Discussion Paper No. 1168

EFFICIENCY AND INFORMATION AGGREGATION IN AUCTIONS

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First Draft: December 1995
This Draft: October 1996

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Abstract

There is an underlying tension between allocative efficiency and information aggregation in markets. We explore this in the context of an auction in which k objects are auctioned off to n bidders. The objects are identical, but of unknown quality. In addition, bidders differ in their taste for an object of any given quality. The k highest bidders get an object and pay a price equal to the $k+1$st highest bid. Bidders receive a binary signal that gives some information about the value of the objects, and know their own tastes. We find conditions under which in the limit, objects are allocated efficiently to those with the highest tastes, and price converges in probability to the value of an object to the marginal taste type.

1 Introduction

In many market settings, there is both a private and common value component to values: buyers have some private information about the quality of the objects for sale but in addition, differ in their reservation price for an object of any given quality. Consider for example, the new car market. Potential buyers may differ in their information about how well built cars of a particular model are, and in addition differ in how much they like the styling and features of that model. As an example in an auction setting, consider the sale of timber harvesting contracts on public forests.[1] Firms may differ in their cost structures, for example, because their current capacity utilizations differ. In addition, they may possess private information about the quality of tracts from a particular forest.

[1]See Haile (1996) for a description of timber auctions.

A common intuition in the economics literature is that if there are many market participants with the same information about the quality of what is being sold, then the equilibrium price in the market should aggregate that information. So, if there are many firms bidding on tracts from a given forest, then price should reflect what is known about the quality of that forest.

Another standard intuition in the economics literature is that if there are many participants, then the market should bring about an efficient allocation (assuming that there are no externalities and all market participants are strategically small). In the timber example, tracts should be sold to those firms with the lowest marginal costs.

In an auction setting, these two intuitions are clearly in tension: If bids depends on private information about the quality of the forest, then there will be allocative inefficiencies. If firm 1 has substantially more favorable information about the quality of the forest than does firm 2, then firm 1 will bid more than firm 2 even if firm 2 has somewhat lower costs. On the other hand, if bids do not depend on private information, then of course the outcome of the auction cannot reflect such information.

In the standard competitive rational expectations model, this problem is essentially assumed away: In a fully revealing rational expectations equilibrium (REE) the price function depends on the *aggregate* information in the economy. The price function is an equilibrium if it clears the market given that all consumers can observe the price. Since buyers can infer the true quality of the object from the price, their demands are independent of their private information. Price thus aggregates private information by assumption, and allocative efficiency follows automatically since demands depend only on tastes.

However, as in all competitive models, there is no explanation given in the REE model of how this price came about. A full analysis of the tension between information aggregation and allocative efficiency must consider a model where price is a function of individual buyer behavior, and where this behavior in turn depends only on the individual's private information (and not on the information contained in the equilibrium price). And, in any such model, it seems likely that efficiency and information aggregation will be in tension: for price to reflect quality, the actions of market participants must depend on their information about quality. This in turn generates allocative inefficiencies.

Of course, the intuitions about information aggregation and efficiency depended on the market having many participants. And, as the number of participants grows, it becomes less clear that the tension between allocative efficiency and information aggregation must persist. If the action of each participant depends less and less on her private information about quality, then in the limit, there need be no allocative inefficiency. On the other hand, as the number of participants grows, it may well be that any given participant's behavior becomes less and less informative about quality, but the information available from the aggregate behavior of all market participants still grows increasingly precise.

This is merely the observation that as a market grows large there need not necessarily be a conflict between efficiency and information aggregation. It says nothing about whether the incentives in the market process actually lead to either efficiency or aggregation. It does however, suggest that it is interesting to consider the limiting properties of a model in which market participants have both differing tastes for objects of any given quality and differing information about quality and in which price is the result of a non-cooperative game played among the market participants.

In this paper, we address this question using a simple model of a uniform price auction. Consider an auction with k objects and n bidders. The objects are identical, but of unknown quality q. In addition however, players have idiosyncratic taste parameter t_i. Player i places value

$$u_i = q + t_i$$

on winning a single object and no value on further objects. Each buyer knows his own t_i and, in addition, receives either a good signal (G) or a bad signal (B) about q, where the probability of receiving G is strictly increasing in q, conditionally independent across players given q, and independent of the t_i's. Each buyer submits a bid on the basis of his observed signal and his taste parameter.[2] The k highest bidders receive an object and pay the $k+1$st highest bid, with ties broken by symmetric randomizations.[3] Bids therefore determine both the allocation and the equilibrium

[2]So, our players have a two dimensional type space. This is a departure from the vast majority of the existing auction literature. We discuss the motivations for, and implications of, this departure in Section 2.1.

[3]For technical reasons, we will use a somewhat unusual tie breaking rule. This is discussed below.

price.

For fixed k and n, there is indeed the described tension between information aggregation and allocative efficiency: the more sensitive are bids to private information, the greater the allocative inefficiency, while the less sensitive are bids to private information, the less information about quality is reflected by the equilibrium price.

However, to gain insight into the validity of our intuitions about efficiency and information aggregation, the key is to understand the behavior of the equilibria of these auctions as both k and n grow large. We obtain the following result:

> Consider a sequence of auctions of the type described. If both k and n go to infinity (and k/n remains bounded away from 0 and 1), then in the limit of any sequence of symmetric equilibria there is both allocative efficiency and full information aggregation. That is, in the limit, objects are allocated to the players with the highest t_i's, and price reflects the true value of an object to the marginal taste type.

We already argued that in the limit, as k and n go to infinity, efficiency and information aggregation become both feasible. And indeed, the equilibrium behavior of players satisfies both in the limit: bidders pay less and less attention to their own information so that in the limit allocative efficiency results, but pay sufficient attention to their information that the market clearing price comes to reflect quality exactly!

The major impetus for our work is to understand the general conflict between information aggregation and allocative efficiency, rather than the limiting behavior of large auctions themselves. Auction models are convenient for this task because they are a tractable non-cooperative model of price setting. However, even for auctions, our results are of some relevance: many real auctions do have a large supply and set of potential buyers. And, as we argue below, auctions which combine a private and common component of values probably should be viewed as more the norm than the exception. The model seems to fit the timber example reasonably well. For general markets it would be desirable to consider strategic sellers as well as strategic buyers. Such a generalization is left for future research.

Grossman and Stiglitz (1976) point out the following paradox in rational expectations models with a small cost of acquiring information: if the equilibrium price reveals information, then there is no value to acquiring information. But, if no information is acquired, then price is uninformative, and then there is an individual incentive to acquire information.

In our setting, there is no cost to acquiring information: the buyer receives the signal for free. Despite this, the basic forces underlying the Grossman-Stiglitz paradox remain: if price is very informative about quality, then individuals have strong incentives to let their idiosyncratic preferences override their signal about quality. But, then price cannot be informative. So, the paradox remains even if information is free: it is not enough that consumers have information, they must want to use it. One way to interpret this is that while there is not cost to acquiring information, there is a cost to *using* information excessively: letting one's bid depend sensitively on one's own information results in too few purchases when bad information is received, and too many when good information is received.

However, as we will see, the forces which generate a paradox in the rational expectations model are precisely those which imply that in the limit there is both allocatively efficiency and full information aggregation. Much like in Grossman-Stiglitz, if the market does not do a good job of revealing true quality, then there is a strong incentive to use information, contradicting that the market fails to aggregate information in the limit. Conversely, because the market is in the limit doing a good job of aggregating information, it does not make sense for an individual to incur much of a cost in terms of potential misallocations to use his own information. So, in the limit, this misallocation must also go to zero.

In a companion paper (Pesendorfer and Swinkels (1996)), we analyze information aggregation in a pure common values setting. There we show that information aggregation holds if and only if $k_r \to \infty$ and $n_r - k_r \to \infty$. The result here is weaker in two significant ways. First, in this paper we require that k_r/n_r remains bounded away from 0 and 1 along the sequence. The result for the case when k_r/n_r headed to a boundary in the pure common value setting depends on a fuller characterization of the equilibrium than we can achieve in this setting. The second weakness is more serious: in the pure common value case, we are able to show that there exists a unique symmetric equilibrium, fully characterize it, and then show that it has the properties needed for information aggregation. In this setting, we are unable to show existence of a symmetric equilibrium, or provide a full description of the equilibrium. Rather, we are able to show only that if symmetric equilibria exist, then they must have the properties necessary for our results.

The present paper is also related to work by Feddersen and Pesendorfer (1996). They analyze two candidate elections in which voters have different preferences and have private information about the quality of the candidates. Similar to the $k+1$st price auction analyzed here, in a voting model the action of a player (the vote) only matters when he is pivotal and thus a voter (like a bidder in an auction) has to condition on being pivotal. As in the present model, there is a tension between information aggregation and the willingness of agents to use their private information: if the election aggregates the private information of agents effectively then voters can infer the quality of the candidates from being pivotal. But then most voters will ignore their private information when making their vote choice. Feddersen and Pesendorfer give conditions under which the election nevertheless fully aggregates the private information of voters.

