Discussion Paper No. 1179

EXPLAINING POSITIONAL VOTING PARADOXES I; THE SIMPLE CASE

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January 1997

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ABSTRACT. A theory is developed to explain all possible (single profile) positional voting paradoxes. This includes all pairwise voting cycles, problems with agendas, conflict between the Borda and Condorcet winners, and differences among positional outcomes (such as the plurality and antiplurality methods). I show how to construct profiles to illustrate all of these paradoxes. Among the new conclusions contradicting accepted belief is that rather than being a standard for the field, the Condorcet winner has serious flaws. This paper discusses three candidates; the companion paper [25] handles $n \geq 3$ candidates.

1. INTRODUCTION

Over the last two centuries considerable attention has focussed on the properties of positional voting procedures. These commonly used approaches are where points are assigned to candidates according to how each voter positions them. The standard plurality method, for instance, assigns one point to a voter's top-ranked candidate and zero to all others while the Borda Count (BC) assigns $n-1, n-2, \ldots, n-n=0$ points, respectively, to a voter's first, second, ... , nth ranked candidate. While the importance of these procedures derives from their wide usage, their appeal comes from their mysterious paradoxes (i.e., counterintuitive conclusions) demonstrating complex outcomes. Indeed, by introducing doubt about the meaning of election outcomes these paradoxes raise the legitimate concern that, inadvertently, we can choose badly.

As these methods serve as prototypes for procedures which aggregate agents' preferences, they identify potential issues for economics and other areas. This is illustrated by the connection between the manipulation of decision procedures and the subsequent incentive literature. Another example is the connection between types of voting paradoxes (Saari [16]) and the extension of the Sonnenschein [32, 33], Mantel [9], Debreu [4] aggregate excess demand result from their limited setting of the single set of n commodities to all subsets of two or more commodities (Saari [17, 18]).

Not only are positional methods interesting in their own right, but their outcomes are needed for other choice procedures. A runoff, for instance, is held among the top-ranked candidates from a first election. An agenda, a tournament, the Copeland Method ([26, 10]), and Kemeny's rule ([8, 27]) are among the many procedures using pairwise voting outcomes. Other methods, such as the controversial Approval Voting [3] and the enigmatic rules of figure skating, use positional outcomes in complicated ways. There even is a connection between positional methods and nonparametric statistics (Haunsperger [6]).

1.1. **Complexity of analysis.** Although important, positional procedures have proved to be formidable to analyze. This severe difficulty is manifested by the fact (Saari [21]) that by using different ways to tally the ballots, a single ten-candidate profile can generate over 84

This research was supported by an NSF grant and my Arthur and Gladys Pancoe Professorship. Draft versions were written while visiting CREME, Universite de Caen, in May of 1995 and of 1996; my thanks to my host M. Salles.

million different election rankings. The voters do not change their opinions, but varying the choice of positional methods leads to millions upon millions of contradictory outcomes where each candidate wins with some procedures but is bottom-ranked with others. All of these conflicting outcomes cannot reflect the voters' true opinions, so which is the correct one?

A consequence of this complexity, which severely limits what we knew about positional methods, is a resigned attitude aptly captured by Riker's assertion [15] that "[t]he choice of a positional voting method is subjective." Even stronger are recent comments by another acknowledged expert who stressed the importance of *social choice* – where we only consider the winner of an election – over the derivation of a social ordering of the candidates. He argues that "[g]iven all the logical barriers that have to be scaled to even come close to making a coherent social choice, demanding a full ordering is a tall order." He confesses that trying to find a full ordering is "something that most of us long ago gave up on as impossible and/or incoherent." His thoughts probably reflect the general sense of the choice community

No longer is this true. Rather than being an impossible dream, this paper and its companion provide ways to understand and choose among the different winners, rankings, and positional voting procedures.[1] By emphasizing an analytic (rather than a normative) argument and by removing the technical complexity, objective criteria for the choice of procedures emerge.

1.2. **Profile decomposition.** To understand positional procedures and their derivative methods, a first step is to characterize all possible paradoxes that can occur with a single profile. (A *profile* lists each voter's ranking of the candidates.) This is done, and the results (see Saari [16] and its references) prove that positional procedures admit significantly more paradoxes and more kinds of them than previously suspected. The next steps, to explain each and every paradox and to construct illustrating examples, are described in these two papers. Fortunately, but unexpectedly, the answers for these two-century old challenges are surprisingly natural and simple with the *profile decomposition* introduced here.

My profile decomposition mimics standard data problems of "noise" where, to obtain accurate outcomes, all contaminating noise is separated from the informative portion of the data. While profiles are not troubled by "noise" in the traditional sense of extraneous signals coming from unrelated sources, a decomposition still applies. Here, "profile noise" is the portion of a profile which should not affect the final outcome. It is convenient to identify the noise with profile portions which should generate a neutral, completely tied outcome; e.g., it could be where voters' votes should cancel. So, adding or dropping noise from a profile should not affect the candidates' ordinal ranking — but it does with certain procedures.

Once all noise is removed from a profile, what remains is the "basic portion." If my assertions about profile noise are correct, then all procedures should agree on the basic profile. They do; as shown, *the rankings* and the tallies *of all positional methods and the pairwise elections agree on the basic profile.* An immediate corollary is that *all paradoxes and difficulties of election procedures are completely caused and explained by the profile noise.* So, the long-standing goals of explaining all possible paradoxes while constructing illustrating examples are attained by characterizing all profile noise and the concomitant reaction of positional procedures.

My approach differs from the literature in that I emphasize profiles rather than procedures. But, all properties and peculiarities of all possible positional procedures and all derivative procedures (e.g., runoffs, figure skating, etc.) quickly follow from this profile division. This is because the profile decomposition defines a dual decomposition of positional procedures.

[1] I introduce the approach with three candidate elections; the case of $n \geq 3$ candidates is in (Saari [25]).

Namely, each noise direction (in profile space) defines an associated direction of positional procedures which react to this particular noise. By understanding both decompositions and their interactions, all basic properties are determined. In this manner answers to historical and contemporary concerns from social choice are found. This includes the issues central to the Borda and Condorcet debates of the 1780s which inaugurated this field of social choice.

1.3. **Borda–Condorcet debates and basic conclusion.** To provide a flavor of the kinds of results which follow from this decomposition, I preview the Sect. 4.4 conclusions about the Borda-Condorcet debate; a debate which introduced and continues to shape the social choice area. To review, the academic study of voting started in 1770 (see [12, 13] for details) when Borda constructed a profile to cast doubt on the wisdom of using the plurality vote. He then showed how the BC avoids this particular difficulty. About 15 years later, Condorcet introduced a competing method where his *Condorcet Winner* is the candidate who wins all pairwise elections. Arrow [1], 165 years later, developed his "binary independence" conditions and impossibility theorem which significantly extend Condorcet's notions.

With its natural, intuitive appeal, Condorcet's method quickly became the widely accepted standard for choice theory. Condorcet distinguished his approach by creating profiles where *all positional methods fail to elect the Condorcet winner.* An accompanying corollary, asserting that the Condorcet winner need not be top-ranked in a Borda election, continues to be cited as a fatal BC flaw.

By examining these historically important examples with the profile decomposition, the surprising conclusion is that *for any conflict between the BC and Condorcet rankings, all examples support Borda's approach while raising serious doubts about Condorcet's method – the standard of the field.* As shown, the conflict resides in failings of the pairwise vote – not the BC. This conclusion, which contradicts what has been accepted for two centuries, completely reverses what Condorcet intended.

The source of this surprising assertion is that the BC ignores a profile noise that distorts the pairwise election outcomes; it forces the pairwise vote *to lose the critical assumption that voters have transitive preferences.* This same phenomenon explains Arrow's impossibility theorem; it turns out that binary independence devalues all information about individual rationality. (See Saari [24].) So, Arrow's result and the problems of pairwise voting just reflect the obvious fact that distorted outcomes must be expected from procedures that ignore the rationality of voters. An equally surprising assertion is that rather than serving as a standard, the Condorcet winner must be held in suspect.[2]

This preview of the Borda-Condorcet analysis indicates how new conclusions follow from this approach. Voting paradoxes manifest the behavior of procedures on the noise portions of profiles. By identifying consequences of the noise (e.g., for the pairwise vote, it is the loss of individual rationality), paradoxes can be explained, procedures compared, and illustrating profiles constructed.

1.4. **Removing bias.** The noise bias is not an anomaly; it is nearly an omnipresent difficulty. As I show, *it is more likely to have a completely tied plurality election than to avoid biased election tallies.* Thus, while we must counter the profile noise difficulties, the traditional "profile restriction" approach is useless because it imposes restrictions so severe that only highly unlikely settings remain.

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[2] Observe the implications for the large literature comparing procedures in terms of the Condorcet standard. If the "standard" ignores the rationality of voters, then maybe the standard – not the compared procedure – is at fault when there is a disagreement.

A more feasible approach is to remove the distorting noise components from a profile before computing outcomes. This allows the outcome to be determined by the "basic profile" terms alluded to earlier where all disagreement and conflict disappear. In this desired harmonious state, any procedure can be used because they all agree. In turn, this underscores an important conclusion of these two papers; *the BC is the only positional procedure which ignores all profile noise.* But as the BC tally of the original profile is what other procedures obtain only after the laborious process of removing all profile noise, it follows that using the BC with the original profile is an efficient, pragmatic way to remove all bias.

2. Notation and division of procedures

I introduce the profile decomposition with the important setting of three candidates $\{A, B, C\}$. The $3! = 6$ voter types are

Type	Ranking	Type	Ranking
1	$A \succ B \succ C$	4	$C \succ B \succ A$
2	$A \succ C \succ B$	5	$B \succ C \succ A$
3	$C \succ A \succ B$	6	$B \succ A \succ C$

(2.1)

2.1. **Terminology and voting vectors.** A *profile* specifies the number of voters of each type. Using the labeling of Table 2.1, the integer profile $(0,5,0,3,4,0)$ has five voters of type-two ($A \succ C \succ B$), four of type-five ($B \succ C \succ A$), and three of type-four ($C \succ B \succ A$).

