t|^2$ at t. If X is a Markov process and Y_t is the number of particles emitted due to heat during $[0, t]$, then (X, Y) is a Markov additive process.

Another, more detailed, example will be given in Section 3. There, X_t will stand for "the time of year and the prevailing season at the absolute time t" and Y_t will be the cumulative input into a water reservoir during $[0, t]$.

Such applications to real phenomena are not the only reasons for the interest in Markov additive processes. Motivation comes also from their applications to the theory of continuous regeneration, and through that theory, the applications to the boundary theory of Markov processes and random time changes of stochastic processes. More will be said on these later.

The organization of the paper is as follows. In the next section we will give a fairly precise definition of Markov additive processes, and will list several of their more important structural properties. In Section 3 we will discuss two special cases in some detail; it is hoped that this will clarify the otherwise complex looking structure of our processes. In Section 4 we will point out the manner in which Markov additive processes arise in such theoretical applications as random time changes of Markov processes, semi-Markov processes, and the theory of continuous regeneration.

Section 5 will introduce an important analytical tool in discussing the infinitesimal behavior of a Markov additive process: that tool is provided by Lévy systems for such processes. In Section 6 we show the interrelationships between those Lévy systems and the infinitesimal generators and resolvent. In particular, we derive there a remarkable resolvent equation which also arises in the generalized Dirichlet problem.

Many of the applications we mention are not yet realized in sufficient depth. It is hoped, therefore, that this paper generates more interest in such problems. This is a young area, and is in need of the talents of many more people.

We end this introduction with a few words on notations and terminology to be employed in the remainder of this paper.

Our notations and terminology will, in general, follow those of BLUMENTHAL and GETOOR [1] and DELLACHERIE [7]. The following are a few particulars. We write $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $\overline{\mathbb{R}}_+ = [0, \infty]$, $\overline{\mathbb{R}} = [-\infty, \infty]$.

By $\underline{\underline{\mathbb{R}}}_+$, $\underline{\underline{\mathbb{R}}}$, ... we denote the set of all Borel subsets of $\mathbb{R}_+$, $\mathbb{R}$, ... Throughout the paper E will be a locally compact space with a countable base, and $\underline{\underline{E}}$ will denote the σ-algebra generated by its open subsets.

If $(F, \underline{\underline{F}})$ and $(G, \underline{\underline{G}})$ are measurable spaces, then the mapping $N : F \times \underline{\underline{G}} \to \overline{\mathbb{R}}_+$ will be said to be a <u>transition kernel</u> from $(F, \underline{\underline{F}})$ into $(G, \underline{\underline{G}})$ provided that $x \to N(x, A)$ is $\underline{\underline{F}}$-measurable for each fixed A in $\underline{\underline{G}}$ and $A \to N(x, A)$ is a σ-finite measure on $\underline{\underline{G}}$ for each fixed x in F.

If Ω is a sample space and $\underline{\underline{M}}$ is a σ-algebra on Ω, then by a <u>history</u> on $(\Omega, \underline{\underline{M}})$ we mean an increasing family of sub-σ-algebras of $\underline{\underline{M}}$. A history $(\underline{\underline{H}}_t)$ is said to be right continuous if

$$\underline{\underline{H}}_t = \bigcap_{s > t} \underline{\underline{H}}_s$$

for every $t \in \mathbb{R}_+$. If $(\underline{\underline{H}}_t)$ is a history, by $\underline{\underline{H}}$ we denote the σ-algebra generated by all the $\underline{\underline{H}}_t$. A history $(\underline{\underline{H}}_t)$ will be said to be complete with respect to a family of measures $\underline{\underline{P}}$ on $\underline{\underline{M}}$ provided that $\underline{\underline{H}}$ be complete wrt $\underline{\underline{P}}$ and that $\underline{\underline{H}}_0$ (and therefore every $\underline{\underline{H}}_t$) contain all the negligible sets of $\underline{\underline{H}}$.

2. MARKOV ADDITIVE PROCESSES

In this section, we shall define Markov additive processes and concentrate on their decomposition along the lines of Lévy's decomposition of additive processes. This is a summary of Sections 1 and 2 of [3].

Let $(\Omega, \underline{\underline{M}})$ be a measurable space, $(\underline{\underline{M}}_t)$ a history on $(\Omega, \underline{\underline{M}})$, (θ_t) a family of shift operators on $(\Omega, \underline{\underline{M}})$, (X_t) a stochastic process taking values in $E_\Delta = E \cup \{\Delta\}$ where Δ is a point not in E, (Y_t) a stochastic process taking values in $\mathbb{R} = (-\infty, \infty)$, and $(P^x)_{x \in E_\Delta}$ a family of probability measures on $\underline{\underline{M}}$.

(2.1) DEFINITION. The object

$$(X, Y) = (\Omega, \underline{\underline{M}}, \underline{\underline{M}}_t, \theta_t, X_t, Y_t, P^x)$$

is said to be a <u>standard Markov additive process</u> with state space $E \times \mathbb{R}$ provided that the following hold.

a) $X = (\Omega, \underline{\underline{M}}, \underline{\underline{M}}_t, \theta_t, X_t, P^x)$ is a standard Markov process with state space $(E, \underline{\underline{E}})$ (augmented by a point Δ) in the sense of BLUMENTHAL and GETOOR [1].

b) Almost surely, the mapping $t \to Y_t$ is right continuous, has left-hand limits, satisfies $Y_0 = 0$ and $Y_t = Y_{\zeta-}$ for $t \geq \zeta = \inf\{s : X_s = \Delta\}$.

c) For each t, Y_t is $\underline{\underline{M}}_t$-measurable.

d) For each t and s in $\mathbb{R}_+$, $Y_{t+s} = Y_t + Y_s \circ \theta_t$ almost surely.

