Discussion Paper No. 1176R

On Transversals and Systems of Distinct Representatives*

by

L.Hurwicz** and S. Reiter***

January 1997, revised January 2000

Math Center web site:
http://www.kellogg.nwu.edu/research/math

---

* Formerly entitled "On representing Classes of Sets"

** Departments of Economics, University of Minnesota, Minneapolis, MN 55455.

*** Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL 60208. This research was supported by NSF Grant No. IRI-902070.

Abstract

A transversal generated by a system of distinct representatives (SDR) for a collection of sets consists of an element from each set (its representative) such that the representative uniquely identifies the set it belongs to. Theorem 1 gives a necessary and sufficient condition that an arbitrary collection, finite or infinite, of sets, finite or infinite, have an SDR. The proof is direct, short, and does not use transfinite induction. A Corollary to Theorem 1 shows explicitly the application to matching problems.

In the context of designing decentralized economic mechanisms, it turned out to be important to know when one can construct an SDR for a collection of sets that cover the parameter space characterizing a finite number of economic agents. The condition of Theorem 1 is readily verifiable in that economic context.

Theorems 2-5 give different characterizations of situations in which the collection of sets is a partition. This is of interest because partitions have special properties of informational efficiency.

Keywords: Distinct Representatives, Transversals, Partitions and Coverings

AMS Classification Numbers: 05-A05, 05-A18, 90-XX

Introduction

A class of sets is said to be *representable* if there is a function that assigns to each set in the class an element of that set in such a way that no two sets are assigned the same element. Such a representation is called a *system of distinct representatives* (SDR). In this paper we present necessary and sufficient conditions for the existence of an SDR covering cases in which both the class of sets and the sets in the class may be infinite.

Our interest in systems of distinct representatives arises from mechanism design, more specifically, from our construction of an algorithmic procedure for designing decentralized mechanisms to realize a given goal function. Ordinarily to come up with a mechanism that solves a given design problem requires the designer to have some insight into the problem—an idea for a mechanism that will meet the given requirements. And ordinarily the designer must then show that her mechanism does in fact work. An algorithmic procedure for constructing mechanisms relieves the designer of both of these burdens. Following the steps of the algorithm results in a mechanism that is guaranteed to work. The algorithm for constructing mechanisms makes use of systems of distinct representatives. This is discussed in more detail after the concepts and technical machinery have been introduced. The algorithmic construction is presented immediately following Theorem 1. The construction relies on both the result stated in Theorem 1, and part of its proof.

Research on mechanism design divides into two main branches. One branch focuses on incentive effects arising from distributed or asymmetric information, ignoring issues of informational feasibility or efficiency, while the other focuses on the problem of informationally efficient coordination arising from distributed or asymmetric information, while ignoring incentive issues. There are a few papers that address both issues together. (Hurwicz [10a], Reichelstein [15a], Reichelstein and Reiter [15b].) Our algorithm focuses on constructing decentralized mechanisms with desirable informational properties that realize a given goal function, ignoring incentive issues.

Roy Radner's work on mechanism design contains important contributions to this branch of mechanism design theory, notably his work on Team Theory, (Radner [14a], [14b], [14c]; Marschak and Radner [1972] and our joint work with Roy on the B-process. [1]

Denote by C a collection of subsets K of a set W. (Thus, the K's are subsets of W but elements, sometimes called members, of C.) A collection C has an SDR (is representable) if there is a function Λ that assigns to each set K in C an element of K, so that $\Lambda(K) \in K$, and Λ satisfies the condition that if $K' \neq K''$ then $\Lambda(K') \neq \Lambda(K'')$.

Not every class of sets is representable, as the following example shows. Let the underlying set W consist of two elements, a and b. Let the collection C consist of three sets, $K_1 = \{a\}$, $K_2 = \{b\}$, $K_3 = \{a,b\}$. Clearly the collection C is not representable because it is impossible to have three different representatives drawn from a set containing only two elements.

P. Hall [8] gave a necessary and sufficient condition that a class be representable when the class contains a finite number of finite sets. Hall's condition is that each union of n elements of C contain at least n elements of W. In

[1] The B-process [Hurwicz, Radner and Reiter1975] is a decentralized stochastic mechanism that realizes the Walras correspondence in classical and non-classical environments – environments that can include indivisibilities and non-convexities, but not externalities.

the above example, Hall's condition is violated because the union of the three sets has only two elements.

The following example shows that Hall's 'counting' (cardinality) argument fails when the collection C of sets is infinite (even just denumerably infinite).

Let the underlying set W be the set $N = \{1, 2, \ldots, \text{ad inf}\}$ of natural numbers. Let the collection C consist of the singleton sets, $\{n\}_{n=1,2,\ldots}$ together with the set N itself, i.e., $C = \{N, \{1\}, \{2\}, \ldots, \text{ad inf}\}$. Then the cardinality of C and N are the same, but it is clear that there can be no SDR, because each singleton $\{n\}$ must be represented by its sole element n; thus all the elements of N are used up representing the singletons and there is no distinct element of N left to represent N itself. This is a (well-known) counterexample to an analogue, based on cardinality, of Hall's theorem for the case of an infinite collection of sets. Note that not all of the sets in C are finite.

An SDR for a family of sets is closely related to the concept of a transversal. If Λ is an SDR for C, then the set $\Lambda(C)$ is a *transversal* for C. [2]Mirsky [11] comments,

"In the transfinite form of Hall's theorem [referring to Everett and Whaples theorem mentioned below], we operate with families of finite sets. This restriction is extremely irksome as it greatly narrows the field of possible applications of Hall's theorem, but it is not easy to see how it might be relaxed."

---

[2] This concept of a transversal is less restrictive than some used in other parts of mathematics, e.g., the concept of a transversal to the sets making up a (differentiable) foliation.