A three-candidate positional election is defined by *voting vector* $\mathbf{w}^3_s = (w_1 = 1, w_2 = s, w_3 = 0)$ where s is a specified value satisfying $0 \le s \le 1$. In tallying a ballot, w_j points are assigned to the voter's jth ranked candidate, $j = 1, 2, 3$. To illustrate with the $(0,5,0,3,4,0)$ profile, its plurality voting procedure $\mathbf{w}^3_0$ outcome is $A \succ B \succ C$ supported by the $5:4:3$ plurality tally. The $\mathbf{w}^3_s$ *vector tally* is $(\tau_s(A), \tau_s(B), \tau_s(C))$ where $\tau_s(K)$ is K's tally. So, the ranking associated with vector $(70, 20, 90)$ is $C \succ A \succ B$.

My normalization of voting vectors requires the top-ranked candidate to receive one point. Thus the BC, given by $\mathbf{B}^3 = (2,1,0)$, has the normalized form $\mathbf{b}^3 = \frac{1}{2}\mathbf{B}^3 = (1, \frac{1}{2}, 0)$. Similarly, an election tallied by assigning six, five, and zero points, respectively, to a voter's top, second, and bottom ranked candidate has the normalized form $(\frac{6}{6}, \frac{5}{6}, 0)$.

An important relationship (probably due to Borda but definitely known by Nanson [14]) between the pairwise and the BC tallies can be described by computing how a voter with preferences $A \succ B \succ C$ votes in pairwise elections.

Candidates	$\{A\}$	$\{B\}$	$\{C\}$
$\{A, B\}$	1	0	–
$\{A, C\}$	1	–	0
$\{B, C\}$	–	1	0
Total	2	1	0

(2.2)

The sum of points this voter provides a candidate over all pairwise elections equals what he would assign her in a BC election. This means (along with neutrality[3] and the fact that each pair is tallied with the same voting vector) that a candidate's BC election tally is the sum of her pairwise tallies. (See Saari [20].) Thus the pairwise tallies 31:29 for $A \succ B$, 30:20 for $A \succ C$, and 40:20 for $B \succ C$ define the BC outcome $B \succ A \succ C$ with the BC tally $(40 + 29) : (30 + 31) : (20 + 20)$. The normalized $\mathbf{b}^3$ vector tally is $\frac{1}{2}(61, 69, 40)$.

[3] Neutrality is where interchanging the names of the candidates interchanges the election tallies.

The three-candidate division of voting vectors is simple. It consists of the (1,0) methods used to tally pairwise elections which, as described above, define the $\mathbf{b}^3$ tally. All remaining $\mathbf{w}_s^3$ methods are represented as a sum of $\mathbf{b}^3$ and the *derived* vector $\mathbf{d}^3 = (0,1,0)$.[4]

Theorem 1. *All three candidate voting vectors can be expressed as*

$$\mathbf{w}_s^3 = (1, s, 0) = \mathbf{b}^3 + (s - \frac{1}{2})\mathbf{d}^3 \quad 0 \le s \le 1 \tag{2.3}$$

Proof. This is a simple algebraic relationship □

Let $F(\mathbf{p}, \mathbf{w}_s^3)$ represent the $\mathbf{w}_s^3$ election tally for profile $\mathbf{p}$. To describe the linearity of F in the $\mathbf{w}_s^3$ variable, suppose the $\mathbf{B}^3 = (2,1,0)$ tally of an election is $(20,40,30)$ and the plurality tally is $(9,8,13)$. Because $(7,2,0) = 2\mathbf{B}^3 + 3(1,0,0)$, the $(7,2,0)$ tally for these same voters is $2(20,40,30) + 3(9,8,13) = (67,104,99)$ with a $B \succ C \succ A$ ranking. The following assertion extends this statement to normalized voting vectors. (The proof is an immediate consequence of Eq. 2.3 and the linearity of F.)

Theorem 2. *The $\mathbf{w}_s^3$ election tally can be expressed as*

$$F(\mathbf{p}, \mathbf{w}_s^3) = F(\mathbf{p}, \mathbf{b}^3) + (s - \frac{1}{2})F(\mathbf{p}, \mathbf{d}^3) \tag{2.4}$$

The line of election outcomes defined by Eq. 2.4 is called the *procedure line.* (Saari, [19, 20].)

2.2. **Geometry.** To obtain a geometric representation for rankings and profiles, assign each candidate a vertex of an equilateral triangle. (See Saari [19, 20].) The *ordinal ranking* of a point in the triangle comes from its distances to the vertices where "closer is better." Points equidistant between two vertices represent indifference. In this manner, the "representation triangle" is divided into "ranking regions." (The numbers in the left triangle of Fig. 1 identify the region's Eq. 2.1 voter type.) Represent a profile by placing the number of voters of each type in its ranking region as illustrated on the right in Fig. 1.

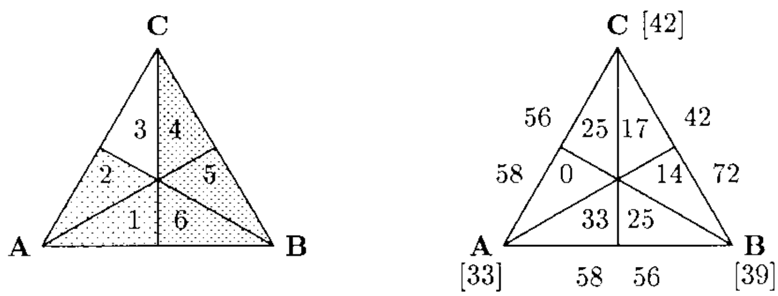


Fig. 1 Representation triangle

The representation triangle geometry makes it easy to compute the plurality, BC, pairwise, $\mathbf{d}^3$, and (with Eq. 2.4) $\mathbf{w}_s^3$ tallies. (I recommend using this method to understand the profile decomposition and to compute tallies for the examples.) To tally the pairwise elections notice that the central vertical line is equidistant between the A and B vertices; it is the $A \sim B$ indifference line. Thus, the $\{A, B\}$ pairwise tally is the sum of voters on each side of the line; e.g., in the left triangle of Fig. 1, B's tally is the sum of the profile entries in the darker shaded region. With the profile on the right in Fig. 1, A beats B by $25+33 = 58$ to $17+14+25 = 56$. The tallies for the other pairs, listed next to the appropriate edges of the triangle, crown A as the Condorcet winner.

[4]Sieberg [31] uses $\mathbf{d}^3$ to capture a statistical "variance" in election outcomes.

A candidate's BC tally is the sum of her pairwise tallies, so the $\mathbf{b}^3$ vector tally $(58, 64, 49)$ defines the BC ranking $B \succ A \succ C$ which conflicts with the pairwise rankings. In particular, A, the Condorcet winner, is not BC top-ranked.

A candidate's plurality tally is the number of voters who have her top-ranked, so it is the sum of the profile entries in the two ranking regions sharing the candidate's vertex. In the left triangle of Fig. 1, A's tally is the sum of entries in the lightly shaded region. These tallies for the profile in the right triangle, given by the bracketed numbers near the vertices, define the plurality ranking $C \succ B \succ A$. For this profile A is the Condorcet winner, B is the BC winner, and C is the plurality winner; who is the voters' true top-choice?

Finally, A's $\mathbf{d^3}$ tally is the sum of the entries of the two ranking regions midway from the A vertex to the opposing edge; it is the sum of the entries in regions 3 and 6. Thus the A, B, C tallies with $\mathbf{d^3}$ are, respectively, $25 + 25 = 50$, $17 + 33 = 50$, $0 + 14 = 14$. According to Eq. 2.4 and the $\mathbf{b^3}$ tally, the $\mathbf{w_s^3}$ election tally is

$$\begin{aligned} F(\mathbf{p}, \mathbf{w_s^3}) &= (58, 64, 49) + (s - \frac{1}{2})(50, 50, 14) \\ &= (33 + 50s, 39 + 50s, 42 + 14s),\ s \in [0, 1]. \end{aligned} \tag{2.5}$$

3. Profile decomposition

For a quick analysis of a three-candidate profile, I recommend the approach described in (Saari [20]). But this earlier approach, designed to offer new insight into voting problems using only elementary tools, fails to address many issues. The following profile decomposition offers an accurate analysis for all profiles.

3.1. Profile differential.

The profile decomposition uses the difference between profiles.

Definition 1. *A* profile differential *is the difference between two profiles involving the same number of voters. Equivalently, a listing of the number of voters of each type is a profile differential if and only if the sum is zero.*

Profile differentials define the basis for various subspaces of profiles. For two of the subspaces, I specify three vectors even though any two suffice. As a profile differential involves negative numbers of voters, we can convert it into an "actual" profile (which requires a non-negative number of voters of each type) by adding a "neutral" profile. So, with profile differential $\mathbf{p_d} = (1, 0, -2, 0, 1, 0)$, add $(2, 2, 2, 2, 2, 2)$ (a profile forcing completely tied elections) to obtain the profile $(3, 2, 0, 2, 3, 2)$.

3.2. Decomposition.

The profile decomposition has four components. The *kernel* portion has no effect on any procedure. The *Condorcet portion* is the profile noise which affects only pairwise votes; e.g., it explains all differences between the pairwise and BC outcomes. The *reversal* portion is the profile noise causing all differences in positional outcomes. The *basic* portion is where all procedures agree.

