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A THEORY OF THE FIRM WITH NON-BINDING EMPLOYMENT CONTRACTS

by

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This version: June 1996

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I worked on this paper while visiting the Center for Rationality and Interactive Decision Theory and the Department of Economics of the Hebrew University of Jerusalem, and the Institute of Economic Analysis at the University Autonoma di Barcelona. I would like to acknowledge helpful comments by Henrik Horn, Sergiu Hart, John Hillas, Joram Mayshar, and Yoram Weiss.

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Abstract

This paper analyzes a dynamic model of a firm in which the wage of each employee is determined in separate bilateral negotiations with the firm. The contracts between the firm and its employees are non-binding in the sense that they can be repeatedly renegotiated to adjust to changing situations. The bargaining power of an employee stems from the threat of quitting that will deprive the firm of this worker's marginal contribution and will put the firm in a weaker position against the remaining workers. This threat is offset to some extent by the replacement opportunities that the firm has, but these are only imperfect in the sense that replacement of quits requires time and effort. The paper characterizes a class of equilibria for this scenario and examines their features. These include a sharp decline of the wage at the firm's target employment level, a mark-up of the wage over the employees' reservation wage and over-employment.

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1. Introduction

This paper analyzes a dynamic model of a firm in which the wage of each employee is determined in separate bilateral negotiations with the firm. The contracts between the firm and its employees are non-binding in the sense that they can be repeatedly renegotiated to adjust to changing situations. The bargaining power of an employee stems from the threat of quitting that will deprive the firm of this worker's marginal contribution and will put the firm in a weaker position against the remaining workers. This threat is offset to some extent by the replacement opportunities that the firm has, but these are only imperfect in the sense that replacement of quits requires time and effort. Employment relations with some of these features are quite wide spread, especially in industries that employ highly skilled labor.

The model builds on an earlier model by Stole and Zwiebel (1995). They develop and analyze a model of such a firm, under the assumption that workers who disagree with the firm and leave cannot be replaced. The present paper develops a dynamic version of this scenario that incorporates the possibility of worker replacement. Initially, the replacement opportunities are modelled as an exogenous stochastic process that brings candidates to the firm. Later on, this process is endogenized by letting the firm affect at a cost the arrival rate of new prospective employees. From a substantive point of view, the possibility of worker replacement is one of the central elements in such a situation. So its incorporation into the model constitutes a significant step in the analysis of this scenario. From a modeling point of view, the incorporation of this feature also involves a non-trivial step, since it entails the construction and analysis of a dynamic model in place of the static model considered before.

The model features a firm which uses labor as its only variable input. Two processes take place over time. First, a stochastic process brings new potential employees to the firm and, second, the wages of the employees are constantly renegotiated. In the absence of binding contracts, the firm and a worker can only agree on the worker's current wage rate. But in so doing they take into account the expected value of their continued relations. The bargained wage splits equally the surplus of this value over the value of the disagreement outcomes. The disagreement value for the worker is the value of his outside opportunities; the disagreement value for the firm is the continuation value of the hiring and renegotiation processes without the quitting worker. The behavior of the firm is described by an employment policy which specifies its employment decisions as a function of the history of the process. An equilibrium consists of an employment policy from which the firm does not wish to deviate after any history. Attention is restricted to target level policies. These policies are such that, after any history, the firm aims at hiring up to some target employment level (which might change following deviations from the original plan). We characterize the set of equilibria supportable by these policies and draw some specific qualitative insights on the nature of the equilibrium employment and wages. We also consider two variations on the basic model. In one of these variations the arrival rate of new prospective employees is controlled by the firm at a cost. In the other variation the firm is also constantly losing employees through a "death" process. Without going into the description of the equilibrium set and the distinctions between the different equilibria, let us mention three of the qualitative insights that emerge (the model has multiple equilibria and some qualifications will apply to these insights). First, the behavior of the wage as a function of employment fits into two distinct phases. In the build-up phase in which the employment level is short of its target level, the wage is relatively high. In the terminal phase, in which the firm's employment is at its target level, the wage is distinctly lower (in a continuous labor version of the model, the wage function is discontinuous at this point). This wage profile owes to the absence of binding wage contracts. Earlier hires have a stronger position, but since they anticipate their wages to be renegotiated downwards as more workers are hired, they have to receive a relatively higher wage initially. Second, the wage paid in the terminal phase, after the target employment level was reached, exhibits a mark-up over the workers' reservation wage which represents their alternative opportunities. Since the employment level is determined in the model, this observation is not a straightforward consequence of the fact that wages are determined in bargaining and indeed it does not appear in the static version of this model. Third, the firm over-employs workers in the sense that, in the long-run employment level, the value of the marginal product of labor is lower than the wage. The explanation is that additional workers reduce the bargaining power of their colleagues, so that the contribution of a marginal worker to the profit exceeds the value of this worker's contribution to production. The last insight already appears in the work of Stole and Zwiebel, but the analysis here can also relate the extent of the over-employment to the worker substitution possibilities. The first two insights are generated by the dynamic replacement opportunities and hence are absent from the static version.

The model is not a fully specified non-cooperative game. The employment decisions of the firm are modelled in the usual way for non-cooperative game models, but the wage bargaining component is left in a black box. It is assumed directly that the bargained wage splits equally the between each worker and the firm the surplus associated with the employment of that worker. The advantage of this approach is that it reduces the complexity of the model by avoiding the detailed strategic modeling of the bargaining processes. However, this approach raises a conceptual issue which will be discussed in detail in Section 8. To assure the reader we also outline there a strategic bargaining component that can be plugged into the model and will yield in equilibrium the same outcomes.

The immediately related literature consists only of the above mentioned work of Stole and Zwiebel (1995) on whose model the present paper builds. The extensive literature on wage determination in unionized firms is related of course through the feature that wages are determined in bilateral bargaining, but it considers a rather different situation in which the workers are cooperating. The contributions to that literature that consider firms who face two separate unions (see, Davidson (1988) Horn and Wolinsky (1988)) are somewhat closer in that they resemble the special case of a firm with two workers in the present model.

The plan of the paper is as follows. Section 2 presents the basic model. Section 3 contains the equilibrium analysis. Section 4 discusses the insights emerging from the equilibrium analysis. Section 5 derives the limit equilibrium outcomes for the case in which workers are negligible relative to the size of the firm. Section 6 endogenizes the worker replacement opportunities by letting the firm affect the arrival of new potential workers through costly recruiting policy. Section 7 discusses another extension whereby the work-force of the firm is also being affected by a stochastic departure process. Section 8 discusses the mixed cooperative/non-cooperative modeling approach and outlines an alternative fully non-cooperative model. Section 9 brings concluding remarks.

2. The Model

The actors in this model are a firm that uses labor as its only variable input and workers who are identical in their preferences and productivity. The events in the model take place over time. The time dimension is discrete and denoted by $t=1,2,\cdots$. The value of the output at any period t is $f(n_t)$, where n_t is the number of employees in that period and f is increasing and concave.

A worker's utility of being employed (by this firm) for the next T periods at wages $w_1,\ldots,w_T$ and not after that is

$$\sum_{t=1}^{T}\delta^{t-1}w_t + \delta^T W_U,$$

where $\delta\in(0,1)$ is the common discount factor, and W_U is a worker's (exogenously given) utility of not being employed by this firm. We shall assume that, once a worker is separated from the firm, he will not be employed by it again, so that only employment profiles of the above form are relevant. The firm's profit given a profile $\{n_t,c_t\}_{t=1,\ldots,\infty}$ of employment levels n_t and costs c_t is

$$\sum_{t=1}^{\infty}\delta^{t-1}[f(n_t)-c_t]$$

To assure that the model is not degenerate, it is assumed that $f(1)>(1-\delta)W_U$. Since $(1-\delta)W_U$ is the reservation wage of the workers (i.e., the minimal constant wage w at which the worker's utility of being employed in perpetuity, $w/(1-\delta)$, is just equal the utility of being unemployed, W_U), this assumption simply guarantees the there is room for profitable employment of at least one worker.

At the beginning of period t the firm faces m_t potential employees who are the n_{t-1} who were employed in period t-1 plus at most one new prospective employee who may arrive through an exogenous arrival process. The per-period probability of a new arrival is α, so that

$$\text{Prob}(m_t|n_{t-1}) = \begin{cases}\alpha & \text{if } m_t=n_{t-1}+1\\ 1-\alpha & \text{if } m_t=n_{t-1}\end{cases}$$

Out of the potential employees, the firm chooses the n_t it actually wants to employ in period t. Then the wages are determined in interaction between the workers and the firm. This interaction is not modelled as a non-cooperative game. Rather its outcome is assumed to split equally between each worker and the firm the surplus associated with the employment of that worker. Finally, production takes place, the wages are paid and the period ends.

The intention of the model is to capture a scenario in which the firm and the workers are not bound by any contract, the wage of each worker is set through individual and separate bargaining with the firm and can be renegotiated at will[1].

Definition: A history at period t, h_t, is a sequence of the form

$$h_t = m_1, n_1, \ldots, m_{t-1}, n_{t-1}, m_t$$

such that $n_i \leq m_i$ and, for $i \geq 2$, $m_i = n_{i-1}$ or $n_{i-1}+1$.

Let H_t denote the set of all the possible histories at t.

Definition: An employment policy is a sequence of functions $\{n_t\}$ each specifying a possible employment level as a function of history[2], i.e., $n_t: H_t \rightarrow \{0,1,\ldots,m_t\}$.

We shall assume that when an employment policy is implemented, the following rules always apply. First, the n_{t-1} employees from the past period have precedence in employment over the new arrival, so the latter is employed only if $n_t > n_{t-1}$. Second, the veteran employees are treated symmetrically: if $n_t < n_{t-1}$, they all have the same dismissal probability of n_t/n_{t-1}.

Without specifying yet how it is determined, let the wage at period t be given by a function $w_t(n_t,h_t)$. Now, after any history h_t, the sequences of functions $\{n_s\}_{s\geq t}$ and $\{w_s\}_{s\geq t}$ determine a stochastic process of wages and employment. Define:

$W_{E,t}(n,h_t,\{n_s\},\{w_s\})$ = The expected utility at period t of a worker who is employed at t by this firm, given the history h_t and any employment level $n \leq m_t$.

$\Pi_t(n,h_t,\{n_s\},\{w_s\})$ = The expected profit of the firm at period t, given the history h_t and any employment level $n \leq m_t$.

$$(2.1)\ W_{E,t}(n,h_t,\{n_s\},\{w_s\}) = w_t(n,h_t) + \delta\alpha\{\min(1,n_{t+1}(h_{t+1})/n)W_{E,t+1}(n_{t+1}(h_{t+1}),h_{t+1},\{n_s\},\{w_s\})$$
$$+ [1-\min(1,n_{t+1}(h_{t+1})/n)]W_U\} + \delta(1-\alpha)\{\min(1,n_{t+1}(\hat{h}_{t+1})/n)W_{E,t+1}(n_{t+1}(\hat{h}_{t+1}),\hat{h}_{t+1},\{n_s\},\{w_s\})$$
$$+ [1-\min(1,n_{t+1}(\hat{h}_{t+1})/n)]W_U\}$$

$$(2.2)\ \Pi_t(n,h_t,\{n_s\},\{w_s\}) = f(n)-nw_t(n,h_t)$$
$$+ \delta[\alpha\Pi_{t+1}(n_{t+1}(h_{t+1}),h_{t+1},\{n_s\},\{w_s\}) + (1-\alpha)\Pi_{t+1}(n_{t+1}(\hat{h}_{t+1}),\hat{h}_{t+1},\{n_s\},\{w_s\})],$$

where $h_{t+1}=(h_t,n,n+1)$ and $\hat{h}_{t+1}=(h_t,n,n)$ are the possible continuation histories at the beginning of t+1 which differ from one another in the arrival of a new worker at t+1.

According to (2.1), $W_{E,t}$ consists of the current wage, $w_t(n,h_t)$, plus the discounted value of continuation, which is $W_{E,t+1}$ or W_U depending on whether the worker is retained. The probability that this worker will be retained is $\min(1,n_{t+1}/n)$. Both this probability and $W_{E,t+1}$ depend on the history up to the beginning of t+1. Equation (2.2) is explained similarly.

Definition: The wage function w_t satisfies the equal split bargaining condition if for all t, h_t and $n \leq m_t$:

$$(2.3)\ W_{E,t}(n,h_t,\{n_s\},\{w_s\}) - W_U = \max[\Pi_t(n,h_t,\{n_s\},\{w_s\})- \Pi_t(n-1,h_t,\{n_s\},\{w_s\}),0]$$

The equal split condition says that the wage is determined so as to equate the worker's utility gain from continued employment (the LHS) to the firm's gain from retaining this worker (the RHS). As was mentioned before, the strategic interaction that determines the wage is not modelled explicitly, and this condition only characterizes its outcome. We shall return to discuss this issue in Section 8.

Definition: An employment policy $\{n_t\}$ is an equilibrium if for all t and any possible history h_t,

$$n_t(h_t) = \text{Argmax}_n \Pi_t(n, h_t, \{n_s\}, \{w_s\})$$

where the functions Π_t, $W_{E,t}$ and w_t satisfy (2.1-3).

That is, at any t, it is in the best interest of the firm to choose employment level $n_t(h_t)$, given the common expectation that in the future employment will continue to be governed by $\{n_t\}$.

3. Equilibrium Analysis

Wage and profit with target level employment policies

The first step of the analysis is characterization of the wage and profit functions arising for a class of simple employment policies. The results of this analysis will be important for the construction of the equilibria to be presented later and for the derivation of the main results. Given an employment policy $\{n_t\}$, let $n_{max}(h_t)$ denote the maximal employment level over the history h_t.

