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ASYMPTOTIC EFFICIENCY FOR DISCRIMINATORY PRIVATE VALUE AUCTIONS

by

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ABSTRACT

We consider discriminatory auctions for multiple identical units of a good. Players have private values, possibly for multiple units. None of the usual assumptions about symmetry of players' distributions over values or of their equilibrium play are made. Because of this, equilibria will typically involve inefficiency: objects may not end up in the hands of those who value them most. We show that, none the less, such auctions become arbitrarily close to efficient as the number of players, and possibly the number of objects, grows large.

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JEL Classification Codes: C72, D44, D82

1. Introduction

Consider the allocation of a number of identical units of some good by a generalized first price or *discriminatory* auction. This is one in which each player submits one or more bids simultaneously, the k highest bids win an object, and winning bidders pay the amount of their winning bid(s).If values are private, if each potential purchaser desires at most one object, if valuations are drawn from distributions that are symmetric across players, and if equilibrium maps from values to bids are also symmetric across players, then the auction will be efficient in the sense that whatever the actual realization of values, in equilibrium objects go to the players who desire them most (see Milgrom and Weber (1982) and Weber (1983)).

The symmetry assumptions are entirely crucial to this result. To begin, consider an auction for a single unit with two potential buyers, players 1 and 2. Valuations are drawn from distributions that are continuous but asymmetric across players. In this setting, it simply cannot be an equilibrium for players 1 and 2 to use the same strictly increasing map $b(.)$ from values to bids, and so equilibrium must involve a positive probability of one player winning when the other has the higher valuation.[1] Several examples of auctions with these characteristics are provided by Marshall et al (1994). Existence results for such auctions are provided by Maskin and Riley (1995). A characterization of equilibrium for single unit first price auctions with asymmetric valuations is given by Lebrun (1994).

With multiple unit demands, problems become even more severe. Even if the distribution from which player 1's set of values is drawn is the same as that for player 2, the presence of multiple unit demands introduces a form of endogenous asymmetry: consider an auction in which 2 objects are available, and each of two players has value for 2 objects. Then, even if valuations are determined symmetrically, the optimization problems faced by player 1 in determining his highest and second highest bids are inherently different, since player 1's highest bid wins anytime that it is greater than player 2's second highest bid, while player 1's second highest bid wins only when it is greater than player 2's highest bid. This idea is explored by Katzman (1995) and Ausubel and Cramtom (1995).

So, once one drops the standard symmetry and single unit demand assumptions, inefficiency becomes essentially unavoidable in discriminatory auctions. In this paper, we examine the extent to which this inefficiency disappears as the auction becomes "large." We show that as the number of players grows large, discriminatory auctions become asymptotically efficient. This holds regardless of whether or not supply also grows large. And since in the limit, objects are allocated efficiently, an asymptotic version of the revenue equivalence result holds: revenue per unit is in expectation of the value of the object to the highest unfilled unit of demand.

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[1] This is fairly immediate from consideration of the first order conditions. See Maskin and Riley (1995) for a proof.

Finally, despite the complexities of equilibria in finite asymmetric auctions, limiting behavior in this setting takes a very simple form: bidders are essentially price takers at a price that converges to the competitive equilibrium price in a market with supply curve vertical at the number of objects available and demand curve given by the *expected* number of objects bidders would want at any given price in a price taking environment.

For the present paper, we look at settings in which, as the number of players and perhaps objects grows, there is in the limit no uncertainty about aggregate demand and supply. Swinkels (1996a) considers discriminatory auctions where uncertainty about demand or supply persists in the limit. Swinkels (1996b) considers asymptotic efficiency for uniform price auctions.[2]

It should be made clear what we do not do. We do not solve for optimal mechanisms, either from a social surplus or seller revenue point of view (on the later problem, see, for example, Maskin and Riley (1989)). Rather we consider the efficiency properties of a mechanism that we actually see. Nor do we provide much insight into the form of equilibria in auctions that are not large or into the conditions under which equilibria exist for these auctions.

We begin in Section 2 by laying out the auction framework. Section 3 discusses the precise meaning of asymptotic efficiency. "Asymptotic efficiency" will mean that as the auction becomes large, almost all feasible surplus will actually be realized. The key question is whether the computation should be made in an ex-ante or ex-post way. We show by example that neither version implies the other, and so derive results for both versions. Section 4 states and proves the main result. Section 5 concludes with a discussion of the issue of rates of convergence.

2. THE AUCTION FRAMEWORK

We begin by laying out the auction framework. Because we are interested in limiting results, we consider a sequence $\{A^n\}_{n=1}^{\infty}$ of auctions. The n^{th} auction, A^n, is defined by:

Supply: There are k^n identical indivisible objects available for sale.

Players: There are n potential buyers, labeled $1,\ldots,n$. The seller is not a strategic actor.

Demands: Each player $i = 1,\ldots,n$, can desire up to m objects, where m is a positive integer that does not depend on n. Player i's demand in A^n is given by an m-vector $v_i^n = (v_i^n(1),\ldots,v_i^n(m))$, with the interpretation that $v_i^n(h)$ is the marginal value of the h^{th} object to player i, so that player i places dollar value $\Sigma_{h=1}^{H} v_i^n(h)$ on obtaining H objects. The vector v_i^n is drawn according to a probability measure G_i^n on $\mathbb{R}^m$. We make four

[2]Uniform price (highest rejected bid) auctions are efficient in the single unit demand case, but cease to be when players can desire more than one object. On this, see Engelbrecht-Wiggans and Kahn (1996), Katzman (1995) and Swinkels (1996b).

assumptions on G_i^n :

(1) *Diminishing marginal utility*: $G_i^n(Y) = 1$, where Y is the set of all $x \in \mathbb{R}^m$ such that $x_h \geq x_{h+1}$ for all $h = 1, \ldots, m - 1$.

(2) *Independent values across players*: G_i^n, $i = 1,2,\ldots,n$ are independent. Of course, while values are independent across players, the various values of any given player are correlated.

(3) *Bounded values*: There is $\bar{v} < \infty$ such that $G_i^n([0,\bar{v}]^m) = 1$ for all i and n. That is, all values are at least 0 and at most $\bar{v}$.

(4) *Atomlessness*: G_i^n puts zero weight on lower dimensional subsets of $\mathbb{R}^m$, except possibly for subsets characterized by $v_i^n(h) = 0$ for $h \geq H$ for some $H \geq 0$. That is, for any $H \geq 0$, it can be a positive probability event that i places no value on more than H objects.

For $h = 1,\ldots,m$, define $G_i^n(h)(.)$ as the cumulative for i's h^{th} value in auction environment n. Note that by (4), $G_i^n(h)$ is atomless except possibly at 0. Let $g_i^n(h)(.)$ be the density for $G_i^n(h)(.)$ at points other than 0.[3]

Each player is allowed to submit up to m bids. To simplify notation, we assume that players submit exactly m bids, but are able to freely dispose of excess objects if they should happen to win more objects than they desire (free disposal also lets us model demand as possibly having atoms at 0 rather than negative values). Given a set of m bids for player i, let $b_i^n(h)$ be i's h^{th} highest bid $h = 1,2,\ldots,m$. Players submit their bids simultaneously. One object is awarded to each player for each bid she made that is unambiguously among the k^n highest.[4] In the event that for some b, there are less than k^n bids above b, but more than k^n bids of b or more, the following tie breaking procedure is used. First allocate one object to each bid above b. Take one of the remaining objects. Find the set of players for whom the highest unfilled bid is b, and allocate the object with equal probability to each such player. Repeat this procedure until the objects are gone.[5]

If player i is awarded h objects, then she pays the sum of her h highest bids.

Strategies: Strategies are measurable mappings from valuation vectors to sets of m bids (or possibly to mixtures over sets of m bids).

Payoffs: Players are risk neutral, so that the payoff in any given realization is just the value of objects

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[3]Since G is non-decreasing, it is differentiable almost everywhere. Since G is continuous except at 0, $G(v) - G(0) = \int_0^v G'(x)dx$. Thus, the density function of G exists.