In the next section, we lay out the model, and describe the key properties of the equilibrium for fixed k and n that we will need for our asymptotic results. We also briefly discuss the role of our two dimensional type space. Section 3 is the heart of the paper. It examines the behavior of the equilibrium as k and n grow large. Finally, our asymptotic efficiency and information aggregation results allow us to characterize asymptotic bidding behavior precisely. This is despite the fact that equilibria are extremely difficult to solve for in the finite setting. Section 4 provides this characterization. Proofs of all results are contained in Section 5.

2 The Model and Equilibrium for Fixed Market Size

Assume there are n buyers and k objects for sale. Buyer i's utility from a single object is

$$u_i = q + t_i,$$

and any further object gives a utility of 0. The quality of the object, q, is common across players, and drawn according to a distribution $F(.)$, with support $[0,1]$. F has continuous density $f(.)$,where $f(q) > \gamma > 0, \forall q \in [0,1]$. Bidder i's taste parameter t_i

is drawn independently across bidders from probability distribution W with support $[0, \bar{t}]$. W has continuous density $w(\cdot)$ where $w(t) > \gamma > 0$.

Bidder i knows t_i, and receives a signal $s_i \in \{B, G\}$ about the quality, q. We assume that s_i and t_i are independent. Conditional on q, the signals s_i are independent across players, with a probability $\pi_G(q)$ of a good signal G, and $\pi_B(q) = 1 - \pi_G(q)$ of a bad signal B, where π_G is continuous, bounded away from 0 and 1, and strictly increasing.

Each bidder i submits a bid b_i as a function of his taste-signal pair (t_i, s_i). The k highest bidders receive the object and pay the $k + 1$st highest bids. Since two or more bidders may be tied at a bid we have to specify a tie breaking rule: A bid b wins the object with probability one if fewer than $k - 1$ other bidders submit a bid greater than or equal to b. If b is the k-th highest bid and k or more other bidders submit a bid larger than or equal to b we assume that the probability of winning the object is equal to z, where $0 < z < 1$. The probability z is independent of the number of bidders above b or at b. This specification of the tie-breaking rule is non-standard. Below we discuss its implications. Here it suffices to note that we show below that in any symmetric equilibrium ties occur with probability 0 and hence in equilibrium the tie breaking rule is never used.

We consider symmetric Nash equilibria. In the following we describe the equilibrium strategies from the perspective of bidder 1. Since we assume symmetric strategies this describes the whole equilibrium profile. Let d denote the kth highest element in the set $(b_2, ..., b_n)$, i.e., d denotes the kth highest bid among all bidders except bidder 1.

Our first proposition shows that symmetric equilibria can be described by a pure strategy with the property that the bid function $b(t, s)$ is strictly increasing in t. In addition, a bidder with taste t bids more when he receives the good signal than when he receives the bad signal.

Proposition 1 *Any symmetric equilibrium can be described by a bidding function $b(t, s)$, where $b(t, s)$ is strictly increasing in t with $b(t, G) > b(t, B)$ for all $t \in [0, \bar{t}]$. Moreover,*

$$b(t, s) = t + E\left(q \,|d = b(t, s), s\right) \tag{1}$$

for all s and t.

Thus, in equilibrium, each bid b made is such that if the type of player 1 who bid b is on the margin between winning and losing (i.e., if $d = b$), then the expected value of the object to him is equal to price.

To give intuition for this result, note first that it is easy to show that if for a given signal s, b is optimal for taste t, then every optimal bid with signal s and taste $t' > t$ is at least b. This is so as t contains no information about either q or other players' actions, and so increasing t simply makes any given increase in bid strictly more attractive. But then, it must be that for almost all t, there is a unique optimal bid.[4] Since the distribution of t is atomless, we can thus without loss of generality restrict attention to equilibria in pure strategies. That is, for every equilibrium, there is a realization equivalent equilibrium in pure strategies (simply take a selection at the zero measure set of points where the best response is not unique). So, an equilibrium can be characterized by a function $b(t, s)$ which for each s is non-decreasing in t.[5]

To give an intuition for equation (1) consider first a bid b such that for each s, $b(t, s)$ is strictly increasing as it passes through b (that is, neither $b(t, G)$ nor $b(t, B)$ has a flat spot at b). Then, the distribution over d has no mass points in a neighborhood of b and for small $\varepsilon > 0$, conditional on $d \in [b, b+\varepsilon]$, there is probability 1 that a bid of b by player 1 loses while a bid of $b + \varepsilon$ by player 1 wins. Since this is a uniform price auction, the only time increasing $1's$ bid by ε matters is when $d \in [b, b+\varepsilon]$. So, the change in payoffs has the same sign as

$$E(q|d \in [b, b+\varepsilon], s) + t - E(p|d \in [b, b+\varepsilon], s), \tag{2}$$

where p is the random variable describing the equilibrium price. Arguing similarly for $\varepsilon < 0$, and taking limits as $\varepsilon \to 0$, we conclude that (1) must hold. Note that the above argument was independent of the tie braking rule because a bid of $b + \varepsilon$ wins with probability 1 and a bid of b loses with probability 1 conditional on $d \in [b, b+\varepsilon]$.

Consider next the case where $b(s, t) = b$ for some interval of t for at least one s. Then, there is a positive probability of a tie at b and therefore there is a positive probability both of winning and of losing with a bid of b when b is pivotal. So, unlike above, it is not the case that whenever $d \in [b, b+\varepsilon]$, there is probability 1 that a bid

[4] To see this, let $y(t)$ be a selection from the best response correspondence for some given s. Then, $y(t)$ is a non-decreasing function, and $y(t)$ jumps at every t where the best response is non-unique. So, there are at most a countable set of such points.

[5] Note that this argument does not depend on the non-standard tie breaking rule.

of b by player 1 loses. Now the change in payoffs has the same sign as

$$E(q|d \in [b, b+\varepsilon], b \text{ loses}, s) + t - E(p|d \in [b, b+\varepsilon], b \text{ loses}, s). \tag{3}$$

The standard tie breaking rule specifies that when there is a tie at b, the probability that 1 wins with a bid of b is equal to $(k - \#\text{bids above } b)/(\#\text{bids equal to } b)$. Under this rule, when $d = b$, whether player 1 wins or loses with a bid of b contains information about the number of bids above b and equal to b. This in turn may contain information about the signals other players have received. Therefore, (3) need not be the same as (2).

It is precisely for this reason that we specify our unusual tie breaking rule. Under our tie breaking rule, supply is always adjusted in the event of a tie so that the probability of winning an object conditional on a tie is z independent of how many bids are either above or equal to b. So, conditioning on losing at b does not provide any further information about q or p once one conditions on the event that b is pivotal, and so (2) and (3) are equivalent. A similar argument applies when $\varepsilon < 0$ for the event that $d = b$ and a bid of b wins. Taking ε to 0, we conclude that (1) must again hold. But, since (1) can hold for at most one t for each s, this in fact rules out the possibility of $b(s, t)$ being constant over any range of t's. Therefore, $b(t, s)$ is strictly increasing in t.

Similarly, since the probability of receiving signal G is strictly increasing in q,

$$E(q|d = b, G) > E(q|d = b, B).$$

and hence if the bid b is made both by the type (t, G) and by the type (t', B) then equation (1) implies that $t' > t$. In the Appendix we demonstrate that this argument can be generalized to imply that $b(t, G) > b(t, B)$ for all t.

Our tie breaking rule is non-standard since it requires the auctioneer to deliver the good with a fixed probability z whenever two or more bidders are tied at the kth highest bid. If the auctioneer sets $z = \frac{1}{n}$ then this tie breaking rule is always feasible. However, this will typically result in fewer than k objects being sold in the case of a tie. Since we show that in equilibrium ties occur with probability zero, this is not a problem.

Note that the only time the tie breaking rule is used in the proof of Proposition 1 is to rule out flat spots of the bidding function. Thus, any symmetric equilibrium of the game with the non-standard tie breaking rule will remain an equilibrium when the standard tie breaking rule is used. And, a non-atomic equilibrium of the standard game is of course an equilibrium of the game with the non-standard tie breaking rule. We have not been able to show, however, that with the standard tie breaking rule other equilibria (involving atoms) cannot exist.[6]

2.1 Comments on the Two Dimensional Type Space

Our model has the property that each bidder is characterized by two parameters: his taste and his signal about quality. It is this two-dimensionality of the type space that gives rise to the conflict between information aggregation and allocative efficiency discussed in the introduction.

In contrast to our model, the vast majority of the existing auction literature works with a one-dimensional type space: each player receives a single signal $x_i \in \Re$. Milgrom and Weber (1982) show that despite this one dimensional type space, the model is rich enough to include pure private values (a player's utility from the object depends only on his own signal), pure common values (a player's utility from the object depends symmetrically on all signals) and models which are intermediate between private and common values (a player's utility depends on all signals, but weighs signals of other players differently than his own).