Mirsky goes on to present a theorem of Rado and Jung, discussed in Rado [15] which allows an infinite number of sets, just one of which is infinite, but requires a condition to exclude the counterexample mentioned above.

M. Hall [8] showed that Hall's condition holds for an infinite collection of finite sets. Everett and Whaples [6] also generalized Hall's theorem to the case when the collection of sets may be infinite, but the member sets (the K's) are all finite. Their approach involves representing the collection C of sets as an indexed family, so that each member of the collection might be counted more than once. The cardinality of the index set is not restricted. Their proof is by transfinite induction, and relies on the finiteness of the sets in C to provide the bound needed for Zorn's lemma. Folkman [7] studied the case of infinite families with finitely many infinite sets, as did Brualdi and Scrimger [3]. The problem can also be formulated as the "marriage problem" in societies consisting of men and women, and studied in the setting of bipartite graphs. Damerell and Milner [4] gave a criterion for deciding whether a countable family of sets has a transversal; an alternative criterion was given by Podewski and Steffens [13] and Nash-Williams [12]. Shelah [17] provided an inductive criterion which together with the other results resolved the issue for the case of countable collections of countable sets. Aharoni, Nash-Williams and Shelah [1], working in the setting of bipartite graphs (matching theory) gave necessary and sufficient conditions that an infinite collection of infinite sets have a transversal.

P. Hall's criterion for the finite case involves a property of subcollections of the given family of sets, namely, that the union of every subcollection have as many elements as there are sets in the subcollection. Hall's proof of sufficiency uses an inductive argument. The generalization presented in [1] follows the pattern of Hall's argument in both respects. Their result is that a society has a solution to the marriage problem if and only if it does not contain any one of a certain set of structures in its subsocieties. Their proof is by transfinite induction.

In Theorem 1 we provide a necessary and sufficient condition that an arbitrary (finite or infinite) collection C of (finite or infinite) subsets of a set W have an SDR. The Corollary to Theorem 1 covers the case of an indexed family of sets, a case that more naturally accommodates matching problems. The condition requires the existence of a correspondence U (whose range is C and whose domain is the union of the members of C, denoted by Θ) that (i) *generates* C, in the sense that for each $\theta \in \Theta$, $U(\theta) \in C$, and for each element K in C there is an element $\theta \in \Theta$ such that $U(\theta) = K$, and (ii) is *self-belonging*, i.e., $\theta \in U(\theta)$. The proof of Theorem 1 is direct, brief and does not involve transfinite induction. It does use the Axiom of Choice.

In the first of the examples above, our condition is violated because there is no self-belonging correspondence defined on the two-element union $\{a,b\}$ of the members of $C = \{K_1, K_2, K_3\}$ that generates all three sets. The second example does not contradict Theorem 1, because the collection $C = \{N, \{1\}, \{2\}, \ldots, \text{ad inf}\}$ cannot be generated by a self-belonging correspondence whose domain is N, (which is the union of the members of C).

Because our criterion is equivalent to the existence of a transversal for a family of sets, and so is the criterion given in [1], the two criteria are logically equivalent. However, they may not be equally useful in application. In our work on the design of decentralized economic mechanisms it turned out to be important to know when it is possible to construct an SDR for a covering C of a given underlying set W. In models of a decentralized economy of N agents, C is a covering of $W = \Theta$, $\Theta = \Theta^1 \times \cdots \times \Theta^N$, and the elements of Θ^i are vectors of parameters characterizing the i-th agent. Hence $W = \Theta$ is called the *parameter space*. The problem of designing decentralized mechanisms involves a given goal function $F : \Theta \to Z$, where Z is the space of outcomes or actions. The problem is to design decentralized mechanisms that for each $\theta \in \Theta$ produce the outcome prescribed by the goal function for that environment θ. We have developed an algorithmic procedure that constructs decentralized mechanisms (with desirable informational properties) for a given goal function. The procedure involves two stages. The first, called the *Rectangles Method* (RM), constructs a covering of the contour sets of the goal function F and therefore also a covering of the space Θ, by product sets, called rectangles.

The second stage, called the *Transversals Method* (TM), involves constructing a transversal for the covering C. In the RM construction the covering C is generated by a correspondence $U : \Theta \to\!\!\to \Theta$. In many cases of interest, the set $W = \Theta$ is infinite and the covering C is an infinite collection of infinite sets.[3]

The case where the covering is a partition is of particular interest is the analysis of information efficiency of mechanisms, and it is helpful to know which correspondences generate partitions. A characterization of partitions in terms of SDR's is provided in Theorem 2. We define a property of correspondences, called *block symmetry*, and show in Theorem 3 that the covering C generated by a self-belonging correspondence U is a partition if and only if U is block symmetric. Block symmetry is a strengthening of the property of symmetry of a relation.

---

[3] A more detailed description of the algorithmic procedure is given in the Appendix.

Symmetry of the generating (self-belonging) correspondence is not sufficient to ensure that the covering generated is a partition.

In Definition 6 we introduce the concept of redundant sets in a covering and the related concept of irreducibility of a covering. These concepts are of interest in connection with informational efficiency. A member set that can be eliminated while the remaining sets still constitute a covering is called redundant. A covering is irreducible if it contains no redundant sets. Clearly a covering that is a partition has no redundant sets and hence is irreducible. Theorem 4 characterizes partitions in terms of symmetry of the generating correspondence and irreducibility of the covering it generates. Theorem 5 summarizes the equivalencies of the combinations of conditions in Theorems 2, 3 and 4.

Returning to the concept of reducibility of coverings, it is clear that every finite covering contains an irreducible subcovering--one with no redundant sets. Dugundji [5, p. 161] has given an example of an infinite covering of the nonnegative real line that does not have an irreducible subcovering. However, that example is a nested family of sets in which each set in the covering is a subset of other sets. A family of sets constructed by the RM procedure cannot have sets that are subsets of others in the family. Therefore Dugundji's example does not settle the question whether an infinite collection of sets constructed by RM has an irreducible subcovering. At this point the question whether infinite RM coverings have irreducible subcoverings is open.

