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THE VECTOR MAXIMIZATION PROBLEM: PROPER EFFICIENCY AND STABILITY†‡

by

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ABSTRACT

Vector maximization problems arise when more than one objective function is to be maximized over a given feasibility region. While the concept of efficiency has played a useful role in the analysis of such problems, a slightly more restricted concept of efficiency, that of proper efficiency, has been proposed in order to eliminate efficient solutions of a certain anomalous type. In this paper necessary and sufficient conditions for an efficient solution to be properly efficient are developed. These conditions relate the proper efficiency of a given solution to the stability of certain single-objective maximization problems. The conditions are useful both in verifying that certain efficient solutions are properly efficient and in identifying efficient solutions that are not proper. An immediate corollary of the theory is that all efficient solutions in linear vector maximization problems are properly efficient. Examples are given to illustrate our results.

1. INTRODUCTION

Vector maximization problems arise when $p \geqq 2$ noncomparable criterion functions are to be simultaneously maximized over a given feasibility region. The concept of efficiency has played a useful role in the analysis of such problems. A slightly restricted definition of efficiency, that of proper efficiency, has been proposed, which eliminates certain efficient points that exhibit an undesirable anomaly. Kuhn and Tucker [8], after proposing the original definition, gave a specific example problem with two criteria in which an improperly efficient solution allows for a first-order gain in one criterion at the expense of but a second-order loss in the other. In this way, the marginal gain-to-loss ratio of the two criteria can be made arbitrarily large. Subsequently, Klinger [7] demonstrated that every improperly efficient solution has this property. Geoffrion [5] noted that efficient solutions may exist which demonstrate a similar anomaly but are proper in the sense of Kuhn and Tucker. In order to exclude all such undesirable solutions, Geoffrion reformulated the definition of a properly efficient solution as one in which, for each criterion, at least one potential marginal gain-to-loss ratio is bounded from above. Using this definition, Isermann [6] proved that each efficient solution of a linear vector maximization problem is properly efficient.

In § 3 of this paper we develop necessary and sufficient conditions for an efficient solution to a vector maximization problem to be properly efficient. These conditions relate the proper efficiency of a given solution to the stability of certain single-objective maximization problems. A direct consequence of this theory is that any efficient solution for a linear vector maximization problem is properly efficient. The use of these results in identifying properly efficient solutions to vector maximization problems is demonstrated by the examples presented in § 4. Conclusions are presented in § 5.

2. THE VECTOR MAXIMIZATION PROBLEM: BASIC DEFINITIONS AND PRELIMINARIES

Consider a vector-valued criterion function

$$f(x) = [f_1(x), f_2(x), \ldots, f_p(x)]$$

defined over a set X given by

$$X = \{x \in \mathbb{R}^n | g_i(x) \geqq 0 \quad \forall i \in I\}$$

where $p \geqq 2$, $f_j: X \subseteq \mathbb{R}^n \to \mathbb{R}$ $\forall$ $j \in J = \{1,2,\ldots,p\}$, $I = \{1,2,\ldots,m\}$, and $g_i: \mathbb{R}^n \to \mathbb{R}$ $\forall$ $i \in I$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_\ell \geq 0, \ \ell = 1,2,\ldots,n\}$ and for any $k \in j$, let J_k denote the set $(J - \{k\})$. Then, the vector maximization problem

$$\text{VMAX:} \quad f(x) \quad \text{subject to } x \in X \qquad \text{(P)}$$

is the problem of finding all solutions that are efficient in the sense of definition 1:

DEFINITION 1. x^0 is said to be an <u>efficient</u> solution of (P) if $x^o \in X$ and $f_i(x\ ) > f_i(x^0)$ for some $x \ \in X$ and some $i \in J$ implies that there exists at least one $j \in J_i$ such that $f_j(x\ ) < f_j(x^0)$.

In order to both exclude all efficient points of a certain anomalous type and to allow for a more satisfactory characterization, Geoffrion [5] has proposed that properly efficient points be singled out for study, where his definition of a properly efficient point for (P) is equivalent to the following:

DEFINITION 2. x^0 is said to be a <u>properly efficient</u> solution of (P) when it is efficient for (P) and there exists a scalar $M > 0$ such that for each $i \in J$ and each $x \in X$ satisfying $f_i(x) > f_i(x^0)$, there exists at least one $j \in J_i$ with $f_j(x) < f_j(x^0)$ and $[f_i(x) - f_i(x^0)]/[f_j(x^0) - f_j(x)] \leqq M$.