In the one-dimensional environment there can be no-conflict between information aggregation and efficiency: a higher estimate of quality always also implies a higher taste parameter. We are thus unable to use the standard Milgrom Weber (1982) framework in this paper. More generally, we believe that most auction settings do contain both common and private values components. And, unless tastes and information are perfectly correlated, this requires a type space with more than one

[6] Doing so would involve establishing that with the standard tie breaking rule there is a winner's curse at an atom: once one has conditioned on b being pivotal, the additional news that one has won should at least weakly reduce one's beliefs about quality. It is possible to establish that in any equilibrium, involving atoms or not, bids are at least weakly increasing in signals. Winning at b tends to suggest that there were not too many people who bid strictly above b (so that there are lots of objects left for people who bid b) and not too many who bid b (so that not too many people are competing for the objects available at b). So, intuitively, winning at b ought to be bad news about the quality. However, while bids are easily shown to be increasing in s_i, we are unable to show that they are affiliated with s_i, which is what one would need to apply for example Theorem 5 of Milgrom and Weber (1982). In the pure common value setting, we were able to establish this result. See Pesendorfer and Swinkels (1996).

dimension. For example, bidders on an oil lease may differ in their current cost of exploration and drilling activity, and in addition, have different information about the amount of oil which a certain tract might contain. It seems highly artificial to assume that good information about the oil-content of a tract also implies that the company has lower costs of exploration.

With a single dimensional type space, there is a natural guess about the form of the equilibrium: bids will be strictly increasing in type (and symmetric across players). Having made this guess, inference problems about equilibrium behavior are reduced to inference problems about underlying parameters. So for example, the question "what would I infer if I knew my bid was tied with the highest bid by my opponents" reduces to "what would I infer if I knew my signal was tied with the highest signal by my opponents." One can then easily derive first order conditions on what the bid of any given signal type must look like. Integrating these first order conditions yields a candidate equilibrium, and the assumption of affiliation allows one to verify that the candidate is indeed an equilibrium.

With a two dimensional type space, there is no "natural" complete ordering on the type space. Guessing an order on the type space involves guessing which pairs t_G and t_B go together: e.g., for any given type t_B, what type t_G has the property that $b(t, G) < b(t_B, B)$ if and only if $t < t_G$? But this question cannot be answered independent of the equilibrium strategies. It is equivalent to guessing what a bidder infers from $d = b$ which in turn depends on t_B and t_G.

As a consequence of the two-dimensional type space we have not been able to prove existence of equilibria. Our results instead hinge on a partial characterization of what equilibria must look like in the limit, if they exist.

3 Efficiency and Information Aggregation

3.1 Asymptotic Efficiency

Efficiency requires that the players with the k highest values are those who win objects. This maximizes the gains from trade across buyers and the seller, and, given our assumption of quasi-linear utility functions, is the unique Pareto optimal allocation. Of course, given that players who observe G always bid more than those who observe B, exact efficiency is unattainable for any finite auction. In particular, since $b(t, G) > b(t, B)$ it follows that there is a positive probability that a bidder with type (t, G) wins the object while a bidder with type $(t', B), t' > t$ does not win. In the following we will define a measure for the degree of inefficiency.

It is convenient to first define the inverse of the bidding function. For a given auction and equilibrium, and $s \in \{B, G\}$, define the function $t_s(b)$ so that $b(s, t) > b$ for all $t > t_s(b)$ and $b(s, t) < b$, for all $t < t_s(b)$. So, for bids b which are actually made with signal s, $t_s(b)$ is the taste parameters that makes bid b, while if b is not made with signal s, then $t_s(b)$ is the taste parameter at which bidding jumps past b. If b is larger than any bid made then we set $t_s(b) = \bar{t}$, and similarly $t_s(b) = 0$, for b smaller than any bid made. Note that since $b(s, t)$ is strictly increasing in t, $t_s(b)$ is unique, and is continuous. Figure 2 illustrates $t_B(b)$ and $t_G(b)$ for a particular equilibrium.

Figure 2 about here.

Assume the price is b in some equilibrium of this auction. The smallest type t who may get the object in equilibrium is $t_G(b)$. Similarly, the largest type t who may not get the object in equilibrium is $t_B(b)$. Thus the maximum potential loss from a misallocation of the objects if the price is b is $t_B(b) - t_G(b)$. Therefore, an upper bound for the loss in utility due to the misallocation of objects in an equilibrium is given by

$$\sup_b [t_B(b) - t_G(b)]. \tag{4}$$

In the following we consider a sequence of auctions with k_r objects and n_r bidders, where both k_r and n_r go to infinity as $r \to \infty$. Along this sequence of auctions we keep the information structure, F, π_s, and W fixed. To indicate that we are working with elements of a sequence we will subscript strategies and inverse bidding functions by r.

We say that a sequence of equilibria is *asymptotically efficient* if

$$\sup_b [t_{Br}(b) - t_{Gr}(b)] \to 0$$

as $r \to \infty$.

Note that this implies a strong form of asymptotic efficiency. First, as discussed above, any misallocated object goes to a player whose utility for the object is almost as high as that of any player who does not receive an object. Second, if k_r/n_r is bounded away from zero, then the expected fraction of objects which is misallocated also converges to zero. This follows since $\sup_b[t_{Br}(b)-t_{Gr}(b)] \to 0$, and therefore if the equlibrium price is p a vanishing fraction of bidders will typically have a parameter t between $t_{Br}(p)$ and $t_{Gr}(p)$ in the limit.

We now turn to the question of when asymptotic efficiency holds. It turns out that a key assumption is that the ratio of objects to bidders (k_r/n_r) stays away from zero and one along the sequence.[7]

Assumption 1 *$n_r \to \infty$ as $r \to \infty$. Moreover, there is a $\beta > 0$ such that $1-\beta > k_r/n_r > \beta$ for all r.*

We then have:

Theorem 1 *If the sequence of auctions satisfies Assumption 1 then any associated sequence of symmetric equilibria is asymptotically efficient.*

To gain an intuition for the result assume that for some bid b $t_{Br}(b) - t_{Gr}(b) > \varepsilon$ for all r. Suppose, for simplicity, that the bid b is made by type $(t_{Br}(b), B)$ and by type $(t_{Gr}(b), G)$. Then Proposition 1 implies that

$$t_{Br}(b) + E(q|d_r = b, B) = t_{Gr}(b) + E(q|d_r = b, G) \tag{5}$$

But then it must be the case that for all r

$$E(q|d_r = b, G) - E(q|d_r = b, B) > \varepsilon \tag{6}$$

Thus it must be the case that the inference about quality made by a bidder with a good signal is significantly different from the inference made by a bidder with a bad signal when conditioning on the event $d_r = b$. We will now argue that (6) cannot be satisfied for large r.

To see this first note that the probability that a bidder with a signal B bids above b is strictly less than the probability that a bidder with signal G bids above b.

[7] In a companion paper (Pesendorfer and Swinkels (1996), we were able to explicitly characterize the equilibrium, and this allowed us to derive an information aggregation result even when k_r/n_r went to 0 or 1. Here, we are only able to derive some properties of the equilibrium: this weaker characterization requires the stronger assumption on k_r/n_r. Whether our results would go through when k_r/n_r does go to 0 or 1 is an open question.

Moreover, since $t_{Br}(b) - t_{Gr}(b) > \varepsilon$ these probabilities stay bounded away from each other. If $d_r = b$ then exactly k_r of the $n_r - 1$ bids are above b. If n_r and k_r are large then we can infer with great precision what fraction of the buyers must have received a good signal in order for a $k_r/(n_r - 1)$ fraction of bids to be above b. If $H_{rs}(b)$ denotes the probability that a bidder with signal s bids below b then, for large r, the bidders estimate of quality conditional on $d_r = b$ must be very close to the unique q that satisfies

$$\begin{aligned} & (1 - H_{Gr}(b))\pi_G(q) + (1 - H_{Br}(b))(1 - \pi_G(q)) \\ = \; & (1 - H_{Br}(b)) + (H_{Br}(b) - H_{Gr}(b))\pi_G(q) = \frac{k_r}{n_r - 1} \end{aligned}$$

But this implies that the private signal s cannot change the bidder's estimate of the quality of the object significantly once he conditions on the event $d_r = b$. Therefore we have shown that Inequality (6) cannot be satisfied.

3.2 Full Information Aggregation

As a benchmark for full information aggregation we use the "full information" market, i.e., the environment where all buyers know the true quality of the object q. In that case the bidding behavior in any symmetric equilibrium is for each bidder simply to bid their valuation $t_i + q$. Thus the equilibrium price will be equal to the $k_r + 1$st highest valuation. Define $t_r^* = W^{-1}(\frac{n_r - k_r - 1}{n_r})$. In expectation, a fraction $(k_r + 1)/n_r$ of bidders have t above t_r^*. The law of large numbers then implies that the equilibrium price of the full information game converges in probability to $t_r^* + q$ as r grows large.

We demonstrate in Theorem 2 that the equilibrium price of the market where quality is unknown also converges to $t_r^* + q$ in probability. Thus, the equilibrium price in a large market is the same (with high probability) whether or not the quality of the object is known to buyers.