[4]So in particular, we are ruling out reserve prices.

[5]We specify the tie breaking procedure in this way to avoid the following problem: assume a single object is for sale, but for some reason, players can submit more than one bid. If the tie breaking rule paid attention to how many bids of b a player had made, then a player who only wanted 1 object might none the less submit several bids of b to improve her chances of being the one to win the tie.

received less payments made, and the payoff from any given bid set is just the expectation of this amount. The seller is risk neutral, and places no value on the objects, so that his (expected) payoff is just the (expectation of the) sum of the payments received.

Equilibrium: We consider Nash equilibria. To simplify the exposition, we require that each player submit only payoff maximizing bid sets for each possible realization of values (ex-ante optimality requires only that bid sets are almost always optimal for almost all realizations of values). Note that in a discriminatory price auction, there exist strategy profiles to which there is no best response. For example, if there is one object available and all players except i bid 4 with probability 1, then i's best bid for any $v_i > 4$ is the "smallest" number greater than 4. So part of the definition of equilibrium is the condition that best responses exist.

For $p \in [0,\bar{v}]$, define $R_i^n(p)$ as the random variable giving the number of elements of v_i^n that are at least p. That is, $R_i^n(p) = H \in \{0,1,\ldots,m\}$, where $v_i^n(h) \geq p$ for $h \leq H$ and $v_i^n(h) < p$ for $h > H$. Define $R^n(p) = \sum_{i=1}^n R_i^n(p)$, so that for each p, $R^n(p)$ gives the number of valuations that are at least p. $R^n(.)$ can be interpreted as i's demand curve. Because each $G_i^n(h)(.)$ is atomless except possibly at 0, $E(R^n(.))$ is non-increasing, continuous except possibly at 0, and satisfies $E(R^n(0)) = nm$ and $E(R^n(\bar{v}))= 0$.

For $b \geq 0$, define $B^n(b)$ as the random variable giving the number of bids above b given A^n and an associated equilibrium. Define $B^n_{-i}(b)$, $B^n_{-M}(b)$ and $B_i^n(b)$ as the number of bids above b by players in $\{1, 2, \ldots, n\}\backslash i$, $\{1,2,\ldots,n\}\backslash M$ and $\{i\}$ respectively, where M is some subset of the players in A^n.

Order statistics are inherently untidy objects: in the symmetric case, one must deal with the combinatoric term representing all the different subsets of the player valuations or bids that might be above a particular value. In the asymmetric case, each different set of player valuations or bids has a different probability of being above a particular value. In addition, there are dependencies across the valuations or bids of any given player. The nice feature of R^n and B^n is that they are the sum of many independent random variables, and so laws of large numbers apply to them. This will allow us to derive the limiting behavior of the relevant order statistics straightforwardly.

3. WHAT IS EFFICIENCY ?

We wish to prove results to the effect that equilibria of discriminatory auctions are "efficient" in the limit. Our first task is to define efficiency. Consider an equilibrium of A^n. Then, an *outcome* of the auction specifies a realization $v_1^n,\ldots,v_n^n$ of the players' valuation vectors and an allocation of the k^n objects among the players. We evaluate outcomes according to the sum of the utilities across buyers and the seller. Define the *actual surplus* in a given outcome as

$$a^n(v^n, k^n) = \sum_{i=1}^{n} \sum_{h=1}^{H_i} v_i^n(h),$$

where H_i is the number of objects i wins in this outcome (if $H_i = 0$, $\sum_{h=1}^{H_i} v_i^n(h)$ is taken to be zero also). Define the *feasible surplus* as

$$f^n(v^n, k^n) = \sum_{i=1}^{n} \sum_{h=1}^{K_i} v_i^n(h),$$

where K_i is the number of objects for which i's value is among the k^n highest in this realization. Note that payments from buyers to sellers do not enter either of these calculations, since such payments are purely transfers.[6]

For each n, choose an equilibrium of A^n. One efficiency criterion might then take the form

$$E(f^n) - E(a^n) \to 0,$$

where, for each n, expectations are taken with respect to the distributions generated by auction A^n along with the selected equilibrium of A^n. However, this seems impossibly demanding: even though the number of objects for sale may get arbitrarily large, the absolute loss must become arbitrarily small. To have any hope of positive results, we need to somehow compare losses to the "size" of the auction. One possibility would be to calculate losses per person. However, one type of auction we would like to understand is auctions in which demand is large relative to supply, as, for example, for a unique painting. As the number of potential buyers for the painting grows large, then (as long as there is an upper bound on the value of the painting), the expected loss on a per person basis gets small even if the painting is simply thrown away. So, we will instead compare losses to feasible surplus.

Once one decides to compare losses to feasible surplus, there is some ambiguity about how to proceed. One possibility is to require

$$\frac{E(a^n)}{E(f^n)} \to 1,$$

[6] So, the key efficiency question is "do the objects end up with the right buyers," not "are the right number of objects sold." In Rustichini et al.(1994)'s work on double auctions the first question is ruled out by fiat (in particular, by a symmetry assumption on buyers' and sellers' play), and the focus is on the second question. In our world, the second question simply does not arise because sellers place no value on the objects, and because reserve prices are ruled out. We are currently working on the extension of our techniques to double auctions with asymmetric and multiple unit demands *and* supplies. In that world, both issues are relevant. For a further discussion of the connections between this line of research and Rustichini et al.'s work, see Swinkels (1996b).

so that the ratio of expected actual surplus to expected feasible surplus grows to 1 as n becomes large.[7] We formalize this with

DEFINITION 1: Fix a sequence $\{A^n\}$ of discriminatory auctions. We say that the discriminatory mechanism is *asymptotically ex-ante efficient* along $\{A^n\}$ if for each $\varepsilon > 0$, there is n^* such that for $n > n^*$ and any equilibrium of A^n,

$$\frac{\mathrm{E}(a^n)}{\mathrm{E}(f^n)} > 1 - \varepsilon,$$

where expectations are taken with respect to the distribution over outcomes generated by A^n and the given equilibrium.

The "ex-ante" is to emphasize that the expected actual surplus is compared to the ex-ante expected feasible surplus. In contrast, consider the requirement

$$\mathrm{E}\left(\frac{a^n}{f^n}\right) \to 1,$$

so that the expectation of the ratio of actual surplus to feasible surplus in any given outcome grows to 1 as n becomes large. We formalize this with

DEFINITION 2: Fix a sequence $\{A^n\}$ of auctions. We say that the discriminatory mechanism is *asymptotically ex-post efficient* along $\{A^n\}$ if for each $\varepsilon > 0$, there is n^* such that for $n > n^*$ and any equilibrium of A^n,

$$\mathrm{E}\left(\frac{a^n}{f^n}\right) > 1 - \varepsilon,$$

where expectations are taken with respect to the distribution over outcomes generated by auction A^n and the given equilibrium.

The "ex-post" is to emphasize that losses are compared to the ex-post feasible surplus. Note that our definitions are demanding in that they are required to hold no matter what equilibria are chosen for the auctions

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[7] Under the assumptions we make about valuations, comparing losses to feasible surplus and to the number of objects available are equivalent. That is, one could equivalently require $(\mathrm{E}(f^n)-\mathrm{E}(a^n))/k^n \to 0$.

in the sequence.

It is not clear which of these requirements is the "right" one. The first says that most of the potential gains are realized by the discriminatory auction mechanism, so that ex-ante there is little incentive to build a better mechanism. The second says that after the fact, it is rare to find that most of the potential gains have not been exploited, i.e., most of time, most of the feasible surplus is realized.

To see the distinction, consider the scenario in Fig. 1. In each A^n, there are two events e_1 and e_2, where as n grows large, e_2 becomes increasingly unlikely, but none the less accounts for a growing fraction of ex ante feasible surplus. Each A^n is assumed to have two equilibria. The columns headed $a_1^n(e)$ and $a_2^n(e)$ indicate the actual surplus achieved by the two equilibria as a function of e.