Definition: A target level employment policy $\{n_t\}$ is characterized by an employment level N as follows: $n_t(h_t) = \text{Min}[m_t, N]$, for all t and h_t such that $n_{max}(h_t) \leq N$.

That is, at least as long as the target level N was not exceeded, the firm plans to hire up to N and stop there. Given $\{n_s\}$, the evolution of h_s and hence $n_s(h_s)$ for $s \geq t+1$ depends only on n_t. Therefore, the value functions $W_{E,t}$ and Π_t and the wage function w_t also depend only on n_t. These functions must satisfy the following system (where the subscript t is suppressed to indicate independence of it).

(3.1) $W_E(n,N) = w(n,N) + \delta[\alpha W_E(\min\{n+1,N\},N) + (1-\alpha)W_E(n,N)],\ \ 1 \leq n \leq N$

(3.2) $\Pi(n,N) = f(n) - w(n,N)n + \delta[\alpha\Pi(\min\{n+1,N\},N) + (1-\alpha)\Pi(n,N)],\ \ 0 \leq n \leq N$

(3.3) $W_E(n,N) - W_U = \mathrm{Max}[\Pi(n,N) - \Pi(n-1,N),0]\ \ 1 \leq n \leq N$

<u>Definition</u>: N is <u>feasible</u> if in the solution to (3.1-3) $\Pi(n,N) \geq \Pi(n-1,N) \geq 0$, for $1 \leq n \leq N$.

Thus, feasibility assures that, when the firm plans to hire up to the level N, it indeed wants to follow this plan in every step on the way.

Let $\Delta f(n) = f(n) - f(n-1)$, the value of the marginal product when there are n workers. Let

$$\Psi(n) = \frac{2}{n(n+1)}\sum_{i=1}^{n} i\Delta f(i)\ .$$

Since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ the term $\frac{2}{n(n+1)}\sum_{i=1}^{n} i\Delta f(i)$ is a weighted sum of the marginal products with higher weights assigned to the marginal products of later units[3]. Since f is concave, Ψ is a decreasing function of n. Let N^0 denote the maximal n such that $\Psi(n) \geq (1-\delta)W_U$. Since $f(1) > (1-\delta)W_U$, such a positive N^0 exists. Assume that $\Psi(N^0)$ is <u>exactly</u> equal to $(1-\delta)W_U$. This assumption will simplify the presentation of later arguments by avoiding "integer problems."

<u>Proposition 1</u>: (i) For $N \leq N^0$, the solution to system (3.1-3) is

$$(3.4)\ w(n,N) = \begin{cases} [\Psi(n)+(1-\delta)W_U]/2 & n<N \\ [(N+1)(1-\delta)\Psi(N)+[(N+1)(1-\delta)+2\delta\alpha](1-\delta)W_U]/2[(N+1)(1-\delta)+\delta\alpha] & n=N \end{cases}$$

(ii) N is feasible iff $N \leq N^0$.

The proof as well as all subsequent proofs are relegated to the appendix. Both at $n<N$ and at $n=N$, the wage $w(n,N)$ is a weighted average of $\Psi(n)$ and the reservation wage $(1-\delta)W_U$, with a higher weight on $(1-\delta)W_U$ when $n=N$. Since $\Psi(n)$ is monotonically decreasing in n, so is $w(n,N)$. Since $\Pi(N,N)=[f(N)-w(N,N)]/(1-\delta)$, (3.4) also yields an explicit expression for $\Pi(N,N)$. Let $\pi(n)$ denote the profit flow accruing to a firm who employs n workers at the workers' reservation wage $(1-\delta)W_U$,

$$\pi(n) = f(n) - n(1-\delta)W_U.$$

$$(3.5) \quad \Pi(N,N) = \frac{1}{1-\delta}\{\frac{\delta\alpha}{(N+1)(1-\delta)+\delta\alpha}\pi(N) + \frac{(N+1)(1-\delta)}{(N+1)(1-\delta)+\delta\alpha}[\frac{1}{N+1}\sum_{i=1}^{N}\pi(i)]\}$$

Thus, $(1-\delta)\Pi(N,N)$ -- the flow equivalent of $\Pi(N,N)$ -- is a weighted average of $\pi(N)$ and the arithmetic average of $\pi(i)$ for $i=0,\ldots N$.

Figure 1 should be placed about here

Figure 1 depicts the relationship between $(1-\delta)\Pi(n,n)$, $\pi(n)$ and $\frac{1}{n+1}\sum_{i=1}^{n}\pi(i)$. The concavity of $\pi(n)$ assures that the two solid curves are as shown. N^S is the maximizer of $\pi(n)$, and N^* is the maximizer of $\Pi(n,n)$. Clearly, $N^S<N^0$ since N^S is the largest n such that $\Delta f(n) \geq (1-\delta)W_U$. The following proposition confirms some of the properties shown in the figure.

Proposition 2: (i) The curves π, $(1-\delta)\Pi(n,n)$ and $\frac{1}{n+1}\sum_{i=1}^{n}\pi(i)$ intersect at N^0, where the latter has its unique maximum. (ii) $\Pi(n,n)$ has a unique local and global maximum (or two adjacent maxima) at N^*. (iii) $N^S \leq N^* < N^0$.

Equilibria in target level policies

This model has many equilibria. We shall focus on a subset of these equilibria in which the employment policies are of the target level variety (on and off the path). That is, at any point the firm plans to keep hiring up to a certain target level (which might change following deviations from the plan) and to remain at that level thereafter.

Definition: A stationary target level policy is characterized by an employment level N as follows: $n_t(h_t)=\min[m_t,N]$, for all t and all h_t.

The difference between this and the earlier definition of a target level policy is that here the fixed target is supposed to be followed after any history.

Proposition 3: The unique equilibrium in stationary target level policies is with target level N^0.

The equilibrium of Proposition 3 is self-sustaining in the sense that, given the common expectations that the firm will behave as dictated by the target level N^0, it is in the best interest of the firm to do so after any history. At $n<N^0$, the firm is induced to continue hiring by the declining profile of wages between n and N^0. Even when $w(n,N^0)>\Delta f(n)$ so that an additional worker costs more than the value of his marginal product, as is the case for $n>N^S$, the reduction of the total wage bill due to additional hiring is profitable. At $n>N^0$, the workers have to be paid at least their reservation wage $(1-\delta)W_U$, since the firm is expected to return to N^0 where the workers' continuation utility will be W_U. This means that hiring beyond N^0 does not reduce the total wage bill, so it must be unprofitable as $\Delta f(n)<(1-\delta)W_U\leq w(n,N)$.

Definition: A semi-stationary target-level employment policy $\{n_t\}$ is such that:

(i) After any history h_t, there is a target level $N(h_t)$ such that $n_t(h_t)=\min[m_t,N(h_t)]$.

(ii) If $n_t \leq N(h_t)$, then $N(h_t,n_t,m_{t+1})=N(h_t)$.

The semi-stationary target level policies are a broader class: after any history, the firm still aims at reaching a certain target level, but the target level may change if the firm deviated from the policy and exceeded the prevailing target level.

Proposition 4: Let $\underline{N}$ be the minimal n such that $\Pi(n,n) \geq \Pi(N^0,N^0)$. There exists an equilibrium in a semi-stationary target level policy such that $N(m_1)=N$, for all m_1, iff $N \in [\underline{N},N^0]$.

As pointed out above, The equilibrium of Proposition 3 is self-sustaining. In contrast, the equilibria of Proposition 4 are sustained by a threat of triggering another less profitable target level as a punishment, in the event that the original equilibrium target level is exceeded. The proof employs the N^0 equilibrium of Proposition 3 as the punishment phase that supports all these equilibria. Consider an equilibrium target level N. At $n<N$ the firm is induced to continue hiring by the declining profile of wages between n and N, just as was explained above. However, if at $n=N+1$, the firm was still expected to return to N, the argument given for the N^0 equilibrium would not work since here $W_E(N,N)>W_U$ and the firm would achieve a reduction of the wage bill by such additional hiring. Instead, such additional hiring is deterred by triggering the less profitable N^0 equilibrium. This also explains why all the equilibria that are sustained in this way are at least as profitable as the N^0 equilibrium.

The abrupt change of the target level following a deviation is perhaps not too appealing as description of behavior: the firm hires an additional worker and it is immediately expected to continue up to N^0 which might be substantially higher. While there is no fundamental game theoretic reason to disqualify such equilibria, it may still be of interest to inquire about equilibria that rely on policies that permit only more moderate changes of the target level in response to deviations. The policies in the class defined below are such that the change in the target level triggered by a deviation does not exceed the extent of the deviation.

Definition: A gradual semi-stationary policy $\{n_t\}$ is a semi-stationary policy such that, if $N(h_t) \neq N(h_{t-1})$, then $N(h_t) \leq n_{max}(h_t)$.

Proposition 5: There exists an equilibrium in a gradual semi-stationary policy such that $N(m_1)=N$, iff $N \in [N^*,N^0]$.

Consider an equilibrium with target level $N<N^0$. It is supported by a policy embodying expectations with adaptive flavor. If the firm exceeds this target level and hires $N+1$ workers, the expectations of everybody are updated to the new target level $N+1$. Since $\Pi(n,n)$ is decreasing between N^* and N^0, such a deviation is unprofitable. Obviously, this argument fails for target levels below N^*, where $\Pi(n,n)$ is increasing in n.

The subset of the equilibria identified in Proposition 5 will be the focus of our subsequent discussion, although we will also comment about other equilibria.

Specific equilibria

Let us call attention to the two endpoints of the subset of equilibria identified by Proposition 5.

The N^0 equilibrium. This equilibrium has a special status in the model in that it serves as the punishment phase which sustains the other equilibria. Following are three of its most salient features. First, this is the only equilibrium in the simple stationary employment policies. Second, the long run outcome of the N^0 equilibrium is independent from the arrival rate α in the sense that the target level N^0 itself as well as the wage $w(N^0,N^0)$ and profit $\Pi(N^0,N^0)$ do not depend on α. N^0 is the solution to $\Psi(n)=(1-\delta)W_U$, which is obviously independent of α and this immediately implies $w(N^0,N^0)=(1-\delta)W_U$ and $\Pi(N^0,N^0)=\pi(N^0)/(1-\delta)$. Third, at N^0 the outcome is Pareto inferior to the outcomes of some of the other equilibria. As seen in the figure, $\Pi(N^0,N^0)<\Pi(N,N)$ for some $N<N^0$ and $W_E(N^0,N^0)=w(N^0,N^0)/(1-\delta)=W_U$ which is the minimum possible worker utility.

The Pareto inferiority is perhaps unappealing, but it is not a surprising property of an equilibrium. However, the independence of α is somewhat counterintuitive. In the model of Stole and Zwiebel which is a static version of the present model, without worker replacement opportunities, the N^0 equilibrium is the only equilibrium. My initial conjecture was that, in the presence of substitution possibilities, this outcome would be modified and that it would resemble the outcome under a wage setting firm, when the substitution possibilities are nearly perfect (with δ and α are close to 1). As it turns out, this equilibrium has a very robust existence in the model. The N^* equilibrium. This equilibrium is distinguished by being the maximizer of long-run average profit, $N^*=\text{Argmax}\Pi(N,N)$. Note, however, that it does not maximize the firm's profit starting at any initial level. For example, the equilibrium based on the target level $N^{**}=\text{Argmax}_N\Pi(0,N)\leq N^*$ will yield a higher profit starting from $m_1=0$ (the profit will be strictly higher when N^{**} is strictly smaller[4]). But the N^{**} equilibrium does not pass the stricter requirements of Proposition 5 (unless of course $N^{**}=N^*$).

The distinction between N^* and N^{**} is also reflected by the following "renegotiation-proofness" property. Once N^* is reached, there is no m_t on the path (i.e.,

$m_t = N^*$ or N^*+1) and another N equilibrium such that the firm would prefer to restart this other equilibrium with $m_1 = m_t$. The N^{**} equilibrium is not immune to this renegotiation test. At $m_t = N^{**}+1$ the firm would prefer to switch to the $N^{**}+1$ equilibrium. It is easy to verify that the N^* equilibrium is the only one with this property.

Multiplicity of equilibria

Even when attention is restricted to the relatively simple target level policies, the model has multiple equilibria. Obviously, there are many other equilibria outside this class as well. Using the idea of Proposition 4 of triggering the N^0 equilibrium as a punishment phase, it is possible to sustain equilibria with more complicated patterns, in which the employment level will change along the path and will even occasionally exceed N^0. It is also possible to construct equilibria on whose path the employment is maintained at a constant level $N < N^0$, but the wages are higher than in the semi-stationary equilibrium with the same target level. In such an equilibrium, the behavior will be conditioned on the history in a way that is not permitted by the target level policies: a disagreement with any employee will trigger the punishment phase thus increasing the surplus that is subject to the bargaining between the firm and each of its employees. In a sufficiently rich dynamic model as this one, there is no escape from multiplicity of equilibria. We can of course achieve easy "uniqueness" by requiring the full stationarity of Proposition 3. But this will deprive us of the broader range of insights that will be generated by some of the other equilibria.

4. Discussion

Recall that in all of the equilibria considered here the employment settles eventually to a target level N. The following discussion presents some of the salient features of the outcomes arising at these target levels. Throughout the discussion we will occasionally refer to the benchmark case of the wage setting firm, which sets the wages of its workers unilaterally. The wage setting firm always pays its workers exactly their reservation wage, i.e., $w^S = (1-\delta)W_U$. Its discounted profit at any stationary employment level n is $\pi(n)/(1-\delta)$ and its optimal target level is therefore the level $N^S = \text{Argmax}\,\pi(n)$ pointed out before.