Let the random variable p_r denote the equilibrium price in auction r. We can then state:

Theorem 2 *If a sequence of auctions satisfies Assumption 1 then any associated sequence of symmetric equilibria satisfies that for all $\delta > 0$ there is an $\bar{r}$ such that for all $r > \bar{r}$, $\Pr\{|q + t_r^* - p_r| > \delta\} < \delta$.*

Together Theorems 1 and 2 imply that the equilibrium price in a large market is equal to the valuation of the object of the marginal bidder. Thus in the limit any bidder who does not buy an object has a valuation less than the equilibrium price and conversely, every bidder who gets the object has a valuation larger than the equilibrium price. This implies that (in the limit) no bidder "regrets" his bid, i.e., no bidder would want to change the bid once the equilibrium price is announced.

To provide an intuition for Theorem 2 consider some bid b that is made by type $(t_{Br}(b), B)$ and by type $(t_{Gr}(b), G)$. That is, consider a b that is in the support of bids for both signals. By Proposition 1 we know that

$$t_{Br}(b) + E(q|d_r = b, B) = t_{Gr}(b) + E(q|d_r = b, G) \tag{7}$$

By Theorem 1 we also know that

$$t_{Br}(b) - t_{Gr}(b) \to 0 \tag{8}$$

and hence it follows that

$$E(q|d_r = b, B) - E(q|d_r = b, G) \to 0 \tag{9}$$

The last expression says that the conditional expectation of the quality of an object is almost independent of the signal s. But recall that the probability of receiving the good signal, $\pi_G(q)$, is strictly increasing in q and hence the only time the signal does not affect a bidder's beliefs about quality is if he can predict the quality extremely precisely independent of the signal. In other words, (9) implies that the probability distribution over q conditional on $d_r = b$ becomes arbitrarily concentrated around the true value q.

Thus we may conclude that if $d_r = b$ and type (t, s) bids b then $b \approx t + q$ where q is the true quality of the object. We claim that this also implies that if $p_r = b$ then $p_r \approx t + q$. To see this note that $d_r = b$ if $k_r - 1$ of $n_r - 2$ bidders bid above b and one bidder bids b whereas $p_r = b$ if k_r of $n_r - 1$ bidders bid above b and one bidder bids b. For large r the events $p_r = b$ and $d_r = b$ provide therefore virtually identical information about q and hence the claim follows.

To complete the argument now observe that by Theorem 1 and the law of large numbers the marginal bidder has a type close to t_r^* with probability close to one for large r. Therefore the equilibrium price must be close to $t_r^* + q$ with probability close to one for large r.

4 The Limiting Bid Distribution

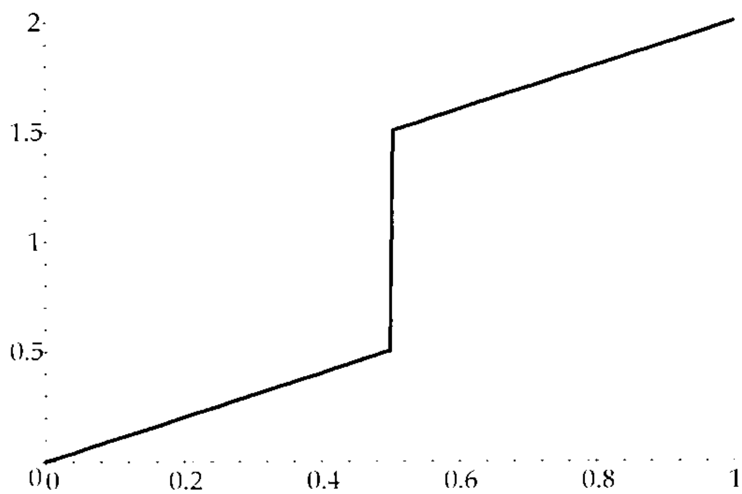
In this section we characterize the bidding behavior in the limit as $r \to \infty$. For simplicity we assume in this section that k_r/n_r converges to constant $\kappa \in (0,1)$. Let $t^* = W^{-1}(1-\kappa)$.

Proposition 2 *For all $\varepsilon > 0$, there exists $\bar{r}$ such that for all $r > \bar{r}$,*

(1) for all $t < t^* - \varepsilon$, $t \le b(t,s) \le t + \varepsilon$.

(2) for all $t > t^* + \varepsilon$, $1 + t + \varepsilon \le b(t,s) \le 1 + t$.

Thus the limiting bidding behavior is easily characterized. Bidders with tastes ε below the pivotal type t^* bid as if the true q of the object were 0 whereas bidders with tastes ε above t^* bid as if the true q of the object were 1. Bidders with these tastes thus essentially ignore their information. Bidders who lie in a very narrow range around t^* behave in a way that depends sensitively on their information. The following graph summarizes Proposition 1. In Figure 1, $q = 1/2$, $\bar{t} = 2$, and the median of W is assumed to be $1/2$.



The limit bid distribution

To give an intuition for Proposition 2 note that since $b(t,s)$ is strictly increasing in t a bidder with $t < t^* - \varepsilon$ expects the equilibrium price (and the pivotal bid) to be larger than his own bid with probability close to one for all values q. This is the case since (by Theorem 2) the fraction of bidders bidding above t is strictly larger than

k_r/n_r for large r. If the unlikely event occurs that t is the marginal bidder then it must be the case that both an unusual distribution of t's and a low value of q have been drawn. A similar argument applies for $t > t^* + \varepsilon$.

This characterization of the equilibrium also makes clearer how the market manages both information aggregation and allocative efficiency. To achieve allocative efficiency, it is enough that the interval of tastes over which bids depends sensitively on information grows narrow. Then, a growing fraction of bidders lie outside of this interval, and thus are acting essentially independently of their information. On the other hand, while the fraction of bidders who use their information a great deal converges to 0, their absolute number grows fast enough that price contains all information about quality in the limit.

The limiting characterization of the equilibrium has an interesting parallel to Feddersen and Pesendorfer (1996). In that paper, voters have different tastes about two candidates and different information about the candidates. Feddersen and Pesendorfer show that as the number of voters grows large, voters who have tastes either a little to the left of center vote for the left candidate regardless of their information, while voters a little to the right of center vote for the right candidate regardless of their information. Only a narrow band of moderates uses their information in their behavior. The analysis in that paper is simplified by the discrete action space (vote left or vote right), which essentially forces players to either use their information a great deal or not at all. Interestingly, our limiting equilibrium approaches that result even though players have available to them strategies which use information only a little bit.

5 Proofs

5.1 Proof of Proposition 1

The proof of Proposition 1 is divided into a sequence of Lemmas. Let $K(b|s)$ denote the probability that a bid b wins in equilibrium given that the bidder has received signal s. Let $\Pi(b, t, s)$ denote the payoff of a bidder with taste t and signal s if he bids b. Let H denote an equilibrium bid distribution, i.e., H is a measure on $[0, \bar{t}] \times [0, \infty)$.

Lemma 1 *Given any equilibrium H, there is a function $b(t,s)$ that is weakly increasing in t and such that (1) a player receives the same expected payoff and probability of winning by using $b(t,s)$ rather than his equilibrium bid distribution, and (2), playing according to $b(t,s)$ generates the same joint distribution over qualities, types, allocations, and prices as does H.*

Proof Let b be an optimal bid for a player with type (t,s), and let $b' < b$. Then,

$$\Pr(b \text{ wins}, b' \text{ loses}|\ s)\left(E(q|b \text{ wins}, b' \text{ loses}, s) + t - E(p|b \text{ wins}, b' \text{ loses}, s)\right) \geq 0.$$

Now, if $\Pr(b \text{ wins}, b' \text{ loses}|\ s) = 0$, then b is a best response for s if and only if b' is, and neither the payoffs of the player making the bid nor the overall distribution over equilibrium outcomes is affected by whether the player bids b or b'.