		A^n				
		Pr(e)	$f^n(e)$	$a_1^n(e)$	$a_2^n(e)$
events	e_1	(n - 1)/n	1	1	0
	e_2	1/n	n^2	0	n^2

Fig. 1

Now, for equilibrium 1,

$$\frac{E(a^n)}{E(f^n)} = \frac{\left(\frac{n-1}{n}\right)1 + \frac{1}{n}0}{\left(\frac{n-1}{n}\right)1 + \frac{1}{n}n^2} = \frac{n-1}{n-1+n^2} \to 0,$$

while

$$E\left(\frac{a^n}{f^n}\right) = \left(\frac{n-1}{n}\right)1 + \frac{1}{n}0 \to 1,$$

so that with this equilibrium the discriminatory mechanism is ex-ante disastrous but ex-post efficient. Conversely, for equilibrium 2,

$$\frac{\mathrm{E}(a^n)}{\mathrm{E}(f^n)} = \frac{\left(\frac{n-1}{n}\right)0 + \frac{1}{n}n^2}{\left(\frac{n-1}{n}\right)1 + \frac{1}{n}n^2} = \frac{n^2}{n-1+n^2} \to 1,$$

while

$$\mathrm{E}\left(\frac{a^n}{f^n}\right) = \left(\frac{n-1}{n}\right)0 + \frac{1}{n}1 \to 0,$$

so that ex-ante, most gains are realized, but ex-post there is almost always reason to regret the outcome.

We take both ex-ante and ex-post efficiency as important, and will state our results for both notions. However, most of our arguments will hinge on players' incentives, and these are innately an ex-ante phenomenon: players bid before they know the realization of the auction. So, our primary strategy will be to explore conditions on the distribution of realizations of the auction environment (i.e., of demand) such that ex-ante efficiency implies ex-post efficiency. This essentially involves ruling out situations like that in Fig. 1, where as n grows, feasible surplus in event e_l becomes arbitrarily small compared to expected feasible surplus.

Formally, we use:

CONDITION 3.1: For all $\varepsilon > 0$, there is $\delta > 0$ and n^* such that for all $n > n^*$

$$\Pr(f^n > \delta \mathrm{E}(f^n)) > 1 - \varepsilon .$$

Condition 3.1 will turn out to be quite weak, and will be implied by conditions already needed for other parts of the analysis.

LEMMA 3.1: Under Condition 3.1, ex-ante efficiency implies ex-post efficiency.

PROOF: See Appendix.

When a mechanism is asymptotically both ex-post and ex-ante efficient along a sequence $\{A^n\}$, then for large n most of the possible surplus is realized most of the time (ex-post efficiency) and any failures to realize surplus are small compared to the expected surplus before the realization of the auction (ex-ante efficiency).

4. Asymptotic Efficiency of Discriminatory Auctions

For auctions of the form considered here, laws of large numbers are the driving force. We will show that for each A^n, there is a "price" p^n with the property that as n grows large, bids above p^n win almost surely, and bids below p^n win almost never. Further, p^n will be driven to the market clearing price, and so discriminatory auctions without aggregate uncertainty are asymptotically efficient.

It is a convenient normalization to consider demand and supply on a per capita basis. Let $r^n = \dfrac{k^n}{n}$ be the per capita supply. To avoid trivial cases, we assume $r^n < m$. We have allowed for the possibility that a player may simply have no use for more than some number of objects. However, it remains possible that if a player has use for an object, there is some strictly positive lower bound on his valuation. Let $\underline{v}$ be $\min_{i \in \{1,\dots,n\}} \min \operatorname{supp}(g_i^n(.))$, where for notational convenience, we assume $\underline{v}$ is independent of n. That is, $\underline{v}$ is the lower bound on values conditional on values not being 0.

Consider a competitive market with demand at price p given by $E(R^n(p))/n$ and with supply vertical at r^n. Let p^n be the price at which this market clears. If $E(R^n(\underline{v}))/n > r^n$, then p^n is determined by $E(R^n(p))/n = r^n$, while if $E(R^n(\underline{v}))/n < r^n$, then $p^n = 0$. See Fig. 2.

Figure 2 about here.

We shall show that the equilibrium price in the auction converges to p^n in probability. The key to this will be that as n grows large, the distribution for the $k^{n\,th}$ order statistic on values will essentially collapse to a point mass at p^n. That is, it will be exceedingly unlikely that the $k^{n\,th}$ value is much different than p^n. The reason for this is that $R^n(p)$ is the sum of many independent random variables. So, it is unlikely that $R^n(p)$ will be very far away from $E(R^n(p))$. In terms of Fig. 2, this is the statement that the actual demand curve $R^n(p)/n$ in any realization is unlikely to be very far to the left or the right of the expected demand curve $E(R^n(p))/n$. Of course, this does not imply that vertical distances between $E(R^n(p))/n$ and $R^n(p)/n$ cannot be large, and so, given the vertical supply curve, this does not imply that the market clearing prices for the expected and actual demand curves necessarily converge. However, if the expected demand curve $E(R^n(p))/n$ is bounded away from vertical in the interval $(\underline{v}, \bar{v})$, then being close in horizontal distance will also imply being close in vertical distance. An assumption that achieves this is

Condition 4.1: There is a continuous function $z: [\underline{v}, \bar{v}] \to \Re$ such that $z(x) > 0 \ \forall x \in (\underline{v}, \bar{v})$ and such that

$$\frac{\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^{\,n}(h)(x)}{n} \geq z(x) \;\; \forall\, x \in (\underline{v}, \bar{v}) \text{ and for all } n.$$

The following lemma makes precise the sense in which this condition "bounds the expected demand curve away from vertical".

LEMMA 4.1: Under Condition 4.1, there is $M(.)$ continuous, increasing, and with $M(x) > 0$ for all $x > 0$, such that for all $w, y \in [\underline{v}, \bar{v}]$, where $y \geq w$,

$$\frac{\mathrm{E}(R^{\,n}(w)) - \mathrm{E}(R^{\,n}(y))}{n} \geq M(|w - y|).$$

PROOF: See Appendix.

So, the expected number of valuations that fall in any interval of $(\underline{v}, \bar{v})$ grows as n. Given Lemma 4.1, it is clear that p^n is always uniquely defined. Using Lemma 4.1, we are able to show that the market clearing price in a competitive market with demand $R^n(.)/n$ and supply r^n converges in probability to p^n. The key to our results is to show that in the discriminatory auction, the equilibrium price on almost all units sold must also converge to p^n. Formally, we show:

THEOREM 4.1: Consider any sequence of discriminatory auctions $\{A^n\}$ such that Condition 4.1 is satisfied, and any associated sequence of equilibria. The discriminatory mechanism is asymptotically both ex-post and ex-ante efficient along $\{A^n\}$. The selling price on almost all units converges to p^n, where p^n is the market clearing price in a competitive market with demand $\mathrm{E}(R^n(.))/n$ and supply fixed at r^n.

REMARK: Since the auction is asymptotically efficient, a form of revenue equivalence also holds in the limit. Note in particular that we will show that the distribution for the $k^n + 1^{st}$ order statistic on values collapses to a point at p^n as n goes to infinity (as does the distribution for the k^{nth} order statistic on values). Since selling price also converges to p^n, expected revenues are asymptotically $k^n p^n$. So, on a per unit basis, revenues converge to the expected value of an object to the highest unfilled unit of demand, which is exactly what revenue equivalence predicts.

We develop the proof through a sequence of lemmas. Between lemmas, we motivate and provide intuition for the results.

Step 1: Preliminary lemmas

We begin with two useful implications of Condition 4.1. First, under Condition 4.1, asymptotic ex-ante efficiency implies asymptotic ex-post efficiency.

LEMMA 4.2: Assume that Condition 4.1 is satisfied. Then, Condition 3.1 is satisfied, and so asymptotic ex-ante efficiency implies asymptotic ex-post efficiency.

PROOF: See Appendix.

Second, under Condition 4.1, total feasible surplus is bounded below on a per unit basis.

LEMMA 4.3: There is $\zeta > 0$ such that for all n, $E(f^n) \geq k^n\zeta$.