Preliminaries

Remark 1. Let C be a collection of (non-empty) subsets of some set W.

Let $\Theta = \Theta(C) = \bigcup_{K \in C} K$.

Then C is a covering of Θ.

Definition 1. A *system of distinct representatives* (SDR) for a collection C of subsets is a function $\Lambda: C \to \Theta \subseteq W$ such that

$$\forall K \in C \quad \Lambda(K) \in K, \tag{i}$$

$$(K, K' \in C, \quad K \neq K') \Rightarrow \Lambda(K) \neq \Lambda(K') \tag{ii}$$

Definition 2. A collection C of subsets of W is said to be *generated by a correspondence* if and only if there exists a correspondence $U : \Theta \rightarrow\rightarrow \Theta$ such that

(1) for every $K \in C$, there is $\theta \in \Theta$ such that $K = U(\theta)$, and

(2) for every $\theta \in \Theta$, $U(\theta) \in C$.[4]

Definition 3. A correspondence $V : A \rightarrow\rightarrow B$ is called *self-belonging* if and only if $\forall a \in A, \quad a \in V(a)$.

Distinct Representatives for an Arbitrary Collection of Subsets.

Theorem 1. Let C be an arbitrary collection of subsets of a set W. (By Remark 1, C is a covering of Θ.) C has an SDR if and only if C is generated by a self-belonging correspondence $U : \Theta \rightarrow\rightarrow \Theta$.

[4] Hence for every $\theta \in \Theta$, $U(\theta) \neq \varnothing$.

<u>Proof</u> To prove sufficiency, suppose C is generated by a self-belonging correspondence $U : \Theta \to\!\!\!\to \Theta$. Then for each $K \in C$ there exists $\theta_K \in K$ such that $U(\theta_K) = K$. Define $\Lambda : C \to \Theta$ by $\Lambda(K) = \theta_K$. This establishes (i) of Definition 1. To establish (ii), suppose $\Lambda(K) = \theta_K = \theta_{K'} = \Lambda(K')$. It follows from $\theta_K = \theta_{K'}$ that $U(\theta_K) = U(\theta_{K'})$. Thus $K = K'$.

To prove necessity, suppose C has an SDR $\Lambda : C \to \Theta$. Then by (i) of Definition 1, for every $K \in C$, $\Lambda(K) \in K$. We define the generating correspondence $U : \Theta \to\!\!\!\to \Theta$ in two steps. First, for $\theta \in \Lambda(C)$, let $U_1 : \Lambda(C) \to\!\!\!\to \Theta$ by given by $U_1(\theta) = K$ if and only if $\Lambda(K) = \theta$. Second, for $\theta \in \Theta \setminus \Lambda(C)$ define $U_2 : \Theta \setminus \Lambda(C) \to\!\!\!\to \Theta$ as follows. First, for all $\theta \in \Theta$, let $C_\theta = \{K \in C \mid \theta \in K\}$. Note that C_θ is not empty, because C is a covering of Θ. Let $U_2(\theta) = K$ for some arbitrary $K \in C_\theta$. Now, define the correspondence U by

$$U(\theta) = \begin{cases} U_1(\theta) \text{ if } \theta \in \Lambda(C) \\ U_2(\theta) \text{ if } \theta \in \Theta \setminus \Lambda(C) \end{cases}.$$

Thus, U is a self-belonging correspondence that generates C. The Axiom of Choice is used in both parts of this proof.

The question of existence of an SDR for certain coverings arises naturally in designing decentralized procedures--mechanisms-- to meet ('realize') a given optimality criterion.

The mechanisms we consider are those that verify whether a certain action, represented by an element z of the 'outcome space' Z, is optimal. The criterion of optimality is a goal function $F: \Theta \to Z$, where $\Theta = \Theta^1 \times \cdots \times \Theta^N$; agent i is characterized by a parameter point $\theta^i \in \Theta^i$, known only to that agent, where Θ^i is the individual parameter space of agent i. In an economically important class of cases the spaces Θ and Z are Euclidean, and in simpler subcases the goal function F is real-valued. More generally it is a vector-valued correspondence.

The verification procedure is indirect. It involves an auxiliary space M, called the message space.[5] The procedure is decentralized in the sense that each agent's role requires only the knowledge of its own parameter value, and not those of others.

A mechanism consists of three basic elements: a message space M, a binary verification relation ρ, relating Θ and M,[6] and the outcome function

---

[5] In economic market models, prices are elements of such an auxiliary space. It is auxiliary in the sense that it helps to determine the relationship between the outcomes and the agents' characteristics represented by their parameter values.

[6] "m ρ θ" is read as "m is an equilibrium message for θ."

$h: M \to Z$ specifying the outcome (action) z appropriate for a given message m. The verification relation ρ can be represented by a correspondence $\mu : \Theta \twoheadrightarrow M$, such that $m \in \mu(\theta)$ if and only if $m \rho \theta$. We say the mechanism (M, ρ, h) 'realizes' the goal function F if $F = h \circ \mu$ where the correspondence μ represents the relation ρ. We call the mechanism 'decentralized' if there exist N individual correspondences μ^i such that $m \rho \theta$ is equivalent to $m \in \mu^i(\theta^i), i = 1, \cdots, N$ and $\theta = (\theta^1, \cdots, \theta^N)$. When the mechanism is decentralized, the set $U_m = \{\theta \in \Theta : m \in \mu(\theta)\}$ is the Cartesian product of N sets $U_m^i = \{\theta^i \in \Theta^i : m \in \mu^i(\theta^i)\}$, $i = 1, \cdots, N$, where each U_m^i is a subset of Θ^i.[7]

If a mechanism realizes F, the sets U_m must cover the parameter space Θ. Hence to construct a decentralized mechanism that realizes F we must find a covering of Θ through a process such that the equilibrium relation ρ is verified at a given message m by having each agent i separately check that θ^i satisfies the relation $m \rho^i \theta^i$, i.e., that $m \in \mu^i(\theta^i)$. If all these relations are satisfied, the

[7] In particular, when $N=2$ and each U_m^i is an interval on the real axis, the set U_m is a rectangle.

proposed m qualifies as an equilibrium message, and the outcome function prescribes the corresponding optimal action $z = h(m)$.