For each properly efficient solution of (P), given any criterion, the marginal gain in that criterion relative to a loss in some other criterion is bounded from above. An efficient solution that is not properly efficient is said to be improperly efficient.

The following two definitions will aid in our derivation of necessary and sufficient conditions for an efficient solution to be properly efficient.

DEFINITION 3. x^0 is said to be a kth-entry efficient solution of (P), where $k \in J$, if $x^\circ \in X$ and $f_k(x) > f_k(x^\circ)$ for some $x \in X$ implies that there exists at least one $j \in J_k$ such that $f_j(x) < f_j(x^0)$.

DEFINITION 4. x^0 is said to be a properly kth-entry efficient solution of (P), where $k \in J$, when it is kth-entry efficient for (P) and there exists a scalar $M_k > 0$ such that for each $x \in X$ satisfying $f_k(x) > f_k(x^0)$, there exists at least one $j \in J_k$ with $f_j(x) < f_j(x^0)$ and $[f_k(x) - f_k(x^0)]/[f_j(x^0) - f_j(x)] \leqq M_k$.

The following results follow directly from these definitions.

PROPOSITION 1. A point x^0 is an efficient solution of (P) if and only if it is a kth-entry efficient solution of (P) for each $k \in J$.

PROPOSITION 2. A point x^0 is a properly efficient solution of (P) if and only if it is a properly kth-entry efficient solution of (P) for each $k \in J$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR PROPER EFFICIENCY

Consider the following problem, where $k \in J$,:

$$\max[f_k(x) + \sum_{j \in J_k} u_j f_j(x)] \qquad (P_u)$$

subject to

$$x \in X.$$

Let X_u^* denote the set of optimal solutions for (P_u). The following lemma characterizes properly kth-entry efficient solutions for (P) in terms of solutions to (P_u).

LEMMA 1. (1) <u>If</u> $x^0 \in X_u^*$ <u>for some</u> $u = u^0 \in \mathbb{R}_+^{p-1}$, <u>then</u> x^0 <u>is a properly kth-entry efficient solution for</u> (P).

(2) <u>Suppose that</u> $f_1(x), f_2(x), \ldots, f_p(x)$ <u>are concave functions on the convex set</u> X. <u>Then</u> x^0 <u>is a properly kth-entry efficient solution for</u> (P) <u>if and only if there exists a</u> $u = u^0 \in \mathbb{R}_+^{p-1}$ <u>such that</u> $x^0 \in X_u^*$.

PROOF. (1a). First it will be shown that x^0 is kth-entry efficient in (P). Note that $x^0 \in X$. Let $U_k = \{j | u_j^0 > 0, j \in J_k\}$. Then, either 1) $U_k = \phi$ or 2) $U_k \neq \phi$.

Case 1. $U_k = \phi$. Then $u_j^0 = 0$ for all $j \in J_k$. So x^0 achieves the global maximum of $f_k(x)$ over X. Thus there is no $x \in X$ such that $f_k(x) > f_k(x^0)$. By definition 3, x^0 is kth-entry efficient in (P).

Case 2. $U_k \neq \phi$. Suppose $x \in X$ and $f_k(x) > f_k(x^0)$. Since x^0 is an optimal solution to (P_u) with $u = u^0$, we have

$$f_k(x) + \sum_{j \in J_k} u_j^0 f_j(x) \leqq f_k(x^0) + \sum_{j \in J_k} u_j^0 f_j(x^0).$$

By definition of U_k, this can be written

$$f_k(x) + \sum_{j \in U_k} u_j^0 f_j(x) \leqq f_k(x^0) + \sum_{j \in U_k} u_j^0 f_j(x^0).$$

Since $f_k(x) > f_k(x^0)$, we have that $\sum_{j \in U_k} u_j^0 f_j(x) < \sum_{j \in U_k} u_j^0 f_j(x^0)$. That is,

$\sum_{j \in U_k} u_j^0[f_j(x\ ) - f_j(x^0)] < 0$. Since $u_j^0 > 0$ for all $j \in U_k$, it is necessary that $f_j(x\ ) - f_j(x^0) < 0$ for at least one $j \in U_k$. Thus $f_j(x\ ) < f_j(x^0)$ for some $j \in J_k$. By definition 3, x^0 is kth-entry efficient.