Definition 2. *The* three-candidate profile decomposition *is defined by the following basis vectors for the different subspaces.*

1. *The* kernel *is spanned by the kernel vector* $\mathbf{K}^3 = (1, 1, \dots, 1)$.
2. *The* Condorcet space *is defined by* $\mathbf{Con}^3 = (1, -1, 1, -1, 1, -1)$.
3. *The* reversal vector *for candidate* K, $K = A, B, C$, *is the profile differential with one voter for each type where* K *is top-ranked, one voter for each type where* K *is bottom-ranked, and* -2 *voters for the remaining two voter types (where* K *is middle-ranked).*

The basis vectors for the reversal subspace are

$$\mathbf{R}_A = (1, 1, -2, 1, 1, -2),\ \mathbf{R}_B = (-2, 1, 1, -2, 1, 1),\ \mathbf{R}_C = (1, -2, 1, 1, -2, 1). \tag{3.1}$$

4. *The* basic vector *for candidate K, $K = A, B, C$, is the profile differential with one voter for each type where K is top-ranked and -1 voters where she is bottom-ranked. The basis vectors for the basic subspace are*

$$\mathbf{B}_A = (1, 1, 0, -1, -1, 0),\ \mathbf{B}_B = (0, -1, -1, 0, 1, 1),\ \mathbf{B}_C = (-1, 0, 1, 1, 0, -1) \tag{3.2}$$

The symmetry of these profile differentials is apparent from Fig. 2.

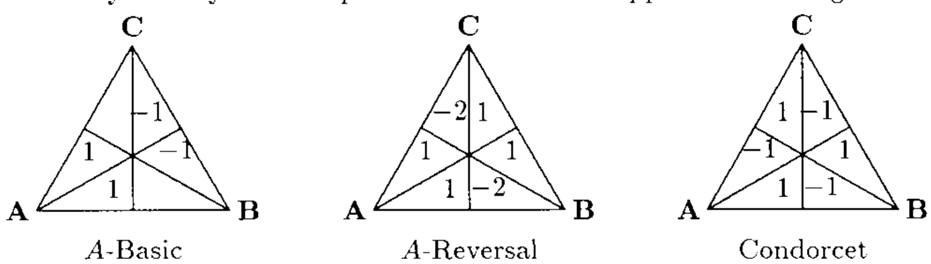


Fig. 2 Profile decomposition

3.3. **Impact of Decomposition.** The value of the decomposition derives from the way voting procedures react to the different subspaces.

Theorem 3. *All profiles can be expressed as*

$$\mathbf{p} = \mathbf{p_K} + \mathbf{p_B} + \mathbf{p_R} + \mathbf{p_C} \tag{3.3}$$

where the profile differentials on the right-hand side come from, respectively, the kernel, the basic, the reversal, and the Condorcet subspaces. The four subspaces are mutually orthogonal. The properties of these profile differentials are as follows.

1. *All pairwise and positional rankings of $\mathbf{K}^3$ are complete ties. The tallies can differ.*
2. *All normalized positional methods have the identical tally for a vector from the basic subspace. The common tally for $a_B\mathbf{B}_A + b_B\mathbf{B}_B + c_B\mathbf{B}_C$ is*

$$(2a_B - b_B - c_B, 2b_B - a_B - c_B, 2c_B - a_B - b_B). \tag{3.4}$$

The pairwise rankings of a basic profile always agree with the common ranking of the positional methods. For $a_B\mathbf{B}_A + b_B\mathbf{B}_B + c_B\mathbf{B}_C$, the $\{A, B\}$, $\{B, C\}$, $\{A, C\}$ tallies are, respectively,

$$(2a_B - 2b_B)(1, -1),\ (2b_B - 2c_B)(1, -1),\ (2a_B - 2c_B)(1, -1). \tag{3.5}$$

3. *For $\mathbf{Con}^3$, all positional methods assign a zero tally to each candidate but the pairwise outcomes define the cycle $A \succ B$, $B \succ C, C \succ A$ with identical $1 : -1$ tallies.*
4. *Each candidate's pairwise and BC tally for a vector from the reversal subspace is zero. All non-BC positional procedures have a non-zero tally for each basis vector. The $\mathbf{w}^3_\mathbf{s}$ tally for $a_R\mathbf{R}_A + b_R\mathbf{R}_B + c_R\mathbf{R}_C$ is*

$$(2s - 1)(2a_R - b_R - c_R, 2b_R - a_R - c_R, 2c_R - a_R - b_R). \tag{3.6}$$

The proof of this theorem is in Sect. 8.

Although Thm. 3 is extensively analyzed in what follows, I preview its impact by constructing a paradoxical example. Suppose we want a profile with the BC ranking $C \succ A \succ B$.

As Thm. 3, part 2 asserts that the BC outcome is strictly determined by the basic vectors, choose the coefficients to satisfy $c_B > a_B > b_B = 0$; e.g., $c_B = 2$, $a_B = 1$ define the profile differential $\mathbf{p_B} = (-1, 1, 2, 1, -1, -2)$ where, according to Thm. 3, all procedures have identical tallies. To introduce conflict, make B the plurality winner by adding $\mathbf{p_R} = 3\mathbf{R_B}$ to obtain $(-7, 4, 5, -5, 2, 1)$. Although B is the plurality winner, part 4 ensures that the BC and pairwise rankings remain untouched. To alter the pairwise rankings, notice that the Condorcet portion causes a pairwise cyclic effect (part 3) without affecting the positional rankings. Adding $\mathbf{p_C} = -2\mathbf{Con^3}$ creates conflict because $\mathbf{p_B} + \mathbf{p_R} + \mathbf{p_C} = (-9, 6, 3, -3, 0, 3)$ requires the previously middle-ranked A to tie the other candidates in pairwise elections. To convert the profile differential into a profile, add $\mathbf{p_K} = 9\mathbf{K^3}$ to obtain $(0, 15, 12, 6, 9, 12)$. By construction, this profile has the BC outcome $C \succ A \succ B$, the plurality outcome $B \succ C \succ A$ and the pairwise outcomes $A \sim B$, $A \sim C$, $C \succ B$.

3.4. **Choice of coefficients.** I use non-negative coefficients in the example because if one coefficient from each set of Eqs. 3.4, 3.6 is zero and the other two non-negative, then the resulting rankings trivially follow from the magnitudes of the coefficients. This choice of coefficients always is possible.

Corollary 1. *The basic and reversal vectors satisfy*

$$\mathbf{B_A} + \mathbf{B_B} + \mathbf{B_C} = \mathbf{R_A} + \mathbf{R_B} + \mathbf{R_C} = \mathbf{0}. \tag{3.7}$$

Consequently, vectors in the basic and reversal subspaces can be represented with two non-negative coefficients and a zero one.

To illustrate, because Eq. 3.7 requires $-\mathbf{B_C} = \mathbf{B_A} + \mathbf{B_B}$, the basic vector $\mathbf{p_B} = -6\mathbf{B_A} - 9\mathbf{B_C} = -6\mathbf{B_A} + 9(\mathbf{B_A} + \mathbf{B_B}) = 3\mathbf{B_A} + 9\mathbf{B_B}$ can be described with $b_B = 9, a_B = 3, c_B = 0$ defining the $B \succ A \succ C$ outcome for all pairs and positional methods.

4. Pairwise voting with the profile decomposition

Theorem 3 identifies the basic and Condorcet vectors, $\mathbf{p_B} + \mathbf{p_C}$, as the *only* portion of a profile which affects pairwise rankings. Consequently, all differences between BC and pairwise rankings, all properties of the Condorcet winner, agendas, cycles, etc. are completely and quickly determined by these profile differentials. This analysis is described here. Combined with the discussion of Sect. 5, where we exploit the Thm. 3 assertion that only the $\mathbf{p_B} + \mathbf{p_R}$ portion of a profile effect positional rankings, we finally understand why different procedures have different societal rankings.

4.1. **Pairwise rankings.** The large pairwise voting literature (see Kelly [7]) considers cycles, properties of the Condorcet winner and loser, properties of procedures based on pairwise election outcomes, the Borda – Condorcet conflict, etc. Stronger results about these and other topics involving a much simpler analysis follow from the Thm. 3 assertion that all properties of the pairwise rankings and tallies are due to the basic and Condorcet profile differentials. Indeed, because Thm. 3 ensures there is no conflict among methods with the basic portion, the previously technically difficult analysis about cycles, agendas, Kemeny's rule, Copeland's method, the Borda and Condorcet debate, etc., etc. reduces to a quick, simple description how the pairwise vote treats the one-dimensional space of Condorcet profiles.

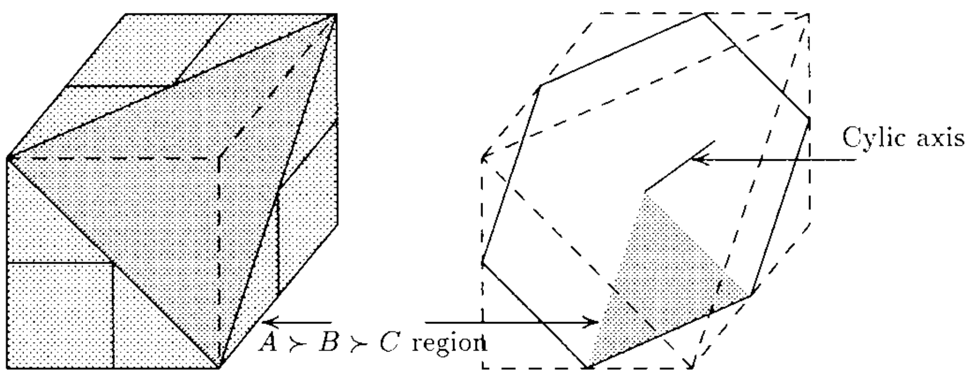
Central to this discussion is the fact that the pairwise election rankings of basic profiles go beyond defining transitive ordinal rankings to require the tallies to satisfy a strong *cardinal transitivity* property. Indeed, as asserted next, the tallies mimic the additive properties of points along the line where $(x_1 - x_2) + (x_2 - x_3) = (x_1 - x_3)$.