To carry out this program we proceed in two stages, RM and TM.

Stage 1. The method of rectangles (abbreviated RM).

We construct a covering of the parameter space Θ by what we call the method of rectangles, abbreviated RM; we do this by associating with each $\theta \in \Theta$ a 'rectangular'[8] contour-contained (abbreviated F-c-c)[9] subset $V(\theta) \subseteq \Theta$ containing the point θ. This construction produces a covering of Θ that is generated by a self-belonging correspondence.[10]

Stage 2. The method of transversals (abbreviated TM).

It follows from Theorem 1 that the covering $C = \{K \subseteq \Theta : K = V(\theta), \theta \in \Theta\}$ has an SDR, i.e., a function $\Lambda: C \rightarrow \Theta$ with the properties specified by Definition 1. However for purposes of mechanism construction we need a 'special' SDR, say Λ^*, satisfying (in addition to the properties required by Definition 1) the condition

(*) $V(\Lambda^*(K)) = K.$

---

[8] I.e., there exist N correspondences V^i on Θ^i, such that for each $\theta \in \Theta$, we have $V(\theta) = V^1(\theta^1) \times \cdots \times V^N(\theta^N)$.

[9] I.e., for each $\theta \in \Theta$, $V(\theta)$ is a subset of the contour set $F^{-1}(F(\theta))$.

The first part of the proof of Theorem 1 shows that such special SDR's exist if the hypothesis of Theorem 1 is satisfied, i.e., if the covering C is generated by a self-belonging correspondence. If the covering is a partition, then (*) is automatically satisfied by any SDR Λ. In general, Λ^* is not unique.

Let $T^* = \Lambda^*(C)$ be a ('special') transversal corresponding to the ('special') SDR Λ^*. This transversal can be used to construct a mechanism $\pi = (M, \mu, h)$ that realizes the goal function F. This is done by the following steps.

(i) We first use the transversal as the message space of the mechanism, i.e., we set $M = T^*$.[11]

(ii) We define the equilibrium correspondence μ from Θ to M by the equivalence

$$m \in \mu(\theta) \text{ if and only if } m = \Lambda^*(V(\theta)) \text{ for some } \Lambda^* \text{ satisfying (*)},$$

where V is the self-belonging F-c-c correspondence on Θ that generates C.

(iii) We define the outcome function $h: M \to Z$ by the relation

[10] It also has the properties (rectangularity and contour-containment) needed to make the resulting mechanism decentralized and one that realizes F.

[11] Typically the message space M provided by the transversal T^* is 'smaller' than the parameter space Θ; e.g., T^* has smaller dimension than Θ in cases where they each have dimension. Hence the use of such a message space increases the informational efficiency of the mechanism.

$$h(m) = F(m) \text{ for all } m \in M.$$

The mechanism $\pi = (M, \mu, h)$ so constructed can be shown to realize the given goal function F. I.e., given any $\theta \in \Theta$, there exists a message m such that $m \in \mu(\theta)$, and for any $m \in M, z \in Z$, and $\theta \in \Theta$,

if $m \in \mu(\theta)$ and $z = h(m)$ then $z = F(\theta)$.

Notice that this construction does not depend on how the covering and the transversal are obtained. If the covering has the F-c-c property, and a transversal, then the mechanism constructed from that transversal realizes the goal function F.

This mechanism is decentralized when the correspondence V is rectangular, i.e., when $V(\theta)$ is the Cartesian product of subsets $V^i(\theta)$ of the individual parameter spaces $\Theta^i, i = 1, \cdots, N$, which is the case in the RM construction. In the case of a covering that is not obtained by RM, the mechanism is decentralized when the sets of the covering are rectangular.

Families of Sets

It is not clear that Theorem 1 applies to matching problems.

A formulation in which matching problems can naturally be stated involves a generalization of the framework used in Theorem 1. We present the definitions and a Corollary to Theorem 1 that applies to matching problems.

The marriage problem is a well-known example of a matching problem. In one version the marriage problem consists of a set of men, a set of women together with a specification of the women who are possible marriage partners for a given man, the requirement that each man marry a woman who is a possible marriage partner, and that no person be married to more than one partner. It is also required that all men be married, but not that all women be married. The problem is: Does there exist an assignment of women to men that meets these requirements in which all men are married? The Corollary to Theorem 1 gives necessary and sufficient conditions for the existence of such a solution. The proof of the Corollary "constructs,"--using the Axiom of Choice-- a solution.

Definition 4. A <u>family of sets</u> is a set I (the index set) and a correspondence $i \to K_i$, where for each $i \in I$, where for each $i \in I, K_i$ is a subset of an underlying set W. We write $\mathcal{K} = \left\{ K_i \mid i \in I \right\}$. In the case where I is the set of natural numbers K is a sequence of subsets of W.

Definition 1.1. An SDR for a family of sets $\mathcal{K}$ is a (single valued) function $\Lambda : \mathcal{K} \to \bigcup_{i \in I} K_i$ such that

i) $\forall i \in I \ \Lambda\left(K_i\right) \in K_i$,

ii) $i, j \in I, i \neq j$, implies $\Lambda\left(K_i\right) \neq \Lambda\left(K_j\right)$.