The target level effect

In all the equilibria except the N^0 equilibrium the instantaneous wage w(n,N) changes sharply at the target employment level N. Inspection of the formulas reveals this in two ways. First, w(n,N) depends on the workers' arrival rate α at n=N, but is independent of α at n<N. Second, for any feasible n and N such that n<N, w(n,n)<w(n,N). That is, at a given n, the wage is lower if this n happens to be the target level than it would be otherwise[5].

The source of this gap is the absence of binding contracts. Since the wages of infra marginal workers will be renegotiated downwards when new workers arrive, they include some bonus upfront to compensate them for the eventual decline. After the target level N is reached, the wage will remain constant and no such upfront bonus is needed. To understand this argument more precisely, suppose for a moment that wage contracts were binding and let v(n,N) denote the perpetual wage agreed upon in this case. The appropriate version of the bargaining equation (3.3) is then

$$v(n,N)/(1-\delta) - W_U = \Pi(n,N)-\Pi(n-1,N), \quad \text{for all } n \leq N.$$

Hence,

$$v(n,N) = (1-\delta)W_U + (1-\delta)[\Pi(n,N)-\Pi(n-1,N)], \quad \text{for all } n \leq N.$$

In contrast, using (3.1) and (3.3) in the model of this paper,

$$w(n,N) = \begin{cases} (1-\delta)W_U+(1-\delta)[\Pi(n,N)-\Pi(n-1,N)]+\delta\alpha\{[\Pi(n,N)-\Pi(n-1,N)]-[\Pi(n+1,N)-\Pi(n,N)]\}, & n<N \\ (1-\delta)W_U + (1-\delta)[\Pi(n,N)-\Pi(n-1,N)], & n=N \end{cases}$$

Thus, $w(N,N)$ is expressed by the same formula as $v(N,N)$. But for $n<N$, there is an extra term $\delta\alpha\{[\Pi(n,N)-\Pi(n-1,N)]-[\Pi(n+1,N)-\Pi(n,N)]\}$ which is absent from $v(n,N)$. This term captures the compensating bonus added to wages that are going to be renegotiated downward and it accounts for the gap between $w(n,n)$ and $w(n,N)$.

The fact that, for $n<N$, $w(n,N)$ is independent of α is a consequence of the uniformity of the arrival rate across states. If we allow these rates to differ and let $\alpha(n,N)$ denote the arrival rate at employment level n, then $w(n,N)$ might depend on the α's as well. In this case the basic difference equation for $w(n,N)$ is

$$w(n,N) = \begin{cases} [\Delta f(n)+(n-1)w(n-1,N)+(1-\delta)W_U+\delta[\alpha(n,N)-\alpha(n-1,N)][\Pi(n,N)-\Pi(n-1,N)]/(n+1) & n<N \\ [\Delta f(N)+(N-1)w(N-1,N)+(1-\delta)W_U-\delta\alpha(N-1,N)(\Pi(N,N)-\Pi(N-1,N)]/(N+1) & n=N \end{cases}$$

Thus, when $\alpha(k,N)\neq\alpha(k-1,N)$, $w(n,N)$ will depend on the arrival probabilities, but as the difference between the $n<N$ and the $n=N$ branches suggests, the target level effect will be still present for the reasons explained above. Obviously, when $\alpha(n,N)=\alpha(n-1,N)=\alpha$, as we assumed throughout, the term $\Delta\alpha(n,N)\Delta\Pi(n,N)$ vanishes. It is interesting to note, however, that in the model of Section 6 below in which arrival rates are selected optimally by the firm at each instant, the independence of $w(n,N)$ of the arrival rates emerges as a result again.

The wage mark-up

At any $N<N^0$ the wage embodies a mark-up over the reservation wage, $w(N,N)>(1-\delta)W_U$. This can be directly observed in Figure 1 from the fact that the $(1-\delta)\Pi(N,N)$ curve is strictly below the $\pi(N)$ curve. For a fixed N, the existence of the mark-up is perhaps not too surprising when the wage is determined in bargaining. But the fact that there is a mark-up at the endogenously determined level N^* is less obvious, and it is a consequence of the downward jump of the wage at the target level.

To see this point, assume that $w(N^*+1,N^*+1)>(1-\delta)W_U$ as well (this will be clearly the case if the "size" of the individual worker is small enough). It then follows from (3.1-3) that $\Pi(N^*+1,N^*+1)-\Pi(N^*,N^*+1) = W_E(N^*+1,N^*+1)-W_U = [w(N^*+1,N^*+1)-(1-\delta)W_U]/(1-\delta) > 0$. Now, in the absence of a target level effect, we would have $w(N^*,N^*+1)=w(N^*,N^*)$ implying $\Pi(N^*,N^*+1)= [(1-\delta)\Pi(N^*,N^*)+\delta\alpha\Pi(N^*+1,N^*+1)]/[1-\delta+\delta\alpha]$. It would then follow that $\Pi(N^*+1,N^*+1)>\Pi(N^*,N^*)$, contrary to the optimality of N^*. Therefore, in the absence of a target level effect, N^* would be incompatible with the existence of a wage mark-up.

Now, owing to the target level effect, $w(N^*,N^*+1)>w(N^*,N^*)$ hence $\Pi(N^*,N^*+1)<\Pi(N^*,N^*)$ and the previous argument breaks down. In other words, the downward jump of $w(n,N)$ at $n=N$ introduces a wedge between $\Pi(N-1,N-1)$ and the firm's disagreement payoff at N, $\Pi(N-1,N)$, which implies that the appropriation of the workers' surplus, as reflected by the wage mark-up, through a larger employment level is not necessarily profitable[6].

The role of α

At target levels $N<N^0$, the extent of the mark-up, $w(N,N)-(1-\delta)W_U$, as well as the extent of the downward jump of $w(n,N)$ at $n=N$, depend on the relative sizes of α and $(1-\delta)$. The ratio $(1-\delta)/\alpha$ captures the effective cost of replacing an employee, where α measures the speed with which the firm's work-force is built up, while $1-\delta$ measures the cost of time. When $(1-\delta)/\alpha$ is small, the mark-up $w(N,N)-(1-\delta)W_U$ is relatively small and the gap $w(N,N+1)-w(N,N)$ is relatively large. When $(1-\delta)/\alpha$ is large, the wage mark-up is relatively large and the gap is small.

The level $N^{*}=\text{Argmax}\Pi(N,N)$ also depends on the ratio $(1-\delta)/\alpha$. When $(1-\delta)/\alpha$ is small, the curve $(1-\delta)\Pi(N,N)$ is close to the curve $\pi(N)$, N^{*} is near N^{S} and $w(N^{*},N^{*})\approx(1-\delta)W_U$. When $(1-\delta)/\alpha$ is large, the $(1-\delta)\Pi(N,N)$ curve is close to the $\frac{1}{N+1}\sum_{i=0}^{N}\pi(i)$ curve, N^{*} is near N^{0}, and $w(N^{*},N^{*})\approx w(N^{0},N^{0})=(1-\delta)W_U$. The disappearance of the wage mark-up at the different extremes of $(1-\delta)/\alpha$ owes to quite different factors in each of them. An arbitrarily small $(1-\delta)/\alpha$ means that the firm incurs arbitrarily small cost in replacing workers. Its position thus resembles that of a wage setting firm which faces an infinitely elastic labor supply at the reservation wage $(1-\delta)W_U$. Accordingly, its optimal labor force is near the wage setting firm's optimum, N^{S}, where the value of the marginal product is just equal to the reservation wage. In contrast, when $(1-\delta)/\alpha$ is large, the firm is limited in replacing workers who leave. With slow replacement, the opposite of the argument presented in the sub-section on the mark-up shows that N^{*} cannot be associated with a significant wage mark-up. That is, in such a case $\Pi(N,N+1)\approx\Pi(N,N)$, which via the bargaining condition $w(N+1,N+1)/(1-\delta)-W_U=\Pi(N+1,N+1)-\Pi(N,N+1)$ implies that, when $w(N+1,N+1)$ exceeds $(1-\delta)W_U$ significantly, then $\Pi(N+1,N+1)>\Pi(N,N)$. In other words, higher employment would increase the firm's profit at the expense of the workers' surplus, and this is why it is driven to a level near N^{0} where the mark-up vanishes.

Over-employment

All the equilibria with target levels $N\geq N^{S}$ exhibit over-employment in the sense that the value of the marginal product of labor at N is lower than the wage, $\Delta f(N)<(1-\delta)W_U<w(N,N)$. The firm nevertheless continues hiring to meet the target level, since an additional worker not only increases production but also lowers the wage. The depressing effect on the wage is seen through the fact that $\Psi(n)$ is decreasing in n. That is, an

additional worker weakens the bargaining power of the other workers by reducing the surplus of which they will deprive the firm by quitting. At the target level N^* the marginal reduction in the wage bill due to additional hiring is just equal to the cost of employing an additional worker (the shortfall of the value of the marginal product of the wage). This over-employment phenomenon appears already in the model of Stole and Zwiebel, where it is actually more pronounced since the only equilibrium of their model has employment level N^0.

Notice that the equilibria with the lowest target levels, such as $\underline{N}$, may exhibit under-employment. At such an equilibrium, it might be that $\Delta f(N) > w(N,N)$, but the firm is deterred from hiring additional workers by the threat of triggering the N^0 equilibrium. Equilibria of this sort are not included in the narrower subset admitted by Proposition 5.

<u>**Distortion of other inputs**</u>

The employment of other inputs will also be distorted from their efficient levels. Suppose that the firm employs also capital denoted by K. The production function will now be f(n,K), and p_K will denote the rental rate of capital. Thus,

$$\pi(n,K) = f(n,K) - n(rW_U) - p_K K$$

The wage setting firm will obviously employ stock K^S such that $\pi_2(N^S,K^S)=0$, i.e., up to the level where its marginal product is equal its rental rate. At the optimal K of the negotiating firm

$$\frac{\partial \Pi(N,N;K)}{\partial K} \equiv \frac{1}{1-\delta}\left(\frac{\delta\alpha\pi_2(N,K) + (N+1)(1-\delta)\frac{1}{N+1}\sum_{i=1}^{N}\pi_2(i,K)}{(N+1)(1-\delta)+\delta\alpha}\right)=0$$

which means that the weighted average of the marginal products is equal to the rental rate

$$\frac{\delta\alpha f_2(N,K) + (1-\delta)(N+1)\frac{1}{N+1}\sum_{i=0}^{N}f_2(i,K)}{(N+1)(1-\delta)+\delta\alpha} = p_K$$

Thus, if capital and labor are complements, $\Delta f_2 > 0$, then the firm under-invests in capital in the sense that $f_2(N^*,K) > p_k$; if capital and labor are substitutes, $\Delta f_2 < 0$, then the firm over-invests in capital, $f_2(N^*,K) < p_k$. The incentives for investment in capital are distorted since it directly affects the wage: more capital increases or decreases the marginal product of labor and hence the negotiated wage according to whether $\Delta f_2 > 0$ or $\Delta f_2 < 0$.

The distortion in the employment of other inputs is a consequences of determination of the wage in bargaining, rather than any of the other features of the model. Indeed this point already appear in the Stole-Zwiebel model as well as in Grout(1984).

5. Limit versions of the outcomes

At the target levels that the above equilibria eventually reach, the wage and profit are given by (3.4) and (3.5). Before turning to discuss these results further, we derive two limit versions of these outcomes. The first version corresponds to the case in which the length of the time period is relatively short, so that decisions can be updated relatively quickly. The second captures a situation in which each worker is "small" relative to the size of the firm.

The continuous time limit

To derive this version, let the length of a time period be denoted by h. The parameters of the model now take the form

$$\delta = e^{-rh} \quad \text{and} \quad \alpha(h) = 1-e^{-\alpha h},$$

so that r is the discount rate and α is the arrival rate (rather than a probability). In the limit as $h \to 0$, system (3.1-3) becomes:

$$\text{(5.1)} \qquad \Pi(n,N)=[f(n) - nw(n,N) + \alpha\Pi(\min\{n+1,N\},N)]/(r+\alpha) \qquad 0 \leq n \leq N$$

$$\text{(5.2)} \qquad W_E(n,N)=[w(n,N) + \alpha W_E(\min\{n+1,N\},N)]/(r+\alpha) \qquad 1 \leq n \leq N$$

(5.3) $W_E(n,N) - W_U = \mathrm{Max}[\Pi(n,N) - \Pi(n-1,N),0] \quad 1 \leq n \leq N$

The limit the versions of w(n,N), Π(n,N) and W_E(n,N) can be obtained either by solving (5.1-3) or by taking directly the limits of (3.4-5).

$$(5.4) \qquad w(n,N) = \begin{cases} [\Psi(n)+rW_U]/2 & n<N \\ [(N+1)r\Psi(n)+[(N+1)r+2\alpha]rW_U]/[2(N+1)r+2\alpha] & n=N \end{cases}$$

$$(5.5) \qquad \Pi(N,N) = \frac{1}{r}\{\frac{\alpha}{(N+1)r+\alpha}\pi(N) + \frac{r(N+1)}{(N+1)r+\alpha}[\frac{1}{N+1}\sum_{i=o}^{N}\pi(i)]\}$$

<u>The continuous labor limit</u>

The purpose of this part is to derive the limit version of the outcomes for the case in which each the "size" of each worker is negligible relative to size of the firm. Besides the interest in this scenario in its own right, this version will also sometimes simplify future arguments by allowing the use of calculus for optimization. To analyze this case, we maintain the continuous time version summarized in (5.1-5) above, and introduce the notion of an ε-version of the model, where ε is the inverse of a natural number. In an ε-version the size of the individual worker is ε in the sense that, in terms of the contribution to production and the rate of arrival, a batch of $1/\varepsilon$ workers is equivalent to a single worker in the original version, as if each worker was split into $1/\varepsilon$ smaller ones. The data of an ε-version are $f^\varepsilon(m)=f(m\varepsilon)$, $\alpha^\varepsilon=\alpha/\varepsilon$ and $W_U^\varepsilon=W_U\varepsilon$, so that for $\varepsilon=1$ it coincides with the original model. The term <u>labor unit</u> will be used to describe one worker in the original version of the model, and the equivalent batch of $1/\varepsilon$ workers in the ε-version of the model. Thus, in terms of labor units, the basic data remain the same across the different ε-versions. The important distinction between the different versions is that the employment of a labor unit in an ε-version requires negotiations with $1/\varepsilon$ independent workers.