PROOF: See Appendix.

These preliminaries out of the way, we now make two distinct chains of argument to establish the result. The first establishes a bound on the efficiency loss that becomes effective when r^n is small. The second establishes a bound that becomes arbitrarily tight as $n \to \infty$ for any given r^n, but for each n is loose when r^n is very small. Combining the two bounds will establish the general result.

Step 2: The efficiency bound for the case when r^n is small:

We have suggested that the distribution for the k^{nth} order statistic on values will collapse to p^n. So, if the lowest winning bid is anything less than p^n, then there is almost sure to be unfilled demand for units having value greater than the selling price. At first blush, it would thus seem that a simple Bertrand style argument would show that it is unlikely that the lowest winning bid will be much less than p^n. However, while this argument can be established in the case where individuals desire only a single unit, it is considerably harder when individuals desire more than one unit (this is discussed further in the Remark following Lemma 4.5). The key to the case where r^n is small is that just players' demands for their highest valuation units will be enough to drive price close to $\bar{v}$.

For $p \in [0, \bar{v}]$, Let $\hat{R}_i^n(p)$ be 1 if i's highest value is p or greater and 0, otherwise. Define $\hat{R}^n(p) = \sum_{i=1}^{n} \hat{R}_i^n(p)$. So, $\hat{R}^n(p)$ gives the number of players in A^n whose *highest* value is p or greater. Then,

LEMMA 4.4: For each $\varepsilon > 0$, there is $n^* < \infty$ and $r^* > 0$ such that for any auction environment A^n with $n > n^*$

and $r^n < r^*$,

$$\Pr\left(\frac{\hat{R}^n(\bar{r} - \varepsilon)}{n} < r^n \right) < \varepsilon.$$

PROOF: By definition of $z(x)$,

$$\frac{\mathrm{E}(R^n(\bar{r} - \varepsilon))}{n} > \int_{\bar{r} - \varepsilon}^{\bar{r}} z(x)dx.$$

Now, whenever $R_i^n(x) \geq 1$, $\hat{R}_i^n(x) = 1$. Since $R_i^n(x) \leq m$, it follows that

$$\frac{\mathrm{E}(\hat{R}^n(\bar{r} - \varepsilon))}{n} > q(\varepsilon),$$

where $q(\varepsilon) = \dfrac{1}{m}\displaystyle\int_{\bar{r} - \varepsilon}^{\bar{r}} z(x)dx$.

Now,

$$\Pr\left(\frac{\hat{R}^n(\bar{r} - \varepsilon)}{n} < r^n \right)$$

$$= \Pr\left(\frac{\hat{R}^n(\bar{r} - \varepsilon) - \mathrm{E}(\hat{R}^n(\bar{r} - \varepsilon))}{n} < r^n - \frac{\mathrm{E}(\hat{R}^n(\bar{r} - \varepsilon))}{n} \right)$$

$$\leq \Pr\left(\frac{\hat{R}^n(\bar{r} - \varepsilon) - \mathrm{E}(\hat{R}^n(\bar{r} - \varepsilon))}{n} < r^n - q(\varepsilon) \right)$$

$$\leq \Pr\left(\left| \frac{\hat{R}^n(\bar{r} - \varepsilon) - \mathrm{E}(\hat{R}^n(\bar{r} - \varepsilon))}{n} \right| > q(\varepsilon) - r^n \right)$$

as long as $r^n < q(\varepsilon)$.

But, the variance of $\hat{R}^n(\bar{r} - \varepsilon)/n$ is at most $1/n$ since $\hat{R}^n(\bar{r} - \varepsilon)$ is the sum of n independent random variables each having variance at most 1. So, by Chebyshev's inequality, the last expression is at most

$$\frac{1}{n\left(q(\varepsilon) - r^n\right)^2}.$$

Since this is increasing in r^n and for any $r^n < q(\varepsilon)$ converges to 0 in n, the claimed r^* and n^* exist, and we are done. ■

LEMMA 4.5 (THE FIRST BOUND): Let $\varepsilon > 0$. Then, there is $n^* < \infty$ and $r^* > 0$ such that any A^n with $n > n^*$ and $r^n < r^*$,

$$\frac{\mathrm{E}(a^n)}{\mathrm{E}(f^n)} \geq 1 \quad \varepsilon$$

for any equilibrium of the discriminatory mechanism.

PROOF: Let $\alpha = \bar{v}\,\varepsilon/(\bar{v} + 2)$. Appealing to Lemma 4.4, choose n^* and r^* such that whenever $n > n^*$ and $r^n < r^*$

$$\Pr(\hat{R}^n(\bar{v} - \alpha) < k^n) < \alpha^2/2m\bar{v}.$$

Choose $n > n^*$ such that $r^n < r^*$, and fix an equilibrium of the discriminatory mechanism for A^n. Let w^n be the random variable giving the lowest winning bid given this equilibrium. Assume that $\Pr(w^n < \bar{v} - 2\alpha) > \alpha$. We will show that this leads to a contradiction. Now, because values and bids are independent across players, any given player can win a single object with probability at least α by submitting a single bid of $\bar{v} - 2\alpha$ and m-1 bids of . So, if i has highest value $\bar{v} - \alpha$, her expected surplus is at least α^2. Define b' as the infimum over bids that are optimal highest bids for some player i with highest value $\bar{v} - \alpha$ or greater. Pick some i for whom b' is the infimum over optimal highest bids when i has value $\bar{v} - \alpha$ or greater. Assume first that there is a positive probability of a bid of b' by players other than i when their value is $\bar{v} - \alpha$ or greater. Then, since there are positive surplus bids available with value $\bar{v} - \alpha$ or greater, b' must win with positive probability and $b' < \bar{v} - \alpha$. Since players with value $\bar{v} - \alpha$ or greater always bid at least b', and since there is a positive probability that more than k^n players have value $\bar{v} \quad \alpha$ or greater, b' wins with probability less than 1. But, then a bid just above b' wins strictly more often than does b' at effectively the same cost. Since $b' < \bar{v} \quad \alpha$, this is a strictly better bid, contradicting that there is a positive probability of a bid of b' by players other than i when their value is $\bar{v} - \alpha$ or greater. So, by submitting a bid set with highest bid b', i wins only if at most $k^n \quad 1$ players other than i have bid above $\bar{v} - \alpha$. By definition of b', a necessary condition for this to happen is that at most $k^n - 1$ of the players other than i have highest value $\bar{v} \quad \alpha$ or greater. And, when this happens, at most k^n of all the bidders have highest value $\bar{v} - \alpha$ or greater, i.e., $\hat{R}^n(\bar{v} - \alpha) < k^n$. By choice of n^* and r^*, $\Pr(\hat{R}^n(\bar{v} - \alpha) < k^n) < \alpha^2/2m\bar{v}$. So, $1 - \alpha^2/2m\bar{v}$ of the time this player wins no objects and earns 0 and, the remaining $\alpha^2/2m\bar{v}$ of the time, her surplus is at most $m\bar{v}$ (since the best that can happen is that m objects are won for free). Thus, i has expected surplus at most $\alpha^2/2$ with this bid set, and so does worse with this bid set than by submitting a single bid of $\bar{v} - 2\alpha$ and all other bids 0. Since b' was chosen as an optimal highest bid, this is a contradiction, and so it must be that $\Pr(w^n > \bar{v} - 2\alpha) > 1 - \alpha$. But, then expected seller surplus is at least $k^n(\bar{v} - 2\alpha)(1 - \alpha) > k^n(\bar{v} - \alpha(\bar{v} + 2)) = k^n(\bar{v} - \bar{v}\varepsilon)$. Since expected buyer surplus is non-negative,

$$\frac{E(a^n)}{E(f^n)} \geq \frac{k^n(\bar{v} - \bar{v}\varepsilon)}{k^n\bar{v}} = 1 - \varepsilon. \qquad \blacksquare$$

REMARK: One might be tempted to try to generalize this argument to the case r^n large. It is easy to show that buyers get surplus at least equal to approximately what they would have earned if they could buy freely at price p^n. The difficulty is in establishing that seller revenue converges to $k^n p^n$. Assume that $\Pr(w^n < p^n - 2\varepsilon) > \varepsilon$. We cannot conclude that if i's h^{th} bid is above $p^n - 2\varepsilon$, it will win with probability ε, because realizations in which $p^n - 2\varepsilon$ wins might be those in which player i's first h bids are particularly low.