Definition 2.1.

a) A family of sets $\mathcal{K} = \{K_i \mid i \in I\}$ <u>is generated by a correspondence</u> if and only if there is a correspondence $\mathcal{U}: \bigcup_{i \in I} K_i \longrightarrow\!\!\!\rightarrow I \times W$ such that

i) for each $w \in \bigcup_{i \in I} K_i$, $\mathcal{U}(w) \in \mathcal{K}$,

ii) for each $K_i \in \mathcal{K}$ there exists $w_i \in \bigcup_{i \in I} K_i$ such that $K_i = \mathcal{U}(w_i)$.

b) A correspondence $\mathsf{U}: \bigcup_{i \in I} K_i \longrightarrow\!\!\!\rightarrow I \times W$ is <u>self-belonging</u> if and only if for each $w \in \bigcup_{i \in I} K_i$, $w \in \mathcal{U}(w)$.

Stating the marriage problem formally, let I be the set of men; let W be the set of women. For each $i \in I$ let K_i be the set of women who are possible marriage partners for Mr. i. The family of sets $\mathcal{K} = \{K_i \mid i \in I\}$ is thereby defined. A solution of the marriage problem is given by an SDR, Λ, for K. Thus, the marriage problem has a solution if and only if the family K has a transversal, $\Lambda(\mathcal{K})$; if

$w \in \Lambda(K_i)$, then woman w will be married to man i.

Corollary to Theorem 1.

Let $\mathcal{K} = \{K_i \mid i \in I\}$ be a family of subsets of W. K has an SDR , Λ, if

and only if $\mathcal{K}$ is generated by a self-belonging correspondence

$\mathcal{U}: \bigcup_{i \in I} K_i \rightarrowtail\rightarrow I \times W$.

The proof of the Corollary parallels the proof of Theorem 1.

Proof: $(\Leftarrow)$ Suppose K is generated by a self-belonging correspondence U. For

each $i \in I$ and $K_i \in \mathcal{K}$, there exists $w_{K_i} \in \bigcup_{i \in I} K_i$ such that

$\mathcal{U}(w_{K_i}) = K_i$. Because U is self-belonging $w_{K_i} \in K_i$. Now, for

$K_i \in \bigcup_{j \in I} K_j$, define $\Lambda(K_i) = w_{K_i}$. Therefore $\Lambda: \bigcup_{i \in I} K_i \rightarrow \mathrm{I} \times \mathrm{W}$. This

establishes $i)$ of Definition 1.1. Next we establish $ii)$ of Definition 1.1.

Suppose $\Lambda(K_i) = w_{K_i} = w_{K_j'} = \Lambda(K_j')$. Then, $\mathcal{U}(w_{K_i}) = K_i$

and $\mathcal{U}(w_{K_j}) = K_j'$. It follows that $K_i = K_j'$.

$(\Longrightarrow)$ We turn now to the converse. Suppose $\mathcal{K}$ has an SDR, Λ. Then for every $K_i \in \mathcal{K}$, $\Lambda(K_i) \in K_i$. Define $\mathcal{U}: \bigcup_{i \in I} K_i \rightarrow\!\!\rightarrow \mathcal{K}$, by

$$\mathcal{U}_1 : \Lambda(\mathcal{K}) \rightarrow\!\!\rightarrow \mathcal{K}$$
$$\mathcal{U}_2 : \bigcup_{i \in I} K_i \setminus \Lambda(\mathcal{K}) \rightarrow\!\!\rightarrow \mathcal{K}$$

where, $\mathcal{U}_1$ is given by $\mathcal{U}_1(w_i) = K_i$ if and only if $\Lambda(K_i) = w_i$. To define $\mathcal{U}_2$, let $w \in \bigcup_{i \in I} K_i \setminus \Lambda(\mathcal{K})$. Let $\mathcal{K}_w = \{K_i \in \mathcal{K} \mid w \in K_i; i \in I\}$. Then, Let $U_2(w) = K_j$ for some arbitrary j such that $K_j \in \mathcal{K}_w$. Define $\mathcal{U}$ by

$$\mathcal{U}(w) = \begin{cases} \mathcal{U}_1(w) \, if \; w \in \Lambda(\mathcal{K}) \\ \mathcal{U}_2(w) \, if \; w \in \bigcup_{i \in I} K_i \setminus \Lambda(\mathcal{K}) \end{cases}.$$

The correspondence $\mathcal{U}(\bullet)$ is self-belonging and generates $\mathcal{K}$. (End of proof.)

Characterizations of Partitions

If a collection of sets C is a covering of Θ, and is a partition, is it generated by a correspondence, $U : \Theta \rightarrow\!\!\rightarrow \Theta$? Clearly, yes, but what conditions must U satisfy if C is a partition? It is obvious that if C is a partition then it has an SDR. Is there any special property that an SDR for a partition has ? Theorems 2, 3 and 4 provide answers to these questions. For the record, Definition 4. A collection C of sets (equivalently a covering C of Θ) is a *partition*, if and only if, for $K, K' \in C$, either $K \cap K' = \varnothing$, or, $K = K'$.

The following characterization of partitions in terms of SDR is straightforward to prove.

Theorem 2. A covering C of Θ is a partition if and only if every function $\Lambda : C \to \Theta$ that satisfies

(A) $\forall K \in C, \ \Lambda(K) \in K,$

is an SDR for C.

Proof. (Necessity) Suppose C is a partition, and suppose $\Lambda : C \to \Theta$ satisfies (A). We show that $K, K' \in C$, $K \neq K'$ implies $\Lambda(K) \neq \Lambda(K')$. Suppose $K, K' \in C$, and $K \neq K'$. Since C is a partition, $K \cap K' = \varnothing$. Since Λ satisfies (A), $\Lambda(K) \in K$, and $\Lambda(K') \in K'$. Hence, $\Lambda(K) \neq \Lambda(K')$. Thus, $\Lambda(\bullet)$ is an SDR for C.