Let $w^\varepsilon(m,M)$, $\Pi^\varepsilon(m,M)$ and $W_E^\varepsilon(m,M)$ denote the solution to (5.1-3) with the basic data of the ε-version. Next, define $w(n,N)=\lim_{\varepsilon\to 0} w^\varepsilon(n/\varepsilon,N/\varepsilon)/\varepsilon$, $W_E(n,N)=\lim_{\varepsilon\to 0} W_E^\varepsilon(n/\varepsilon,N/\varepsilon)/\varepsilon$ and $\Pi(n,N)=\lim_{\varepsilon\to 0}\Pi^\varepsilon(n/\varepsilon,N/\varepsilon)$. Here, n and N are measured in labor units. The actual numbers n/ε and N/ε increase indefinitely as ε approaches 0, but the equivalent numbers of labor units remain constant. Notice that $W_E(n,N)$ is obtained as the limit of the ratio $W_E^\varepsilon(n/\varepsilon,N/\varepsilon)/\varepsilon$, while $\Pi(n,N)$ is the limit of $\Pi^\varepsilon(n/\varepsilon,N/\varepsilon)$ itself. This reflects the fact that, due to the negligibility of the individual worker, W_E is now a density, so it is not anymore of the same order as $\Pi(n,N)$ which remains of finite size[7]. Taking the limits of the ε-versions of (5.4-5) yields

$$
(5.6)\quad w(n,N) = \begin{cases} \frac{1}{n^2}\int_0^n xf'(x)\,dx+\frac{1}{2}rW_U & n<N \\ \frac{Nr}{2(Nr+\alpha)}\left[\frac{2}{N^2}\int_0^N xf'(x)\,dx\right]+\frac{Nr+2\alpha}{2(Nr+\alpha)}rW_U & n=N \end{cases}
$$

Similarly,

$$
(5.7)\quad \Pi(N,N) = \frac{1}{r}\left\{\frac{\alpha}{Nr+\alpha}\pi(N)+\frac{rN}{Nr+\alpha}\left[\frac{1}{N}\int_0^N \pi(x)\,dx\right]\right\}
$$

It is interesting to observe that the continuous limits of w(n,N) and Π(N,N) are essentially of the same form and on the same order as their discrete counterparts in which n and N are actual numbers of workers rather than labor units. In particular, the downward jump of the wage at the target level that was recognized before, turns here into a discontinuity, $\lim_{n\to N} w(n,N)\equiv w(N^-,N)>w(N,N)$.

In the light of the findings of the previous section that, as α approaches ∞, w(n,N) approaches the reservation wage rW_U, this proximity between the discrete case and the continuous labor limit is somewhat surprising. This is because, as ε approaches 0, the arrival rate of individual workers α/ε approaches ∞ and one might expect that in the limit w(N,N) will be equal to rW_U, for $N<N^0$ as well, contrary to (5.6). To reconcile these seemingly conflicting observations, recall that the loss imposed on the firm by a quitting worker consists of this worker's contribution to production and the temporary increase in the wages of the remaining workers. With other things equal, a larger α shortens the duration of this loss and hence reduces the significance of the worker's threat and the bargained wage. But when ε approaches 0, the arrival rate α/ε and the worker's size change simultaneously. The significance of this is that, while the duration of the loss is shortened as above, the magnitude of the temporary increase of the total wage bill becomes very large in relation to the wage of a single worker. Thus, in relation to size of the gains over which the worker bargains the worker's threat is not diminished and the wage is not driven to rW_U.

6. An extension: recruitment efforts

This section extends the model to let the arrival rate of workers be determined by the recruitment efforts of the firm. This is obviously a natural extension which is interesting in its own right. It also provides a robustness test for some of the insights derived above.

The basic model is modified as follows. In the end of each period t the firm chooses the probability α_t that a new worker will arrive in the beginning of period t+1. To induce probability α the firm has to incur cost $c(\alpha)$, where c is increasing and convex, $c(0)=c'(0)=0$ and $c(1)=\infty$. Other than this the model remains unchanged.

The added feature changes the forms of a history and a policy (which will now include the choice of the arrival rate) and hence the equilibrium analysis. We will not reproduce here the appropriate version of the equilibrium analysis for the modified environment. Instead, we assume directly that the firm follows a target level policy of the type that was discussed before. That is, it keeps hiring up to some target level N and in each period in the process it chooses the arrival probability optimally. The arrival probability in each period may therefore depend on the current employment level and on the target and will be denoted by $\alpha(n,N)$. Given such target level policy, the counterpart of system (3.1-3) is

(6.1) $W_E(n,N) = w(n,N) + \delta[\alpha(n,N)W_E(\min\{n+1,N\},N) + (1-\alpha(n,N))W_E(n,N)], \quad 1 \leq n \leq N$

(6.2) $\Pi(n,N) = f(n) - w(n,N)n - c(\alpha(n,N)) + \delta[\alpha(n,N)\Pi(\min\{n+1,N\},N)+(1-\alpha(n,N))\Pi(n,N)],$

$0 \leq n \leq N$

(6.3) $W_E(n,N) - W_U = \text{Max}[\Pi(n,N) - \Pi(n-1,N),0] \quad 1 \leq n \leq N$

An additional condition guarantees that $\alpha(n,N)$ is chosen to maximize $\Pi(n,N)$,

(6.4) $\qquad c'(\alpha(n,N)) = \delta[\Pi(n+1,N) - \Pi(n,N)] \quad 0 \leq n < N$

As before, N is feasible, if the solution to this system satisfies $\Pi(n,N) \geq \Pi(n-1,N) \geq 0$, for all $1 \leq n \leq N$. The following proposition expresses $w(n,N)$ in terms of the parameters of the model and the endogenously determined $\alpha(i,N)$'s. To get a compact expression which can be directly compared to the expressions obtained for the case of an exogenous α, we introduce the following notation. Let

$$\Gamma(n,N) = \frac{1}{n(n+1)} \sum_{i=1}^{n} i\,[c'(\alpha(i-1,N)\,\Delta\alpha(i,N) - \Delta c(\alpha(i,N))]$$

where Δ denotes first difference with respect to i. Let $\tilde{w}(n,N,\alpha)$ denote the wage formula of (3.4), given α. To avoid confusion, note that $\tilde{w}(n,N,\alpha)$ stands for the function $w(n,N)$ of the previous sections, since the notation $w(n,N)$ is reserved here for the wage function arising with endogenous α. Besides the ~ , we also added α as an explicit argument, since now it is not obvious anymore at which fixed α this expression is evaluated (recall, however, that $\tilde{w}(n,N,\alpha)$ actually depends on α only at $n=N$). Similarly, let $\tilde{\Pi}(n,N,\alpha)$

denote the profit formula of (3.5).

Proposition 6: For a feasible N,

$$(6.5)\ w(n,N) = \begin{cases} \tilde{w}(n,N,\alpha) + \Gamma(n,N) & n<N \\ \tilde{w}(N,N,\alpha) + (1-\delta)[(N+1)\Gamma(N-1,N)+c(\alpha)]/[(N+1)(1-\delta)+\alpha] \end{cases}$$

$$(6.6)\quad \Pi(N,N) = \tilde{\Pi}(n,N,\alpha) - [(N+1)\Gamma(N-1,N)+c(\alpha)]/[(N+1)(1-\delta)+\alpha]$$

where $\alpha=\alpha(N-1,N)$.

Notice that, if the individual worker is sufficiently small relative to the firm, the term $\Gamma(n,N)$ is negligible. This is because $\Gamma(n,N)$ is a weighted average of expressions of the form $c'(\alpha(i-1,N))\Delta\alpha(i,N)-\Delta c(\alpha(i,N))$ each of which is approximately zero when $\alpha(i,N)$ is near $\alpha(i-1,N)$.

Before proceeding, it would be useful to get more compact expressions by obtaining the limit version of (6.5-6) for the negligible workers' case (the continuous time-continuous labor case). As before, first let $\delta(h)=e^{-rh}$ and $\alpha(h)=1-e^{-\alpha h}$. In addition, let $C(\alpha(h);h)$ denote the cost of inducing arrival probability $\alpha(h)$ when the period length is h. Assume that $C(\alpha(h);h)=hc(\alpha)$, where c is convex with $c(o)=c'(0)=0$. Note that the argument of c is the arrival rate α, while the argument of C is the probability $\alpha(h)=1-e^{-\alpha h}$. Next consider the ε-versions of the model. The data of an ε-version are $f^{\varepsilon}(m)=f(m\varepsilon)$ and $W_U^{\varepsilon}=W_U\varepsilon$, as before, and in addition $c^{\varepsilon}(\alpha^{\varepsilon})=c(\alpha^{\varepsilon}\varepsilon)$. Here $\alpha^{\varepsilon}\varepsilon$ is the arrival rate of a labor unit (the arrival rate of a single worker, α^{ε}, divided by the number of workers, $1/\varepsilon$, making up a labor unit). The function c^{ε} is so defined to assure that, in terms of labor units, the cost of inducing a given arrival rate remains constant across the different versions (this is analogous to the way f^{ε} preserves the same productivity in terms of labor units). Now, letting superscript ε,h index the values corresponding to an ε-version with period length h, the limiting values are $w(n,N)=\lim_{\varepsilon\to 0}\lim_{h\to 0}w^{\varepsilon,h}(n/\varepsilon,N/\varepsilon)/\varepsilon$, $W_E(n,N)=\lim_{\varepsilon\to 0}\lim_{h\to 0}W_E^{\varepsilon,h}(n/\varepsilon,N/\varepsilon)/\varepsilon$, $\Pi(n,N)=\lim_{\varepsilon\to 0}\lim_{h\to 0}\Pi^{\varepsilon,h}(n/\varepsilon,N/\varepsilon)$ and $\alpha(n,N)=\lim_{\varepsilon\to 0}\alpha^{\varepsilon}(n/\varepsilon,N/\varepsilon)\varepsilon$.

As above, let $\tilde{w}(n,N,\alpha)$ and $\tilde{\Pi}(n,N,\alpha)$ denote the wage and the profit formulas for the case of exogenous α, which for the continuous labor case are given by (5.6-7).

<u>Proposition 7</u>: In the continuous labor case

$$w(n,N) = \begin{cases} \tilde{w}(n,N,\alpha) & n<N \\ \tilde{w}(n,N,\alpha) + c(\alpha)/[nr+\alpha] & n=N \end{cases} \tag{6.7}$$

$$\Pi(N,N) = \tilde{\Pi}(n,N,\alpha) - Nc(\alpha)/[nr+\alpha] \tag{6.8}$$

where $\alpha=\alpha(N^-,N)\equiv\lim_{n\to N}\alpha(n,N)$ is given by

$$c'(\alpha) = \frac{N}{2\,(Nr+\alpha)}\,[\frac{2}{N^2}\int_0^N xf'(x)\,dx-rW_U] + \frac{c(\alpha)}{Nr+\alpha} \tag{6.9}$$

There are some clear similarities as well as some differences between (6.7-8) and their counterparts (5.6-7) for the case of an exogenous α. First, for $n<N$, $w(n,N)$ remains the same as in the case of an exogenously given α. In particular, it is entirely independent of the arrival process. In contrast, $w(N,N)$ includes now the extra term $c(\alpha)/(Nr+\alpha)$, where $\alpha=\alpha(N^-,N)$ is the recruiting intensity just before the target level is reached. Consequently, $\Pi(N,N)$ is also reduced by a term depending on $c(\alpha)$.

Recall that, with exogenously given α, $w(n,N)$ exhibits a discontinuity at $n=N$ for all $N<N^0$, $\lim_{n\to N}w(n,N)\equiv w(N^-,N)>w(N,N)$. It follows from (6.7) and (6.9) that

$$w(N^-,N)-w(N,N) = [c'(\alpha(N^-,N))\alpha(N^-,N)-c(\alpha(N,N))]/N.$$

Since c is convex, $w(N^-,N)-w(N,N)>0$ if and only if $\alpha(N^-,N)>0$. Thus, the discontinuity of $w(n,N)$ at N will be present here for N such that $\alpha(N^-,N)>0$.

Proposition 8: $\alpha(N^{-},N)>0$ for $N<N^0$ and is 0 at N^0.

Thus, the discontinuity of the wage function and the associated insights, such as the existence of a mark-up over the reservation wage, are robust to the endogenization of the arrival rate.

Let us now focus on the target level $N^{*}=\mathrm{Argmax}\Pi(N,N)$. The following proposition relates the location of N^{*} and the magnitude of $\alpha(N^{*-},N^{*})$ to the steepness of the cost function. To quantify this, consider the family of cost functions indexed by k, $c(\alpha;k)=kc(\alpha)$, where $k>0$ and $c(\alpha)$ is a cost function with the properties assumed above and with the additional property that $c''(\alpha)$ is bounded away from 0 for all α. Thus, smaller values of k are associated with more moderately sloped cost functions in this family.