(1b). Now it will be shown that x^0 is actually properly kth-entry efficient for (P), with $M_k = (p-1) \max_{j \in J_k} u_j^0$. Assume, to the contrary, that there exists an $x \in X$ such that $f_k(x\ ) > f_k(x^0)$ and $f_k(x\ ) - f_k(x^0) >$ $M_k[f_j(x^0) - f_j(x\ )]$ for all $j \in J_k$ such that $f_j(x\ ) < f_j(x^0)$. By definition of M_k, we have

$$f_k(x\ ) - f_k(x^0) > (p-1)u_j^0\ [f_j(x^0) - f_j(x\ )] \text{ for all } j \in J_k.$$

This last inequality holds even for those $j \in J_k$ such that $f_j(x\ ) \geqq f_j(x^0)$ because, in these cases, the right-hand side is nonpositive, and, in all cases, the left-hand side is positive. Dividing both sides by (p-1) and summing over all J_k, we obtain

$$f_k(x\ ) - f_k(x^0) > \sum_{j \in J_k} u_j^0\ [f_j(x^0) - f_j(x\ )]$$

which, upon rearranging, becomes

$$f_k(x\ ) + \sum_{j \in J_k} u_j^0 f_j(x\ ) > f_k(x^0) + \sum_{j \in J_k} u_j^0 f_j(x^0).$$

This contradicts $x^0 \in X_u^*$ and completes the proof to (1).

(2) The proof of the "if" part of this statement is provided in the proof of (1) above.

Assume x^0 is a properly kth-entry efficient solution for (P). Then there exists an $M_k > 0$ such that the following system of inequalities admits no solution in X:

$$f_k(x) - f_k(x^0) > 0$$

$$f_k(x) + M_k f_j(x) - f_k(x^0) - M_k f_j(x^0) > 0 \text{ for all } j \in J_k\ .$$

By the Generalized Gordan Theorem [9] there exists $u \in \mathbb{R}^p_+$, $u \neq 0$, such that

$$u_k[f_k(x) - f_k(x^0)] + \sum_{j \in J_k} u_j[f_k(x) + M_k f_j(x) - f_k(x^0) - M_k f_j(x^0)] \leqq 0,$$

or equivalently

$$[\sum_{j=1}^{p} u_j][f_k(x) - f_k(x^0)] + \sum_{j \in J_k} (M_k)(u_j)[f_j(x) - f_j(x^0)] \leqq 0,$$

for all $x \in X$. Dividing through by $C = [\sum_{j=1}^{p} u_j] > 0$ and rearranging, we obtain that

$$f_k(x) + \sum_{j \in J_k} (M_k u_j/C) f_j(x) \leqq f_k(x^0) + \sum_{j \in J_k} (M_k u_j/C) f_j(x^0)$$

for all $x \in X$. Since M_k, $C > 0$ and $u_j \geqq 0$ $\forall j \in J_k$, we have $x^0 \in X^*_{u^0}$, where $u^0_j = (M_k u_j/C) \geqq 0$ $\forall j \in J_k$. Thus, $u^0 \in \mathbb{R}^{p-1}_+$ and the proof of (2) is complete. ||

With the aid of Lemma 1, we will derive necessary and sufficient conditions for a kth-entry efficient solution for (P) to be properly kth-entry efficient. Consider the problem

$$\begin{aligned} &\max\ f_k(x) \\ &\text{subject to} \\ &\quad f_j(x) - b_j \geqq 0 \quad \forall j \in J_k \\ &\quad x \in X, \end{aligned} \qquad (P_{b,k})$$

where $k \in J$ and $\{b_j \mid j \in J_k\}$ is any set of (p-1) real numbers. The following definitions have been adapted from Geoffrion [4], where they were given in association with a nonlinear minimization, rather than maximization, problem. We assume throughout that X is a nonempty set.