Corollary 2. *The pairwise rankings of a basic profile are transitive, and the tallies from any two pairwise elections uniquely determine the tally for the remaining pairwise election. More specifically, if $\tau_B(X,Y)$ denotes the difference between X's and Y's basic pairwise tallies, then*

$$\tau_B(A,B)+\tau_B(B,C)=\tau_B(A,C) \tag{4.1}$$

This Eq. 4.1 feature of basic vectors means that the pairwise tallies measure how strongly one candidate beats another. In an "idealized world," we expect the pairwise election rankings of $A \succ B$ and $B \succ C$ to imply the $A \succ C$ election outcome. Even stronger, we expect A's victory over C to be larger than over B. But these assertions are false in general because we cannot even ensure ordinal transitivity. This idealized setting, however, holds for basic vectors because the $\tau_B(C,B)$ plus the $\tau_B(B,A)$ election margins equals the $\tau_B(C,A)$ margin. (Clearly, Eq. 4.1 holds when the names of the candidates are interchanged.)

So, going beyond the transitivity of the pairwise rankings, the point totals for basic profiles capture the intuitive sense that a wider point spread – even for just one pairwise election – signals a stronger candidate. Theorem 3 places all blame for the failure of these desirable properties to hold in general on the Condorcet portion of a profile. The basic profile tallies preserve cardinal transitivity (Eq. 4.1); it is the Condorcet portion which disrupts the cardinal and ordinal transitivity.



a. Representation cube **b. Transitive Plane**

Fig. 3. Pairwise outcomes

4.2. **Transitivity plane.** To represent Eq. 4.1 geometrically, I use the *representation cube* of (Saari [19, 20]). Here the difference between pairwise tallies (not just the basic terms) $\tau(X,Y)$ is replaced with the fraction of voters voting in a particular manner; i.e., with v voters $x_{X,Y}=\tau(X,Y)/v$. So, $-1\le x_{X,Y}\le 1$ and $x_{X,Y}=-x_{Y,X}$. Point $(x_{A,B},x_{B,C},x_{C,A})$ in R^3 defines the marginal outcomes for all pairs. The $-1\le x_{X,Y}\le 1$ restriction forces these values into the *orthogonal cube* centered at the origin. Six of the eight vertices of this cube correspond to unanimity profiles. For instance, if all voters prefer $A \succ C \succ B$, then the unanimity outcomes $x_{A,B}=-x_{B,C}=-x_{C,A}=1$ defines the vertex $(1,-1,-1)$.

The representation cube is the convex hull of the six unanimity vertices. This shaded region depicted in Fig. 3-a is the set of points in the orthogonal cube between the planes

$$x_{A,B}+x_{B,C}+x_{C,A}=\pm 1. \tag{4.2}$$

The importance of the representation cube is that all (rational) points are the pairwise election outcome for some profile. Conversely, all pairwise election outcomes are in this cube. (See [19, 20].)

The *transitivity plane* passing through the origin is given by

$$x_{A,B} + x_{B,C} + x_{C,A} = 0. \tag{4.3}$$

According to Eq. 4.1, this plane, represented in Fig. 3.b, contains all basic pairwise outcomes. Perpendicular to the plane is the axis connecting the cyclic rankings of the two vertices $\pm(1,1,1)$; call this the cyclic axis. An outcome of a $\mathbf{Con}^3$ term is this direction.

It is immediate that any point in the representation cube – any pairwise election tally – can be described by its distance along the cyclic axis and in the transitivity plane. I call this the *transitive plane coordinate representation*. To connect this coordinate system with the profile decomposition, notice that the basic profile uniquely determines the transitive plane coordinate while the Condorcet part uniquely determine the cyclic axis distance. Indeed, (with algebra) the profile $a_B\mathbf{B}_A + b_B\mathbf{B}_B + \gamma\mathbf{Con}^3 + k\mathbf{K}^3$ defines

$$\frac{2}{3k}(a_B - b_B, b_B, -a_B) + \frac{\gamma}{3k}(1,1,1) \tag{4.4}$$

where the first vector is the transitive plane component and the second is the cyclic axis component. Conversely, if a point in the transitivity plane coordinates is $(q_1^T, q_2^T, q_3^T) + \mu(1,1,1)$ (so $q_1^T + q_2^T + q_3^T = 0$), we have from algebra that

$$a_B = -\lambda q_3^T/2,\ b_B = \lambda q_2^T/2,\ \gamma = \lambda\mu,\ k = \lambda/3 \tag{4.5}$$

for any $\lambda > 0$. Equation 4.5 provides an immediate relationship between the geometry of pairwise outcomes and the profile decomposition.

This geometric representation of the pairwise outcomes offers new insight into classical concerns. To illustrate, recall that an agenda is a listing of the candidates, say $< A, B, C >$, meaning that the majority winner of the $\{A, B\}$ election is matched against C. In the idealized world of Cor. 2, the outcome is independent of the choice of an agenda. In general, however, the winning candidate can depend upon the agenda. This converts the choice of the agenda into a strategic variable.

This phenomenon never occurs with the basic portion of a profile (with its outcome in the transitive plane) because the transitivity forces the same outcome with any agenda. The cycle portion of an outcome, on the other hand, always elects the agenda's last listed candidate. So, according to Thm. 3 (and the transitive plane coordinate representation), this "agenda manipulation" phenomenon is strictly a consequences of the $\mathbf{p_C}$ portion of a profile. Indeed, the problem occurs iff the Condorcet portion is sufficiently strong to force the election outcome into a cyclic region of the representation cube.

Similarly, all problems of any "reasonable" ranking or choice procedure based on pairwise rankings are completely attributed to the Condorcet portion of a profile. By "reasonable" I mean that if the pairwise rankings and tallies satisfy the "idealized world" conditions described above, then the outcome is the expected conclusion. For instance, the Condorcet winner, the Copeland Method, Kemeny's rule, and agendas are "reasonable procedures." As Cor. 2 requires a reasonable procedure to be well behaved on the basic portion $\mathbf{p_B}$, all problems, paradoxes, weaknesses, and inadequacies of these procedures are strictly due to the $\mathbf{p_C}$ Condorcet portion. Consequently, a quick but complete analysis of these procedures only involves understanding their behavior on $\mathbf{Con}^3$.

4.3. **The Condorcet portion.** As all conflict with pairwise rankings and tallies comes from the Condorcet portion of a profile, we need to analyze the pairwise vote on this one-dimensional space. I do so with the traditional three-voter profile

$$A \succ B \succ C,\ B \succ C \succ A,\ C \succ A \succ B \tag{4.6}$$

given by $\frac{1}{2}(\mathbf{K}^3 + \mathbf{Con}^3)$. Here, each candidate is in first, second, and last place exactly once, so (along with neutrality and anonymity) it is easy to argue that no candidate has an advantage; in particular, these voters' votes should cancel. This natural outcome of a complete tie for the Condorcet triplet $\frac{1}{2}(\mathbf{K}^3 + \mathbf{Con}^3)$ holds for all positional methods. But, a pairwise vote (Thm. 3, part 3) yields the $A \succ B$, $B \succ C$ $C \succ A$ cycle.

Any analysis of the pairwise vote of $\mathbf{Con^3}$ must explain why the cyclic outcome replaces the complete tie. In particular, we must understand why individual transitive preferences cause a non-transitive cyclic outcome. I do this by showing that *the pairwise vote applied to* $\mathbf{p}_C$ *drops the critical assumption of individual rationality.*

Clearly, when procedures emphasize different information from a profile, we must expect different outcomes. So, a way to analyze procedures is to identify what profile information a procedure retains, and what information it devalues. The pairwise vote (along with Arrow's IIA [1] and Sen's Minimal Liberalism [30]) drops all information concerning the individual rationality of voters (Saari [24]). Clearly, a procedure which devalues information about individual transitivity cannot be expected to have transitive outputs.

To explain, if we only know that a voter prefers $A \succ C$ from $\{A, B, C\}$, then it is impossible to determine whether his full preferences are rational or irrational. This is because transitivity involves specific sequencing conditions on the three pairwise rankings. Similarly, a procedure which ignores this sequencing discards information about the individual rationality of voters. This happens with the pairwise vote as it solely concentrates on how voters rank a particular pair when determining that pair's societal ranking. All information about the relative rankings of other pairs is ignored.

To illustrate, the irrational voters described by the cyclic preferences

Number	**Pairwise Rankings**
2	$\{A \succ B,\ B \succ C,\ C \succ A\}$
1	$\{B \succ A,\ C \succ B, A \succ C\}$

cannot vote in a $\mathbf{w_s^3}$ election. This is because to use a $\mathbf{w_s^3}$ procedure voters need a transitive ranking (for $0 < s < 1$), or at least a top (for $s = 0$) or a bottom-ranked (for $s = 1$) candidate, but cyclic voters fail to meet these minimal conditions. However, because a pairwise vote ignores information about individual rationality, the pairwise vote can tally the ballots for these irrational voters defining the expected cycle $A \succ B$, $B \succ C$, $C \succ A$ with $2:1$ tallies.

When information about the individual rationality of voters is ignored, we must expect with a sufficiently heterogeneous society that the pairwise vote cannot distinguish whether the voters have transitive or irrational preferences. Theorem 3 asserts that this is true and that $\mathbf{Con^3}$ defines the required heterogeneity. Indeed, notice how the combination of anonymity[5] and the ignored sequencing information makes it impossible for the pairwise vote to distinguish between the Eq. 4.6 and Eq. 4.3 profiles; the only relevant information is the number of voters with each ranking of each pair. Consequently, the pairwise vote cannot distinguish between the Condorcet triplet of Eq. 4.6 and any irrational way voters rank pairs that generates the same tallies.

[5] Here, anonymity means that the procedure does not check the names of the voters for any input; it only determines the number of voters with each ranking.