Proposition 9: (i) For any $\underline{\alpha}>0$ and $\varepsilon>0$, there exists $K>0$ such that, for all $k<K$, $\alpha(N^{*-},N^{*})>\underline{\alpha}$, $|\Pi(N^{*},N^{*})-\pi(N^{S})/r|<\varepsilon$ and $|N^{*}-N^{S}|<\varepsilon$.
(ii) For any $\underline{\alpha}>0$ and $\varepsilon>0$, there exists $K>0$ such that, for all $k>K$, $\alpha(N^{*-},N^{*})<\underline{\alpha}$, $|\Pi(N^{*},N^{*})-\pi(N^{0})/r|<\varepsilon$ and $|N^{*}-N^{0}|<\varepsilon$.

Among other things, the proposition tells us that, if the cost function of the recruiting effort is not too steep, then the arrival rate induced near N^{*}, $\alpha(N^{*-},N^{*})$, is bounded away from zero. As was mentioned above, this means that the target level effects discussed in Section 4 are present at N^{*}.

For a wage setting firm which has the same recruiting technology, the optimal intensity, $\alpha^{S}(n,N)$, satisfies an analogous condition, $c'(\alpha^{S}(n,N))=\Pi_1^{S}(n,N)$. Now, since $\Pi_1^{S}(N^{-},N)=\pi'(N)/r$, it follows from $\pi'(N^{S})=0$ that $\alpha^{S}(N^{S-},N^{S})=0$. That is, in the neighborhood of its optimum, the wage setting firm exerts only little effort on recruiting.

This contrasts with the above observation that, near the maximizer N^* of the wage negotiating firm's profit, $\alpha(N^{*-},N^*)>0$. The source of this difference between the two regimes is the wedge between $w(N^-,N)$ and $w(N,N)$ which opens up when $\alpha(N^-,N)$ is positive. The wage negotiating firm boot straps itself as follows: a positive $\alpha(N^{*-},N^*)$ creates a wedge between $w(N^{*-},N^*)$ and $w(N^*,N^*)$ which implies $\Pi_1(N^{*-},N^*)>0$, which in turn induces $\alpha(N^{*-},N^*)>0$. In other words, $\alpha(N^{*-},N^*)$ is an equilibrium of the firm's decisions: its depressing effect on $w(N^*,N^*)$ has to be just sufficient to induce the firm to choose $\alpha(N^{*-},N^*)$.

Finally, the proposition also relates the location of N^* to the shape of the recruiting cost function: for sufficiently low cost and sufficiently high costs, N^* is close to N^S and N^0 respectively. This observation is of course related to the effect of α on the location of N^* in the exogenous α case.

7. An extension: departure of workers

This section extends the basic model by adding a process of departure whereby employees may leave ("die") the firm for exogenous reasons. Specifically, the evolution of m_t is as follows

$$(7.1)\qquad Prob(m_t=m\,|\,n_{t-1}=n) = \begin{cases} \alpha & \text{if } m=n+1 \\ n\beta & \text{if } m=n-1 \\ 1-\alpha-n\beta & \text{if } m=n \end{cases}$$

That is, as before, in each period the pool of the potential employees may change by at most one employee, but now this pool can both increase and decrease. In addition, it is assumed that all workers have equal probability of departure (i.e., conditional on the occurrence of a departure at some period, the probability of each of the n employees to depart is β). To assure that (7.1) is well defined, it is assumed that the number of employees will always be below some bound $\hat{n}$, where $\alpha+\hat{n}\beta<1$. The expected utility of a worker who departs through the exogenous departure process is W_D which might but need not be equal to W_U.

The notions of history, policy and equilibrium are just as they were defined in Section 2. As before, a stationary target level employment policy $\{n_t\}$ is characterized by a number N so that $n_t = \text{Min}[m_t,N]$. The following system is the counterpart of system (3.1-3). It describes the dynamic behavior of the wage and the profit, given the N policy.

(7.2) $\Pi(n,N) = [f(n)-nw(n,N)+\delta\alpha\Pi(\max\{n+1,N\},N)+\delta n\beta\Pi(n-1,N)]/(1-\delta+\delta\alpha+\delta n\beta),\ \ 0\leq n\leq N$

(7.3) $W_E(n,N) = [w(n,N)+\delta\alpha W_E(\max\{n+1,N\},N)+\delta(n-1)\beta W_E(n-1,N)+\delta\beta W_U]/(1-\delta+\delta\alpha+\delta n\beta),$

$1\leq n\leq N$

(7.4) $W_E(n,N) - W_U = \text{Max}[\Pi(n,N) - \Pi(n-1,N),0]$

Equations (7.2-4) are explained just as (3.1-3) above, except that they account now for the possibility of departures as well. N is feasible if the solution to this system satisfies $\Pi(n,N)\geq\Pi(n-1,N)\geq 0$, for all $1\leq n\leq N$. Recall that $\Psi(n) = \frac{2}{n(n+1)}\sum_{i=1}^{n} i\Delta f(i)$. Consider a modified version of system (7.2-4) in which the RHS of (7.4) is replaced by $\Pi(n,N) - \Pi(n-1,N)$. This is a linear system with 3N+1 independent equations and 3N+1 unknowns and as such has a unique solution. Let N^{00} denote the minimal n such that in the solution to that system $W_E(n,n)\leq W_U$. To avoid integer problems, assume that $W_E(N^{00},N^{00})=W_U$.

Proposition 10: (i) For any feasible N,

$$(7.5)\ w(n,N) = \begin{cases} [\Psi(n)+[(1-\delta)W_U+\delta\beta(W_U-W_D)]]/2 & n<N \\ [\Psi(N)+[(1-\delta)W_U+\delta\beta(W_U-W_D)]]/2 - \delta\alpha[\Pi(N,N)-\Pi(N-1,N)]/(N+1) & n=N \end{cases}$$

(ii) N is feasible iff $N\leq N^{00}$.

For $n<N$, $w(n,N)$ is the same as it was in the absence of departures except for the changed reservation wage which is now $(1-\delta)W_U+\delta\beta(W_U-W_D)$. (This is the constant perpetual wage w at which the present value of being employed, $(w+\delta\beta W_D)/(1-\delta+\delta\beta)$, is just equal to the value of being unemployed W_U.) Note that (7.5) gives $w(N,N)$ only implicitly. The explicit expression for $w(N,N)$ is substantially more complicated to an extent that its presentation does not provide much useful information.

Proposition 11: There exists an equilibrium with stationary target level policy N^{00}, i.e., $n_t(h_t)=\min[m_t,N^{00}]$.

This equilibrium is the counterpart of the N^0 equilibrium of Proposition 3 and the proof of the proposition is almost identical to the proof there. Recall that N^0 there was the n at which $w(n,n)$ drops to the reservation wage. It was also the level at which $W_E(n,n)$ drops to W_U. In the present environment the level N^0 at which the wage drops to the reservation wage (i.e., the maximal n such that $\Psi(n)\geq(1-\delta)W_U+\delta\beta(W_U-W_D)$) is smaller than the level N^{00} at which $W_E(n,n)$ drops to W_U. The reason is that, when the target level is N^0, the worker will be paid his reservation wage at N^0, but he will be paid a higher wage following departures of his colleagues. More precisely, note that $W_E(n,n)>W_U$ if and only if $n<N^{00}$. Now, it follows from (7.5) that, for all $n<N^0$, $w(n,N^0)>(1-\delta)W_U+\delta\beta(W_U-W_D)$ hence $W_E(N^0,N^0)>W_U$ which implies that $N^0<N^{00}$. It is interesting to note that at the employment levels between N^0 and N^{00}, which occur in equilibrium, the wage is lower than the reservation wage. However, workers accept employment at these levels, since their utility of being employed W_E exceeds the utility of unemployment W_U, owing to their expectations that wages will rise when employment will fall due to future departures.

Now, if there is a feasible n such that $\Pi(n+1,N^{00}) < \Pi(n,n)$, then there is an equilibrium in a semi-stationary target level policy which on the equilibrium path follows the target level n. Such an equilibrium is sustained by the threat that, if the target level n is ever exceeded, then the policy will switch to the stationary N^{00} policy, completely analogously to the equilibria described by Proposition 4 above. It is obvious that, when β is sufficiently small, there are such semi-stationary target level equilibria which yield higher profit than the N^{00} equilibrium. This follows immediately from the fact that, with small β, the Π function here is close to the Π function of Section 3. But in contrast with the analysis of Section 3, we do not have here a complete characterization of the set of feasible n's which satisfy $\Pi(n+1,N^{00}) \leq \Pi(n,n)$[8], and hence we do not have a characterization of the set of semi-stationary equilibria in target level policies.

Finally, it is immediate to verify that, in any equilibrium with target level $N < N^{00}$, the wage exhibits a downward jump at the target level. That is, $w(N,N) < w(N,N+1)$ for any $N < N^{00}$. The definition of N^{00} implies that $\Pi(N,N) > \Pi(N\text{-}1,N)$, for all $N < N^{00}$. Now, this together with (7.5) imply $w(N,N) < w(N,N+1)$. The appearance of this target level effect here is not surprising and is explained in just the same way as it was explained previously.

Remark: The assumptions on the arrival and departure processes as summarized by (7.1) require that the employee pool can change by at most one person per period. An alternative formulation would be that in the end of any period each employee may depart with probability β. The events of departure are independent across employees and are also independent of the arrival process, which continues to bring one new employee with probability α. Thus, the potential employees at the beginning of period t, m_t, are those of the employees at t-1 who did not depart plus the new arrival if such occurred. The distribution of m_t, $p(m|n) \equiv \text{Prob}(m_t = m | n_t = n)$ is given by

$$p(m|n) = \alpha[n!/(n+1-m)!(m-1)!]\beta^{n+1-m}(1-\beta)^{m-1} + (1-\alpha)[n!/(n-m)!m!]\beta^{n-m}(1-\beta)^{m} \qquad 0 \leq m \leq n+1,$$

where (-1)! is defined to be 0.

The counter-part of system (7.2-3) above is as follows.

$$W_E(n,N) = w(n,N) + \delta\left[(1-\beta)\sum_{k=0}^{n} p(k|n-1)\, W_E(\max\{k+1, N\}, N) + \beta W_D\right] \qquad 1 \leq n \leq N$$

$$\Pi(n,N) = f(n) - w(n,N)n + \delta \sum_{k=0}^{n+1} p(k|n)\, \Pi(\max\{k, N\}) \qquad 0 \leq n \leq N$$

Now, this system is more complicated and this is why we chose the other version. But in the limit as the time periods are made short, the two systems coincide.

8. Remarks on the modeling

The basic issue

The model of this paper is not a fully specified non-cooperative game. The employment decisions are described and analyzed just as they would be in a non-cooperative game model, with the notion of an employment policy $\{n_t\}$ as the counterpart of a strategy in non-cooperative games. But the wage bargaining component is left as a black box. Rather than fully describing the moves of the bargainers, it is assumed directly that the bargained wage divides equally between the firm and each worker the surplus associated with the retention of that worker. The advantage of this approach is that it avoids the detailed strategic modeling of the multi-person bargaining processes that take place in each period. This reduces substantially the complexity of the model and allows to focus the attention on what seem as the more central forces in this situation.

But the greater simplicity achieved by adopting this short-cut comes at a cost. If the surplus to be divided in the bargaining were determined exogenously, there would be no conceptual problem involved in analyzing the model as a non-cooperative game with respect to employment decisions, taking the equal split rule as part of the description of the game. One may still prefer to replace the equal split rule with some bargaining procedure (which might be equally arbitrary), but this is more a matter of taste than a conceptual issue. The source of the difficulty with this modeling approach is that the surplus to be divided in the bargaining is not exogenous, but rather depends on future developments which in turn depend on future employment decisions of the firm. In order for the equal split assumption to be meaningful in such a bargaining situation, there has to be complete information about the expected surplus. This implies that the firm and the workers share the same expectations regarding the future developments. But this means that we do not allow the firm to contemplate multiple step deviations from its employment policy. This is because, once the firm embarks on such a deviation, it will no longer share the same expectations with its workers and it is hard to defend the bargaining solution. For this reason, the equilibrium definition in Section 2 requires only immunity against single step deviations by the firm. In a model with discounting and bounded payoffs, this limitation itself is harmless. This is in the sense that, if the bargaining component is modified to allow consideration of multi-step deviations by the firm from its employment policy, it would still be sufficient to consider only single step deviations from it[9]. But this is nevertheless a limitation of the model which should be understood.

Strategic bargaining

In the remainder of this section we outline a strategic bargaining procedure that can replace the bargaining component of the present model whereby turning it into a fully specified non-cooperative game model. First, the game to be described will have sequential equilibria whose outcomes coincide with the equilibrium outcomes of the above model, and in addition the bargaining outcomes at each employment level yield the equal split assumed in the above model. This should provide an additional assurance to the reader that the results of the above analysis are not an artifact of the particular modeling approach we adopted. Second, the description of the strategic bargaining component will illustrate the substantial complexity that was avoided by the modeling approach that was adopted above.

As is well known, there are relatively simple strategic bargaining models that yield equal surplus sharing in equilibrium. However, the model outlined below is somewhat more complicated because of its multi-person nature and because it is designed to give deterministic outcomes rather than stochastic ones, as would arise in a simple take-it-or-leave-it game with random selection of proposers. Suppose that at the beginning of each period t, after the firm has chosen n_t, the workers and the firm play the following bargaining game. First, simultaneously all workers submit wage demands to the firm and the firm makes an offer to each one of them. If the firm's offer to each of the workers is greater or equal to that worker's demand, the bargaining for that period is concluded with the wages offered by the firm. If one or more of the workers demanded a higher wage than the firm's offered them, then in each one of these worker-firm pairs, one of the parties (the worker or the firm) is chosen randomly with probability 1/2 to make an offer to which the other responds with acceptance or rejection. If all these interactions end with acceptances, the bargaining for this period is concluded with wages agreed upon in the first or second rounds respectively. If any of the second round offers was rejected, the workers involved depart, all the other agreements are canceled and the bargaining resumes in the same way with the remaining workers. The informational assumptions are as follows. A worker who did not agree with the firm in the first round does not know whether or not there are other workers who continue to the second round. If there is some disagreement in the second round so that some workers leave, all remaining workers learn all the information before the bargaining resumes. Note that this bargaining procedure is a take-it-or-leave-it procedure with random selection of proposers to which we appended a preliminary stage of simultaneous offers. The role of this stage is to facilitate an equilibrium with no uncertainty in the outcome.