e) For each $t \in \mathbb{R}_+$, $A \in \underline{\underline{E}}_\Delta$, $B \in \underline{\underline{R}}$, the mapping

$$x \to P^x\{X_t \in A, Y_t \in B\}$$

is $\underline{\underline{E}}$-measurable.

f) For each $t, s \in \mathbb{R}_+$, $x \in E_\Delta$, $A \in \underline{\underline{E}}_\Delta$, $B \in \underline{\underline{\mathbb{R}}}$,

$$(2.2) \qquad P^x\{X_s \circ \theta_t \in A, Y_s \circ \theta_t \in B \mid \underline{\underline{M}}_t\} = P^{X(t)}\{X_s \in A, Y_s \in B\}.$$

Condition (2.1a) contains a number of requirements: $(\underline{\underline{M}}_t)$ is a right continuous and complete history; $P^x\{X_0 = x\} = 1$; $t \to X_t$ is almost surely right continuous, and has left-hand limits; (X_t) is progressively measurable, strong Markov, and quasi-left-continous on $[0, \zeta)$.

Condition (2.1b) is a regularity condition on the paths of (Y_t); we may and do assume that the stated properties are true for all ω (instead of almost all).

Condition (2.1c) makes (Y_t) adopted to $(\underline{\underline{M}}_t)$, and by the right continuity of (Y_t), (Y_t) is progressively measurable with respect to $(\underline{\underline{M}}_t)$. In view of this condition, the Markov property of X is stronger than the usual cases where $\underline{\underline{M}}_t$ is merely the σ-algebra generated by $(X_s, s \le t)$ alone.

Condition (2.1d) makes (Y_t) additive by requiring that the increment $Y_{t+s} - Y_t$ depend on "the history between t and t + s".

Condition (2.1e) sets up the stage for (2.1f) by seeing to it that the right-hand side of (2.2) is measurable with respect to $\underline{\underline{M}}_t$.

Finally, (2.1f) lists the most important condition: the future of (X, Y) is conditionally independent of the past $\underline{\underline{M}}_t$ given X_t. In particular, this implies that (X_t, Y_t) is a Markov process, but, here the future of (Y_t) is conditionally independent of the past once the present value of (X_t) is known.

The definition above is slightly restrictive: in ÇINLAR [2], a most general definition is given which allows (X_t) to be an arbitrary Markov process and (Y_t) can take values in $\overline{\mathbb{R}}^n$. In [3], (X_t) is as in here, but (Y_t) is allowed to take values in $\overline{\mathbb{R}}^n$. The definitions of EZHOV and SKOROKHOD [8] and JACOD [9] are closer to that of ÇINLAR [3].

In working with Markov additive processes, the point of view adopted is that the Markov process (X_t) is well known and that we are interested in the structure of the second component (Y_t) conditioned on that of (X_t). To make such statements precise, we first define the following. We let $(\underline{\underline{K}}_t)$ denote the canonical history generated by (X_t); that is, $(\underline{\underline{K}}_t)$ is the minimal right continuous and complete history such that $\underline{\underline{K}}_t \supset \sigma(X_s;\ s \le t)$ for every t. Similarly, let $(\underline{\underline{L}}_t)$ denote the canonical history generated by (X_t, Y_t). Recall that

$$\underline{\underline{K}} = \bigvee_t \underline{\underline{K}}_t \qquad \underline{\underline{L}} = \bigvee_t \underline{\underline{L}}_t \quad .$$

Then, $\underline{\underline{K}} \subset \underline{\underline{L}} \subset \underline{\underline{M}}$ and $\underline{\underline{K}}_t \subset \underline{\underline{L}} \subset \underline{\underline{M}}_t$ for every t.

The following proposition states that, roughly speaking, conditional upon the knowledge of $\underline{\underline{K}}$, the process (Y_t) has independent increments. We will make this more precise in Theorem (2.7) by choosing a proper version of "the conditional probability given $\underline{\underline{K}}$".

(2.3) PROPOSITION. For any s, $t \in \mathbb{R}_+$ and $B \in \underline{\underline{\mathbb{R}}}$,

$$P^x\{Y_s \circ \theta_t \in B \mid \underline{\underline{M}}_t \vee \underline{\underline{K}}\} = P^x\{Y_s \circ \theta_t \in B \mid \underline{\underline{K}}\} . \tag{2.4}$$

Moreover, there exists a $\underline{\underline{K}}_s$-measurable random variable W such that

$$(2.5) \qquad P^x\{Y_s \circ \theta_t \in B \mid \underline{\underline{K}}\} = W \circ \theta_t .$$

Evidently, W is a "nice" version of the conditional probability $P^y\{Y_s \in B \mid \underline{\underline{K}}_s\}$ which is further independent of y. Putting (2.5) into (2.4), we see that, given $\underline{\underline{K}}$, $Y_{t+s} - Y_t$ is independent of the past $\underline{\underline{M}}_t$, and further, depends only on the path of (X_t) during $[t, t+s]$.

The following is the first step in making (2.3) more precise:

(2.6) PROPOSITION. There is a transition probability P from $(\Omega, \underline{\underline{K}})$ into $(\Omega, \underline{\underline{L}})$ such that, for any $x \in E_\Delta$, $K \in \underline{\underline{K}}$, and $L \in \underline{\underline{L}}$,

$$\int_K P^x(d\omega)\ P(\omega,L) = P^x(K \cap L) .$$

In other words, P is a regular version of the conditional probability $P^x\{\cdot \mid \underline{\underline{K}}\}$ for all x. Now, fix $\omega_0 \in \Omega$ and write $P_0(L)$ for $P(\omega_0,L)$. Consider the probability space $(\Omega, \underline{\underline{L}}, P_0)$, and consider the stochastic process (Y_t) on that probability space. Proposition (2.3) implies the following important

(2.7) THEOREM. Considered as a stochastic process on the probability space $(\Omega, \underline{\underline{L}}, P_0)$, $(Y_t)_{t \in \mathbb{R}_+}$ has independent increments.