DEFINITION 5. The <u>perturbation function</u> $v: \mathbb{R}^{p-1} \to \mathbb{R}$ associated with $(P_{b,k})$ is defined as

$$v(y) = \sup_{x \in X} \{ f_k(x) \mid f_j(x) - b_j \geqq y_j \quad \forall j \in J_k \}.$$

Notice that when the optimal value of $(P_{b,k})$ exists, it is equal to $v(0)$.

DEFINITION 6. $(P_{b,k})$ is said to be <u>stable</u> when $v(0)$ is finite and there exists a scalar $M > 0$ such that, for all $y \neq 0$

$$[v(y) - v(0)]/\|y\| \leqq M.$$

If $(P_{b,k})$ is not stable, the ratio of the improvement in its optimal value to the amount of perturbation can be made as large as desired. We note that the choice of the particular norm $\|\cdot\|$ used to measure the amount of perturbation is arbitrary.

The next lemma shows that if concavity holds the stability of a problem of the form $(P_{b,k})$ constitutes a necessary and sufficient condition for a kth-entry efficient solution for (P) to be properly kth-entry efficient for (P).

LEMMA 2. <u>Assume</u> $f_1(x), f_2(x), \ldots, f_p(x)$ <u>are concave functions on the nonempty convex set</u> X. <u>Suppose</u> x^0 <u>is a kth-entry efficient solution for</u> (P). <u>Then</u> x^0 <u>is a properly kth-entry efficient solution for</u> (P) <u>if and only if</u> $(P_{\bar{b},k})$, <u>where</u> $\bar{b}_j = f_j(x^0) \quad \forall j \in J_k$, <u>is stable</u>.

PROOF. By definition 3, $x^0 \in X$. Then, by the definition of $\bar{b}$, x^0 is feasible in $(P_{\bar{b},k})$. In fact, x^0 is an optimal solution for $(P_{\bar{b},k})$. For an assumption to the contrary implies the existence of at least one vector $x \in X$ satisfying $f_k(x) > f_k(x^0)$, $f_j(x) \geqq f_j(x^0)$ $\forall j \in J_k$, which contradicts that x^0 is kth-entry efficient for (P).

(a) Assume $(P_{\bar{b},k})$ is stable and define the dual problem of $(P_{\bar{b},k})$ by

$$\varphi_k = \min_{u \geqq 0} \{ \max_{x \in X} [f_k(x) + \sum_{j \in J_k} u_j (f_j(x) - f_j(x^0))]\} \qquad (D_{\bar{b},k})$$

where u is a (p-1)-vector of dual variables. Adapting a result of Geoffrion (theorem 3, [4, p. 9]), we have that since $(P_{\bar{b},k})$ is stable, $(D_{\bar{b},k})$ has an optimal solution u^0, and x^0 is an optimal solution for

$$V_k = \max_{x \in X} \{f_k(x) + \sum_{j \in J_k} u_j^0 [f_j(x) - f_j(x^0)]\}. \qquad (P'_u)$$

Since $\{-\sum_{j \in J_k} u_j^0 f_j(x^0)\}$ is constant for all $x \in X$, x^0 is an optimal solution for (P_u) with $u = u^0 \in \mathbb{R}_+^{p-1}$. By Lemma 1, x^0 is a properly kth-entry efficient solution for (P).

(b) Assume x^0 is a properly kth-entry efficient solution for (P). By Lemma 1 (2), there exists a $u = u^0 \in \mathbb{R}_+^{p-1}$ such that $x^0 \in X_u^*$. This implies that x^0 is an optimal solution to (P'_u), where (P'_u) is as given in (a) above. Thus $V_k = f_k(x^0)$. Note that (x^0, u^0) is thus feasible in $(D_{\bar{b},k})$. $(D_{\bar{b},k})$ has $f_k(x^0)$ as its objective value when $(x,u) = (x^0, u^0)$. Thus $\varphi_k \leqq f_k(x^0)$. Furthermore, since x^0 is an optimal solution to $(P_{\bar{b},k})$ the optimal objective value of $(P_{\bar{b},k})$ is $f_k(x^0)$. By the weak duality theorem for nonlinear programming, we have $\varphi_k \geqq f_k(x^0)$. Thus $\varphi_k = f_k(x^0)$ and (x^0, u^0) is an optimal solution to $(D_{\bar{b},k})$. By Geoffrion [4] $(P_{\bar{b},k})$ is stable. ||

Lemma 2 can be used to derive an analogous result that gives necessary and sufficient conditions for an efficient solution for (P) to be properly efficient. This is our main result and is stated in the following theorem.