(Sufficiency) Suppose that every function $\Lambda : C \to \Theta$ that satisfies (A) is an SDR for C. Because the sets $K \in C$ are not empty there are many such functions. We choose one such function, $\overline{\Lambda}$, if necessary, using the Axiom of Choice. Thus, $\overline{\Lambda}$ is an SDR for C.

If C is a partition, there is nothing to prove. So suppose C is not a partition. Then there exist two sets K' and K'' in C such that

$$K' \cap K'' \neq \varnothing, \qquad (a)$$

$$K' \neq K''. \qquad (b)$$

By (a), there is a point $\theta' \in \Theta$, such that $\theta' \in K'$ and $\theta' \in K''$. Now define the function $\Lambda' : C \to \Theta$ by

$$\Lambda'(K) = \overline{\Lambda}(K), \text{ for all } K \in C \setminus \{K', K''\},$$

and

$$\Lambda'(K') = \Lambda'(K'') = \theta'. \qquad (*)$$

Then, for all $K \in C$, $\Lambda'(K) \in K$. Therefore (1) is satisfied. But, by (b), $K' \neq K''$, and by $(*)$, $\Lambda'(K') = \Lambda'(K'')$. Therefore, Λ' is not an SDR for C. This completes the proof.

We next give a characterization of partitions in terms of the generating correspondence. First, we define a property that we show is a property of correspondences that generate partitions.

Definition 5 Let $\overline{\theta}, \theta', \theta''$ denote points of Θ. A correspondence $U : \Theta \to\to \Theta$, is *block symmetric* if and only if

$$\left[\theta' \in U(\bar{\theta}) \text{ and } \theta'' \in U(\bar{\theta})\right] \Rightarrow \left[\theta' \in U(\theta'') \text{ and } \theta'' \in U(\theta')\right]. \quad \text{(B)}$$

We show below that block symmetry is a strengthening of the concept of symmetry of relations. The term 'block symmetric' is used because, when (B) is satisfied, there is a permutation of the elements of Θ such that the graph of U consists of blocks, ('squares') with the 'northeast' and 'southwest' vertices on the diagonal of $\Theta \times \Theta$.

<u>Theorem 3</u>: A covering C of Θ is a partition if and only if C is generated by a block symmetric, self-belonging correspondence $U : \Theta \rightarrow\!\!\rightarrow \Theta$.

Proof (Necessity): Suppose C is a partition of Θ. then C has an SDR. To see this, define $\Lambda(K)$ to be any element in K. Because C is a partition, $K \neq K'$ implies $\Lambda(K) \neq \Lambda(K')$.

Because C has an SDR, it follows from Theorem 1, that C is generated by a self-belonging correspondence $U : \Theta \rightarrow\!\!\rightarrow \Theta$. It remains to show that U is block symmetric.

Let $\bar{\theta}, \theta', \theta''$ be elements of Θ satisfying the hypothesis of (B), i.e., let

$$\theta' \in U(\bar{\theta}) \text{ and } \theta'' \in U(\bar{\theta}). \quad \text{(i)}$$

To prove (B) we show that

$$\theta' \in U(\theta''), \text{ and } \theta'' \in U(\theta'). \tag{ii}$$

The relations (i) and the self-belonging property of U yield

$$\theta' \in U(\bar{\theta}) \cap U(\theta') \tag{iii.a}$$

and

$$\theta'' \in U(\bar{\theta}) \cap U(\theta''). \tag{iii.b}$$

Since C is generated by U, there exist $\bar{K}, K', K'' \in C$ such that

$$\bar{K} = U(\bar{\theta}),\ K' = U(\theta'),\ K'' = U(\theta''). \tag{iv}$$

Since C is a partition,

$$\bar{K} \cap K' = \varnothing, \text{ or } \bar{K} = K'$$

and (v)

$$\bar{K} \cap K'' = \varnothing \text{ or } \bar{K} = K''.$$

But the relations (iii.a) and (iii.b) rule out the emptiness of the intersections $\bar{K} \cap K'$ and $\bar{K} \cap K''$. Hence $\bar{K} = K'$ and $\bar{K} = K''$. By (iv), this yields

$$U(\bar{\theta}) = U(\theta')$$

and

$$U(\bar{\theta}) = U(\theta'').$$

Using each of these relations in (i) yields the corresponding relation in (ii), and hence U is block symmetric.

This concludes the proof of necessity.

(Sufficiency) Suppose that C is generated by a block symmetric, self-belonging correspondence $U : \Theta \to\!\to \Theta$. We show that C is a partition, i.e., that for every $K, K' \in C$ either $K \cap K' = \varnothing$, or $K = K'$.

Let K and K' be elements of C. If $K \cap K' = \varnothing$ there is nothing to prove. So, suppose there is $\tilde{\theta} \in K \cap K'$. Then, since C is generated by U, there are elements, $\hat{\theta}$ and $\hat{\hat{\theta}}$ in Θ such that $K = U(\hat{\theta})$, and $K' = U(\hat{\hat{\theta}})$. Thus,

$$\tilde{\theta} \in U(\hat{\theta}) \text{ and } \tilde{\theta} \in U(\hat{\hat{\theta}}).$$

It follows from (B) and $\tilde{\theta} \in U(\bar{\theta})$ that

$$\forall \theta \in U(\hat{\theta}),\ \theta \in U(\tilde{\theta}).$$

Therefore,

$$U(\hat{\theta}) \subseteq U(\tilde{\theta}).$$

Now, since $\hat{\theta} \in U(\tilde{\theta})$, because $\hat{\theta} \in U(\hat{\theta})$, by self-belonging, and $U(\hat{\theta}) \subseteq U(\tilde{\theta})$ as just shown, it follows from (B) of Definition 5 (with $\hat{\theta}$ here corresponding to θ' in (B), θ to θ'', and $\tilde{\theta}$ to $\overline{\theta}$ in (B)) that

$$\forall \theta \in U(\tilde{\theta}),\ \theta \in U(\hat{\theta}).$$

Thus,

$$U(\hat{\theta}) \supseteq U(\tilde{\theta}).$$

Therefore

$$U(\hat{\theta}) = U(\tilde{\theta}).$$

The same argument applied to $U(\hat{\hat{\theta}})$ and $U(\tilde{\theta})$ shows that

$$U(\hat{\hat{\theta}}) = U(\tilde{\theta}).$$

Therefore,

$$K = U(\hat{\theta}) = U(\hat{\hat{\theta}}) = K'.$$

This concludes the proof.