Assume $\Pr(b \text{ wins}, b' \text{ loses}|\ s) > 0$. Then, it must be that

$$E(q|b \text{ wins}, b' \text{ loses}, s) + t - E(p|b \text{ wins}, b' \text{ loses}, s) \geq 0.$$

But then, for any $t' > t$,

$$E(q|b \text{ wins}, b' \text{ loses}, s) + t' - E(p|b \text{ wins}, b' \text{ loses}, s) > 0,$$

and so b is a strictly better bid than b' for type (t', s). So, let $t' > t$, and let b be a bid which is a best response for with type (t,s). Then, every bid $b' < b$ such that $\Pr(b \text{ wins}, b' \text{ loses.}|\ s) > 0$ is a strictly worse bid than b for (t',s). If $b' < b$ and $\Pr(b \text{ wins}, b' \text{ loses.}|\ s) = 0$, then b is at least as good as b' for (t',s) and there is no difference in the outcome if (t',s) bids b instead of b'. ■

Lemma 2 *For each $s = B, G$, $b(t,s)$ is strictly increasing in t.*

Proof By the previous Lemma, $b(t,s)$ is weakly increasing. Suppose that for some s, $b(s,t) = b$ for all $t \in (t_1, t_2)$. Then, $\Pr(d = b,\ b \text{ loses}, s) > 0$, and so $E(q|d = b, b$ loses, $s)$ is well defined. Consider a type (t,s) who bids b. By bidding just above b a bidder with type (t,s) changes his payoff by

$$[E(q|d = b, b \text{ loses}, s) + t - b] \Pr(d = b, b \text{ loses}, s) \tag{10}$$

whereas by bidding just below b the payoff changes by

$$-[E(q|d=b,b \text{ wins}, s)+t-b]\Pr(d=b,b \text{ wins}, s)$$

But, the probability of winning is the same and equal to $z, 0<z<1$, for each event in which the bidder is pivotal and therefore

$$E(q|d=b,b \text{ loses}, s)=E(q|d=b,b \text{ wins}, s)=E(q|d=b,s). \tag{11}$$

Moreover, $\Pr(d=b,b \text{ loses})>0$ and $\Pr(d=b,b \text{ wins})>0$ and hence if b is optimal for (t,s) then

$$E(q|d=b,s)+t=b$$

which can hold at most for one t. Thus we have a contradiction and the Lemma follows. ■

For a (Lebesgue) measurable subset $X\subset[0,\infty)$ let $H_s(X)$ be the probability of a bid $b\in X$ by a player with signal s. Note that $H_s([0,b])=W(t_s(b))$. Let $\mathcal{H}_s=\text{supp}H_s(.)$. Since each $t_s(b)$ is increasing function, $t_s(b)$ is differentiable almost everywhere. Let $\mathcal{B}$ be the subset of all $b\in\{\mathcal{H}_G\cup\mathcal{H}_B\}\backslash\{\min\{\mathcal{H}_G\cup\mathcal{H}_B\}\cup\max\{\mathcal{H}_G\cup\mathcal{H}_B\}\}$ for which $t_B(b)$ and $t_G(b)$ are both differentiable and either $t'_B(b)\neq0$ or $t'_G(b)\neq0$. Since $t_s(b)$ is continuous and increasing, $H_s(\mathcal{B})=1, s=B,G$.

Clearly if both $t_B(.)$ and $t_G(.)$ are differentiable at b, then $K(.|s)$ is also differentiable at b, so that $K(b|s)$ is differentiable for all $b\in\mathcal{B}$. If either $t'_B(b)>0$ or $t'_G(b)>0$, then $K'(b|s)>0$. Since $H_s(\mathcal{B})=1, s=B,G$, it follows that $\text{supp}K(.|s)=\mathcal{H}_G\cup\mathcal{H}_B$ Finally, note that for $b\in\mathcal{B}$, $E(q|d=b,s)$ is well defined.

Lemma 3 *For all $b\in\mathcal{H}_s$, b is a best response for type $(t_s(b),s)$.*

Proof Since $b(t,s)$ is strictly increasing, $K(.|s)$ is continuous. But then, $\Pi(b,t,s)$ is continuous in b, and trivially continuous in t. Assume that for some $b\in\mathcal{H}_s$, b is not a best response for type $(t_s(b),s)$. Then, there is b' such that $\Pi(b',t_s(b),s)>\Pi(b,t_s(b),s)$. But then by continuity, $\Pi(b',t_s(b),s)>\Pi(b(s,t),t,s)$ for t sufficiently close to $t_s(b)$. But then, for all such t, $b(t,s)$ is sub-optimal, contradicting that in a Nash equilibrium, players must use a best response with probability 1. ■

Lemma 4 *If $b\in\mathcal{H}_s\cap\mathcal{B}$ then $E(q|d=b,s)+t_s(b)=b$.*

Proof Consider a point $b \in \mathcal{H}_s \cap \mathcal{B}$. Since $b \in \mathcal{B}$ it follows that $E(q|d = b, s)$ is well defined and hence

$$\frac{\partial \Pi(b,t,s)}{\partial b} = [E(q|d = b,s) + t - b]K'(b|s).$$

Since $b \in \mathcal{B}$, $K'(b|s) > 0$. And, since $b \in \mathcal{H}_s$, b is optimal for type $t_s(b)$, and so it follows that $E(q|d = b,s) + t_s(b) - b = 0$. ■

The following Lemma says that as long as the probability distribution over q is not entirely concentrated at one point, the expectation of q conditional on receiving the signal G is larger than, and uniformly bounded away from, the expectation of q conditional on receiving signal B.

Lemma 5 *Let Z be any distribution on $[0,1]$. Then for all $\delta > 0$, if $\Pr(|q - E(q)| > \delta) > \delta$ there is an $\eta > 0$, independent of Z, such that*

$$E(q|G) - E(q|B) > \eta,$$

where probabilities and expectations are taken with respect to Z.

Proof Since $\Pr(|q - E(q)| > \delta) > \delta$, it must be the case that either $\Pr(q > E(q) + \delta) > \delta/2$ or that $\Pr(q < E(q) - \delta) > \delta/2$. Consider the first case (the other is entirely analogous). Then, since there is at least $\delta/2$ mass δ above $E(q)$, and since $q \in [0,1]$, there must at least $\delta^2/2$ mass below $E(q)$. Let $W_1 = [0, E(q)]$, $W_2 = (E(q), E(q) + \delta)$, and $W_3 = [E(q) + \delta, 1]$.

$$E(q) = \sum_{i=1}^{3} \Pr(q \in W_i)\, E(q|W_i)$$

and,

$$E(q|G) = \sum_{i=1}^{3} \Pr(q \in W_i|G) E(q|W_i, G).$$

Clearly $E(q|W_i, G) > E(q|W_i)$, $E(q|W_i, G)$ is increasing in i, and $E(q|W_3, G) > E(q|W_1, G) + \delta$. Since the signal has the MLRP, $\Pr(q \in W_i|G)$ stochastically dominates $\Pr(q \in W_i)$. To establish the existence of our η, it is thus enough to find ν depending only on δ such that $\Pr(q \in W_3|G) - \Pr(q \in W_3) > \nu$, and $\Pr(q \in W_1) - \Pr(q \in W_1|G) > \nu$ (because then $Z(.|G)$ can be thought of as obtained from $Z(.)$ by a series of transformations which at least weakly increased the expectation of q, and by shifting a mass ν a distance at least δ to the right). Since π_G is strictly increasing, there is $\mu > 1$ such that $\pi_G(q)/\pi_G(w) > \mu$ for all $q \in W_3, w \in W_1$, and so, $\frac{\Pr(q \in W_3, G)}{\Pr(q \in W_1, G)} > \mu \frac{\Pr(q \in W_3)}{\Pr(q \in W_1)}$. Since each of $\Pr(q \in W_3)$ and $\Pr(q \in W_1)$ was at least $\delta^2/2$, we are done. ■

The following Lemma shows that bidders who receive a signal G will bid more than bidders who receive a signal B.

Lemma 6 *For all t, $b(t, G) > b(t, B)$.*

Proof We will demonstrate that for all $b \in \mathcal{B}$, $t_G(b) < t_B(b)$. Since $\mathcal{B}$ is dense on $\mathcal{H}_G \cup \mathcal{H}_B$ the lemma follows from the continuity of $t_s(b)$.

Case 1 Consider first any $b \in \mathcal{H}_G \cap \mathcal{H}_B \cap \mathcal{B}$. Then by Lemma 4,

$$E(q|d = b, B) + t_B(b) = b$$

and

$$E(q|d = b, G) + t_G(b) = b.$$

Since the signal satisfies the monotone likelihood ratio property it follows that

$$E(q|d = b, G) > E(q|d = b, B)$$

and therefore it must be that $t_B(b) > t_G(b)$.

Case 2 Consider any b such that $b \in \mathcal{H}_B \cap \mathcal{B}$ but $b \notin \mathcal{H}_G$. Since $b \in \mathcal{B}$, at least one of $t_G(b)$ or $t_B(b)$ is strictly positive. If $t_G(b) = 0$, we are thus done. Assume $t_G(b) > 0$. Define $\underline{b} = \max_{b' < b} \mathcal{H}_G$. Since $t_G(\underline{b}) = t_G(b)$ it follows from Lemma 3 that $\underline{b}$ is a best response for a player with type $(t_G(b), G)$.