So, we might instead attempt to show that the probability that the probability of more than m winning bids being less that $p^n - 2\varepsilon$ is small. Then, we could conclude that no matter how many bids i makes above $p^n - 2\varepsilon$, they all have at least an ε chance of winning. However, the problem here is deeper: even if raising one's h^{th} bid from b to $p^n - 2\varepsilon$ does raise the probability of that bid winning from effectively 0 to ε or more, this may not raise one's surplus. As an example, for some i and n, let $b_i^n(2) = b$ and $b_i^n(1) = b'$ for some v_i^n where $v_i^n(2) \geq p^n - \varepsilon$, assume that b wins with probability close to 0, and assume that $b < b' < p^n - 2\varepsilon$. Then, replacing b by $p^n - 2\varepsilon$ is equivalent to raising $b_i^n(2)$ from b to b' and $b_i(1)$ from b' to $p^n - 2\varepsilon$. The first change needs not significantly increase the probability of winning a second object, since $b' < p^n - 2\varepsilon$. The second change may significantly lower surplus, because it may have been the case that i was already often winning one object, and so the primary effect of raising $b_i^n(1)$ from b' to $p^n - 2\varepsilon$ is to raise the amount paid in cases where one was already winning. Even if one raises both bids to $p^n - 2\varepsilon$, the gain from the increase in the probability of winning a second object can be outweighed by the loss in paying more for the first in cases where one would have won the first anyway.

The preceding analysis did not run into this problem because when r^n becomes small, even players' largest valuations are enough to drive the "market clearing price" to $\bar{v}$. And, the problem we have just described does not arise for ones highest bid.

To deal with the case r^n large, we need to turn to a different line of proof.

Step 3: The efficiency bound for the case when r^n is not small:

When r^n does not become small, we cannot appeal just to players' highest valuations. However, if n becomes big and r^n does not become small, then k^n becomes very large. This allows us to apply large numbers arguments to the number of bids above key thresholds.

Consider

$$q^n = k^n - (k^n)^{.6} - m.$$

The key to this rather odd looking quantity is that as k^n grows large, q^n and k^n become far apart in absolute terms, but arbitrarily close in proportionate terms.

Recall that $B^n(b)$ is the random variable giving the number of bids above b in A^n. Define ψ^n by

$$\psi^n = \min\{b \geq 0 \mid E(B^n(b)) \leq q^n\}.$$[8]

So, in expectation, there are q^n bids at or above ψ^n. The key to our argument is to show that because $B^n(\psi^n)$ is the sum of n independent random variables, and because k^n grows roughly as n, the probability that more than k^n bids are greater than ψ^n goes to 0. Using this, we will establish (Lemma 4.6 below) that in the limit, expected buyer surplus is at least as great as their consumer surplus in a market with price ψ^n and demand curve $E(R^n(.))$, or (on a per capita basis) the shaded region in Fig. 3.

Figure 3 about here.

On the other hand, we will show that it is also unlikely that the number of bids greater than or equal to ψ^n is much less than q^n. And $q^n/k^n \to 1$ by construction. So we can establish (Lemma 4.7 below) that expected seller surplus converges to at least $k^n\psi^n$, or to $r^n\psi^n$ on a per capita basis. But $E(f^n)$ is of course simply the area under the demand curve $E(R^n(.))$ for quantities up to k^n. If ψ^n is in the limit either above or below p^n, then our limiting bounds add to more than f^n, a contradiction. This is illustrated in Fig. 4. So, in the limit, ψ^n must converge to p^n, and $\dfrac{E(a^n)}{E(f^n)}$ converges to 1.

---

[8] Note that $E(B^n(.))$ is non-increasing, continuous from the right and satisfies $E(B^n(\bar{v})) = 0$, and therefore ψ^n is well defined. We define ψ^n in the way we do rather than simply by $E(B^n(\psi^n)) = q^n$ to account for the possibility of atoms in the distribution of bids.

Figure 4 about here.

Finally, to show the price converges to p^n in probability, note that we have already argued that it is in the limit very unlikely for the seller to receive less than ψ^n on any given unit. Since buyer surplus is converging to her consumer surplus in a market with price ψ^n, it must thus also be very unlikely that any given unit sells for more than ψ^n.

Now we formalize this argument:

LEMMA 4.6: For each A^n each player i's expected surplus in A^n when he has value vector $v_i^n(.)$ is at least

$$(1 - u^n) \Sigma_{k=1}^{n} \mathrm{E}(\max(v_i^n(h) - \psi^n, 0)), \tag{1}$$

where $u^n = \dfrac{m^2}{(r^n)^{1.2} n^{.2}}$.

PROOF: Consider any player i, and consider his probability of winning when he makes m bids of ψ^n or more given equilibrium play by players other than i. For any realization of the auction such that the equilibrium number of bids above ψ^n by all players would have been $k^n - m$ or less, then even when player i deviates, at most k^n bids will be above ψ^n, and so all m of player i's bids will be winners. That is, all m bids of ψ^n win with probability at least $\Pr(B^n(\psi^n) < k^n - m)$. Subtracting q^n from both sides, and using that $q^n = k^n - (k^n)^{.6} - m \geq \mathrm{E}(B^n(\psi^n))$, we see that

$$\Pr\left(B^n(\psi^n) < (k^n) \quad m\right)$$

$$\geq \Pr\left(B^n(\psi^n) - \mathrm{E}(B^n(\psi^n)) < (k^n)^{.6}\right)$$

$$\geq \Pr\left(\left|B^n(\psi^n) \quad \mathrm{E}(B^n(\psi^n)\right| < (k^n)^{.6}\right) \geq 1 - \frac{\mathrm{var}(B^n(\psi^n))}{\left((k^n)^{.6}\right)^2}, \tag{2}$$

where the last inequality is Chebyshev's.

But, $B^n(\psi^n)$ is the sum of n independent random variables each of which has variance at less than m^2, and so $\mathrm{var}(B^n(\psi^n)) < nm^2$. Thus the right hand side of (2) is at least

$$1 - \frac{nm^2}{\left((r^n n)^{.6}\right)^2} = 1 - \frac{m^2}{(r^n)^{1.2} n^2} = 1 - u^n .$$

So, consider the strategy for player i of bidding ψ^n for each h such that $v_i^n(h) \geq \psi^n$, and 0 for all other h. We have established that bids of ψ^n win at least $1 - u^n$ of the time. Since this strategy never results in an ex-post loss, it thus makes at least the amount in (1). ■

LEMMA 4.7: At least $1 - u^n$ of the time, at least $q^n - (k^n)^{.6}$ bids are ψ^n or greater. The expected revenue of the seller is thus at least $(1 - u^n)\psi^n(q^n - (k^n)^{.6})$.

PROOF: If $\psi^n = 0$, the result is obvious. Assume $\psi^n > 0$. Then, for any $b < \psi^n$, $E(B^n(b)) > q^n$. So

$$\Pr\left(B^n(b) < q^n - (k^n)^{.6}\right)$$

$$< \Pr\left(B^n(b) < E(B^n(b)) - (k^n)^{.6}\right)$$

$$< \Pr\left(\left|B^n(b) - E(B^n(b))\right| > (k^n)^{.6}\right)$$

$$< \frac{\operatorname{var}(B^n(b))}{((k^n)^{.6})^2}$$

$$< u^n.$$

As before, $B^n(b)$ is the sum of n independent random variables having variance less than m^2. So at least $1 - u^n$ of the time, at least $q^n - (k^n)^{.6}$ bids are above b. As this is true for all $b < \psi^n$, at least $1 - u^n$ of the time, at least $q^n - (k^n)^{.6}$ bids are ψ^n or greater. Expected seller surplus is thus at least

$$(1 - u^n)\psi^n(q^n - (k^n)^{.6}).$$ ■

We are now ready to give an efficiency bound for the case in which r^n is not small.