We may now appeal to the theory of processes with independent increments given, say, in DOOB's book in Chapter VIII. Recalling that independence with respect to P_0 is the same as "conditional independence given $\underline{\underline{K}}$ with respect to P^x for all x," we obtain the following main result. This is the equivalent of Lévy-Khinchine decomposition for processes with independent increments.

(2.8) THEOREM. We may decompose Y as

$$Y = A + Y^f + Y^c + Y^d \tag{2.9}$$

where (A_t), (Y_t^f), (Y_t^c), (Y_t^d) are conditionally independent given $\underline{\underline{K}}$ with respect to P^x for all x, and where the components satisfy the following:

(2.10) a) (A_t) is an additive functional of X.

b) (Y_t^f) is a purely discontinuous process whose jump times are <u>fixed</u> by $\mathbf{X}$; $(\mathbf{X}, Y^f)$ is a Markov additive process; there is a sequence (T_n) of stopping times of $(\underline{\underline{K}}_t)$ which exhausts the jumps of (Y_t^f); if for some stopping time T of $(\underline{\underline{K}}_t)$ the value $Z = Y_T^f - Y_{T-}^f$ is $\underline{\underline{K}}$-measurable, then $Z = 0$ almost surely.

c) (Y_t^c) is a continuous process; (X, Y^c) is a Markov additive process.

d) (Y_t^d) is conditionally stochastically continuous; in fact, if T is $\underline{\underline{K}}$-measurable, then $Y_{T-}^d = Y_T^d$ almost surely; (Y_t^d) is a compensated sum of jumps; $(\mathbf{X}, Y^d)$ is a Markov additive process.

The following is the analytic version of the preceding result. Let

$$M_t^\lambda = E^x[\exp(i \lambda Y_t) \mid \underline{\underline{K}}] . \tag{2.11}$$

(2.12) COROLLARY. For each $t \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$, $\omega \in \Omega$,

$$(2.13)\qquad M_t^\lambda(\omega) = \{\prod_j F_j^\lambda(\omega)\, 1_{[0,t]}(\tau_j(\omega))\}$$

$$\cdot \{\exp[i\,\lambda\, A_t(\omega) - \tfrac{1}{2}\lambda^2\, C_t(\omega)$$

$$+ \int_{\mathbb{R}} (e^{i\lambda x} - 1 - \frac{i\lambda x}{1+x^2})\, \frac{1+x^2}{x^2}\, B_t(\omega, dx)]\}$$

where

a) for each j, τ_j is a stopping time of $(\underline{\underline{K}}_t)$, and F_j^λ is a characteristic function in λ and is $\underline{\underline{K}}$-measurable for fixed λ;

b) (A_t) is an additive functional of X;

c) (C_t) is a continuous increasing additive functional of X;

d) for each t and ω, $A \to B_t(\omega, A)$ is a finite measure on $\mathbb{R}\setminus\{0\}$; and for fixed $A \in \underline{\underline{\mathbb{R}}}$, $(B_t(A))_{t\in\mathbb{R}_+}$ is a continuous increasing additive functional of X.

This corollary was obtained by EZHOV and SKOROKHOD [8] by direct analytical techniques under the assumption of continuity for $t \to M_t^\lambda$. Then, in (2.13), the first factor disappears, and $t \to A_t$ must be continuous.

We end this section with several remarks on the further properties of (Y_t). As all the results above, these can be found in [3].

Suppose that (Y_t) is increasing. Then, the component Y^c vanishes and each one of the remaining components is increasing. The additive functional A may be further decomposed as

$$(2.14)\qquad A = A^c + A^p + A^q$$

where A^c is a continuous additive functional of X, A^p is a purely discontinuous predictable additive functional of X, and A^q is a purely discontinuous quasi-left-continuous additive functional of X.

A similar further decomposition can be given for Y^f:

$$Y^f = Y^{fp} + Y^{fq} \tag{2.15}$$

where the jump times of Y^{fp} are exhausted by a family of predictable stopping times of $(\underline{\underline{K}}_t)$, and those of Y^{fq} are by a family of totally inaccessible stopping times.

In some situations it is convenient to represent Y (we are still assuming that Y is increasing) as

$$Y = A^c + Y^p + Y^q + Y^d \tag{2.16}$$

by defining

$$Y^p = A^p + Y^{fp}, \qquad Y^q = A^q + Y^{fq} \quad . \tag{2.17}$$

Then, A^c is continuous; each one of Y^p, Y^q, Y^d is purely discontinuous. The jump times of Y^p and Y^q are fixed by X, the jump times of Y^d are <u>not</u> fixed by X. Y^p jumps only when X is continuous; Y^q jumps only when X jumps. Y^d is conditionally stochastically continuous given X.

The behavior of Y^d is of greater interest. In the case when X has only one state, it is well known that Y^d is the limit of a sequence of compound Poisson processes. A similar result holds in general, and one is able to give a decomposition for Y^d along the lines of ITO's decomposition for additive processes. We refer to Sections 4 and 5 of [3] for results.

3. TWO SPECIAL CASES

In this section we discuss two special cases in some detail. First, we consider a Markov additive process (X, Y) where X is a Markov process with only two states. This example essentially covers all processes (X, Y) where X is a regular step process. Secondly, we give an example where X is essentially continuous; in this example we are interested in pointing out an interesting model of seasonal behavior.

3a. MARKOV ADDITIVE PROCESSES WITH $E = \{a, b\}$. Suppose that the state space of X is $E = \{a, b\}$, and further suppose that the lifetime is infinite (so that Δ does not enter the picture). Such a process alternates between the two states. Sojourn times are independent random variables, have exponential distributions, with means depending on the state being occupied.