A computation proves there are four ways to combine the pairs from Eq. 4.6 to define new profiles. Three of them have two voters with transitive rankings that reverse each other (so, they cancel each others vote) while the third has the cyclic ranking $A \succ B, B \succ C, C \succ A$. As this third voter's preferences should break the tie to determine the outcome, the cyclic ranking is a natural conclusion. The final profile is Eq. 4.3 where, again, the cyclic outcome is reasonable.

In summary, the pairwise vote cannot distinguish between the Condorcet profile of transitive preferences where the arguable outcome is a complete tie, and four other profiles involving irrational voters where the cyclic outcome is the "correct" one. The cyclic pairwise outcome merely manifests the pairwise vote's attempt to reflect the beliefs of potential irrational voters. (Geometric support is in (Saari [20]).) Stated in another way, when the pairwise vote is applied to $\mathbf{Con^3}$, it loses the assumption that individual preferences are transitive. Thus all non-transitive arrangements of pairwise outcomes – quasi-transitive rankings, acyclic rankings, cyclic rankings, tallies violating cardinal transitivity (Eq. 4.1) – are due to this $\mathbf{Con}^3$ portion of a profile where the pairwise vote loses the assumption of individual rationality.

To see why a basic profile avoids these difficulties, notice that $\mathbf{B_A}$ has one voter with $A \succ B \succ C$ and another with $A \succ C \succ B$ causing the $\{B, C\}$ comparisons to cancel. (The same cancellation holds for the $\mathbf{B_A}$ rankings associated with negative numbers of voters.) This cancellation accentuates A's role while treating equally all other candidates with a tie vote. Consequently, the pairwise ranking of a general basic profile $a_B\mathbf{B_A} + b_B\mathbf{B_B} + c_B\mathbf{B_C}$ strictly manifests the ordering properties of the a_B, b_B, c_B coefficients. But, rather than reflecting desirable properties of the procedure, the transitivity of the pairwise vote is preserved by the nature of the basic profiles.

4.4. **Borda-Condorcet comparison.** As noted, all procedures based on pairwise voting such as Copeland's method, agendas, Kemeny's rule, etc. can be quickly and completely analyzed with this decomposition.[6] To indicate how to do this, I derive properties of the Condorcet winner along with comparisons with the BC.

The key point is that while the BC and pairwise rankings agree on the basic portion of a profile, the $\mathbf{Con^3}$ portion has no impact upon the BC outcome but it distorts the pairwise tally. Using the representation cube, the transitive plane component of a point uniquely defines the BC ranking (Thm. 3), while the component in the cyclic axis direction determines the deviation from transitivity caused by the Condorcet portion. Thus all conflict between the BC and the pairwise rankings and tallies is completely explained by the pairwise vote's treatment of $\mathbf{Con}^3$; it is due to the pairwise vote's dismissal of individual rationality. This effect, then, explains all differences between the BC and Condorcet winners. Since the Condorcet portion of a profile is one-dimensional, it is easy to analyze these difficulties and to construct illustrating examples.

Example 1. I am unaware of any necessary and sufficient conditions on profiles ensuring that the Condorcet winner is *not* BC top-ranked. The profile decomposition reduces such previously difficult issues to elementary algebra.

Without loss of generality, use the BC ranking $A \succ B \succ C$ which occurs (Thm. 3) if and only if the basic profile coefficients satisfy $a_B > b_B > c_B = 0$; the $\mathbf{b^3}$ tally is $(2a_B - b_B, 2b_B - a_B, -(a_B + b_B))$. The pairwise and BC rankings agree where the $\{A, B\}, \{B, C\}, \{A, C\}$ pairwise tallies are, respectively,

$$2(a_B - b_B) : -2(a_B - b_B),\ 2b_B : -2b_B,\ 2a_B : -2a_B \qquad (4.7)$$

[6] A purpose of (Saari and Merlin [27]) is to show how the previously difficult analysis of Kemeny's rule is easy with the decomposition; e.g., for $n = 3$ candidates, all problems emerge just by considering $\mathbf{Con^3}$.

It is well known (and derived below) that a Condorcet winner cannot be BC bottom ranked, so our example must crown B as the Condorcet winner. According to Thm. 3, B is the Condorcet winner for appropriate γ values of $\gamma\mathbf{Con^3}$. The γ conditions come from the $\{A, B\}, \{B, C\}, \{A, C\}$ pairwise tallies for the augmented profile which are, respectively,

$$2(a_B - b_B) + \gamma : -2(a_B - b_B) - \gamma,\ 2b_B + \gamma : -2b_B - \gamma,$$
$$2a_B - \gamma : -2a_B + \gamma. \tag{4.8}$$

To make B the Condorcet winner by creating $B \succ A, B \succ C$ pairwise rankings, we need

$$2(a_B - b_B) + \gamma < -2(a_B - b_B) - \gamma, \quad 2b_B + \gamma > -2b_B - \gamma.$$

Theorem 4. *A necessary and sufficient condition for the BC ranking $A \succ B \succ C$ to be accompanied with B as the Condorcet winner is that the coefficients of the basic profile satisfy $a_B > b_B > c_B = 0$ and the coefficient of the Condorcet term $\gamma\mathbf{Con^3}$ satisfies*

$$2(a_B - b_B) < -\gamma < 2b_B. \tag{4.9}$$

A necessary and sufficient condition that the BC ranking is accompanied by a pairwise cycle is that γ satisfies one of the inequalities

$$2a_B < \gamma, \quad -2\ Max((a_B - b_B), b_B) > \gamma \tag{4.10}$$

The first inequality defines the cycle with $A \succ B$; the second the cycle with $B \succ A$

Proof. The proof only involves solving the appropriate inequalities from Eq. 4.8 for γ. □

The choices $a_B = \frac{5}{3}, b_B = \frac{4}{3}, \gamma = -\frac{5}{3}$ define the profile differential $(0, 2, -3, 0, -2, 3)$ or the profile (by adding $3\mathbf{K^3}$) $(3, 5, 0, 3, 1, 6)$. As this profile does not have a reversal component, all $\mathbf{w_s^3}$ rankings are $A \succ B \succ C$ while the pairwise rankings are $B \succ A, B \succ C, A \succ C$.

To see this conflicting behavior geometrically, start with a point in the transitive plane yielding the $A \succ B \succ C$ ranking. To force a pairwise outcome with B as a Condorcet winner, we need a component in the $-(1, 1, 1)$ direction caused by the Condorcet portion of the profile. If this component is too strong, the outcome will end in the cyclic region; this is a geometric explanation for the upper bound on $-\gamma$ in the theorem.

These descriptions show how to create profiles illustrating Condorcet's assertion that the Condorcet winner need not be top-ranked by any positional procedure. But B's status as the Condorcet winner requires adding a $\mathbf{Con^3}$ profile component - a component which vitiates the assumption of individual rationality. Thus, it is the BC, not the Condorcet outcome, that is to be trusted. Although this assertion contradicts accepted belief, the proof is a trivial consequence of Thm. 3 or the transitive plane geometry of representation cube.

To create an example with pairwise cycles, choose $a_B = 2, b_B = 1, c_B = 0$ to define the basic profile $(2, 1, -1, -2, -1, 1)$ with its universal $A \succ B \succ C$ election ranking. The accompanying $A \succ B, B \succ C, C \succ A$ pairwise cycle occurs with $\gamma = 5 > 2a_B = 4$ to define the profile differential $(7, -4, 4, -7, 4, -4)$ or a profile $(14, 3, 11, 0, 11, 3)$. To construct an example where the BC ranking is accompanied with a $B \succ A, A \succ C, C \succ B$ cycle, choose $\gamma = -3 < -2\ \mathrm{Max}((a_B - b_B), b_B) = -2$ to obtain the profile differential $(-1, 4, -4, 1, -4, 4)$ or the profile $(3, 8, 0, 5, 0, 8)$. It is instructive to consider these examples with the transitive plane geometry of the representation cube. □

It has been known since the days of Borda that there exist relationships between the BC and the pairwise rankings, but the reasons for the conclusions have not been well understood. These earlier results use the Eq. 2.2 condition that a candidate's BC tally is the sum of her pairwise tallies. It now follows from Thm. 3 that this summation cancels the tallies from the Condorcet portion of the profile. Alternatively, with the transitive plane coordinates, the summation cancels all effects in the cyclic axis direction leaving only the basic profile terms to influence the BC outcome. So, the profile decomposition leads to extensions and new proofs of known statements.

Theorem 5. *Assume there are $n = 3$ candidates.*

1. *For any profile, there exists a unique γ value so that by removing γ points from each of $\tau(A, B)$, $\tau(B, C)$, $\tau(C, A)$, the reduced tallies satisfy the cardinal transitivity condition of Eq. 4.1. The removed cyclic terms from the tally correspond to the* $\mathbf{Con}^3$ *portion of the profile, while the reduced tally is due to the basic portion. The ranking from the reduced, or transitive plane tally, agrees with the BC ranking.*
2. *If all pairwise tallies have a complete tie, then the BC outcome is a complete tie. If the BC outcome is a complete tie, then either all pairwise elections are tied, or they define a cycle with the same victory margin in each pairwise election.*
3. *The Condorcet winner cannot be BC bottom-ranked. The Condorcet loser cannot be BC top-ranked. The Condorcet winner is BC strictly ranked above the Condorcet loser.*

Proof. The proof of part 1 follows from Thm. 3. Because the first part of part 2 requires the pairwise tallies to satisfy cardinal transitivity, the profile has no Condorcet portion. (The point is in the transitive plane.) This means that each pairwise outcome for the basic profile is zero, so, according to Thm. 3, the BC ranking also is a tie. In the opposite direction, a BC complete tie requires a zero basic portion for the profile. Consequently, as the pairwise vote is strictly determined by the Condorcet portion, the outcome is cyclic.