LEMMA 4.8 (THE SECOND BOUND): For any n,

$$\frac{E(a^n)}{E(f^n)} \geq (1 - u^n)\left(1 - \frac{2\bar{v}}{(nr^n)^{.4}\zeta} - \frac{\bar{v}m}{nr^n\zeta}\right). \tag{3}$$

where $u^n = \dfrac{m^2}{(r^n)^{1/2} n^{1/2}}$, and $\zeta > 0$ is chosen by Lemma 4.3 such that for all n, $\mathrm{E}(f^n) \geq k^n \zeta$.

PROOF: We essentially add up the expected surplus of the buyer and the sellers, and show that it converges to an arbitrarily large fraction of expected feasible surplus.

By Lemma 4.6, ex-ante expected surplus for player i is at least

$$(1 - u^n) \textstyle\sum_{h=1}^{m} \mathrm{E}(\max(v_i^n(h) - \psi^n, 0)) = (1 - u^n) \sum_{h=1}^{m} \int_{\psi^n}^{\bar{v}} (x - \psi^n) g_i^n(h)(x)dx.$$

So, the sum across players of ex-ante surplus is at least

$$
\begin{aligned}
&(1 - u^n) \int_{\psi^n}^{\bar{v}} (x - \psi^n) \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx \\
&= (1 - u^n)\Big(\int_{\psi^n}^{p^n} (x - \psi^n) \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx \\
&\qquad + \int_{p^n}^{\bar{v}} (x - \psi^n) \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx \Big) \\
&= (1 - u^n)\Big(\int_{\psi^n}^{p^n} (x - \psi^n) \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx \\
&\qquad + \int_{p^n}^{\bar{v}} x \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx - k^n \psi^n \Big),
\end{aligned}
$$

using that

$$\int_{p^n}^{\bar{v}} \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx = \mathrm{E}(R^n(p^n)) = k^n.$$

Adding seller surplus and using Lemma 4.7,

$$
\begin{aligned}
\mathrm{E}(a^n) \geq (1 - u^n)\Big(&\int_{\psi^n}^{p^n} (x - \psi^n) \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx \\
&+ \int_{p^n}^{\bar{v}} x \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx - \psi^n\big(2(k^n)^{.6} \cdot m\big)\Big).
\end{aligned}
\tag{4}
$$

Now

$$\int_{\psi^n}^{p^n} (x - \psi^n) \textstyle\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx \geq 0$$

always, since for all x in the range of integration, $x - \psi^n$ has the same sign as $p^n - \psi^n$. So

$$\mathrm{E}(a^n) \geq (1 - u^n)\left(\int_{p^n}^{\bar{v}} x \sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx - \psi^n\left(2(k^n)^{.6} - m\right)\right)$$

Using that $\psi^n < \bar{v}$,

$$\mathrm{E}(a^n) \geq (1 - u^n)\left(\int_{p^n}^{\bar{v}} x \sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx - \bar{v}\left(2(k^n)^{.6} - m\right)\right) \tag{5}$$

Now, $\mathrm{E}(f^n)$ is given by

$$\int_{\underline{v}}^{\bar{v}} x \left[\sum_{i=1}^{n} \sum_{h=1}^{m} \Pr(x \text{ is one of } k^n \text{ highest values when it is } i\text{'s } h^{th} \text{ highest})\, g_i^n(h)(x)\right] dx$$

$$= k^n \int_{\underline{v}}^{\bar{v}} x \left[\frac{\sum_{i=1}^{n} \sum_{h=1}^{m} \Pr(x \text{ is one of } k^n \text{ highest values when it is } i\text{'s } h^{th} \text{ highest})\, g_i^n(h)(x)}{k^n}\right] dx \tag{6}$$

The bracketed term integrates to 1. But,

$$\int_{p^n}^{\bar{v}} \left[\frac{\sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)}{k^n}\right] dx = 1 \tag{7}$$

by the definition of p^n. And, for any x,

$$\sum_{i=1}^{n} \sum_{h=1}^{m} \Pr(x \text{ is one of } k^n \text{ highest values when it is } i\text{'s } h^{th} \text{ highest})\, g_i^n(h)(x)$$

$$\leq \sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)$$

and so the density function bracketed in (6) is stochastically dominated by that in (7). So,

$$\mathrm{E}(f^n) \leq \int_{p^n}^{\bar{v}} x \sum_{i=1}^{n} \sum_{h=1}^{m} g_i^n(h)(x)dx. \tag{8}$$

Combining (5) and (8),

$$\frac{\mathrm{E}(a^n)}{\mathrm{E}(f^n)} \geq (1 - u^n)\left(1 - \frac{\bar{v}\left(2(k^n)^{.6} - m\right)}{\mathrm{E}(f^n)}\right).$$

But, by choice of ζ, $E(f^n) \geq k^n\zeta$, and therefore,

$$\frac{E(a^n)}{E(f^n)} \geq (1 - u^n)\left(1 - \frac{\bar{v}\left(2(k^n)^{.6} - m\right)}{k^n\zeta}\right)$$

$$= (1 - u^n)\left(1 \quad \frac{\bar{v}\left((2nr^n)^{.6} - m\right)}{nr^n\zeta}\right)$$

$$= (1 \quad u^n)\left(1 - \frac{2\bar{v}}{(nr^n)^{.4}\zeta} \quad \frac{\bar{v}m}{nr^n\zeta}\right).$$

■

Step 4: Combining the Bounds:

We can now combine our two bounds to derive the efficiency result and complete the proof of Theorem 4.1.

PROOF OF THEOREM 4.1: Choose $\varepsilon > 0$. By Lemma 4.5, there are n^* and r^* such that for all $n > n^*$ and $r \leq r^*$,

$$\frac{E(a^n)}{E(f^n)} \geq 1 - \varepsilon.$$

So, consider

$$\left(1 - \frac{m^2}{r^{*\,\text{[illegible]}}n^2}\right)\left(1 \quad \frac{2\bar{v}}{(nr^*)^{.4}\zeta} - \frac{\bar{v}m}{nr^*\zeta}\right). \tag{9}$$

Clearly, there is n^{**} such that for $n > n^{**}$, (9) is at least $1 - \varepsilon$. Also, note that the right hand side of (3) is increasing in r^n. So, for all $n > n^{**}$, and $r > r^*$, the right hand side of (3) is at least $1 - \varepsilon$.

Let $n' = \max(n^*, n^{**})$, and consider any $n > n'$. If $r^n < r^*$, then by the first bound, $E(a^n)/E(f^n) \geq 1$ ε, while if $r^n > r^*$ then by Lemma 4.8 , $E(a^n)/E(f^n) \geq 1 - \varepsilon$, and we have established ex-ante efficiency.

Finally, let us establish that selling price converges in probability to p^n. If r^n is small, this is clear from the argument underlying the first bound. Consider the case r^n large. Assume there is $\varepsilon > 0$ such that $|\psi^n - p^n| > \varepsilon$ for all n along some subsequence. Consider the first term inside the brackets in (4): if $|\psi^n - p^n| > \varepsilon$, then this term is at least $\varepsilon nM(\varepsilon/2)/2$ (this term corresponds to the area X in Fig. 4), where $M(.)$ is as defined in Lemma 4.1. This is so as $|v \quad \psi^n|$ is at least $\varepsilon/2$ over half of the range of integration. When we ignored this term, $E(a^n)/E(f^n)$ converged to 1. So, now it converges to at least $1 + \varepsilon n M(\varepsilon/2)/2E(f^n) \geq 1 + \varepsilon M(\varepsilon/2)/2m\bar{v} > 1$, which is a contradiction. So, $\psi^n \to p^n$.