(3.1) ADDITIVE FUNCTIONALS. Any continuous additive functional of X has the form

$$C_t = \int_0^t f(X_s)\, ds \ , \tag{3.2}$$

where f(a) and f(b) are any two numbers in $\mathbb{R}$.

Any purely discontinuous additive functional of X has the form

$$B_t = \sum_{s \le t} g(X_{s-}, X_s)\, I_{\{X_{s-} \neq X_s\}} \tag{3.3}$$

where g is any function on $E \times E$. Note that X jumps only finitely often (almost surely) in $[0,t]$; therefore, the sum in (3.3) has in fact finitely many terms. In words, $t \to B_t$ jumps only when $t \to X_t$ does. If X jumps from a to b at time t, for instance, then B jumps from B_{t-} to $B_t = B_{t-} + g(a,b)$.

The most general additive functional of X has the form

$$A = B + C \tag{3.4}$$

where B has the form (3.3) and C has the form (3.2).

(3.5) COMPONENT Y^f. For a process **X** of the present type, the component Y^{fp} in the decomposition (2.15) vanishes. So, we have that $Y^f = Y^{fq}$. Y^{fq} jumps only when **X** does; in this regard it is similar to B given by (3.3). However, the magnitude of the jump of $Y^f = Y^{qf}$ is a random variable whose distribution depends on the left-hand and right-hand values of X at the instant of that jump. More precisely, if τ is a jump time of Y^f, τ is also a jump time of X and

$$P^x\{Y^f_\tau - Y^f_{\tau-} \in B \mid \underline{K}\} = F(i, j, B) , \quad B \in \mathbb{R}, \tag{3.6}$$

on $\{X_{\tau-} = i, \ X_\tau = j\}$. Here, $F(i,j,\cdot)$, $i, j \in E$, are arbitrary distributions on $\mathbb{R}$. This is the most general Y^f possible in the present case.

(3.7) COMPONENT Y^c. The most general form of it is

$$Y^c_t = Z(C_t) \tag{3.8}$$

where Z(t) is a Brownian motion independent of X, and where (C_t) is an increasing continuous additive functional of **X**. If (C_t) is represented as in (3.2), we may explain (3.8) as follows. While X is in state a, Y^c behaves as a Gaussian process with stationary and independent increments with covariance function f(a)t. Similarly, while X is in state b, Y^c has s.i.i. with covariance function f(b)t.

(3.9) COMPONENT Y^d. Its most general form is as follows. Let Y^a and Y^b be two processes which are independent of each other and of X. Suppose Y^a is an additive process (that is, Y^a has stationary and independent increments), such that

$$E^x[\exp(i\lambda Y_t^a)] \tag{3.10}$$

$$= \exp\{t \int_{\mathbb{R}} (e^{i\lambda x} - 1 - \frac{i\lambda x}{1+x^2}) \frac{1+x^2}{x^2} \nu(a, dx)\}$$

where $\nu(a,\cdot)$ is a finite measure on $\mathbb{R}\setminus\{0\}$. Similarly for Y^b with $\nu(b, \cdot)$.

Define

$$L_t^a = \int_0^t 1_a(X_s)\, ds \; ; \qquad L_t^b = \int_0^t 1_b(X_s)\, ds \; ; \tag{3.11}$$

that is, L_t^a and L_t^b are the amounts of time spent respectively in a and b during $[0,t]$ by **X**. Finally, define

$$Y_t^d = Y^a(L_t^a) + Y^b(L_t^b) \, . \tag{3.12}$$

This is the most general from of Y^d.

In other words, while X is in state a, Y^d behaves as Y^a; while X is in state b, Y^d behaves as Y^b.

Finally we consider a slight rearrangement of the terms. We may write

$$Y = A^c + (A^q + Y^{fq}) + Y^c + Y^d \tag{3.13}$$

where A^c is the continuous component of A; (recall that $A^p = 0$ and $Y^{fp} = 0$).

The component $Y^q = A^q + Y^{fq}$ jumps only when X does. We now concentrate on the remaining three terms A^c, Y^c, Y^d . Suppose

$$(3.14) \qquad A_t^c = \int_0^t h(X_s)\, ds \; ;$$

and suppose Y^c is as in (3.8) and Y^d is as in (3.12) . Let Z^a be an additive process with

$$(3.15) \qquad E[\exp(i\lambda Z_t^a)] = \exp\{i\lambda t\, h(a) - \tfrac{1}{2} f(a)\lambda^2 t + t \int (e^{i\lambda x} - 1 - \frac{i\lambda x}{1+x^2}) \frac{1+x^2}{x^2} \nu(a, dx)\} ,$$

and suppose Z^b is an additive process with a Fourier transform as in (3.15) but with drift term h(b), variance term f(b), and "Lévy measure" $\nu(b,\cdot)$. Suppose Z^a and Z^b are independent of each other and of X. Define L^a and L^b as before by (3.11). Put

$$(3.16) \qquad Z_t^c = Z^a(L_t^a) + Z^b(L_t^b) .$$

Then, Z_c is the most general form of $A^c + Y^c + Y^d$.

* * *

3b. ADDITIVE PROCESSES IN RANDOM ENVIRONMENTS. This is to illustrate the case where X has a continuous state space, and also to provide an example of a rather interesting practical application. The Markov additive process to be described now was proposed in ÇINLAR [4] as a model for the input into a water reservoir in the presence of yearly cyclicity and seasonal variations. Here X_t is to stand for "environmental conditions at time t," and

Y_t will be the cumulative input during $[0,t]$.

Since Y is increasing, we shall use the decomposition (2.16) for it.