We consider two sub cases:

Case 2a $H_B((\underline{b}, b)) > 0$. Then, of course $K(b|G) - K(\underline{b}|G) > 0$. Consider the payoff change to a player with type $(t_G(b), G)$ of bidding b instead of $\underline{b}$. Since $\mathcal{B} \cap (\underline{b}, b) \subseteq \mathcal{H}_B$, Lemma 2 implies that for every $b' \in \mathcal{B} \cap (\underline{b}, b)$

$$E(q|d = b', B) - b' + t_B(b') = 0.$$

The distribution over q conditional on the event $d = b$ conveys (possibly noisy) information about $n - 1$ signals and hence satisfies the assumption in Lemma 5 for some $\delta > 0$. Thus, for some $\varepsilon > 0$,

$$E(q|d = b', G) \geq E(q|d = b', B) + \varepsilon$$

But, since $b' \leq b$, $t_B(b') \leq t_B(b)$. So, if $t_G(b) \geq t_B(b)$, then

$$\begin{aligned}
\Pi(b, t_G(b), G) - \Pi(\underline{b}, t_G(b), G) &= \int_{\mathcal{B} \cap (\underline{b}, b)} (E(q|d = b', G) - b' + t_G(b))\, dK(b'|G) \\
&\geq \int_{\mathcal{B} \cap (\underline{b}, b)} (E(q|d = b', B) - b' + \varepsilon + t_G(b))\, dK(b'|G) \\
&\geq \int_{\mathcal{B} \cap (\underline{b}, b)} (E(q|d = b', B) - b' + \varepsilon + t_B(b'))\, dK(b'|G) \\
&= \varepsilon(K(b|G) - K(\underline{b}|G)) > 0
\end{aligned}$$

which contradicts that $\underline{b}$ is a best response for a player with type $(t_G(b), G)$. Thus, $t_G(b) < t_B(b)$.

Case 2b $H_B((\underline{b}, b)) = 0$. Then, since every neighborhood $\underline{b}$ has non-empty intersection with $\mathcal{H}_G$, and since $\mathcal{B}$ is dense on $\mathcal{H}_G$, it must be that for t' arbitrarily close to $t_G(b)$, and b' arbitrarily close to $\underline{b}$,

$$E(q|d = b', G) + t' - b' = 0.$$

And, of course,

$$E(q|d = b, B) + t_B(b) - b = 0$$

However, since $H_B((\underline{b}, b)) = 0$, the event $d = b'$ conveys arbitrarily close to the same information about bidders who bid either strictly more than b or strictly less than b as does the event that $d = b$. And, conditional on $d = b$, the opposing bidder who bid b must have seen the signal B, while conditional on $d = b'$, the opposing bidder might have seen G. So, for b' close enough to $\underline{b}$, $E(q|d = b', G) > E(q|d = b, G) - \epsilon/2 > E(q|d = b, B) + \epsilon/2$. Thus, it must be that $t' < t_B(b) - \epsilon/2$, and so $t_G(b) < t_B(b)$.

Case 3 Consider any b such that $b \in \mathcal{H}_G \cap \mathcal{B}$ but $b \notin \mathcal{H}_B$. This is entirely analogous to case 2. ■

5.2 Proof of Theorem 1

In this section we show that every sequence of equilibria is *asymptotically efficient*. To this end we first need to establish a central information aggregation Lemma.

Denote by $X_r(b)$ the event that $k_r - 1$ of bidders $\{3, \ldots, n_r\}$ bid above b and $n_r - k_r$ bidders bid below b. Note that because the equilibrium is symmetric, the event $d_r = b$ can be replaced for all relevant purposes by the event $X_r(b) \cap \{b_2 = b\}$.

The following Lemma says that if players who receive a good signal are more likely to bid above b than players who receive a bad signal, then the event $X_r(b)$ allows a very precise prediction about the common value component q of the object.

Lemma 7 *There is $r_1(.)$ such that for all $\delta > 0$ and for all $r > r_1(\delta)$ and for all b such that $H_{Br}(b) - H_{Gr}(b) > \delta$.*

$$\Pr\left(|q - E(q|X_r(b))| > \delta | X_r(b)\right) < \delta$$

and

$$E(q|X_r(b), s_2 = G, s_1 = G) - E(q|X_r(b), s_2 = B, s_1 = B) < \delta.$$

Proof Fix $\delta > 0$, and b such that $H_{Br}(b) - H_{Gr}(b) > \delta$. Let

$$x(q) = H_{Br}(b) - \pi_G(q)(H_{Br}(b) - H_{Gr}(b))$$

be the probability that any given bidder bids less than b given q. Let

$$\alpha_r(q) = x(q)^{\frac{n_r - k_r + 2}{n_r - 2}}\left[1 - x(q)\right]^{\frac{k_r - 2}{n_r - 2}}.$$

Since $H_{Br}(b) - H_{Gr}(b) > \delta$ and $\pi_G(q)$ is strictly increasing, $x(q)$ is strictly decreasing in q. Thus $\alpha_r(q)$ is a single peaked function of q. Let $q_r^* = \arg\max_q \alpha_r(q)$. By the hypothesis of the lemma $H_{Br}(b) - H_{Gr}(b) > \delta$. This implies that for every $\epsilon > 0$ there is an $L(\epsilon) > 0$ such that for all r and for all q with $|q - q_r^*| \geq \epsilon$, $|\alpha_r'(q)| \geq L(\epsilon)$. Let $f(q|X_r(b))$ denote the density of q conditional on $X_r(b)$:

$$f(q|X_r(b)) = \frac{f(q)\alpha_r(q)^{n_r-2}}{\int f(q')\alpha_r(q')^{n_r-2}dq'}$$

But this implies that for any q', q such that $q' + \epsilon < q < q_r^* - \epsilon$,

$$\frac{f(q_r'|X_r(b))}{f(q_r|X_r(b))} = \frac{f(q')}{f(q)}\left(\frac{\alpha_r(q')}{\alpha_r(q)}\right)^{n_r-2} \leq (1 - \epsilon L(\epsilon))^{n_r-2}\gamma^{-2}.$$

Similarly, for q', q such that $q' - \epsilon > q > q_r^* + \epsilon$,

$$\frac{f(q'|X_r(b))}{f(q|X_r(b))} = \frac{f(q')}{f(q)}\left(\frac{\alpha_r(q')}{\alpha_r(q)}\right)^{n_r-2} \leq (1 - \epsilon L(\epsilon))^{n_r-2}\gamma^{-2}.$$

Choosing r large enough, the first part of the lemma follows.

To see the second part of the lemma note that from the first part of the lemma. that for all $\zeta > 0$.

$$\Pr(|q - E(q|X_r(b))| > \zeta | X_r(b)) < \zeta$$

for sufficiently large r. So. for sufficiently large r.

$$E(q|X_r(b). G. G) < \frac{(1-\zeta)(E(q|X_r(b)) + \zeta) \cdot \pi_G(0)^2 \min f(q) + \zeta \cdot 1 \cdot \pi_G(1)^2 \max f(q)}{(1-\zeta)\pi_G(0)^2 \min f(q) + \zeta \cdot \pi_G(1)^2 \max f(q)}$$

and

$$E(q|X_r(b). B. B) > \frac{(1-\zeta)(E(q|X_r(b)) - \zeta) \cdot (1-\pi_G(1))^2 \min f(q) + \zeta \cdot 0 \cdot (1-\pi_G(0))^2 \max f(q)}{(1-\zeta)(1-\pi_G(1))^2 \min f(q) + \zeta \cdot (1-\pi_G(0))^2 \max f(q)}.$$

Since $\pi_s(.)$ is bounded away from zero and one and $f(.)$ is bounded away from zero and infinity. each of the last two rhs expressions can be driven arbitrarily close to $E(q|X_r(b))$ by choice of ζ. And of course. $E(q|X_r(b). G. G) > E(q|X_r(b). B. B)$. The second part of the Lemma follows. ■

Finally. the following Lemma demonstrates that $t_{Br}(b) - t_{Gr}(b) \to 0$ uniformly for all b. This proves Theorem 1.

Lemma 8 *There is $r(.) > 0$ such that for all $\epsilon > 0$. for all $r > r(\epsilon)$. and for all $b \in \mathcal{H}_{Gr} \cup \mathcal{H}_{Br}$. $t_{Br}(b) - t_{Gr}(b) \leq \epsilon$.*

Proof Let

$$\delta = \min_{x \in [0, \bar{t} - \epsilon]} (W(x+\epsilon) - W(x))$$

and let $r(\epsilon) = r_1(\delta)$. where $r_1(.)$ is as given by Lemma 7. Note that $\delta \leq \epsilon$ and if $t_{Br}(b) - t_{Gr}(b) > \epsilon$ then $H_{Br}(b) - H_{Gr}(b) \geq \delta$.

Choose $r > r(\epsilon)$ and assume that there is $b \in \mathcal{H}_{Gr} \cup \mathcal{H}_{Br}$ such that $t_B(b) - t_G(b) > \epsilon$. We suppress the r subscript for the remainder of this proof.

Define $\bar{b}_s = \min_{b' > b} \mathcal{H}_s$ whenever this is well defined and set $\bar{b}_s = b$ otherwise.[8] Define $\underline{b}_s = \max_{b' < b} \mathcal{H}_s$ if this is well defined and set $\underline{b}_s = b$ otherwise.[9] Also note that if $b \in \mathcal{H}_G \cap \mathcal{H}_B$ then $\bar{b}_s = \underline{b}_s = b$.

[8] Note that if $\bar{b}_s$ is not well defined then it must be that $s = B$.

[9] Note that if $\underline{b}_s$ is not well defined then it must be that $s = G$.

Case (i) If $\bar{b}_G \geq \bar{b}_B$ define $\bar{b} = \bar{b}_G$ and $\underline{b} = \max_{b' < b} \mathcal{H}_B$.