By Lemma 4.7, at least $1 - u^n$ of the time, at least $q^n - (k^n)^{.6}$ bids are above b. Since $\frac{q^n - (k^n)^{.6}}{k^n} \to 1$, this establishes that almost always, almost all units sell for at least ψ^n. But, by Lemma 4.6, it cannot be the case that the average selling price per unit remains bounded above ψ^n, since in particular, the buyer surplus converges to their consumer surplus when they can buy freely at a price ψ^n. But then, since expected price per unit is driven to ψ^n, and since in the limit, almost all units sell for at least ψ^n almost all the time, the fraction of objects selling for strictly more than ψ^n also goes to 0 almost all the time. So the selling price on almost all units converges in probability to ψ^n. Since $\psi^n \to p^n$, this establishes the result.

$\blacksquare\blacksquare$

5. Conclusion

An issue not addressed by this paper is the rate at which the equilibrium converges to efficiency. It is readily seen that the bounds established in this paper take hold very slowly. This is so for two reasons. First, at various points in the exposition, we choose bounds for simplicity rather than tightness. For example, most applications of Chebyshev's inequality in this paper can, with some effort, be replaced by the central limit theorem, and multiplicative constants are vigorously ignored.

However, the bounds derived converge slowly for a more fundamental reason. A cost of the rather general nature of our setting is that it is difficult to establish much of a characterization of equilibrium in the finite numbers case. So, our results work off a very coarse description of equilibrium play. Note in particular that for the case where supply per capita remains large, the argument essentially begins by throwing away a fraction of the supply, so that one can be sure that with high probability, the realized number of bids above ψ is at least as large as the remaining fraction. However, even if one uses a normal approximation to the number of bids above ψ, one still needs to throw away 2 standard deviations (in the distribution of the number of bids above ψ) worth of supply to be sure that there is a 97% chance of this. And, for the case of for example 1,000 bidders each submitting 2 bids on a set of 1,000 objects, this still involves throwing away approximately 6% of the objects. So, even with a more careful approach to tightness, the bound in Lemma 4.7 still involves an efficiency loss bounded below by around 9% for this case!

On the other hand however, it is important to note that the cause identified for the slowness in the approach to efficiency is due entirely to the coarseness of the equilibrium characterization, and does not seem to reflect any real feature of the auction mechanism. So, it seems quite possible, and even probable, that with a fuller characterization, we would discover much faster convergence. Very fast convergence to efficiency has been shown in other auction settings, most notably by Rustichini, Satterthwaite and Williams (1994) in the context of large double auctions. However, their set up is sufficiently simple (single unit demands, symmetric valuations, symmetric equilibria) as to allow them to utilize a first order approach in their calculations, and this is critical to their method of analysis.[9]

[9]Katzman (1995), working in a symmetric setting, is indeed able to show asymptotic efficiency of the discriminatory auction in which players may desire multiple objects, and does so using a first order approach.

APPENDIX: *Miscellaneous Proofs*

PROOF OF LEMMA 3.1:

Fix a sequence $\{A^n\}$ of auctions. Assume that for all $\alpha > 0$, $\Pr(a^n > (1 - \alpha)f^n) > 1 - \alpha$ for all n sufficiently large and for all equilibria of A^n. Then, since $a^n \geq 0$ always,

$$E\left(\frac{a^n}{f^n}\right) \geq (1 - \alpha)^2$$

for n large. Since this holds for all α, the discriminatory mechanism is asymptotically ex-post efficient along $\{A^n\}$.

So, if the discriminatory mechanism is not asymptotically ex-post efficient along a sequence $\{A^n\}$, then there is $\alpha > 0$, a subsequence of $\{A^n\}$ and an equilibrium for each A^n in the subsequence such that

$$\Pr(a^n < (1 - \alpha)f^n) > \alpha$$

for all n along the subsequence. By Condition 3.1, there is $\delta > 0$ such that

$$\Pr(f^n > \delta E(f^n)) > 1 - \alpha/2$$

for all sufficiently large n. Then, at least $\alpha/2$ of the time, $a^n < (1 - \alpha)f^n$ and $f^n > \delta E(f^n)$ so with probability $\alpha/2$

$$f^n - a^n \geq f^n - (1 - \alpha)f^n = \alpha f^n \geq \alpha\delta E(f^n).$$

But, then since $f^n - a^n \geq 0$ always,

$$E(f^n - a^n) \geq (\alpha/2)\alpha\delta E(f^n)$$

and so

$$\frac{E(a^n)}{E(f^n)} \leq 1 - \frac{\alpha^2\delta}{2},$$

and thus the mechanism is not asymptotically ex-ante efficient along $\{A^n\}$ either. That is, under Condition 3.1, asymptotic ex-ante efficiency implies asymptotic ex-post efficiency. ■.

PROOF OF LEMMA 4.1: For each $\alpha > 0$, define

$$M(\alpha) = \min_{y \in [0, \bar{v} - \alpha]} \int_y^{y+\alpha} z(x)dx.$$

Since the minimand is continuous in y, and positive for each y, and since y is chosen from a compact set, $M(\alpha) > 0$. So, let $w, y \in [\underline{v}, \bar{v}]$ where $y \geq w$. Then,

$$\frac{\mathrm{E}(R^n(w)) - \mathrm{E}(R^n(y))}{n} = \frac{\sum_{i=1}^{n} \sum_{h=1}^{m} \int_w^y g_i^n(h)(x)dx}{n}.$$

$$\geq \int_w^y z(x)dx \geq M(y - w)$$ ■

PROOF OF LEMMA 4.2: Let $\varepsilon > 0$. We must find $\delta > 0$ such that

$$Pr(f^n > \delta \mathrm{E}(f^n)) > 1 - \varepsilon.$$

for n sufficiently large. Let $v^* = \dfrac{\underline{v} + \bar{v}}{2}$. Using Condition 4.1, $\mathrm{E}(R^n(v^*)) > n\Theta$, where

$$\Theta = \int_{v^*}^{\bar{v}} z(x)dx.$$

For each n, let $\hat{k}^n$ be the largest integer less than or equal to both k^n and $n\Theta$. By Lemma 4.1, for all $\alpha > 0$,

$$\mathrm{E}(R^n(v^* - \alpha)) - \mathrm{E}(R^n(v^*)) \geq nM(\alpha)$$

and so since $\mathrm{E}(R^n(v^*)) > n\Theta \geq \hat{k}^n$,

$$\mathrm{E}(R^n(v^* - \alpha)) \geq \hat{k}^n + nM(\alpha).$$

But,

$$\Pr(R^n(v^* - \alpha) < \hat{k}^n)$$

$$= \Pr\left(R^n(v^* - \alpha) - E(R^n(v^* - \alpha)) < \hat{k}^n - E(R^n(v^* - \alpha))\right)$$

$$< \Pr\left(\left|R^n(v^* - \alpha) - E(R^n(v^* - \alpha))\right| < \left|\hat{k}^n - E(R^n(v^* - \alpha))\right|\right)$$

$$\leq \Pr\left(\left|R^n(v^* - \alpha) - E(R^n(v^* - \alpha))\right| < nM(\alpha)\right)$$

$$< \frac{\text{var}(R^n(v^* - \alpha))}{(nM(\alpha))^2}$$

by Chebyshev's inequality. And since $R^n(v^* - \alpha)$ is the sum of n independent random variables each having variance at most m^2 (since in particular, $0 \leq R_i^n(v^* - \alpha) < m$ for all i), $\text{var}(R^n(v^* - \alpha)) \leq nm^2$, and so as $n \to \infty$,

$$\Pr(R^n(v^* - \alpha) < \hat{k}^n) \to 0.$$

So, for n large enough, the top $\hat{k}^n$ valuations are at least $v^* - \alpha$ at least $(1 - \alpha)$ of the time . Therefore, with probability $1 - \alpha$, $f^n \geq \hat{k}^n(v^* - \alpha)$. But, as $E(f^n) < k^n\bar{v}$,

$$\frac{f^n}{E(f^n)} > \frac{\hat{k}^n}{k^n}\frac{v^* - \alpha}{\bar{v}}$$

at least $1 - \alpha$ of the time. If $\hat{k}^n = k^n$, then $\frac{f^n}{E(f^n)} > \frac{v^* - \alpha}{\bar{v}}$. Otherwise, $k^n \leq nm$, and $\hat{k}^n \geq n\Theta - 1$, and so

$$\frac{f^n}{E(f^n)} \geq \frac{n\Theta - 1}{nm}\frac{v^* - \alpha}{\bar{v}}.$$

For n large and $\alpha < v^*$, both expressions are bounded away from 0, and so, since this can be done for any α, we are done. ■