(3.17) THE PROCESS X. Take a circle of unit circumference and let A, B, C, D be four points on it ordered clockwise. Let the arc ABCD be painted green and the arc CDAB yellow; note that the green and yellow overlap over the arcs AB and CD. Let E be the union of the green and the yellow points (with colors preserved) with the natural topology they bring; (the result is a compact space). Consider now the position and color of a chameleon moving on the circle clockwise with unit speed; if it started at a green point, it stays on the green arc for a while and then, somewhere between C and D, crosses over to the yellow arc; it stays on the yellow arc for a while and then, somewhere between A and B, crosses over to the green arc; and so on. The probability law of the motion is described by two distribution functions: one for the additional random time which the chameleon stays on the green arc after reaching C, and one for the additional time it stays on the yellow arc after reaching A. The resulting position-color process is a Markov process. (We omit the formal construction of X but only mention that the state space we described cannot be made simpler: it is tempting to take E to be the Cartesian product of a circle with the set $\{1,2\}$, but then there is no way of defining the measures P^x for all x and still have a normal Markov process.)

For this environment process, the colors may be thought as the two seasons which alternate almost deterministically except that the changes of seasons are allowed to take place at random times (which, however, are restricted to certain intervals).

By the essential cyclicity of the process, it is sufficient to describe Y_t for $0 \le t \le 1$.

(3.18) CONTINUOUS ADDITIVE FUNCTIONAL A^c. Let a_y and a_g be two functions defined on the yellow and green arcs, respectively. If at instant t the environmental process is at position x and color green, then $d\,A_t^c = a_g(x)\,dt$; similarly on the yellow arc. That is,

(3.19) $$A_t^c = \int_0^t f(X_s)\,ds$$

where $f(g,x) = a_g(x)$ and $f(y,x) = a_y(x)$.

(3.20) PREDICTABLE COMPONENT Y^p. Let $x_1, x_2, \ldots$ be predetermined points on the circle. For each j, let W_j be a random variable with distribution φ_j . Then,

(3.21) $$Y_t^p = \sum_j W_j\, 1_{[o,t]}(x_j), \qquad 0 \le t \le 1$$

is an example of the predictable component. Here, we may think of Y_t^p as the cumulative input during $[0,t]$ which have been scheduled at the times $x_1, x_2, \ldots$ during the year.

(3.22) COMPONENT Y^q. Let U and V be two random variables independent of each other and of X. Let S be the time X crosses from yellow to green, and let T be the time X crosses from green to yellow. Then,

(3.23) $$Y_t^q = U\, I_{\{S \le t\}} + V\, I_{\{T \le t\}}, \qquad 0 \le t \le 1,$$

is the general shape of Y^q. In other words, Y^q is the cumulative input due to rains accompanying the changings of seasons.

(3.24) COMPONENT Y^d. This is the limit of a sequence of increasing compound Poisson processes. Its conditional probability law is described by a kernel H from $(E, \underline{\underline{E}})$ into $(\mathbb{R}_0, \underline{\underline{\mathbb{R}}}_0)$ satisfying

$$\int_{\mathbb{R}_0} H(z, dy)(y \wedge 1) < \infty .$$

The interpretation for $H(z, A)$ is the following:

Consider a small time interval $(t, t + dt)$ and suppose the value of the environment at t is $X_t = z$. Then, $H(z, A)$ is the rate (per unit time) at t with which inputs having some magnitude y in A arrive. So,

$$E^x[\exp(-\lambda Y_t^d) \mid \underline{\underline{K}}] = \exp\left[-\int_0^t ds \int_0^\infty H(X_s, dy)(1 - e^{-\lambda y})\right] .$$

4. RANDOM TIME CHANGES, SEMI-MARKOV PROCESSES, AND THE THEORY OF CONTINUOUS REGENERATION

In this section we shall discuss the manner in which Markov additive processes arise in theoretical investigations.

4a. RANDOM TIME CHANGES. Let (Z_t) be a Hunt process with state space F, and let (A_t) be an increasing continuous additive functional of Z. Define the random time associated with A by

$$Y_t = \inf \{s : A_s > t\} , \tag{4.1}$$

and let (X_t) be the time-changed process:

$$X_t = Z_{Y_t} . \tag{4.2}$$

Then, MAISONNEUVE [13] has shown that the process (X_t, Y_t) is a Markov additive process. This observation should enable one to obtain a finer analysis of the "time change" involved.

In particular, if E is a compact subset of F, and if (A_t) is defined by

$$A_t = \int_0^t 1_E(Z_s)\, ds , \tag{4.3}$$

then the time changed process (X_t) has the state space E. In the literature, when studying the boundary behavior of Markov processes (Z_t), this case is made use of in the following manner. First, a "simple" set E is chosen, on which the process (Z_t) behaves "nicely". Then, A is defined by (4.3) and (X_t) by (4.2) and (4.1). The resulting process (X_t) is in general easier to work with than (Z_t). The transition semi-group of (X_t), and/or the resolvent of (X_t), are then obtained. Finally, these results are used to obtain an "approximate picture" of the process (Z_t) by noting that, if $E \subset F$ is close enough to F, then (X_t) should be close to (Z_t).

The first one to make use of this idea was NEVEU [16], [17]. He took F to be a countable set with the discrete topology, and let E be a finite subset of F. Then, choosing a sequence E_n increasing to F, one obtains a sequence of simple Markov processes (X_t^n) which converges to the original process (Z_t).

4b. SEMI-MARKOV PROCESSES. These are processes (Z_t) which are not Markovian, but they enjoy the strong Markov property at those stopping times T whose graphs are contained in the set of discontinuities of (Z_t), i.e., for almost all ω, $T(\omega)$ is a time of discontinuity for $t \to Z_t(\omega)$. Such processes were introduced by LÉVY [12] and still are far from being understood in sufficient detail. (See also SMITH [18] for the same notion.)