Strategies for the full game are defined in the obvious way. The notion of a history will now include the employment history introduced before and the history of the bargaining moves. A strategy for the firm will consist of an employment policy as before, as well as the specification of the offers and the responses as functions of the relevant histories at the different stages of the bargaining components of each period; a strategy for a worker will prescribe his moves in the bargaining interaction of each period as a function of the history. This game has sequential equilibria whose outcomes coincide with the equilibrium outcomes presented in this paper.

Consider, for example, the equilibrium with the stationary employment policy $n_t=\min\{m_t,N^0\}$ discussed above. In the sequential equilibrium that yields the same outcome in the extended model, the strategies are as follows. First, in all t, the firm will choose $n_t=\min\{m_t,N^0\}$.

<u>After $n_t \leq N^0$</u>.

In the first stage of the bargaining game of period t:

(i) Each worker will demand $w(n_t,N^0)$.

(ii) The firm will offer each of the workers $w(n_t,N^0)$.

If the second stage of t is reached:

(iii) A worker who proposes demands $w(n_t,N^0)+\Pi(n_t,N^0)-\Pi(n_t-1,N^0)$;

a worker who responds accepts any wage greater than or equal to

W_U-$[W_E(n_t,N^0)$-$w(n_t,N^0)]$.

(iv) The firm proposes W_U-$[W_E(n_t,N^0)$-$w(n_t,N^0)]$;

it accepts all demands if $f(n_t)$ minus the total wage bill upon acceptance exceeds $\Pi(n_t\text{-}1,N^0)$;

it rejects the demand of exactly one worker.

If the second stage ends with departures of k workers, the above process is repeated with $n_t'=n_t$-k workers.

<u>After $n_t>N^0$.</u>

In the first stage of the bargaining game of period t:

(v) The workers will demand $(1\text{-}\delta)W_U$.

(vi) The firm will offer less than $(1\text{-}\delta)W_U$.

In the second stage of the bargaining in t:

(vii) The workers will demand $(1\text{-}\delta)W_U$ and will agree to any offer greater or equal to $(1\text{-}\delta)W_U$.

(viii) The firm will choose its offers and rejections to get rid of precisely n_t-N^0 workers.

If the second stage ends with departures of k workers, the above process is repeated with $n_t'=n_t$-k workers.

9. Concluding Remarks

Employment relations which take the form of individual contracts where the wage is decided in bilateral negotiations between each worker and the employer are rather widespread. The literature on which this model builds tries to gain some understanding of the determination of wages and employment under such circumstances. The specific contribution of this paper is the explicit introduction of substitution possibilities that were missing from the previous work. Still there are important aspects that have been ignored here and should be considered in further research. First, the above analysis focused on homogenous labor force. But the heterogeneity of workers both with respect to their skills and with respect to their substitution possibilities are probably important particularly in employment relations involving highly skilled labor which are also often governed by individual contracts of the general type modelled here. Second, the internal organization of the firm might both affect and be affected by employment relations of this type. Jackson and Wolinsky (1995) consider this point briefly in the context of a simple network model of a firm, but this topic requires much more attention.

Finally, the analysis of this paper might also suggest a direction in a more abstract direction. Stole and Zwiebel show that the solution that emerges in their model (which is the current model without arrival and departure) coincides with the Shapley Value of a game in coalitional form which is defined naturally from the data of the problem. The addition of arrival and departure processes in the present paper modifies the solution as explained throughout the paper. One might consider the introduction of a similar extension back into a more abstract model of a game in a coalitional form, thus obtaining a class of solutions of which the Shapley Value is one extreme point.

Appendix

Proposition 1: (i) For $N \leq N^0$, the solution to system (3.1-3) is

$$
(3.4)\ w(n,N) = \begin{cases} [\Psi(n)+(1-\delta)W_U]/2 & n<N \\ [(N+1)(1-\delta)\Psi(N)+[(N+1)(1-\delta)+2\delta\alpha](1-\delta)W_U]/2[(N+1)(1-\delta)+\delta\alpha] & n=N \end{cases}
$$

(ii) N is feasible iff $N \leq N^0$.

Proof: If N is feasible, the feasible solution to system (3.1-3) also solves the modified system in which the RHS of (3.3) is replaced by $\Pi(n,N)$ - $\Pi(n-1,N)$. This is a linear system of 3N+1 independent equations with the 3N+1 unknowns $w(n,N)$, $W_E(n,N)$, $\Pi(n,N)$, $n=1...N$ and $\Pi(0,N)$. It therefore has a unique solution. Let us first derive this solution. For $n<N$, using the modified (3.3) to substitute for W_E on both sides of (3.2) and rearranging, we get

$$
(1-\delta+\delta\alpha)\Pi(n,N) - \delta\alpha\Pi(n+1,N) = (1-\delta+\delta\alpha)\Pi(n-1,N) - \delta\alpha\Pi(n,N) + w(n,N) - (1-\delta)W_U
$$

Using (3.1) for substitution on both sides we get

$$
f(n) - nw(n,N) = f(n-1) - (n-1)w(n-1,N) + w(n,N) - (1-\delta)W_U
$$

The solution to this system of difference equations is the $n<N$ branch of (3.4) above. Now, to get $w(N,N)$, substitute from (3.1) and (3.2) into (3.3) to get

$$
w(N,N)/(1-\delta) - W_U = [f(N) - Nw(N,N)]/(1-\delta) - \{[f(N-1) - (N-1)w(N-1,N) + \delta\alpha[f(N) - Nw(N,N)]/(1-\delta)]\}/(1-\delta+\delta\alpha),
$$

which upon substituting for $w(N-1,N)$ from above yields the $n=N$ branch of (3.4).

Now, since $\Psi(n)$ is a decreasing function and $\Psi(N^0)=(1-\delta)W_U$, for any $n \leq N^0$, $\Psi(n) \geq (1-\delta)W_U$. This together with (3.4) imply that, for any n and N such that $n \leq N \leq N^0$, $w(n,N) \geq (1-\delta)W_U$ and hence $W_E(n,N) \geq W_U$ and $\Pi(n,N) \geq \Pi(n-1,N)$. Therefore, the solution just found for the system with the modified equation (3.3) is also the solution for the original system.

(ii) The last paragraph already establishes that any $N \leq N^0$ is feasible. Now, suppose to the contrary that $N > N^0$ is feasible. Since the feasible solution solves the modified system, w(N,N) is given by (3.4). But then for $N > N^0$, $w(N,N) < (1-\delta)W_U$ implying $W_E(N,N) < W_U$ and from (3.3) $\Pi(N,N) < \Pi(N-1,N)$, in contradiction to the feasibility. Thus, N is feasible iff $N \leq N^0$. QED

Proposition 2: (i) The π, $(1-\delta)\Pi(n,n)$ and $\frac{1}{n+1}\sum_{i=1}^{n}\pi(i)$ curves intersect at N^0, where the latter has its unique maximum. (ii) $\Pi(n,n)$ has a unique local and global maximum (or two adjacent maxima) at N^*, $N^S \leq N^* < N^0$.

Proof: (i) A straightforward rearrangement yields (see footnote 3)

$$\pi(n) - \frac{1}{n+1}\sum_{i=1}^{n}\pi(i) = [\Psi(n)-(1-\delta)W_U]n/2$$

Since by assumption $\Psi(N^0) = (1-\delta)W_U$, the $\frac{1}{n+1}\sum_{i=1}^{n}\pi(i)$ and the π curves intersect at N^0. Since $\Psi(n)$ is strictly decreasing in n, these curves intersect only once. Clearly, $\frac{1}{n+1}\sum_{i=1}^{n}\pi(i)$ is maximized at N^0, since it is increasing when it lies below π and is decreasing when the opposite is true. The $(1-\delta)\Pi(n,n)$ curve is just a convex combination of the other two, hence it also intersects them at N^0.

(ii) From (3.5) and after some rearrangement we get:

$\mathrm{Sign}\{\Pi(n+1,n+1) - \Pi(n,n)\} =$

$$\mathrm{Sign}\{\delta\alpha[(n+2)(1-\delta)+\delta\alpha]\Delta\pi(n+1) + (1-\delta)^2[(n+1)\pi(n) - \sum_{i=1}^{n}\pi(i)]\}$$

By the definition of N^S, $\Delta\pi(n)$ is positive for $n < N^S$ and negative for $n \geq N^S$. By the definition of N^0, the second term is positive for $n < N^0$ and negative for $n \geq N^0$. Also, for $n > N^S$, the expression on the RHS is strictly decreasing. This can be verified by examining the first difference of the expression on the RHS above,

$$[(n+2)(1-\delta)+\delta\alpha][\delta\alpha\Delta^2\pi(n+2)+(1-\delta)\Delta\pi(n+2)],$$

which is negative for $n > N^S$ since $\Delta^2\pi < 0$ for all n and $\Delta\pi(n) < 0$ for $n > N^S$. Taken together these observations imply that $\Pi(n,n)$ has a unique global and local maximum (or two adjacent maxima) at a level $N^* \in [N^S, N^0)$. QED

<u>Proposition 3</u>: The unique equilibrium in stationary target level policies is with target level N^0.

<u>Proof</u>: Suppose that the stationary employment policy $n_t = \min[m_t, N]$ is an equilibrium. Consider the deviation $n_t = N+1$. Since $m_{t+1} \geq N+1$, the employment policy prescribes $n_{t+1} = N$. Therefore, $W_E(N+1,N) = w(N+1,N) + \delta[W_E(N,N)N/(N+1) + W_U/(N+1)]$. Since the N policy is an equilibrium, $\Pi(N+1,N) \leq \Pi(N,N)$. It follows from (2.3) that $W_E(N+1,N) = W_U$. Therefore,

(#) $w(N+1,N) = (1-\delta)W_U - \delta[W_E(N,N) - W_U]N/(N+1) = (1-\delta)W_U - \delta[w(N,N) - (1-\delta)W_U]N/(N+1)(1-\delta)$.

Hence,

(##) $\Pi(N+1,N) - \Pi(N,N) = f(N+1) - f(N) - (N+1)w(N+1,N) + Nw(N,N) =$

$\Delta f(N+1) - (1-\delta)W_U + [w(N,N) - (1-\delta)W_U]N/(1-\delta) =$

$\{[(N+1)(1-\delta) + \delta\alpha]\Delta\pi(N+1) + [(N+1)\pi(N) - \sum_{i=1}^{N}\pi(i)\ ]\}/[(N+1)(1-\delta) + \delta\alpha]$.

The first equality makes use of the fact that, according to the policy, the continuation profit is $\Pi(N,N)$ in both cases; the second inequality follows from (#); the third equality is obtained by substitution from (3.4).

Observe now that $\Pi(N+1,N) - \Pi(N,N) > 0$ for all $N < N^0$. Clearly, for $N < N^S$ both of the terms in the numerator of the last expression are positive. To see it for $N \in [N^S, N^0)$, rewrite the last expression as

$$\{[\alpha - (N+1)]\delta\Delta\pi(N+1) + [(N+2)\pi(N+1) - \sum_{i=1}^{N+1}\pi(i)\ ]\}/[(N+1)(1-\delta) + \delta\alpha].$$

The first term in the numerator is positive since $N+1 > \alpha$ and $\Delta\pi(N+1) < 0$ for all $N \geq N^S$; the second term is nonnegative for all $N < N^0$ by the definition of N^0.

Thus , if N is the equilibrium target level, then $N \geq N^0$. It was already established by proposition 1 that $N > N^0$ is not feasible, i.e., $\Pi(N-1,N) > \Pi(N,N) < 0$. Therefore, the only candidate for equilibrium is N^0.

It remains to verify that N^0 is indeed an equilibrium. First, observe that the last expression in (##) is negative at $N = N^0$, so that $\Pi(N^0+1,N^0) < \Pi(N^0,N^0)$. An immediate continuation of the same argument shows that $\Pi(n,N^0) < \Pi(N^0,N^0)$, for all $n > N^0$. Second, the feasibility of N^0 implies that, for any $n < N^0$, $\Pi(n+1,N^0) > \Pi(n,N^0)$. Therefore, for any $m_t \leq N^0$, the choice of $n_t = m_t$ maximizes the firm's profit over all $n_t \leq m_t$. QED

<u>Proposition 4</u>: There exists an equilibrium in a semi-stationary target level policy such that $n_1(m_1) = N$, for all m_1, iff $N \in [\underline{N}, N^0]$.

<u>Proof</u>: (i) Let $N \in [\underline{N}, N^0]$ and let $n_{max}(h_t)$ denote the maximal employment level over the history h_t. Define a semi-stationary target level employment policy $\{n_t\}$ as follows. For any m_1, $N(m_1) = N$; for $t > 1$,

$$N(h_t) = \begin{cases} N & \text{if } n_{max}(h_t) \leq N \\ N^0 & \text{if } n_{max}(h_t) > N \end{cases}$$

I.e., as long as the employment never surpassed N, the firm is supposed to follow the simple employment policy characterized by N. If the firm ever deviated and hired more than N workers, it would switch to the simple policy characterized by N^0.