THEOREM 1. *Assume* $f_1(x), f_2(x), \ldots, f_p(x)$ *are concave functions on the nonempty convex set* X. *Suppose* x^0 *is an efficient solution for* (P). *Then* x^0 *is a properly efficient solution for* (P) *if and only if* $(P_{\bar{b},k})$, *where* $\bar{b}_j = f_j(x^0)\ \forall j \in J_k$, *is stable for each* $k \in J$.

PROOF. By Proposition 1, x^0 is kth-entry efficient for each $k \in J$. By Lemma 2, then, x^0 is properly kth-entry efficient for each $k \in J$ if and only if $(P_{\bar{b},k})$ is stable for each $k \in J$. By Proposition 2, x^0 is properly kth-entry efficient for each $k \in J$ if and only if x^0 is properly efficient. Combining the latter two statements yields the desired conclusion. ||

Theorem 1 states that in vector maximization problems with concave criterion functions $f_j(x)$, $j \in J$, defined over a nonempty convex set X, an efficient point x^0 is properly efficient iff, for each $k \in J$, the problem $\{\max f_k(x) | x \in \bar{X}_k\}$ is stable, where $\bar{X}_k = \{x \in X | f_j(x) - f_j(x^0) \geqq 0\ \ \forall j \in J_k\}$. Stability may not be an easy property to demonstrate directly. However, whenever $\bar{X}_k$ satisfies any "constraint qualification" which insures that the Kuhn-Tucker conditions [8] are satisfied at optimality, the problem $\{\max f_k(x) | x \in \bar{X}_k\}$ is stable [4]. In this way Theorem 1 can provide proof, in some cases, that a given efficient point is properly efficient without resorting to the basic definition. For example, if $f_j(x)$ and $g_i(x)$ are linear functions $\forall j \in J$ and $\forall i \in I$, respectively, then for *any* efficient point x^0 and any $k \in J$, Slater's constraint qualification [9] holds for $\bar{X}_k$ by setting $x = x^0$. Thus, all efficient points in linear vector maximization problems are properly efficient. We have proved the following result:

COROLLARY 1. *Assume* $f_j(x)$ *and* $g_i(x)$ *are linear functions for all* $j \in J$ *and for all* $i \in I$, *respectively. Then any efficient solution* x^0 *for* (P) *is a properly efficient solution for* (P).

Hence, our results include those obtained by Isermann [6] as a special case. In the following section, the first example uses Slater's constraint qualification together with Theorem 1 to provide proof of proper efficiency in a nonlinear problem. This example also illustrates how Slater's constraint qualification and Lemma 2 can be used to identify efficient points that may not be properly kth-entry efficient for some k. However, the second example demonstrates that even though Slater's constraint qualification may fail to hold for some $\bar{X}_k$, the associated efficient point x^0 may still be properly kth-entry efficient.

4. EXAMPLES

Example 1

Consider the problem

$$\text{VMAX: } [-x^2 + 4,\ -(x-1)^4] \qquad \text{(P1)}$$

subject to

$$x + 100 \geqq 0.$$

The set of efficient points for problem (P1) is $\{x | 0 \leqq x \leqq 1\}$. Let $E^P = \{x | 0 < x < 1\}$. Suppose $x^0 \in E^P$. Theorem 1 will be invoked to show that x^0 is properly efficient. The functions $f_1(x) = -x^2 + 4$ and $f_2(x) = -(x-1)^4$ are both concave on the nonempty convex set $X = \{x \in \mathbb{R} |$ $x + 100 \geqq 0\}$. For k = 1, $x \ = (x^0/2 + 1/2)$ satisfies $g_1(x\ ) \geqq 0$, $f_2(x\ ) > f_2(x^0)$. For k = 2, $x \ = x^0/2$ satisfies $g_1(x\ ) \geqq 0$, $f_1(x\ ) > f_1(x^0)$. Thus, Slater's constraint qualification holds for $\bar{X}_1 = \{x \in X | f_2(x) - f_2(x^0) \geqq 0\}$ and for $\bar{X}_2 = \{x \in X | f_1(x) - f_1(x^0) \geqq 0\}$. Because this implies that the problems $\{\max f_k(x) | x \in \bar{X}_k\}$, k = 1,2, are both stable, we have, by Theorem 1, that x^0 is properly efficient. Thus, each point in E^P is properly efficient.