Remark 2   Block symmetry of a correspondence is a strengthening of the usual notion of symmetry of a relation applied to the graph of the correspondence. Symmetry may be defined by the condition

$$\theta' \in U(\ddot{\theta}) \Rightarrow \bar{\theta} \in U(\theta'). \qquad \text{(S)}$$

To see that (B) implies (S), suppose U is block symmetric. Suppose $\theta' \in U(\bar{\theta})$. We show that $\bar{\theta} \in U(\theta')$. The hypotheses of (B) in the definition of block symmetry are satisfied for $\theta', \theta'', \bar{\theta}$, where $\theta'' = \bar{\theta}$. Therefore, $\theta'' = \bar{\theta} \in U(\theta')$.

The following example shows that symmetry of the generating (self-belonging) correspondence is not sufficient for the covering it generates to be a partition.

Example 3. Let $\Theta = \{a, b, c\}$, and let $U(a) = \{a, b, c\}$, $U(b) = \{a, b\}$, $U(c) = \{a, c\}$. Then U is self-belonging and symmetric, but the covering it generates is not a partition. However, in this example the covering is reducible in the sense of the following definition.

Definition 6. An element of a covering C of Θ is *redundant* if eliminating that element from C still leaves a covering of Θ. A covering is *irreducible* [12] if it has no redundant elements; otherwise it is *reducible*.

If C is a finite covering, then it has an irreducible subcovering, which might be C itself. If C is not irreducible then it has a redundant element. When C is finite, successive elimination of redundant elements must eventually result in an irreducible subcovering. This is not true when C is infinite, as is shown by Dugundji's example [4. p161].

The covering C in Example 3 can be reduced in two different ways. First to the covering $C' = \{\{a,b,c\}\}$, which is generated by the (constant) correspondence $U'(\theta) = \{a,b,c\}$, for $\theta \in \{a,b,c\}$, and, second, to the covering $C'' = \{\{a,b\},\{b,c\}\}$, which is generated by the correspondence $U''(a) = U''(b) = \{a,b\}$, and $U''(c) = \{b,c\}$. Both C' and C'' are irreducible, and U' is symmetric, while U'' is not. Of course, C' is a partition and C'' is not.

While symmetry is not enough to guarantee that the covering generated by a self-belonging correspondence be a partition, it is the case that if the covering is irreducible, then symmetry ensures that it is a partition. The converse also holds.

<u>Theorem 4</u> Let C be a covering of Θ. C is a partition if and only if (i) C is generated by a self-belonging, symmetric correspondence $U : \Theta \rightarrow\!\!\rightarrow \Theta$, and (ii) C is irreducible.

---

[12] The term "irreducible" applied to a covering was introduced by Dugundji [5] p. 160.

Proof $(\Leftarrow)$ Suppose (i) and (ii) hold. We show that U is block symmetric, and hence, by Theorem 3, that C is a partition. To show that U is block symmetric we must show that U satisfies

$$\left[\theta' \in U(\bar{\theta}) \text{ and } \theta'' \in U(\bar{\theta})\right] \Rightarrow \left[\theta' \in U(\theta'') \text{ and } \theta'' \in U(\theta')\right]. \qquad \text{(B)}$$

So, suppose that $\bar{\theta}$ is an arbitrary point of Θ, and consider $U(\bar{\theta})$. Let

$$E(\bar{\theta}) = \left\{\theta \in U(\bar{\theta}) \colon U(\theta) = U(\bar{\theta})\right\}$$

and

$$D(\bar{\theta}) = \left\{\theta \in U(\bar{\theta}) \colon U(\theta) \neq U(\bar{\theta})\right\}.$$

Note that $U(\bar{\theta}) = E(\bar{\theta}) \cup D(\bar{\theta})$, and $E(\bar{\theta}) \cap D(\bar{\theta}) = \varnothing$.

Either $D = \varnothing$, or $D \neq \varnothing$.

Suppose $D \neq \varnothing$. We shall show that $D \neq \varnothing$ leads to the conclusion that the set $K = U(\bar{\theta})$ is redundant, thereby contradicting (ii).

Consider $\bar{\bar{\theta}} \in U(\bar{\theta})$. Either $\bar{\bar{\theta}} \in E(\bar{\theta})$, or $\bar{\bar{\theta}} \in D(\bar{\theta})$.

If $\bar{\bar{\theta}} \in D(\bar{\theta})$, then $\bar{\bar{\theta}} \subset \bigcup_{\theta \in D(\bar{\theta})} U(\theta)$.

Now suppose that $\overline{\overline{\theta}} \in E(\overline{\theta})$. Then $U(\overline{\overline{\theta}}) = U(\overline{\theta})$. Let $\hat{\theta} \in U(\overline{\overline{\theta}})$. If for all $\overline{\overline{\theta}} \in E(\overline{\theta})$, and all $\hat{\theta} \in U(\overline{\overline{\theta}})$ $\hat{\theta} \in E(\overline{\theta})$, then $D(\overline{\theta}) = \varnothing$. So we may suppose that $\hat{\theta} \in D(\overline{\theta})$. Since, by symmetry, $\overline{\overline{\theta}} \in U(\hat{\theta})$, it follows that $\overline{\overline{\theta}} \in \bigcup_{\theta \in D(\overline{\theta})} U(\theta)$.