Case (ii) If $\bar{b}_G < \bar{b}_B$ and $\underline{b}_G \geq \underline{b}_B$ then define $\underline{b} = \underline{b}_B$ and $\bar{b} = \min_{b' > \underline{b}} \mathcal{H}_G$.

Case (iii) If $\bar{b}_G < \bar{b}_B$ and $\underline{b}_G < \underline{b}_B$ then define $\bar{b} = \bar{b}_G$ and $\underline{b} = \underline{b}_B$.

Note that if $b \in \mathcal{H}_G \cap \mathcal{H}_B$ then $\bar{b} = \underline{b} = b$. The above construction is necessary to account for the possibility of discontinuities in the bid functions. We have the following additional properties of $(\underline{b}, \bar{b})$:

1. $(\underline{b}, \bar{b}) \cap \mathcal{H}_s = \emptyset$ for all s.

2. $t_B(\underline{b}) \geq t_B(b)$.

3. $t_G(\bar{b}) \leq t_G(b)$.

4. $\underline{b} \in \mathcal{H}_B$, $\bar{b} \in \mathcal{H}_G$.

To see (1) in case (i), note that by construction $\mathcal{H}_B \cap (\underline{b}, \bar{b}) = \emptyset$ and also by construction $\mathcal{H}_G \cap (b, \bar{b}) = \emptyset$. Since $\bar{b} \geq \bar{b}_B$ and since $\bar{b}_B \in \mathcal{H}_B$ it follows that $\underline{b} \geq \bar{b}_B \geq b$ and so $\mathcal{H}_G \cap (\underline{b}, \bar{b}) = \emptyset$. Case (ii) is analogous and (iii) is trivial.

To see (2) in case (i) note that as we just observed, $\underline{b} \geq b$ and since $t_B(b)$ is increasing we conclude that $t_B(\underline{b}) \geq t_B(b)$. An analogous argument establishes (3) in case (ii).

To see (2) in case (ii) or (iii), note that in these cases, $\mathcal{H}_B \cap (\underline{b}, b) = \emptyset$ and thus $t_B(\underline{b}) = t_B(b)$. A similar argument establishes (3) in cases (i) and (iii).

Now, by (4), and since $\mathcal{B}$ is dense on $\mathcal{H}_s$, there is $\underline{b}' \in \mathcal{H}_B \cap \mathcal{B}$ arbitrarily close to $\underline{b}$, and $\bar{b}' \in \mathcal{H}_G \cap \mathcal{B}$ arbitrarily close to $\bar{b}$.

By Proposition 1

$$E(q|d = \underline{b}', B) + t_B(\underline{b}') = \underline{b}'$$

and

$$E(q|d = \bar{b}', G) + t_G(\bar{b}') = \bar{b}'$$

So,

$$\begin{aligned}
t_B(\underline{b}') - t_G(\bar{b}') &= (\underline{b}' - \bar{b}') + E(q|d = \bar{b}', G) - E(q|d = \underline{b}', B) \\
&= (\underline{b}' - \bar{b}') + E(q|X(\bar{b}'), b_2 = \bar{b}', G) - E(q|X(\underline{b}'), b_2 = \underline{b}', B) \\
&\leq E(q|X(\bar{b}'), b_2 = \bar{b}', G) - E(q|X(\underline{b}'), b_2 = \underline{b}', B) \\
&\leq E(q|X(\bar{b}'), G, G) - E(q|X(\underline{b}'), B, B) \qquad (12)
\end{aligned}$$

Since this holds for $\underline{b}'$ and $\bar{b}'$ arbitrarily close enough to $\underline{b}$ and $\bar{b}$, it follows that

$$t_B(\underline{b}) - t_G(\bar{b}) \leq E(q|X(\bar{b}), G, G) - E(q|X(\underline{b}), B, B)$$

But since

$$(\underline{b}, \bar{b}) \cap (\mathcal{H}_G \cup \mathcal{H}_B) = \emptyset$$

it follows that

$$X(\underline{b}) = X(\bar{b}) = X(b)$$

Therefore, since $t_B(b) - t_G(b) > \epsilon$, Lemma (7) implies that

$$E(q|X(\bar{b}), G, G) - E(q|X(\underline{b}), B, B) < \epsilon.$$

This is a contradiction since by construction

$$t_B(\underline{b}') - t_G(\bar{b}') \geq t_B(b) - t_G(b) \geq \epsilon. \blacksquare$$

5.3 Proof of Theorem 2

The proof of Theorem 2 proceeds in two parts. The following lemma shows that if b is the pivotal bid then, for large r,there is essentially no uncertainty about the value of q for large r.

Lemma 9 *For all $\delta > 0$ there is an $\bar{r}$ such that for all $r > \bar{r}$ there is a $C_r \subset [0, \infty)$ with the property that $Pr\{d_r \notin C_r\} < \delta$ and $Pr\{|q - E(q|d_r = b)| > \delta | d_r = b\} < \delta$ for every $b \in C_r$.*

Proof For $0 \leq x \leq 1$, let t_x satisfy $W(t_x) = x$. Note that by the strong law of large numbers the fraction of bidders with $t < t_x$ converges to x in probability as $r \to \infty$. By Lemma 8, for any $\varepsilon > 0$, $t_{Br}(\min \mathcal{H}_{Gr}) \leq \epsilon$ for $r > r(\epsilon)$ and $t_{Gr}(\max \mathcal{H}_{Br}) \geq 1 - \epsilon$. If ϵ is chosen so that $W(\epsilon) < \beta/2$ and $W(1-\epsilon) > \beta/2$ (where β is from Assumption 1) then the strong law of large numbers implies that there is an r' such that for $r > r'$

$$\Pr\{d_r \in [\min \mathcal{H}_{Gr}, \max \mathcal{H}_{Br}]\} > 1 - \delta/2$$

Thus choose $C_r = [\min \mathcal{H}_{Gr}, \max \mathcal{H}_{Br}] \cap \mathcal{B}_r$ and let $\bar{r} > r'$.

By Lemma 5 there is an $\eta > 0$ such that if $\Pr(|q - E(q|d = b)| > \delta | d = b) > \delta$ then $E(q|d = b, G) - E(q|d = b, B) > \eta$. Now let $\bar{r} > r(\eta\delta/2)$ and consider any $r > \bar{r}$. (In the following we drop the subscript r.)

Step 1 Consider first any $b \in \mathcal{H}_B \cap \mathcal{H}_G \cap C$. Then

$$E(q|d = b, B) + t_B(b) = b$$

and

$$E(q|d = b, G) + t_G(b) = b$$

Since $r > r(\eta) > r(\eta\delta/2)$ this implies that $t_B(b) - t_G(b) < \eta$ but then

$$E(q|d = b, G) - E(q|d = b, B) < \eta$$

and so

$$\Pr(|q - E(q|d = b)| > \delta|\, d = b) < \delta$$

Step 2 Define

$$Z = \{b \in C : \Pr(|q - E(q|d = b)| > \delta|\, d = b) > \delta\}.$$

By Step 1 we know that $b \in Z$ implies that $b \notin (\mathcal{H}_G \cap \mathcal{H}_B)$. Consider any $b \in Z$, $b \in \mathcal{H}_B$ and $b \notin \mathcal{H}_G$. Define $\bar{b} = \min_{b'>b} \mathcal{H}_G$ and $\underline{b} = \max_{b'<b} \mathcal{H}_G$. Let $\mathcal{B}' = \mathcal{B} \cap \mathcal{H}_B \cap (\underline{b}, \bar{b})$. For every $b' \in \mathcal{H}_B \cap (\underline{b}, \bar{b})$

$$E(q|X(b'), B) - b' + t_B(b') = 0.$$

and thus since $\bar{b} \geq b$,

$$E(q|X(b'), B) - b' + t_B(\bar{b}) \geq 0.$$

By Lemma 3 both $\underline{b}$ and $\bar{b}$ are best responses for G with $t_G(\bar{b}) = t_G(\underline{b})$, and thus

$$\begin{aligned}
0 &= \int_{\mathcal{B}'} \left(E(q|X(b), G) - b + t_G(\bar{b})\right) dK(b|G) \\
&\geq \int_{\mathcal{B}'} \left(E(q|X(b), G) - b + t_B(\bar{b})\right) dK(b|G) - \frac{\delta\eta}{2} \int_{\mathcal{B}'} dK(b|G) \\
&\geq \int_{\mathcal{B}'} \left(E(q|X(b), B) - b + t_B(\bar{b})\right) dK(b|G) - \frac{\delta\eta}{2} \int_{\mathcal{B}'} dK(b|G) + \eta \int_{Z \cdot \mathcal{B}'} dK(b|G) \\
&\geq \eta \int_{Z \cdot \mathcal{B}'} dK(b|G) - \frac{\delta\eta}{2} \int_{\underline{b}}^{b} dK(b|G)
\end{aligned}$$

where the first inequality follows from the fact that $t_B(\bar{b}) - t_G(\bar{b}) \leq \delta/2\eta$, the second uses the fact that on Z, $E(q|X(b), G) - E(q|X(b), B) > \eta$ and the final inequality uses the fact that $E(q|X(b), B) - b + t_B(\bar{b}) \geq 0$ for all $b \in \mathcal{B}'$. Thus we have that

$$\frac{K(Z \cap (\underline{b}, \bar{b}))}{K((\underline{b}, \bar{b}))} \leq \delta/2.$$