Note that for $x^0 = 0$ and k = 2, and for $x^0 = 1$ and k = 1, Slater's constraint qualification does not hold for $\bar{X}_2$ and $\bar{X}_1$, respectively. Hence from Lemma 2, one might suspect that x = 0 and x = 1 although efficient, are not properly kth-entry efficient for k = 2 and for k = 1, respectively. Indeed, a routine check via definition 4 shows that $x^0 = 0$ is not properly 2nd-entry efficient, and that $x^0 = 1$ is not properly 1st-entry efficient. Thus, we have that $E^I = \{0,1\}$ is the set of efficient points for (P1) that are improper.

Example 2

Now consider the problem

$$\text{VMAX: } [-(x-1)^2,\ -x^2,\ -x^3] \qquad \text{(P2)}$$

subject to

$$x \geqq 0.$$

The set of efficient points for problem (P2) is $E = \{x|0 \leqq x \leqq 1\}$. The properly efficient solution set is given by $E^P = \{x|0 < x < 1\}$. Thus, $E^I = \{0,1\}$ is the set of efficient points for (P2) that are improper. Since $f_1(x) = -(x-1)^2$, $f_2(x) = -x^2$, and $f_3(x) = -x^3$ are all concave functions on the nonempty convex set $X = \{x \in \mathbb{R} \, |x \geqq 0\}$, we have, by Theorem 1, that for any $x^0 \in E^P$, $(P_{\bar{b},k})$, where $\bar{b}_j = f_j(x^0)$ $\forall j \in J_k$, is stable for each $k \in \{1,2,3\}$. This is true in spite of the fact that for any $x^0 \in E$, Slater's constraint qualification fails to hold for $\bar{X}_2 = \{x \in X| \; f_1(x) - f_1(x^0) \geqq 0, \; f_3(x) - f_3(x^0) \geqq 0\}$ and for $\bar{X}_3 = \{x \in X|f_1(x) - f_1(x^0) \geqq 0, \; f_2(x) - f_2(x^0) \geqq 0\}$. Thus, in this example, given an $x^0 \in E$, no knowledge can be gained as to whether x^0 is properly efficient or not by testing for stability of problems $(P_{\bar{b},k})$ via Slater's constraint qualification.

In the following, we will demonstrate for problem (P2) by direct computation, the stability or instability of various problems of the form $(P_{\bar{b},k})$. These computations serve as both illustrations and independent verifications of the conclusions implied by Theorem 1.

Since $x^0 = 1$ is an improperly efficient solution to (P2), Theorem 1 implies that at least one of the problems $(P_{\bar{b},k})$, $k \in \{1,2,3\}$, where $\bar{b}_j = f_j(1)$ $\forall j \in J_k$, is unstable. In fact, both $(P_{\bar{b},2})$ and $(P_{\bar{b},3})$ are unstable, and, by Lemma 2, $x^0 = 1$ is improperly 2nd-entry and improperly 3rd-entry efficient.

Consider problem $(P_{\bar{b},3})$, for example:

$$\begin{aligned} &\max \quad -x^3 \\ &\text{subject to} \quad -(x-1)^2 \geqq 0 \\ &\qquad\qquad\quad -x^2 + 1 \geqq 0 \\ &\qquad\qquad\qquad\quad x \geqq 0. \end{aligned} \qquad (P_{\bar{b},3})$$