Let (X_t, Y_t) be a Markov additive process. We now think of Y_t as the real time at a local time t. Define

$$L_t = \inf \{s : Y_s > t\} \tag{4.4}$$

as the corresponding local time: that is, when the real time is t, the local time is L_t; when the local time is s, the real time is Y_s. Consider now the process (X_t) in real time: this is the process (Z_t) defined by

$$Z_t = X_{L_t} \quad . \tag{4.5}$$

The process (Z_t) is in general not a Markov process. But, suppose T is a stopping time of (Z_t) such that

$$(4.6) \qquad T(\omega) \in \{s : Y_t(\omega) = s \quad \text{for some } t\}$$

For almost all ω in $\{T < \infty\}$. Then, by the strong Markov property of (X_t, Y_t), the process (Z_t) enjoys the strong Markov property at T. Now, (4.6) holds for any stopping time T which is a time of discontinuity. Hence, (Z_t) is a semi-Markov process.

There is a nice converse to this, due to JACOD [11], which shows that essentially all right-continuous semi-Markov processes which are constant over their intervals of continuity are obtained in the above manner. We now sketch the main lines of JACOD's result.

Let (Z_t) be a right continuous stochastic process with state space E; and let R_t be the time of first discontinuity to the right of t :

$$(4.7) \qquad R_t = \inf\{s > t : \ u \to Z_u \text{ is discontinuous at } s\} .$$

For each t, R_t is a stopping time of the process (Z_s). Define

$$(4.7) \qquad M = [0] \cup (\bigcup_{t>0} [R_t])$$

where $[T]$ denotes the graph of t : $[T] = \{(t,\omega) : T(\omega) = t\}$. For each ω, the section $M(\omega)$ of M at ω is some subset of $\mathbb{R}_+$. Its complement $\mathbb{R}_+ \setminus M(\omega)$ is a countable union of intervals of form [ ) or ( ): these intervals are said to be <u>contiguous</u> to $M(\omega)$.

Suppose that, for almost all ω,

(4.8) a) $M(\omega)$ is a perfect set (that is, it has no isolated points);

b) $t \to Z_t(\omega)$ is constant over each contiguous interval;

c) for any contiguous interval $[a,b)$, $t \to Z_t(\omega)$ is continuous at a.

The net effect of these axioms is that the process (Z_t) is specified by its values on M. We now suppose that we are given a family of probabilities $(P^x)_{x \in E}$ such that

$$(4.9) \qquad P^x\{Z_o = x\} = 1, \qquad x \in E;$$

and that (Z_t) has the <u>strong Markov</u> property at any stopping time T such that $[T] \subset M$.

The process (Z_t) satisfying all these is said to be a <u>semi-Markov process</u> without branching points and without isolated discontinuities. This is the proper generalization of Lévy's notion of semi-Markov processes to the present case of arbitrary state spaces. It is worth pointing out, however, that in the case where E is countable, LÉVY [12] had envisioned processes which are <u>not</u> right continuous. In that case, there are complexities quite beyond anything which is encountered in the present case. For an introduction to Lévy's thinking (corrected and brought up to date) we refer the reader to ÇINLAR [6].

Going back to the semi-Markov process (Z_t) introduced above, we now list JACOD's theorem, relating it to Markov additive processes.

(4.10) THEOREM. Let (Z_t) be a right continuous semi-Markov process without isolated discontinuities and without branching points. Further, suppose that, for any sequence (T_n) of stopping times of (Z_t), if (T_n) increases to T, then (R_{T_n}) increases to R_T. Then, there is an increasing continuous process (L_t) whose support is the set $\bar{M}$; moreover, if

$$(4.11) \qquad Y_t = \inf\{s : L_s > t\}, \qquad X_t = Z_{Y_t} ,$$

then (X_t, Y_t) is a Markov additive process.

In view of the fact that (Z_t) is hard to investigate because of the lack of Markov property, this theorem is of great significance. Based on it, studying a semi-Markov process is reduced to studying a Markov additive process and then changing times to infer back. However, a detailed examination of this program of study has not been done yet.

4c. THEORY OF CONTINUOUS REGENERATION

We start by describing the simpler case of complete regeneration. Consider a process (Z_t) and a random set M. Suppose that the future $\sigma(Z_{T+s}; s \geq 0)$ and the past $\sigma(Z_s; s \leq T)$ are completely independent for any stopping time T such that $[T] \subset M$. Then, (Z_t) is said to be a <u>regenerative process</u> and M is called its <u>regeneration</u> <u>set</u>.

The simplest case is obtained when

(4.12) $$Z_t(\omega) = 1_{M(\omega)}(t) .$$

The most fundamental result concerning regeneration sets is the one due to MAISONNEUVE [13] which he had obtained earlier (see his paper in Séminaire de Probabilités V (1970), Lecture Notes in Mathematics vol. 191). Here is his result.

(4.13) THEOREM. Every regeneration set M is the image of an increasing additive process.

In other words, for every regeneration set M, there exists an increasing additive process Y (i.e., an increasing process Y with stationary and independent increments and with right continuous paths and $Y_0 = 0$) such that

$$M(\omega) = \{t : Y_s(\omega) = t \quad \text{for some } s\} , \tag{4.14}$$

for almost all ω.

In the case where Y is a compound Poisson process, M is a discrete set; and, if we define $0 = T_0, T_1, T_2, \ldots$ to be the points of M in increasing order, then the process (T_n) is a renewal process. Thus, the present theory is an extension of renewal theory.