Since $\overline{\overline{\theta}}$ is an arbitrary point of $U(\overline{\theta})$, we have shown that $U(\overline{\theta}) \subseteq \bigcup_{\theta \in D(\overline{\theta})} U(\theta)$.

In order to conclude that $K = U(\overline{\theta})$ is redundant, we must show that not every set $U(\theta)$, for $\theta \in D(\overline{\theta})$, is equal to $U(\overline{\theta})$. But this follows immediately from the definition of $D(\overline{\theta})$.

Thus, we have shown that if $D \neq \varnothing$, then $U(\overline{\theta})$ is redundant, contradicting (ii). Therefore, we may conclude that $D = \varnothing$.

It then follows from the hypotheses of (B), i.e., that $\theta' \in U(\overline{\theta})$ and $\theta'' \in U(\overline{\theta})$, that $\theta' \in U(\theta'')$ and $\theta'' \in U(\theta')$, since it follows from $D = \varnothing$, that for every $\theta \in U(\overline{\theta})$, $U(\theta) = U(\overline{\theta})$, and hence $U(\theta') = U(\overline{\theta}) = U(\theta'')$. Thus, (B) is satisfied. Hence U is block symmetric, and by Theorem 3, C is a partition.

($\Rightarrow$). Suppose C is a partition. A covering C is a partition if and only if it is generated by a block symmetric, self-belonging correspondence $U : \Theta \to\!\!\to \Theta$. Since block symmetry implies symmetry, U is symmetric. Finally, if C is a partition, then it is irreducible. This establishes (i) and (ii).

Theorems 2, 3 and 4 may be summarized in Theorem 5.

Theorem 5. The following four propositions are equivalent:

1) A covering C is a partition;

2) Every function $\Lambda : C \rightarrow \Theta$ that satisfies condition (A) is an SDR for C;

3) C is generated by a block symmetric, self-belonging correspondence $U : \Theta \rightarrow\rightarrow \Theta$;

4) C is an irreducible covering generated by a symmetric, self-belonging correspondence $U : \Theta \rightarrow\rightarrow \Theta$.

References

R. AHARONI, C. ST. J. A. NASH-WILLIAMS, AND S. SHELAH, (1983), *A general criterion for the existence of transversals*, Proc. London Math. Soc., 47, pp. 43-68.

C. BERGE, (1963), *Topological Spaces*, Macmillan, New York.

R. A. BRUALDI AND E. B. SCRIMGEN, (1968), *Exchange systems, matchings and transversals*, J. Combin. Theory, 5, pp. 242-257.

R.M. DAMERELL, AND E.C. MILNER, (1974), *Necessary and sufficient conditions for transversals of countable set systems*, J. Combin. Theory Ser. A, 17, pp. 350-379.

J. DUGUNDJI, (1966), *Topology*, Allyn and Bacon, Inc., Boston.

C. J. EVERETT AND G. WHAPLES, (1949), *Representations of sequences of sets*, Amer. Journ. of Math, 71, pp. 287-293.

K. J. FOLKMAN, (1968), *Transversals of infinite families with finitely many infinite members*, RAND Corp. memorandum, RM - 5676 - PR.

M. HALL, JR., (1948), *Distinct representatives of subsets*, Bull. Amer. Math. Soc., 54, pp. 922-926.

P. HALL, (1935), *On representatives of subsets*, J. London Math. Soc., 10, pp. 26-30.

L. HURWICZ, (1976), *Mathematical Models in Economics*, Papers and Proceedings of a U.S.—U.S.S.R. Seminar, Moscow.

L. HURWICZ, R. RADNER AND S. REITER, (1975), *A Stochastic Decentralized Allocation Process: Part I*, Econometrica," Vol. 43, No. 2.

L. HURWICZ, R. RADNER AND S. REITER, (1975), *A Stochastic Decentralized Resource Allocation Process: Part II*, Econometrica, Vol. 43, No. 3.

L. HURWICZ AND S. REITER, (1990), *Constructing decentralized mechanisms by the method of rectangles*, Decentralization Conference, Northwestern University.

L. HURWICZ AND S. REITER, (1993), *Designing mechanisms by the 'method of rectangles,'* Decentralization Conference, University of California, Berkeley.

L. MIRSKY, (1971), *Transversal Theory*, Vol. 75 in Science and Engineering, Academic Press, New York and London.

C. ST J.A. NASH-WILLIAMS, (1978), *Another criterion for marriage in denumerable societies,* Ann. Discrete Math., 3, pp. 165-179.

K.P. PODEWSKI AND K. STEFFENS, (1976), *Injective choice functions for countable families,* J. Combin. Theory Ser. B, 21, pp. 40-46.

R. RADNER, (1972), *Normative theories of organization: an introduction,* C.C.McGuire and R.Radner, Decision and Organization, North Holland/American Elsevier. 177-188.

R. RADNER, (1972), *Teams,* C.C.McGuire and R.Radner, Decision and Organization, North Holland/American Elsevier, 189-216.

R. RADNER, (1972), *Allocation of a scarce resource under uncertainty: an example of a team,* C.C.McGuire and R.Radner, Decision and Organization, North Holland/American Elsevier, 217-236.

S. REICHELSTEIN, (1984), *Incentive Compatibility and Informational Requirements,* Journal of Economic Theory, 32, 384-390.

S. REICHELSTEIN AND S. REITER, (1988), *Game Forms with Minimal Message Spaces,* Econometrica, 56, 3, 661—692.

R. RADO, (1967), *Note on the transfinite case of Hall's theorem on representatives,* J. London Math. Soc., 42, pp. 321-324.

S. SHELAH, Notes on partition calculus, (1973), *Infinite and finite sets* (ed. A. HAJNAL, R. RADO, AND V.T. SOS, Colloq. Math. Soc. Janos Bolyai), pp. 1257-1276.