We can quickly demonstrate the instability of $(P_{\bar{b},3})$ by the following argument. If $(P_{\bar{b},3})$ were stable, since $x^0 = 1$ is the unique optimal solution, there would exist an optimal multiplier $u^0 = (u_1^0, u_2^0) \in \mathbb{R}_+^2$ associated with x^0 such that

$$-x^3 - u_1^0(x-1)^2 - u_2^0(x^2-1) \leqq -1 \qquad (1)$$

for all $x \in X$ (See Geoffrion [4]). Let $x_\epsilon = 1-\epsilon$ where $0 < \epsilon \leqq 1$. Then $x_\epsilon \in X$ so that, by setting $x = x_\epsilon$ in (1) and rearranging, we have that

$$\epsilon^3 - (3+u_1^0 + u_2^0)\epsilon^2 + (3+2u_2^0)\epsilon \leqq 0. \qquad (2)$$

Dividing both sides of (2) by $\epsilon > 0$, we obtain that

$$\epsilon^2 - (3+u_1^0 + u_2^0)\epsilon + (3+2u_2^0) \leqq 0 \qquad (3)$$

for all $0 < \epsilon \leqq 1$. Taking limits on both sides in (3), we have that

$$\lim_{\epsilon \to 0^+} \{\epsilon^2 - (3+u_1^0 + u_2^0)\epsilon + (3+2u_2^0)\} \leqq \lim_{\epsilon \to 0^+} \{0\}. \qquad (4)$$

From (4), we obtain that $(3+2u_2^0) \leqq 0$ which contradicts that $u_2^0 \geqq 0$. In a similar manner, one can demonstrate that $(P_{\bar{b},2})$ is unstable, where $\bar{b}_j = f_j(1)$ for $j = 1,3$.

Theorem 1 also implies that for each $x^0 \in E^P$, problem $(P_{\bar{b},k})$, where $\bar{b}_j = f_j(x^0)$ $\forall j \in J_k$, is stable for each of $k = 1,2,3$. We will show this by direct computation for the case when $x^0 = \frac{1}{2}$ and $k = 3$.

When $x^0 = \frac{1}{2}$, problem $(P_{\bar{b},3})$ becomes

$$\max \; -x^3$$

subject to

$$\begin{aligned} -(x-1)^2 + (\tfrac{1}{4}) &\geqq 0 \\ -x^2 + (\tfrac{1}{4}) &\geqq 0 \\ x &\geqq 0. \end{aligned} \qquad (P_{\bar{b},3})$$

Any optimal multiplier $u^0 = (u_1^0, u_2^0) \in \mathbb{R}_+^2$ associated with the optimal solution $x^0 = \frac{1}{2}$ for $(P_{\bar{b},3})$ must be such that

$$-x^3 - u_1^0[(x-1)^2-(\tfrac{1}{4})] - u_2^0[x^2 - (\tfrac{1}{4})] \leqq -(1/8) \qquad (5)$$

for all $x \in X$. It is easily verified that $u^0 = (3/4,0)$ satisfies (5), so that $(P_{\bar{b},3})$ is, indeed, stable (See Geoffrion [4]). Notice that testing $(P_{\bar{b},3})$ via Slater's constraint qualification yields no information as to its stability and, thus, no information as to the proper or improper 3rd-entry efficiency of $x^0 = \frac{1}{2}$ in problem (P2).

5. SUMMARY

The concept of a properly efficient solution has been proposed in relation to vector maximization problems. The goal of this paper was to derive a set of necessary and sufficient conditions for an efficient point to be properly efficient. In order to derive these conditions, the concepts of kth-entry efficient and properly kth-entry efficient solutions proved useful. Necessary and sufficient conditions for a kth-entry efficient solution to be properly kth-entry efficient were established in Lemma 2. This development led to the set of necessary and sufficient conditions for an efficient solution to be properly efficient which are presented in Theorem 1. The conditions relate the proper efficiency of a given solution to the stability of certain associated single-objective maximization problems. These results show, as a special case, in Corollary 1 that all efficient points in linear vector maximization problems are properly efficient. The first example problem indicated a nonlinear case in which Slater's constraint qualification, together with our results, was useful in confirming that certain efficient points were properly efficient and in identifying efficient solutions that, with further scrutiny, might prove to be improperly kth-entry efficient for some likely k. In the latter case, a routine check demonstrates whether the solution is properly efficient or not. The second example demonstrated that constraint qualifications such as Slater's, however, may not always succeed in confirming the proper efficiency of efficient solutions, since a stable nonlinear programming problem does not necessarily satisfy some constraint qualification.

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