Let us verify that this is indeed an equilibrium. After histories h_t such that $n_{max}(h_t) > N$, $n = \text{Min}\{m_t, N^0\}$ clearly maximizes $\Pi_t(n,h_t,\{n_t\}) = \Pi(n,N^0)$, since by Proposition 3 the N^0 policy is an equilibrium. After histories h_t such that $m_t \leq N$ and $n_{max}(h_t) \leq N$, $n = m_t$ maximizes $\Pi_t(n,h_t,\{n_t\}) \equiv \Pi(n,N)$, since N is feasible and hence $\Pi(n,N) \geq \Pi(k,N)$ for any $k < n \leq N$. The remaining histories h_t are such that $m_t > N$ and $n_{max}(h_t) \leq N$. This means that $m_t = N+1$. Here, for $n \leq N$, $\Pi_t(n,h_t,\{n_t\}) = \Pi(n,N)$, while for $n > N$ $\Pi_t(m_t,h_t,\{n_t\}) = \Pi(n,N^0)$. Since $N \in [\underline{N}, N^0]$, by Proposition 2, $\Pi(N,N) \geq \Pi(N^0,N^0)$.

Since N^0 is feasible, $\Pi(N^0,N^0) \geq \Pi(n,N^0)$. $\Pi(N,N) \geq \Pi(N+1,N^0)$. Therefore, $\Pi(N,N) \geq \Pi(n,N^0)$ for any $n > N$. Finally, since N is feasible $\Pi(N,N) \geq \Pi(n,N)$ for all $n < N$. Therefore, $n = N$ maximizes $\Pi_t(n,h_t,\{n_t\})$ after these histories as well.

(ii) It follows from the proof of Proposition 3 that there does not exist an equilibrium with a target level $N > N^0$, since the deviation to N-1, which does not change the target level, will be profitable. Let N_{min} denote the minimal n such that there exists an equilibrium with $N(m_1) = n$ for all m_1. Suppose to the contrary that $N_{min} < \underline{N}$. Notice that there must be a semi-stationary equilibrium in which $N(h_t) = N_{min}$ for all h_t such that $n_{max}(h_t) \leq N_{min}$. In such an equilibrium, for h_t such that $n_{max}(h_t) = N_{min} + 1$, there must exist $N > N_{min}$ such that $N(h_t) = N$ for all h_t such that $N_{min} + 1 \leq n_{max}(h_t) \leq N$ and $\Pi(N_{min},N_{min}) > \Pi(N_{min}+1,N)$. Otherwise, N_{min} cannot be sustained in equilibrium, since the deviation to $N_{min}+1$ would be profitable. Now, from $N_{min} < \underline{N}$ and $N > N_{min}$ it follows that $\Pi(N,N) > \Pi(N_{min},N_{min})$. This and $N(h_t) = N$ for all h_t such that $N_{min} + 1 \leq n_{max}(h_t) \leq N$ imply together that for $m_1 \geq N$, $n_1 = N$ yield higher profit than $n_1 = N_{min}$ -- contradiction.

Therefore, $N_{min} = \underline{N}$. QED

Remark: If we did not impose the requirement $n_1(m_1) = N$, for all m_1, there would be some equilibria that from some initial m_1's would reach target levels which are even lower $\underline{N}$. Such a target level N would have to satisfy $\Pi(N,N) \geq \Pi(N+1,N^0)$. But if it is lower than $\underline{N}$, it will not be reached in a semi-stationary equilibrium that starts with $m_1 > N+1$.

Proposition 5: There exists an equilibrium in a gradual semi-stationary policy such that $N(m_1) = N$, iff $N \in [N^*,N^0]$.

Proof: For $N \in [N^*,N^0]$, the semi stationary policy with the following target levels

$$N(h_t) = \begin{cases} N & \text{if } n_{max}(h_t) \leq N \\ N^0 & \text{if } n_{max}(h_t) > N \end{cases}$$

is gradual and supports N in equilibrium.

Let N be the minimal target level supportable in equilibrium by a gradual policy. Suppose to the contrary that $N < N^*$. Consider a history h_t which proceeded along the path up to t-1 when the firm deviated to $n_{t-1} = N+1$. Since the policy is gradual and N is minimal, $N(h_t) = N$ or $N+1$. In both cases the deviation to N+1 is profitable. If $N(h_t)=N$, this is shown in the proof of Proposition 3 for any $N < N^0$. If $N(h_t)=N+1$, this follows from the monotonicity of $\Pi(n,n)$ for $n < N^*$ which implies that $\Pi(N+1,N+1) > \Pi(N,N)$. QED

__Proposition 6:__ For a feasible N,

$$\text{(6.5)}\quad w(n,N) = \begin{cases} \tilde{w}(n,N,\alpha) + \Gamma(n,N) & n<N \\ \tilde{w}(N,N,\alpha) + (1-\delta)[(N+1)\Gamma(N-1,N)+c(\alpha)]/[(N+1)(1-\delta)+\alpha] \end{cases}$$

$$\text{(6.6)}\quad \Pi(N,N) = \tilde{\Pi}(n,N,\alpha) - [(N+1)\Gamma(N-1,N)+c(\alpha)]/[(N+1)(1-\delta)+\alpha]$$

where $\alpha=\alpha(N-1,N)$.

__Proof:__ Using (6.1) to substitute for W_E in (6.3) and noting that the feasibility of N implies that the RHS of (6.3) is equal to $\Pi(n,N) - \Pi(n-1,N)$, we get for $n<N$,

$$W_U + \Pi(n,N) - \Pi(n-1,N) = [w(n,N) + \delta\alpha(n,N)W_E(n+1,N)]/[1-\delta+\delta\alpha(n,N)]$$

Using (6.1) again to substitute for $W_E(n+1,N)$,

$$W_U + \Pi(n,N) - \Pi(n-1,N) = [w(n,N) + \delta\alpha(n,N)(W_U + \Pi(n+1,N) - \Pi(n,N))][1-\delta+\delta\alpha(n,N)]$$

Adding and subtracting $\delta[\alpha(n,N)-\alpha(n-1,N)][\Pi(n,N)-\Pi(n-1,N)]/[1-\delta+\delta\alpha(n,N)]$ and rearranging

$$\frac{(1-\delta)W_U}{1-\delta+\delta\alpha(n,N)} + \Pi(n,N) - \frac{\delta\alpha(n,N)\Pi(n+1,N)}{1-\delta+\delta\alpha(n,N)} = \frac{\delta[\alpha(n-1,N)-\alpha(n,N)]}{1-\delta+\delta\alpha(n,N)}[\Pi(n,N) - \Pi(n-1,N)]$$

$$+\frac{1-\delta+\delta\alpha(n-1,N)}{1-\delta+\delta\alpha(n,N)}[\Pi(n-1,N) - \frac{\delta\alpha(n-1,N)\Pi(n,N)}{1-\delta+\delta\alpha(n-1,N)}] + \frac{w(n,N)}{1-\delta+\delta\alpha(n,N)}$$

Using (6.1) for substitution on both sides we get

$$f(n) - nw(n,N) - c(\alpha(n,N)) = f(n-1) - (n-1)w(n-1,N) - c(\alpha(n-1,N)) + w(n,N) - (1-\delta)W_U$$
$$- \delta[\alpha(n,N)-\alpha(n-1,N)][\Pi(n,N)-\Pi(n-1,N)]$$

The solution to this system of difference equations is

$$w(n,N) = [\Delta f(n) + (n-1)w(n-1,N) + (1-\delta)W_U + \Delta\alpha(n,N)\Delta\Pi(n,N) - \Delta c(\alpha(n,N))]/(n+1)$$

where Δ denotes first differences with respect to n, e.g., $\Delta c(\alpha(n,N)) \equiv c(\alpha(n,N))-c(\alpha(n-1,N))$ and so on. Substituting from (6.4) we get

$$w(n,N) = [\Delta F(n) + (n-1)w(n-1,N) + (1-\delta)W_U + \Delta\alpha(n,N)c'(\alpha(n-1,N)) - \Delta c(\alpha(n,N))]/(n+1)$$

and the solution to this system yields for $n<N$,

$$\text{(A.1)} \quad w(n,N) = \frac{\sum_{i=0}^{n} i\,[\Delta f(i) + c'(\alpha(i-1,N)\,\Delta\alpha(i,N) - \Delta c(\alpha(i,N))]}{n(n+1)} + \frac{1}{2}(1-\delta)W_U$$

$$= \bar{w}(n,N,\alpha) + \Gamma(n,N)$$

To get w(N,N), substitute from (6.1) and (6.2) into (6.3) to get

$$w(N,N)/(1-\delta) - W_U = [f(N) - Nw(N,N)]/(1-\delta) - \{f(N-1) - (N-1)w(N-1,N) - c(\alpha(N-1,N))$$
$$+ \delta\alpha(N-1,N)[f(N) - Nw(N,N)]/(1-\delta)\}/[1-\delta+\delta\alpha(N-1,N)]$$

Upon substituting w(N-1,N) from (A.1) and rearranging

$$\text{(A.2)} \quad w(N,N) = \frac{(N+1)(1-\delta) + 2\delta\alpha(N-1,N)}{2[(N+1)(1-\delta) + \delta\alpha(N-1,N)]}(1-\delta)W_U$$

$$+ \frac{(1-\delta)\{\frac{1}{N}[\sum_{i=1}^{N} i\Delta f(i) + \sum_{i=1}^{N-1} i\,[c'(\alpha(i-1,N))\,\Delta\alpha(i,N) - \Delta c(\alpha(i,N))]] + c(\alpha(N-1,N))\}}{(N+1)(1-\delta) + \delta\alpha(N-1,N)}$$

$$= \bar{w}(N,N,\alpha(N-1,N)) + (1-\delta)[(N-1)\Gamma(N-1,N) + c(\alpha(N-1,N))]/[(N+1)(1-\delta) + \delta\alpha(N-1,N)]$$

Finally, $\Pi(N,N)$ is obtained by substituting w(N,N) into (6.2).

QED

Proposition 7: In the continuous labor case

$$(6.7)\quad w(n,N) = \begin{cases} \tilde{w}(n,N,\alpha) & n<N \\ \tilde{w}(n,N,\alpha) + c(\alpha)/[nr+\alpha] & n=N \end{cases}$$

$$(6.8)\quad \Pi(N,N) = \tilde{\Pi}(n,N,\alpha) - Nc(\alpha)/[nr+\alpha]$$

where $\alpha=\alpha(N^-,N)\equiv \lim_{n\to N}\alpha(n,N)$ is given by

$$(6.9)\qquad c'(\alpha) = \frac{N}{2\,(Nr+\alpha)}\,[\frac{2}{N^2}\int_0^N xf'(x)\,dx - rW_U] + \frac{c(\alpha)}{Nr+\alpha}$$

Proof: Consider now the ε versions of (A.1),(A.2) in the proof of Proposition 6 and derive the limits of these expressions as ε goes to 0. In the limit the expressions $c'(\alpha(i-1,N))\Delta\alpha(i,N) - \Delta c(\alpha(i,N))$ vanish, and hence the terms $\Gamma(n,N)$ vanish. Now, from (6.3-4) we get

$$(A.3)\quad c'(\alpha(N-1,N)) = \Pi(N,N) - \Pi(N-1,N) = W_E(N,N) - W_U = w(N,N)/(1-\delta) - W_U$$

As above, consider the ε version of (A.3), derive its limit and substitute from (6.7) to get (6.9). Note that the symbol α in all the above expressions stands for

$\alpha(N^-,N)\equiv \lim_{\varepsilon\to 0}\alpha^{\varepsilon}(N/\varepsilon-1,N/\varepsilon)\varepsilon$.

QED

Proposition 8: $\alpha(N^-,N)>0$ for $N<N^0$ and is 0 at N^0.

Proof: Let $Y(N) = \frac{1}{N}\int_0^N xf'(x)\,dx - \frac{N}{2}rW_U$ and rewrite (6.9) as

$$(A.4)\qquad Nrc'(\alpha) + \alpha c'(\alpha) - c(\alpha) = Y(N).$$

Observe that $Y(N)$ is 0 at $N=0$ and $N=N^0$ and is strictly positive for $N\in(0,N^0)$. Since $Y'(N)=\pi'(N)-Y(N)/N$ it follows that $Y(N)$ is decreasing over $[N^S,N^0)$. Now, the LHS of (A.4) is continuous in α, it is 0 at $\alpha=0$ and is increasing unboundedly in α. Therefore, for each $N\in(0,N^0)$, (A.4) is solved by a positive α, while for N^0 the solution is $\alpha=0$.

QED

Proposition 9: (i) For any $\underline{\alpha}>0$ and $\varepsilon>0$, there exists $K>0$ such that, for all $k<K$, $\alpha(N^{*-},N^*)>\underline{\alpha}$, $|\Pi(N^*,N^*)-\pi(N^S)/r|<\varepsilon$ and $|N^*-N^S|<\varepsilon$.

(ii) For any $\underline{\alpha}>0$ and $\varepsilon>0$, there exists $K>0$ such that, for all $k>K$, $\alpha(N^{*-},N^*)]<\underline{\alpha}$, $|\Pi(N^*,N^*)-\pi(N^0)/r|<\varepsilon$ and $|N^*-N^0|<\varepsilon$.

Proof: The first order condition satisfied by N^* is

$$d\Pi(N,N)/dN \equiv \partial\Pi(N,N)/\partial N + (\partial\Pi(N,N)/\partial\alpha)d\alpha(N^-,N)/dN = 0,$$

Differentiating (6.8) and rearranging using (6.9) (which implies that $\partial\Pi(N,N)/\partial\alpha=0$) yields the following version of the first order condition for N^*

(A.5) $$\alpha\pi'(N) + r[\pi(N) - r\Pi(N,N) - c(\alpha)] = 0$$

(i) Since the LHS of (A.4) is increasing in α and k, for any $\underline{\alpha}>0$, there is a sufficiently small $k>0$ such that (A.4) can hold only if $\alpha>\underline{\alpha}$ and there is a sufficiently large k such that (A.4) can hold only if $\alpha<\underline{\alpha}$.