To prove part 3, notice that a Condorcet winner and/or Condorcet loser requires (from part 1) a nonzero basic portion. On the basic portion, the pairwise and BC rankings agree, and the pairwise tallies satisfy the cardinal transitivity condition Eq. 4.1. The $\tau(X, Y)$ outcomes are

$$\tau(A, B) = \tau_B(A, B) + \gamma,\ \tau(B, C) = \tau_B(B, C) + \gamma,$$
$$\tau(A, C) = \tau_B(A, C) - \gamma \tag{4.11}$$

for any γ. The assertion now follows from simple algebra. □

For a geometric proof, notice that result follows directly from the orientation of the transitive plane. Other results are obtained from properties of the transitive plane. For instance, the plane's orientation restricts the pairwise outcomes generated by $\gamma > 0$ and a basic profile outcome of $A \succ B \succ C$. This restriction determines new BC ranking properties.

4.5. **Summary.** The results of this discussion directly counter basic assumptions for much of choice theory. For instance, it is easy to find criticisms arguing that even though the BC has desirable properties, the BC "violates the binary independence axiom $\cdots$ it is not rationalizable and violates choice theoretic conditions." (Schofield, p. 12 [29].) But, instead of being a BC fault, the real flaw is because the binary independence condition unintentionally drops the crucial assumption of individual rationality; this always happens when this condition is applied to the $\mathbf{p_C}$. An easy proof of this assertion is to note that by dropping the Condorcet portion of the profile, the BC does satisfy binary independence. (This is immediate from Thm. 3.) More generally, *by removing the $\mathbf{p_C}$ portion of a profile before applying IIA, Arrow's theorem now has a positive, rather than the negative dictatorial assertion where (according to Thm. 3) the BC is an admitted procedure.* (The reversal portion of a profile disqualifies all other $\mathbf{w}_s^3$ procedures.) Stated in other words, Arrow's impossibility theorem is completely due to the $\mathbf{p_C}$ portion of a profile; without this profile noise there are no difficulties. In this manner, Arrow's Theorem underscores the importance and utility of the profile decomposition.

Notice how this analysis compromises the many normative and intuitive arguments advanced to support the Condorcet winner, the Copeland winner, agendas, and a host of other procedures based on pairwise outcomes. After all, it is difficult to justify including the Condorcet bias manifesting the violation of individual rationality. A natural way to correct this difficulty is to remove the troubling Condorcet portion of a profile so that the pairwise vote is determined only by the basic portion of the profile. Indeed, removing the Condorcet $\mathbf{p_C}$ portion removes all of the flaws and faults of the pairwise vote. Just as this act reverses the outcome of Arrow's assertion, it permits many of the normative and intuitive arguments to regain their merits.

According to Thm. 3, the BC and pairwise outcomes completely agree in the reduced setting of basic profiles. This equivalence condition means that once the glaring flaws of pairwise voting are removed, its virtues apply to the BC. Thus a pragmatic way to correct the pairwise vote – and all reasonable procedures based on the pairwise vote – is to use the BC. This observation, which is a direct consequence of Thm. 3, reverses what has been generally accepted for two centuries.

5. Positional methods and Reversal bias

Similar to how the Thm. 3 profile decomposition significantly simplifies all three-candidate pairwise voting problems, it also assures us that all of the troubling ranking and choice problems caused by positional methods can be completely analyzed with the basic and reversal portions of a profile. Moreover, because the tallies of all positional methods agree on the basic portion of a profile, all conflict in societal rankings and choice must be attributed to the effect of the reversal portion on non-BC positional procedures. Thus, all of these previously perplexing three-candidate problems – problems central to choice theory – admit a surprisingly simple yet complete analysis.

5.1. **Reversal Symmetry.** Recall how the symmetry of "neutrality" requires vote tallies to be interchanged along with the names of the candidates. For instance, if all voters thought Anni was Rose and Rose was Anni, then the final outcome is corrected by assigning the correct name to a tally. Similarly, if all voters confused instructions by marking their ballots opposite to what was intended, then it seems reasonable to correct the final outcome by reversing it. To illustrate, consider the profile

Number	**Ranking**	**Number**	**Ranking**	
5	$A \succ C \succ B$	5	$B \succ C \succ A$	(5.1)
3	$A \succ B \succ C$	3	$C \succ B \succ A$	

and its $A \succ B \succ C$ plurality ranking with plurality tally 8:5:3. When each voter reverses his ranking, it is reasonable to expect the new outcome to be the reversed $C \succ B \succ A$. It is not; it is the same $A \succ B \succ C$ ranking with an identical 8:5:3 tally.

To understand why the plurality vote violates the natural conclusion, notice that in each row of Table 5.1, each voter's preferences are reversed by the other voter. This suggests that their ballots cancel leading to the $A \sim B \sim C$ tied election outcome. While this occurs with the BC and pairwise votes, no other $\mathbf{w}_s^3$ voting procedure honors this natural cancellation and reversal symmetry.

Definition 3. Reversal symmetry *is where when each voter reverses his preference ranking, the ranking of the candidates also is reversed.*

Theorem 6. *The pairwise vote and the BC satisfies reversal symmetry. All positional procedures satisfy reversal symmetry on the Condorcet and basic portions of a profile. The non-BC procedures do not satisfy reversal symmetry on the reversal portions of a profile.*

Proof. This is a simple computation. □

For a reversal profile differential $\mathbf{R_K}$, when each voter reverses his ranking, we obtain the same profile. Indeed, the reversal subspace consists of all profile differentials with this property. So, those procedures failing to give the natural outcome of a complete tie for these profiles are plagued with distorted outcomes. For instance, the two-voter profile $\frac{1}{3}[\mathbf{K^3} - \mathbf{R_A}]$ has one voter with preference $B \succ A \succ C$ and one with $C \succ A \succ B$; the profile involves both preferences where A is middle-ranked. But instead of cancelling opposing voters' votes, the $\mathbf{w_s^3}$ vector tally is $(2s, 1, 1)$ with the $B \sim C \succ A$ ranking for $s < \frac{1}{2}$ and the reversed $A \succ B \sim C$ ranking for $s > \frac{1}{2}$.[7] The important point is that when each voter reverses preference rankings, we obtain the same profile. A similar explanation holds for the profile of Table 5.1 which is $\frac{1}{3}[-5\mathbf{R_C} - 3\mathbf{R_B} + 8\mathbf{K^3}]$) with the $\mathbf{w_s^3}$ tally $5(1, 1, 2s) + 3(1, 2s, 1) = (8, 5 + 6s, 3 + 10s)$. Again, one outcome holds for $s < \frac{1}{2}$, the reversed outcome holds for $s > \frac{1}{2}$, and the BC has a tied vote. The explanation of this behavior is that the profile symmetry ensures that when each voter reverses preferences, the same profile emerges.

5.2. **Different outcomes.** In Sect. 4, I showed how the fact that the pairwise vote devalues information about individually rationality completely explains all pairwise voting difficulties. Similarly, the fact that all non-BC $\mathbf{w_s^3}$ procedures devalue reversal symmetry completely explains *all differences in positional election outcomes.*[8] Activating the reversal profile is the $\mathbf{d^3} = (0, 1, 0)$ component of a voting vector (Thm. 2) because it recognizes only voters' second-ranked candidates.

To illustrate with the two-voter profile $\frac{1}{3}[\mathbf{K^3} - \mathbf{R_A}]$, A is the only second ranked candidate; B and C are symmetrically ranked at the top by one voter and at the bottom by the other. Thus, the $\mathbf{d^3}$ vector tally is $(2, 0, 0)$. According to Thm. 2 and Eq. 2.4, the $\mathbf{w_s^3}$ tally is $(1, 1, 1) + (s - \frac{1}{2})(2, 0, 0) = (2s, 1, 1)$. This tally, then, identifies the value placed upon the second ranked candidate by the different $\mathbf{w_s^3}$ procedures. Only the BC (where $s = \frac{1}{2}$) recognizes and honors the reversal symmetry of this profile.

This phenomenon is illustrated with Eq. 2.5 and the profile of Fig. 1 where the $\mathbf{b^3}$ ranking is $B \succ A \succ C$ with a $(58, 64, 49)$ tally. Theorem 3 ensures this ranking for all procedures when restricted to the basic portion of the profile. Consequently all differences caused by the $(s - \frac{1}{2})(50, 50, 14)$ term of Eq. 2.5 come from the reversal portion of the profile. To show how wildly this reversal term changes the ranking, simple algebra shows that the resulting profile admits *five* different rankings as s varies. Indeed, by solving the obvious inequalities,

[7]This reversal behavior is an example of symmetry breaking from mathematics. When a natural symmetry is broken, such as reversal symmetry, its effects are symmetrically distributed over the breaking procedures. Here, the symmetry is captured by similar $|s - \frac{1}{2}|$ values.

[8]Because of the analytic theme of these two papers, I resisted expanding this observation into a normative argument. But, normative justifications are not needed because just the fact that the reversal portion of a profile totally answers a leading choice theory issue ensures that it is an crucial concept.

we have the following.

s values	Ranking	s values	Ranking
$0 \le s < \frac{1}{12}$	$C \succ B \succ A$	$s = \frac{1}{12}$	$C \sim B \succ A$
$\frac{1}{12} < s < \frac{1}{4}$	$B \succ C \succ A$	$s = \frac{1}{4}$	$B \succ C \sim A$
$\frac{1}{4} < s = 1$	$B \succ A \succ C$		

These conflicting rankings are completely understood with the reversal portion of the profile.

It now is easy to construct examples illustrating all possible differences with positional methods. To illustrate how to do this, I derive necessary and sufficient conditions for a profile to have the BC ranking $B \succ A \succ C$ with plurality winner ($s = 0$) A and anti-plurality winner ($s = 1$) C. The resulting profile, then, allows each candidate to win with an appropriate $\mathbf{w}^3_s$ choice. But, as we now know, this previously perplexing behavior is completely explained by the reversal portion of the profile.