A similar argument can be made for $b \in \mathcal{H}_G, b \notin \mathcal{H}_B$. Thus we conclude that the probability that the pivotal bid is in Z is less than $\delta/2$ which proves the proposition. ■

Proof of Theorem 2 By Lemma 9 it follows that whenever $b \in C_r$, $F(q|d_r = b)$ is concentrated around its expectation. Now consider the probability distribution of q conditional on the *price* being b. Note that the price is b if k_r bidders bid above b, one bidder bids b, and $n_r - k_r - 1$ bidders bid below b. Thus $F(q|p_r = b) = F(q|d_r = b, b_1 \geq b)$ and hence

$$|F(q|d_r = b) - F(q|p_r = b)| \leq \max_{s \in \{B,G\}} |F(q|d_r = b) - F(q|d_r = b, s)|$$

Since the distribution $F(q|d_r = b)$ is arbitrarily concentrated around its mean for large r, adding one additional (noisy) signal only changes the distribution by a very small amount. More precisely, for all $\epsilon > 0$ there is an r' such that for $r > r'$

$$|F(q|d_r = b) - F(q|p_r = b)| < \epsilon$$

and

$$|E(q|d_r = b) - E(q|p_r = b)| < \epsilon$$

with probability larger than $1-\epsilon$. Thus, since by Lemma 9 $F(q|d_r = b)$ is concentrated around the true value q^* it follows that $F(q|p_r = b)$ is also concentrated around the true value q^*.

Consider a bid $b \in C_r$. If a bidder with type (t, s) made the bid b and if the equilibrium price is equal to b then by the preceding argument

$$p_r \in [t + q^* - (\delta + \epsilon), t + q^* + (\delta + \epsilon)]$$

with probability $1 - \epsilon - \delta$, where q^* denotes the true q. Further note that by Theorem 2 it must be the case that for large r the bidder who makes the $k_r + 1^{st}$ highest bid has a t very close to t_r^* with high probability. Thus for r sufficiently large we have that

$$p_r \in [t_r^* + q^* - (2\delta + \epsilon), t_r^* + q^* + (2\delta + \epsilon)]$$

with probability larger than $1 - \epsilon - 2\delta$.

To prove the Theorem it is now sufficient to show that

$$\Pr\{p_r \notin C_r\} \to 0$$

as $r \to \infty$. But $\{p_r \in C_r\}$ denotes the event that the $k_r + 1$-st highest bid of n_r bids is in C_r.

By construction, $W[t_{Br}(\min \mathcal{H}_{Gr})] \leq \beta/2$ and $W[t_{Gr}(\max \mathcal{H}_{Br})] \geq 1 - \beta/2$. Thus the strong law of large numbers implies that

$$\Pr\{p_r \in [\min \mathcal{H}_{Gr}, \max \mathcal{H}_{Br}]\} \to 1,$$

which proves the final claim. ■

5.4 Proof of Proposition 2

Proof of Proposition 2 We will demonstrate that $b_r(t, G) \to t$ uniformly for all $t \leq t^* - \varepsilon$. Since $t \leq b_r(t, B) \leq b_r(t, G)$ this proves the result for $t \leq t^* - \varepsilon$.

Consider $t \leq t^* - \varepsilon$, and $b_r(t, G) \in C_r$ (where C_r is defined as in Lemma 9). By Lemma 9 we know that

$$\Pr\{|q - E(q|d_r = b_r(t, G))| > \delta |\, d_r = b_r(t, G)\} \to 0$$

uniformly for all $b_r(t, G) \in C_r$ which in turn implies that

$$\Pr\{|q - E(q|X(b_r(t, G))| > \delta |\, X(b_r(t, G))\} \to 0$$

uniformly for all $b_r(t, G) \in C_r$ since the information about q contained in the event $\{d_r = b_{Gr}\}$ differs from the information about q contained in the event $X(b_r(t, G))$ by at most one signal. Thus, uniformly for all $b_r(t, G) \in C_r$,

$$f(q|X(b_r(t, G))) \equiv \frac{f(q)\alpha_r(q, b_r(t, G))^{n_r}}{\int_0^1 f(w)\alpha_r(w, b_r(t, G))^{n_r} dw} \tag{13}$$

converges to a density that has all its mass concentrated at some $\hat{q}_r$. Recall that is a single peaked function of q. Since $f(q) > \gamma > 0$ and bounded, $\hat{q}_r$ must satisfy

$$|\hat{q}_r - \arg\max_q \alpha_r(q, b_r(t, G))| \to 0$$

uniformly for all $b_r(t, G) \in C_r$.

By Theorem 2,

$$H_{Br}(b_r(t, G)) - H_{Gr}(b_r(t, G)) \to 0 \tag{14}$$

and

$$H_{Br}(b_r(t, G)) - \pi_G(q)(H_{Br}(b_r(t, G)) - H_{Gr}(b_r(t, G))) \to W(t)$$

for all q. Therefore, there is a $\delta > 0$ such that for sufficiently large r and $t \leq t^* - \varepsilon$

$$\begin{aligned} & H_{Br}(b_r(t,G)) - \pi_G(q)(H_{Br}(b_r(t,G)) - H_{Gr}(b_r(t,G))) \qquad (15) \\ \leq \; & W(t) + \delta/2 \\ \leq \; & W(t^* - \varepsilon) + \delta/2 \\ \leq \; & 1 - \frac{k_r}{n_r} - \delta/2. \end{aligned}$$

But then (14) and (15) imply that uniformly for all $b_r(t,G) \in C_r$ with $t \leq t^* - \varepsilon$

$$\arg\max_q \alpha_r(q, b_r(t,G)) \to 0$$

and consequently

$$\hat{q}_r \to 0.$$

Thus, $b_r(t,G) \leq t + E(q|X_r(b),G) \to t$ as $r \to \infty$ for all $b_r(t,G) \in C_r$ with $t \leq t^* - \varepsilon$. Note that C_r contains all but a countable number of the bids of types (t,G) with $t \leq t^* - \varepsilon$. Thus we have established that for all but countably many t with $t \leq t^* - \varepsilon$, $\lim b_r(t,G) = t$. But since $b_r(t,G)$ is strictly increasing in t for all r,the fact that $\lim b_r(t,G) = t$ on a dense subset of the support of G implies that $\lim b_r(t,G) = t$ for all $t \leq t^* - \varepsilon$.

The argument for t, such that $t^* + \varepsilon \leq t$ is exactly analogous and therefore omitted. ■

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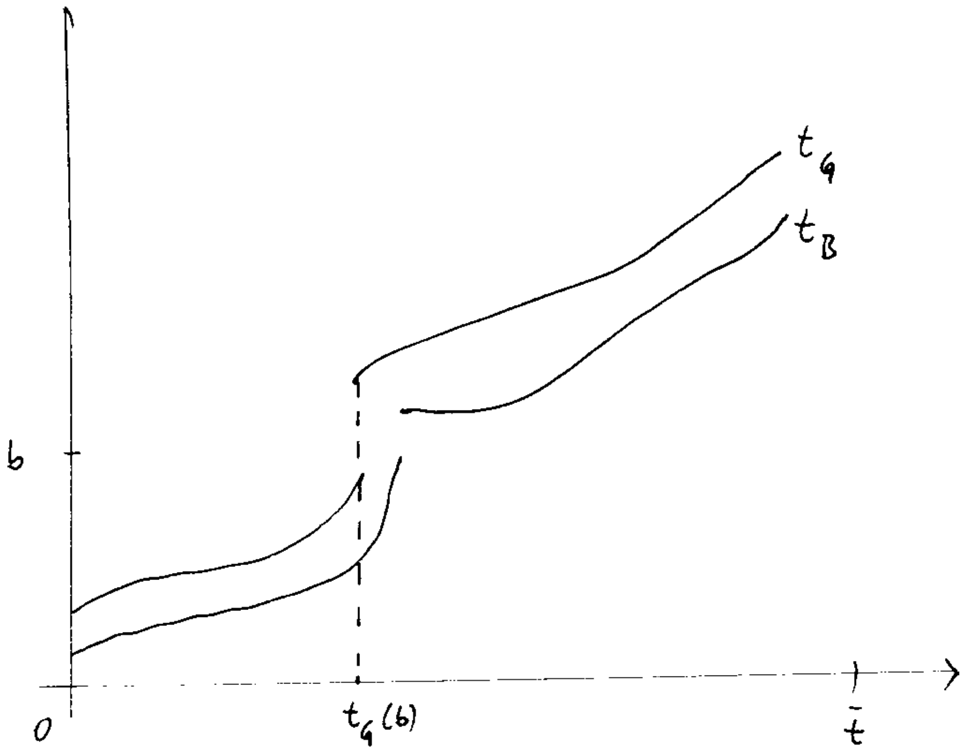

Figure 2