Next, observe that there exists $M>0$ such that $c(\alpha;k)<M$ for all k. To verify this, suppose to the contrary that for some N there is a sequence of k's such the associated sequence of $c(\alpha)$'s is increasing without a bound. This implies that $c'(\alpha;k)$ is also increasing without bound along the sequence, since otherwise a contradiction will result by noting that the assumption that $c''(\alpha)$ is bounded away from 0, implies $(c'/c)'=[c''c-(c')^2]/(c)^2>0$ far enough in the sequence. But then (A.4) will fail to hold far enough in the sequence.

Consider now the first order condition (A.5). Since the term $[\pi(N) - r\Pi(N,N) - c(\alpha)]$ is bounded and since α increases without a bound when k decreases to 0, (A.5) can hold for a sufficiently small k only if $\pi'(N)$ is sufficiently small. This means that N^* is sufficiently close to N^S. Also, when α increases without a bound while $c(\alpha;k)$ is bounded, the RHS of (6.8) approaches $\pi(N)/r$. Therefore, $\Pi(N^*,N^*)$ approaches $\pi(N^S)/r$.

(ii) It again follows from (A.4) that, for any $\underline{\alpha}>0$, there is a sufficiently large k such that (A.4) can hold only if $\alpha<\underline{\alpha}$.

Next observe that $c(\alpha;k)$ approaches 0 when k becomes arbitrarily large. This follows from the fact that, for large k, $dc(\alpha;k)/dk$ is negative and bounded away from 0.

$$dc(\alpha;k)/dk = c(\alpha) - kc'(\alpha)d\alpha/dk = c(\alpha) - kc'(\alpha)(Nr+\alpha)kc''(\alpha)/Y(N)$$

$$< kc'(\alpha)[\alpha/k - (Nr+\alpha)kc''(\alpha)/Y(N)],$$

where Y(N) is defined in the proof of Proposition 8, the expression substituted for $d\alpha/dk$ is obtained by differentiating (A.4), and the last inequality follows from the convexity of $c(\alpha)$. Now, for sufficiently large k, the last expression is negative and bounded away from 0, since α is arbitrarily small, c'' is bounded away from 0, and to satisfy (A.4) $kc'(\alpha)$ has to be bounded away from 0.

Since, for a large k, α is small and $c(\alpha;k)$ is small, it follows immediately from (A.5) and (6.8) that N^* is near N^0 and $\Pi(N^*,N^*)$ is near $\pi(N^0)/r$. QED

Proposition 10: (i) For any feasible N,

$$(7.5)\ w(n,N) = \begin{cases} [\Psi(n)+[(1-\delta)W_U+\delta\beta(W_U-W_D)]]/2 & n<N \\ [\Psi(n)+[(1-\delta)W_U+\delta\beta(W_U-W_D)]]/2 - \delta\alpha[\Pi(N,N)-\Pi(N-1,N)]/(N+1) & n=N \end{cases}$$

(ii) N is feasible iff $N\leq N^{00}$.

Proof: (i) If N is feasible, the RHS of (7.4) is always $\Pi(n,N) - \Pi(n-1,N)$, so that (7.2-4) is a linear system. For $n<N$, using the (7.4) to substitute for W_E in (7.3) and rearranging, we get

$$(1-\delta+\delta\alpha+\delta n\beta)\Pi(n,N) - \delta\alpha\Pi(n+1,N) - \delta\beta n\Pi(n-1,N) = (1-\delta+\delta\alpha+\delta(n-1)\beta)\Pi(n-1,N) - \alpha\Pi(n,N) - \delta\beta(n-1)\Pi(n-2,N) + w(n,N) - [(1-\delta)W_U+\delta\beta(W_U-W_D)]$$

Using (7.2) for substitution on both sides we get

$$f(n) - nw(n,N) = f(n-1) - (n-1)w(n-1,N) + w(n,N) - [(1-\delta)W_U+\beta(W_U-W_D)]$$

The solution to this difference equation is the $n<N$ branch of (7.5). For $n=N$ we get

$(1-\delta+\delta\beta N)\Pi(N,N) - \delta\beta N\Pi(N-1,N) = (1-\delta+\delta\alpha+\delta\beta(N-1))\Pi(N-1,N) - \delta\alpha\Pi(N,N) - \delta\beta(N-1)\Pi(N-2,N) + w(N,N) - [(1-\delta)W_U+\delta\beta(W_U-W_D)] + \delta\alpha[\Pi(N,N)-\Pi(N-1,N)]$.

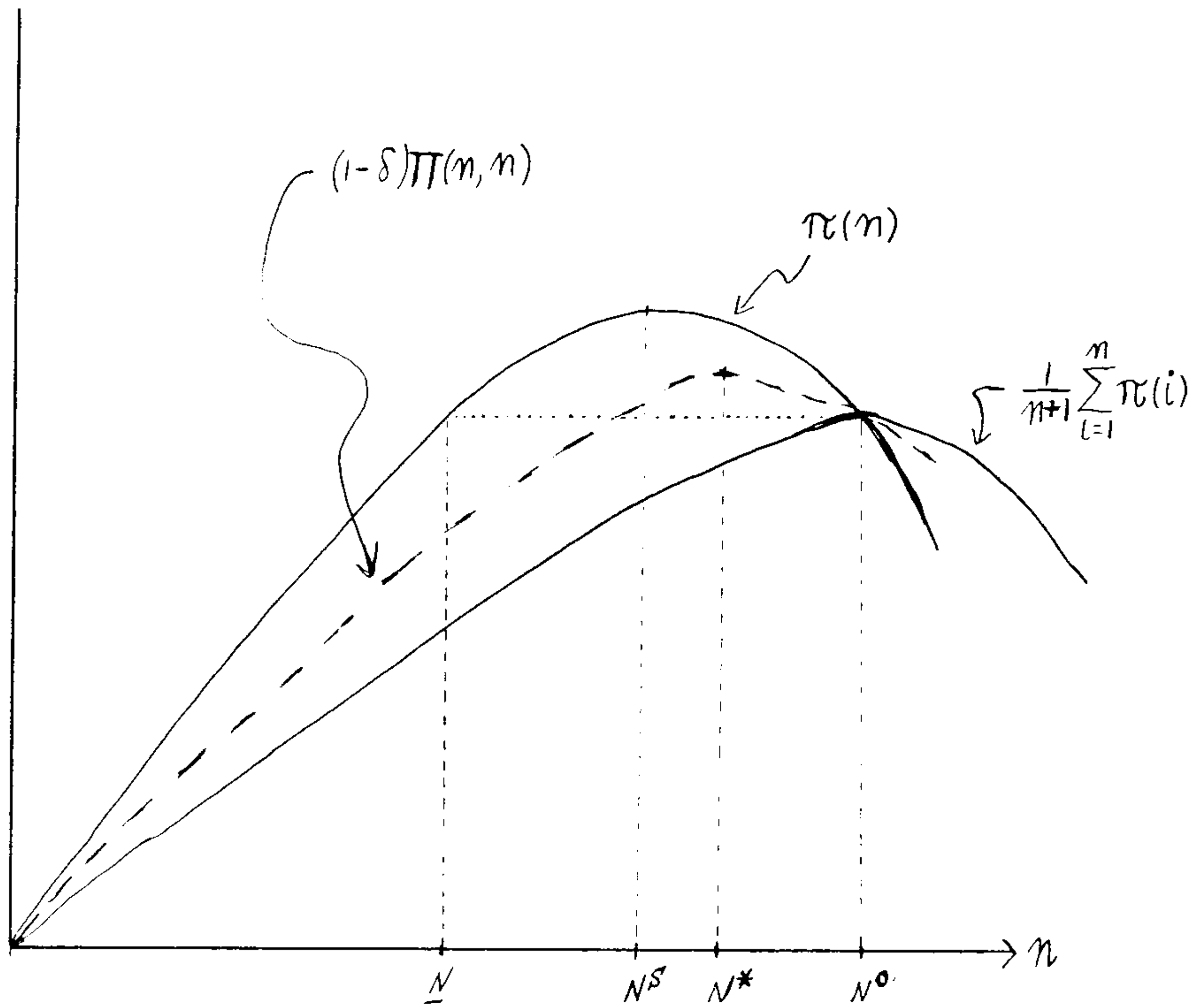
Using (7.2) for substitution on both sides we get the difference equation

$f(N) - Nw(N,N) = f(N-1) - (N-1)w(N-1,N) + w(N,N) - [(1-\delta)W_U+\delta\beta(W_U-W_D)] + \delta\alpha[\Pi(N,N)-\Pi(N-1,N)]$, which yields the above formula for $w(n,N)$.

(ii) Recall that $\Pi(N^{00},N^{00})=\Pi(N^{00}-1,N^{00})$. Together with (7.5) this implies that $w(N^{00},N^{00})=\{\Psi(N^{00})+[(1-\delta)W_U+\delta\beta(W_U-W_D)]\}/2$ and together with (7.4) it implies $W_E(N^{00},N^{00})=W_U$. Consider the solution to the modified system with the RHS of (7.4) replaced by $\Pi(n,N)-\Pi(n-1,N)$. By the definition of N^{00}, $W_E(N,N)\geq W_U$ and hence $\Pi(N,N)\geq\Pi(N-1,N)$, for all $N\leq N^{00}$. Now, since $w(n,N)$ in that solution is given by (7.5), it follows that $W_E(n,N)\geq W_U$ for all $n\leq N\leq N^{00}$. This is because $w(n,N)$ is strictly decreasing in n and hence $W_E(n,N)\geq W_E(N,N)\geq W_U$. This implies through (7.4) that $\Pi(n,N)\geq\Pi(n-1,N)$, for all $n\leq N\leq N^{00}$. In addition, from (7.2), $\Pi(1,N)\geq\Pi(0,N)$ implies $\Pi(0,N)\geq 0$. Therefore, for $N\leq N_{00}$, the solution to the modified system solves the original system and N is feasible.

Consider now $N>N^{00}$. If N is feasible, $w(n,N)$ is given by (7.5). But since $w(n,N)$ is strictly decreasing in n, $W_E(N,N)<W_E(N^{00},N^{00})=W_U$, in contradiction to the feasibility of N. QED

FIGURE 1



Footnotes

1. Stole and Zwiebel coined the term "at will" firm to describe such a firm.

2. The notion of an employment policy here is the counterpart of the notion of strategy in a non-cooperative game. We do not use here the term strategy to avoid confusion.

3. Rearrangement yields the alternative formula

$$\Psi(n) = \frac{2}{n}\left[f(n) - \frac{1}{n+1}\sum_{i=1}^{n} f(i)\right]$$

4. $\text{Sign}[\Pi(0,N+1)-\Pi(0,N)] = \text{Sign}[\Pi(N,N+1)-\Pi(N,N)] = \text{Sign}\{-N[w(N,N+1)-w(N,N)]+\alpha[\Pi(N+1,N+1)-\Pi(N,N)]\}$. Since for all feasible N, $w(N,N+1)-w(N,N)>0$ and for all $N\geq N^*$, $\Pi(N+1,N+1)-\Pi(N,N)\leq 0$, we have $\Pi(0,N+1)<\Pi(0,N)$, for all $N\geq N^*$. Hence, $N^{**}\leq N^*$. Now, if the workers are "small" relative to the size of the firm, so that $\Pi(N+1,N+1)-\Pi(N,N)\approx 0$ in the neighborhood of N^*, then $N^{**}<N^*$.

5. In the continuous labor case to be presented in Section 5 below this is actually translated to discontinuity at N: $w(N,N)<w(N^-,N)\equiv \lim_{n\to N} w(n,N)$.

6. Notice that when $\Pi(N,N+1)<\Pi(N,N)$ the firm cannot simply stop at N and get a higher profit. In such a situation everybody expects the firm to hire up to N+1. The firm can of course "stop at N", but since it would be expected to continue hiring, its profit will not be $\Pi(N,N)$ but rather $\Pi(N,N+1)$.

7. Indeed, observe that the RHS of equation (3.3), which is the one equation that brings together $W_E(n,N)$ and $\Pi(n,N)$, approaches 0 as ε goes to zero. The limit version of (3.3) is $W_E(n,N) - W_U = \partial\Pi(n,N)/\partial n$.

8. The difference between the analysis here and that of Proposition 4 is that there it was relatively simple to characterize the shape of the function $\Pi(n,n)$ and this facilitated characterization of the set of all semi-stationary target level equilibria.

9. This follows from well known results in dynamic programming which have already been applied repeatedly in the repeated games literature.

References

Davidson, C. (1988),"Multi-unit Bargaining in Oligopolistic Industries," Journal of Labor Economics.

Grout, P. (1984), "Investment and Wages in the Absence of Binding Contracts: A Nash Bargaining Approach," Econometrica, Vol. 52, 449-60.

Horn, H. and A. Wolinsky (1988), "Worker Substitutability and Patterns of Unionisation," Economic Journal, Vol. 98, 484-97.

Horn, H. and A. Wolinsky (1988), "Bilateral Monopolies and Incentives for Merger," Rand Journal of Economics, Vol. 19, No. 3, 408-419.

Jackson and Wolinsky (1995), "A Strategic Model of Social and Economic Networks," forthcoming in Journal of Economic Theory.

Stole, L. and J. Zwiebel (1995), "Organizational Design and Technology Choice Under Intrafirm Bargaining," American Economic Review, vol. 86, No. 1, pp. 195-222.