Because the basic profile determines the BC outcome, the first condition requires

$$b_B > a_B > c_B = 0. \tag{5.2}$$

with outcomes

$$(2a_B - b_B, 2b_B - a_B, -(a_B + b_B)) \tag{5.3}$$

The $\mathbf{d}^3$ tally of reversal vector $a_R\mathbf{R_A} + b_R\mathbf{R_B} + c_R\mathbf{R_C}$ is

$$a_R(-4,2,2) + b_R(2,-4,2) + c_R(2,2,-4) \tag{5.4}$$

causing (see Eq. 2.4) the reversal vector tally

$$(2s-1)(-2a_R + b_R + c_R, -2b_R + a_R + c_R, -2c_R + a_R + b_R). \tag{5.5}$$

Adding the reversal and basic vector tallies and using $s = 0$, the respective conditions for the plurality winner A to beat B and C are

$$2a_R - b_R - c_R + 2a_B - b_B > 2b_R - a_R - c_R + 2b_B - a_B$$
$$2a_R - b_R - c_R + 2a_B - b_B > 2c_R - a_R - b_R - b_B - a_B$$

while the $s = 1$ respective conditions for C to beat A and B are

$$-2c_R + a_R + b_R - a_B - b_B > -2a_R + b_R + c_R + 2a_B - b_B$$
$$-2c_R + a_R + b_R - a_B - b_B > -2b_R + a_R + c_R + 2b_B - a_B$$

Solving the inequalities proves the following.

Theorem 7. *Necessary and sufficient conditions for a profile to have a BC ranking $B \succ A \succ C$ while A and C are, respectively, the plurality and anti-plurality winner are that $b_B > a_B > c_B = 0$, $c_R = 0$, $a_R > a_B$, $b_R > b_B$, and $a_R - b_R > b_B - a_B$.*

More generally, choose a BC ranking and a ranking for $\mathbf{w}^3_s \neq \mathbf{b}^3$. There exists a basic and a reversal profile so that the BC and $\mathbf{w}^3_s$ rankings of the combined profile are the chosen ones.

The profile differential defined by $a_B = 1, b_B = 2, a_R = 5, b_R = 3$ is $(0,7,-9,-2,9,-5)$ with an associated profile $(9,16,0,7,18,4)$ where the plurality, BC, and anti-plurality rankings are, respectively, $A \succ B \succ C$, $B \succ A \succ C$, and $C \succ B \succ A$. The reader can show that this reversal portion forces seven different rankings when different $\mathbf{w}^3_s$ methods are used to tally the ballots. In general, the procedure line allows anywhere from one to seven different rankings for a single profile. By mimicking the above analysis, necessary and sufficient conditions for each condition now is easy to derive.

As all differences in positional election rankings result from the reversal portion of a profile, support for a non-BC positional procedure, or for any procedure based on these procedures (such as standard runoffs, the Hare method, Approval Voting, etc.) must justify the bias that is introduced by the procedure ignoring the reversal bias. Similarly, all paradoxes, all flaws of these procedures result from these terms. Namely, by mimicking the earlier illustration of the BC-Condorcet analysis, one can identify all flaws and properties of these procedures.

5.3. **Summary.** Mimicking our comments about pairwise voting, all normative arguments about positional procedures, or procedures based on positional procedures, are compromised by their violation of reversal symmetry. A way to avoid positional voting paradoxes is to remove the reversal portion of the profile. As this requires the procedure's outcome to be determined by the basic portion of the profile, its tally agrees with that of the BC. Again, this means that the BC inherits all of the normative arguments applied to the other procedures. But the BC outcome is not affected by the reversal portion, so a simple, pragmatic way to achieve this state is to use the BC. Stated in another manner, because the real difference between the BC and other positional procedures is that the non-BC approaches violate reversal symmetry, any justification for using these other procedures must address this characteristic.

6. Combined behavior

What makes the profile decomposition a powerful but easily used tool is the assertion that the Condorcet portion does not affect positional rankings while the reversal portion has no effect upon the pairwise rankings. Consequently, we can be separately consider the effects of the $\mathbf{Con^3}$ and reversal portions. This allows us to extend earlier results, say those of Thm. 5, by changing the non-BC positional outcomes to whatever we wish. From this we re-obtain the following known theorem.

Theorem 8. *Choose any ranking of the three candidates and any rankings for the pairs. If $\mathbf{w_s^3} \neq \mathbf{b^3}$, there is a profile where the pairwise and the $\mathbf{w_s^3}$ rankings are as described. The BC is the only procedure where its ranking must be related to the pairwise rankings.*

The original proof of this theorem (Saari [22]) used very different methods. We know from the analysis leading to Thms. 4, 5 that the algebra associated with the $\gamma\mathbf{Con^3}$ term captures all possible relationships between the BC and pairwise outcomes. (This, for instance, prohibits the Condorcet winner from being BC bottom ranked.) On the other hand, we know from Thm. 7 that no such constraint applies to the reversal portion. Thus, the $\mathbf{w_s^3}$ ranking can be anything.

Example 2. To illustrate Thm. 8, recall that Example 1 used $a_B = \frac{4}{3}$, $b_B = \frac{4}{3}$ and $\gamma = -\frac{5}{3}$ to define the profile $(0,2,-3,0,-2,3)$ with the common $\mathbf{w_s^3}$ vector tally $(2,1,-3)$ and the $A \succ B \succ C$ ranking even though B is the Condorcet winner. To further complicate the example, add the reversal tally from Eq. 5.4 to obtain

$$(2+(2s-1)(-2a_R+b_R+c_R), 1+(2s-1)(-2b_R+a_R+c_R), \tag{6.1}$$

$$-3+(2s-1)(-2c_R+a_R+b_R)). \tag{6.2}$$

Clearly, for $s \neq \frac{1}{2}$, reversal coefficients can be chosen to create any desired $\mathbf{w_s^3}$ ranking while leaving the BC and pairwise rankings untouched. For instance, with $s = 0$ (the plurality method), the choices $c_R = 3$, $a_R = b_R = 0$ have the plurality ranking $C \succ A \succ B$, while changing b_R to $b_R = 1$ defines the plurality ranking $C \succ B \succ A$. □

6.1. **Other procedures.** This decomposition can be used to analyze methods involving $\mathbf{w_s^3}$ elections of the three candidates and of pairs. The main point is that when the BC is not used, the procedural outcomes are subject to all difficulties where the $\mathbf{w_s^3}$ outcome is distorted by the $\mathbf{p_R}$ portion of the profile $\mathbf{p} = \mathbf{p_K} + \mathbf{p_B} + \mathbf{p_R} + \mathbf{p_C}$, while the pairwise election outcomes are twisted by $\mathbf{p_C}$. This makes it trivial to explain all single-profile weaknesses of these procedure while constructing illustrating profiles. For instance, with a plurality runoff[9] select an appropriate $\mathbf{p_R}$ term to eliminate the undisputed winner of the basic portion (who is the Condorcet and BC winner) by making her plurality bottom-ranked. Similarly, such examples applied to the Hare method, where two candidates are being elected, demonstrates how the $\mathbf{p_R}$ bias can force the Condorcet winner to lose. Other faults are found with the $\mathbf{p_C}$ variable.

The above examples make it clear how to carry out such an analysis, so I now introduce a new tool to addresses multi-profile issues. I have in mind questions such as manipulation (where the first profile is sincere, the second profile is the strategic action), monotonicity questions (such as when two groups supporting the same candidate join forces, the candidate loses), etc. For other examples, see [20, 10]. Simplifying this analysis is the following assertion.

Theorem 9. *When several profiles are added, then the sum of the basic portions is a basic profile, the sum of Condorcet portions is a Condorcet profile, and the sum of reversal portions is a reversal profile.*

Proof. This follows from the vector space representation of profile differentials. □

To illustrate the power of this assertion, consider two profiles $\mathbf{p}^j = \mathbf{p_K}^j + \mathbf{p_B}^j + \mathbf{p_R}^j$, $j = 1, 2$, where both basic portions support the ranking $A \succ B \succ C$. (As neither profile has a Condorcet portion, this also defines the pairwise rankings.) Using algebra, we can select the two reversal portions so that

1. the $\mathbf{p}^1$ plurality ranking is $B \succ A \succ C$,
2. the $\mathbf{p}^2$ plurality ranking is $C \succ A \succ B$ and
3. the $\mathbf{p}^1 + \mathbf{p}^2$ plurality outcome has A bottom ranked.

(It follows from the algebra that parts one and two are necessary for part three to occur.) In this setting, the Condorcet and basic profile winner A is the runoff winner for both groups. But, when these voters combine to vote as a single group, A is eliminated at the first stage and B is the winner. If you wish C to be the winner, add an appropriate Condorcet portion to the profile.

In this same manner, many other issues can be addressed and illustrating examples constructed. Important to this analysis is that the sum of Condorcet portions can be used to significantly distort the pairwise rankings. Other distortion are introduced by adding reversal portions. It now is easy to understand why so many procedures deny selecting the Condorcet winner, or to have other properties.

The source of the three-candidate paradoxes and conflict in voting and choice procedures, then, is that different procedures use different information from a profile. By being based on different information, we must expect conflicting conclusions. But, the BC is the only procedure which ignores the Condorcet and reversal portions of a profile, so only the BC is immune from these extraneous effects.

6.2. **Probability considerations.** A paradox can be viewed as a profile noise bias that has become so extreme that it is manifested by ordinal rankings. For instance, part of the choice literature worries about the fact that procedures need not elect the Condorcet winner. We now know that this merely manifests a sufficiently large reversal and Condorcet portion of a profile. Prior to this analysis, however, it was not clear what caused the problems. Consequently, for several of these issues, clever and technically difficult methods were introduced to determine the likelihood of the various difficulties. For instance, we may wish to determine how likely

[9] After dropping the plurality loser, the remaining candidates are ranked with a pairwise election

it is for the Condorcet winner to be bottom-ranked in a $\mathbf{w_s^3}$ election. (See Merlin and Tataru [11]. For a small sample of other papers, see [34, 5, 28].)