PROOF OF LEMMA 4.3: Consider a world with mn agents each desiring one object, and such that for each i and h, there is one agent with cumulative $G_i^n(h)$ on values. Consider randomly allocating the k^n objects among these mn agents. Then, total expected surplus is

$$\sum_{i=1}^{n} \sum_{h=1}^{m} \frac{k^n}{mn} \int_{\underline{v}}^{\bar{v}} x g_i^n(h)(k)\, dx.$$

But, by Condition 4.1, this is at least

$$\frac{k^n}{mn}\int_{\underline{x}}^{x} xnz(x)dx = k^n \cdot \zeta,$$

where

$$\zeta = \frac{1}{m}\int_{\underline{x}}^{x} xz(x)dx.$$

But, under efficient allocation, the k^n objects go to the k^n "agents" who value them most, and so $E(f^n)$ is at least $k^n\zeta$. ■

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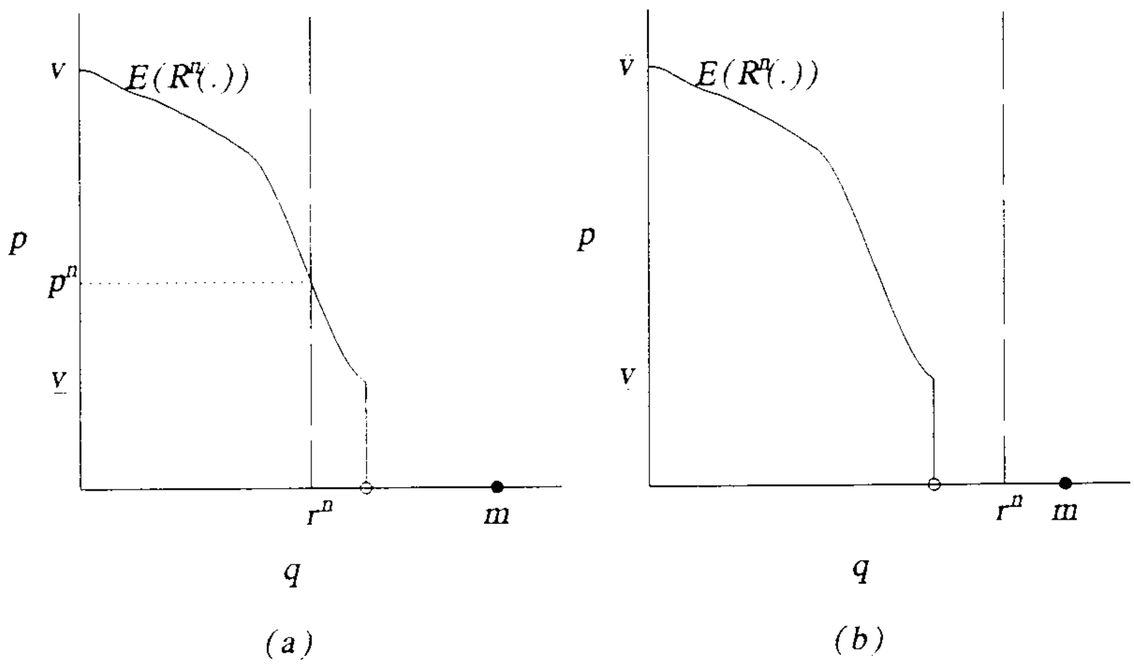

FIG. 2. The normalized supply and expected demand curves.
In (a), $E(R^n(\underline{v}))/n > r^n$, while in (b), $E(R^n(\underline{v}))/n < r^n$, and so $p^n = 0$.

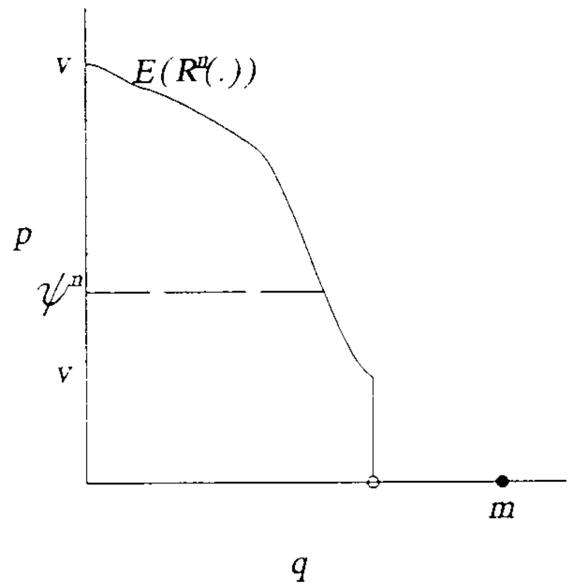

FIG. 3. In the limit, expected buyer surplus is at least the shaded area.

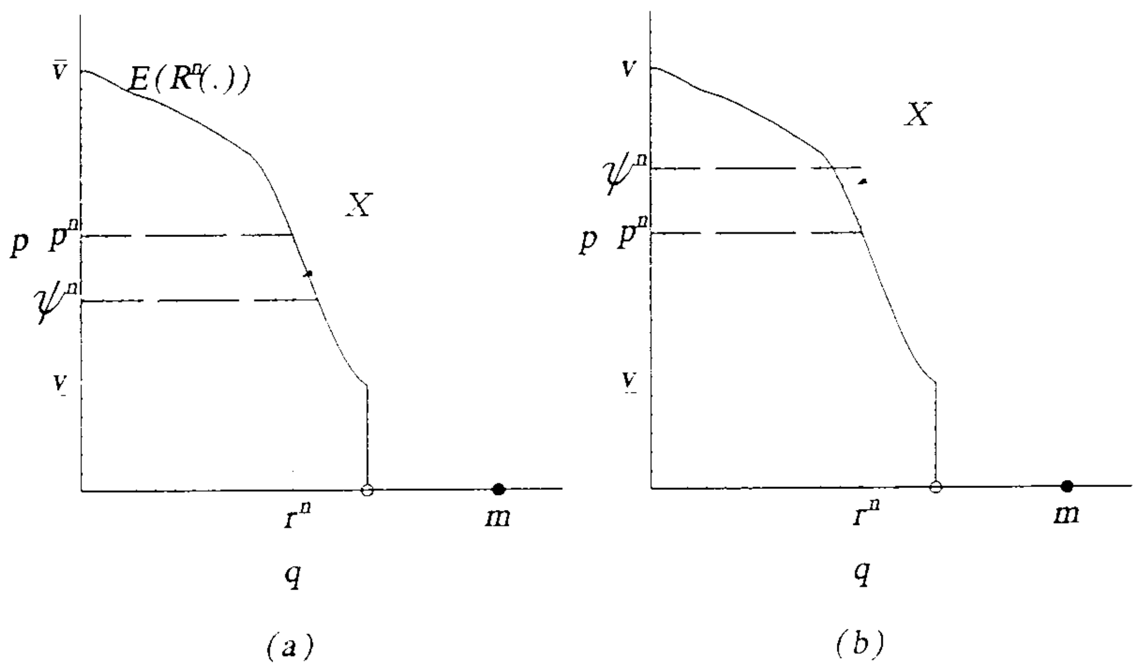

FIG 4. If ψ^n does not converge to p^n in the limit, then the buyer's and seller's expected surplus add to more than feasible surplus. In case (a), $\psi^n < p^n$. In case (b), $\psi^n > p^n$. In either case, expected actual surplus converges to the shaded area, which exceeds expected feasible surplus by the area of the triangle indicated by X.