In general, if Y is not a compound Poisson process, then M has no isolated points. Examples of this latter case are

$$M = \{t : Z_t = 0\} \tag{4.15}$$

where (Z_t) is the Brownian motion on $\mathbb{R}$;

$$M = \{t : Z_t = 0\} \tag{4.16}$$

where (Z_t) is a Chung process with state space $\{0,1,2,\ldots\}$ with 0 instantaneous; and

$$M = \{t : Z_t = x\} \tag{4.17}$$

where (Z_t) is a Markov process with state x a holding point.

We now pass on to the generalization of the concept of regeneration introduced above. Note that, above, the future and the past were completely independent at those stopping times T with $[T] \subset M$; and note again that, then, the regeneration set M is the image of an additive process Y.

Now we replace the complete independence of the past and future by the strong Markov property: We assume only that the future $\sigma(Z_{T+s};\ s \geq 0)$ and the past $\sigma(Z_s;\ s \leq T)$ are conditionally independent given the present $\sigma(Z_T)$ for all those stopping times T such that $[T] \subset M$. Then, (Z_t) is said to be a semi-regenerative process with regeneration set M.

As should be expected, then, M turns out to be the image of a process Y for some Markov additive process (X, Y). This is to show the manner in which Markov additive processes arise in this theory. For a precise introduction to the notion of semi-regeneration we refer to MAISONNEUVE [14] .

5. LÉVY SYSTEMS OF MARKOV ADDITIVE PROCESSES

With this section we resume our studies of Markov additive processes proper. Lévy systems provide us with analytical insights into the infinitesimal behaviors of such processes. The following account is a fast résumé of such results which were obtained in ÇINLAR [5]. Some of these results were also obtained by JACOD [10] for processes (X, Y) such that both X and (X, Y) are Markovian. Our treatment will follow [5] and will concentrate on tying the Lévy systems with the structure outlined in Section 2. Our notations, etc. are those of Section 2. Throughout, (X, Y) is a Markov additive process where Y is increasing.

(5.1) DEFINITION. Let H be an increasing continuous additive functional of (X_t), and let L be a transition kernel from $(E, \underline{\underline{E}})$ into $(E\times\mathbb{R}_+, \underline{\underline{E}}\times\underline{\underline{\mathbb{R}}}_+)$. Then, (H, L) is said to be a Lévy system for (X, Y) provided that, for any positive Borel measurable function f on $E\times E\times\mathbb{R}_+$,

$$(5.2)\qquad E^x\Big[\sum_{s\le t} f(X_{s-}, X_s, Y_s - Y_{s-})I_{\{X_{s-}\neq X_s\}\cup\{Y_{s-}\neq Y_s\}}\Big] = E^x\Big[\int_0^t dH_s \int_{E\times\mathbb{R}_+} L(X_s, dx, dy)f(X_s, x, y)\Big]$$

for all $x\in E$ and $t\in\mathbb{R}_+$.

(5.3) DEFINITION. Let H be an increasing continuous additive functional of X, and let K be a transition kernel from $(E, \underline{\underline{E}})$ into itself. Then, (H, K) is said to be a Lévy system for X provided that, for any positive Borel measurable function on E,

$$(5.4) \qquad E^x\Big[\sum_{s\le t} f(X_{s-}, X_s) I_{\{X_{s-} \neq X_s\}}\Big] = E^x\Big[\int_0^t dH_s \int_E K(X_s, dx) f(X_s, x)\Big]$$

for all $x \in E$ and $t \in \mathbb{R}_+$.

In intuitive terms, supposing $dH_s = ds$, we see that $K(x, dx')$ is the rate at which X jumps from x into dx'; similarly, $L(x, dx', dy)$ is the rate per unit time at which X jumps from x to dx' at the same time that Y jumps by an amount dy.

The following is the main

(5.5) THEOREM. Suppose X is a Hunt process, and Y is increasing and quasi-left-continuous. Then, (X, Y) has a Lévy system (H, L) which further satisfies the following: for any $x \in E$,

$$(5.6) \qquad L(x, \{(x, 0)\}) = 0 \ ;$$

$$(5.7) \qquad \int_{\mathbb{R}_+} L(x, \{x\} \times dy)(y \wedge 1) < \infty \ .$$

Moreover, if K is defined by

$$(5.8) \qquad K(x, A) = L\big(x, (A - \{x\}) \times \mathbb{R}_+\big), \quad x \in E, \ A \in \underline{\underline{E}} \ ,$$

then, (H, K) is a Lévy system for (X_t).

To indicate the manner in which (H, L) is related to the parameters of the process (X, Y) we give the following facts: Consider the decomposition

$$Y = A + Y^q + Y^d \tag{5.9}$$

where A is a continuous (increasing) additive functional of X, Y^q is the purely discontinuous quasi-left-continuous component with fixed jump times (fixed by X); and Y^d is the purely discontinuous component whose jump times are not fixed by X. In terms of the Lévy system (H, L) we have the following:

$$A_t = \int_0^t a(X_s)dH_s \tag{5.10}$$

for some positive Borel measurable function $a: E \to \mathbb{R}$.

$$\begin{aligned} E^x\Big[\sum_{s\le t} f(X_{s-}, X_s, Y^q_s - Y^q_{s-}) I_{\{X_{s-} \ne X_s\}}\Big] \\ = E^x\Big[\int_0^t dH_s \int_E K(X_s, dx) \int_{\mathbb{R}_+} F(X_s, x, dy) f(X_s, x, y)\Big] \end{aligned} \tag{5.11}$$

where $F(x', x, dy)$ is the distribution of the magnitude of a jump of Y^q occurring at an instant when X jumps from x' to x. Moreover,

$$K(x', dx)F(x', x, dy) = L(x', dx, dy) \tag{5.12}$$

for $x \ne x'$.