As demonstrated by Thms. 4, 7, the profile decomposition reduces to algebra the previously difficult task of deriving necessary and sufficient conditions for profiles to exhibit specified behavior. For instance, suppose we are interested in super-majority cycles where a candidate in a pairwise election needs $y > \frac{1}{2}$ of the vote to win. These cycles are in two cones of the representation cube; one is where $x_{A,B}, x_{B,C}, x_{C,A} > 1 - 2y$, and the other is where the terms are negative but the magnitudes satisfy the inequality. It is elementary to find the transitive plane coordinates of all such points. This determines all possible profiles causing such problems. Likelihood estimates now involve elementary calculus.

As this example illustrates, when analyzing paradoxes with the decomposition, the resulting conditions define polygonal regions in the different profile subspaces. Because these subspaces are orthogonal, the likelihood questions reduce to integral calculus problems. While messy, they are solvable with standard techniques or computer programs.

A more interesting question, that previously was impossible to attack, is to understand how likely it is for a procedure to exhibit *any* bias. The profile decomposition allows an easy answer. To explain, normalize profiles so that p_j is the fraction of all voters with the jth preference; $j = 1, 2, \ldots, 6$. The space of normalized profiles is the unit simplex

$$Si(6) = \{\mathbf{x} = (x_1, \ldots, x_6) \mid \sum_{j=1}^{6} = 1,\ x_j \geq 0\}$$

By moving this simplex to the origin (that is, subtract $\frac{1}{6}\mathbf{K^3}$ from each vector in $Si(6)$), we have the five-dimensional normalized space of profiles differentials. (Two dimensions come from the basic space, two from the reversal space, and one from the Condorcet space.) A profile does not exhibit any bias iff it is restricted to the two-dimensional basic subspace; this is highly unlikely. In particular, with any standard measures, this event has a zero likelihood of occurring. For comparison purposes, notice that a completely tied plurality election is the set of profiles where a equal number of voters have A, B, and C top-ranked; it is a three-dimensional subspace of profiles. (Think of this as two equations in five unknowns.) Thus, an even stronger statement comes from comparing these dimensions; it tells us that the likelihood of a profile not exhibiting bias is rarer than a completely tied plurality election.

Determining the bias effect depends upon the procedure. The bias of a $\mathbf{w_s^3} \neq \mathbf{b^3}$ election is affected only by the reversal portion of the profile, so there is zero-likelihood (in the space of normalized profiles) of not exhibiting a bias. But the (dimensional) likelihood of avoiding a paradox is equivalent to having a completely tied election. A pairwise outcome, however, only is influenced by the $\mathbf{p_C}$ portion. So, while over the space of normalized profiles, there is zero probability of avoiding a Condorcet bias, from the dimensional perspective, it is more unlikely to have a completely tied plurality election outcome.

7. Conversion of profiles

It remains to convert profiles between the Eq. 2.1 and profile decomposition descriptions. To change a profile decomposition into the standard representation, use

$$\mathbf{p} = a_B\mathbf{B_A} + b_B\mathbf{B_B} + a_R\mathbf{R_A} + b_R\mathbf{R_B} + \gamma\mathbf{Con^3} + k\mathbf{K^3}. \tag{7.1}$$

From the Eq. 7.1 matrix representation $\mathbf{p} = \mathcal{A}(\mathbf{v})$, we have that matrix $T = \mathcal{A}^{-1}$ converts a standard profile $\mathbf{p}$ into its profile decomposition format.

$$T = \frac{1}{6}\begin{pmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{7.2}$$

The effects of profile $\mathbf{p}$ are determined by $T(\mathbf{p})$.

7.1. **Condorcet's example.** To illustrate with the historically important profiles of Sect. 1.3, Condorcet used $\mathbf{p} = (30, 1, 10, 1, 10, 29)$ to try to discredit the BC. The first two terms of $T(\mathbf{p}) = \frac{1}{6}(68, 76, -28, -20, 19, 81)$ dictates the $B \succ A \succ C$ ranking for the basic portion. The next two terms, which are equivalent to $a_R = 0, b_R = \frac{8}{6}, c_R = \frac{28}{6}$, capture a reversal bias favoring B and C. The change is slight enough to alter only the antiplurality ranking to $A \sim B \succ C$. The important effect of this profile is the $\gamma\mathbf{Con}^3$ coefficient $\gamma = \frac{19}{6}$ and its cyclic distortion which changes the Condorcet winner from B to A. As this cyclic effect reflects the loss of the assumption of individual transitive preferences, rather than supporting the Condorcet winner, this profile identifies a flaw of Condorcet's procedure by demonstrating its susceptibility to the distorting $\mathbf{Con^3}$ portion of a profile.

7.2. **Borda's example.** Another historically important profile is $\mathbf{p} = (0, 5, 0, 3, 4, 0)$ used by Borda in 1770 to show that the pairwise and BC rankings can radically disagree from the plurality ranking. (Some tallies of this profile are computed following its introduction in Sect. 2.1.) His example, which initiated the mathematical investigation of voting procedures, has the profile decomposition $T(\mathbf{p}) = \frac{1}{6}(-5, -4, 9, 6, -4, 12)$.

The converted basic portion $a_B = 0, b_B = \frac{1}{6}, c_B = \frac{5}{6}$ supports the $C \succ B \succ A$ election ranking. What makes his profile effective for his purpose are the $a_R = \frac{9}{6}, b_R = 1$ terms indicating a strong reversal bias favoring A and helping B to create the conflicting $A \succ B \succ C$ plurality ranking. The cyclic coefficient $\gamma = -\frac{4}{6}$, which favors B in the $\{A, B\}$ pairwise election, sharpens its conflict with the plurality ranking.

7.3. **Unanimity profile.** An instructive example is the *unanimity profile* where all voters have the same $A \succ B \succ C$ preference. Intuition suggests that nothing surprising or unusual can occur, but this is not the case. The basic terms of the normalized profile decomposition, $T(\mathbf{p}) = \frac{1}{6}(2, 1, 0, -1, 1, 1)$, do recapture the accurate $A \succ B \succ C$ ranking. Somewhat unexpected are the reversal and Condorcet terms. To explain them, notice that the reversal terms $a_R = c_R = \frac{1}{6}$ captures the conflict between plurality $A \succ B \sim C$ outcome and the unanimity preference. This peculiarity, then, is totally explained by the reversal bias.

While the pairwise outcomes agree with the unanimity ranking, the tallies fail to reflect A's distinct favored status. Compare this with the respective $\{A, B\}, \{B, C\}, \{A, C\}$ basic pairwise outcomes of $(\frac{2}{6}, -\frac{2}{6}), (\frac{2}{6}, -\frac{2}{6}), (\frac{4}{6}, -\frac{4}{6})$ which provide A an healthier spread over C than over B. This diminished respect for A in the profile's standard elections is caused by the Condorcet coefficient $\gamma = \frac{1}{6}$ which introduces enough rotation in C's favor to reduce A's victory margin in their pairwise election. Namely, the cyclic effect even influences unanimity outcomes! Only the BC captures the essence of the unanimity profile.

7.4. **Black's method.** As a final illustration, Black's single-peakedness restriction [2] (which only admits profiles where some candidate never is bottom-ranked) avoids cycles by tempering the magnitude of the $\gamma\mathbf{Con}^3$ coefficient. However, *Black's restriction need not preserve the integrity of the basic portion's pairwise rankings; it still allows bias to creep into the outcome.* For instance, Black's condition is satisfied with the profile where six voters prefer $A \succ B \succ C$, six prefer $C \succ A \succ B$ and one prefers $B \succ A \succ C$. The profile decomposition $a_B = \frac{11}{6}, b_B = 0, c_B = \frac{5}{6}$ requires the $A \succ C \succ B$ basic ranking. The pairwise elections support the conflicting $A \succ B \succ C$ outcome; a discrepancy caused by the $\mathbf{Con}^3$ coefficient of $\gamma = \frac{11}{6}$. (Other examples are easy to create by using the transitive plane coordinates with the representation cube.)

8. Proofs

Proof of Theorem 3. An elementary computation shows that the subspaces are mutually orthogonal. The assertion about $\mathbf{K}^3$ is obvious.

To prove part 3, it suffices from the properties of the procedure line to show that the normalized plurality and BC tallies agree. As they agree on each basis vector, they agree on all vectors in the subspace.

The proof of part 2 for positional methods involves showing that the plurality and $\mathbf{b}^3$ tallies of the basis vectors agree. (For instance, with $\mathbf{B}_A$ they both are $(2, -1, -1)$.) The rest of the conclusion follows from the properties of the procedure line and linear algebra. The pairwise tallies of Eq. 3.5, which is a simple computation, show that the ranking of pairs is identical to the corresponding ranking of a_B, b_B, c_B coefficient values. Indeed, the tallies for any two pairs uniquely determines the tally for the third. (For instance, the pairwise rankings $A \succ B, B \succ C$ hold if and only if $a_B - b_B > 0$, $b_B - c_B > 0$. The sum of these two tallies, $a_B - c_B$, also is positive, so $A \succ C$.) Thus, cycles are impossible. To show that the pairwise rankings always agree with the positional rankings, it suffices to compare one pair. The $\{A, B\}$ pairwise ranking is determined by the ranking of the $\{a_B, b_B\}$ values. The $\{A, B\}$ relative positional ranking is determined by the ranking of $\{2a_B - b_B - c_B, 2b_B - a_B - c_B\}$. As "$c_B$" is common to both sides, the comparison is between $2a_B - b_B$ and $2b_B - a_B$ or between $3a_B$ and $3b_B$. This completes the proof.

The proof of part 4 is a direct computation of the pairwise, BC, and plurality tallies for each basis vector. The assumption about the non-BC positional tallies follows from the non-zero tally of the plurality method and the properties of the procedure line. □

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