Finally, concerning the component Y^d, we have

$$\begin{aligned} E^x\big[\exp(-\lambda Y^d_t) \mid \underline{\underline{K}}\big] \\ = \exp\Big\{-\int_0^t dH_s \int_{\mathbb{R}_+} L(X_s, X_s, dy)(1 - e^{-\lambda y})\Big\}. \end{aligned} \tag{5.13}$$

6. LÉVY SYSTEMS AND INFINITESIMAL GENERATORS

Our purpose here is to sketch the connections between the Lévy systems, infinitesimal generators, and resolvents. Our notations are those of Sections 2 and 5.

Throughout this section we suppose that X is a regular step process with infinite lifetime; that is, every $x \in E$ is a holding point, and the successive jump times $\tau_1, \tau_2, \ldots$ of $t \to X_t$ are such that $\tau_n \to \infty$ almost surely. Furthermore, as in the preceding section, we assume that Y is increasing.

For such a process X, Y is automatically quasi-left-continuous, and Theorem (5.5) holds. Moreover, any increasing continuous additive functional C of X can be written in the form

(6.1) $$C_t = \int_0^t c(X_s)ds$$

for some positive Borel measurable function c on E. It follows that, X has a Lévy system (H, K) such that

(6.2) $$H_t = t$$

identically. Then, the Lévy kernel K is related to the infinitesimal generator G of X by

(6.3) $$Gf(x) = -k(x)f(x) + \int_E K(x, dy)f(y)$$

where

(6.4) $$k(x) = K(x, E)$$

is the parameter of the exponential holding time at x.

(6.5) EXAMPLE. Consider the case where $E = \{a, b\}$ as in Section 3a. Then, the infinitesimal generator G of X is described by a 2×2 matrix

$$G = \begin{bmatrix} -\lambda(a) & \lambda(a) \\ \lambda(b) & -\lambda(b) \end{bmatrix}$$

where $\lambda(a)$ and $\lambda(b)$ are two positive numbers, and

$$Gf(x) = \sum_{y \in E} G(x, y)f(y).$$

In this case, the Lévy kernel K is simply

$$K = \begin{bmatrix} 0 & \lambda(a) \\ \lambda(b) & 0 \end{bmatrix};$$

and the formula (6.3) is obvious. □

The following relates the Lévy system of (X, Y) to the infinitesimal generator of (X, Y). Let (H, L) be the Lévy system of (X, Y) having H satisfying (6.2), and let K be as defined in (5.8). Then, (6.3) holds for the infinitesimal generator G of X. Consider again the decomposition (5.9), and in particular note that now the continuous additive functional A there has the form (see (5.10) also)

(6.6) $$A_t = \int_0^t a(X_s)ds.$$

The infinitesimal generator for (X, Y) will be related to the function $\underline{a}$ and Lévy kernel L. See (5.11), (5.12), and (5.13) again for an interpretation.

Let

$$Q_t(x,\ A \times B) = P^x\{X_t \in A,\ Y_t \in B\}; \tag{6.7}$$

then,

$$Q_t^\lambda(x,\ f) = \int_{E \times \mathbb{R}_+} Q_t(x,\ dx',\ dy) f(x') e^{-\lambda y} \tag{6.8}$$

$$= E^x[f(X_t)\ \exp(-\lambda Y_t)].$$

The
~~Following~~ following is the main result.

(6.9) THEOREM. Let f be continuous and bounded. Then,

$$\lim_{t \downarrow 0} \frac{1}{t} [Q_t^\lambda(x,\ f) - f(x)] = Gf(x) - \lambda N^\lambda f(x) \tag{6.10}$$

where G is the infinitesimal generator of X, and

$$N^\lambda f(x) = \int_{E \times \mathbb{R}_+} N(x,\ dx',\ dy) f(x') e^{-\lambda y} \tag{6.11}$$

with

$$N(x,\ A \times B) = \epsilon_0(B) a(x) I(x,\ A) + \int_B L\Big(x,\ A \times (s,\ \infty)\Big) ds. \tag{6.12}$$

For a proof we refer to [5]. When $\lambda = 0$, Q^λ is the transition semi-group of X; and thus, $\lim_{\lambda \downarrow 0} \lambda N^\lambda = 0$. Hence, Q defines both G and N^λ, and through them, a, K, L. Conversely, given a and L, we may compute K from L by the formula (5.8) of the preceding section, and then we have the infinitesimal generator G directly from K by (6.3). Finally, note that $(Q_t^\lambda)_{t \in \mathbb{R}_+}$ is a transition semi-group. The preceding theorem states that the infinitesimal generator of Q^λ is

$$G^\lambda = G - \lambda N^\lambda. \tag{6.13}$$

Consider finally the potential R^λ corresponding to Q^λ: let R^λ be defined by

$$(6.14) \qquad R^\lambda f(x) = \int_0^\infty Q_t^\lambda f(x)dt$$

$$= E^x\left[\int_0^\infty f(X_t)\,\exp(-\lambda Y_t)dt\right].$$

It follows from the well-known relations between potentials and infinitesimal generators that

$$(6.15) \qquad R^\lambda \left(-G^\lambda\right) = I.$$

Using the formula above for G^λ we now get

$$(6.16) \qquad R^\lambda(\lambda N^\lambda - \mu N^\mu)R^\mu = R^\mu - R^\lambda$$

for all $\lambda, \mu \geq 0$. Note that this becomes the ordinary resolvent equation when $N^\lambda = I$ for all λ, which is the case where

$$(6.17) \qquad Y_t = t$$

identically. Hence, the usual resolvents in the theory of a Markov process X in effect correspond to the Markov additive process (X, Y) where $Y_t = t$. This explains somewhat why, in certain situations in the past, some authors were led to explicitly consider the space-time process (X_t, t).

This remarkable resolvent equation (6.16) was first noticed by NEVEU [16] in the case where E is finite. It has further applications in the theory of generalized Dirichlet problems.

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