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ON THE ROBUSTNESS OF FACTOR STRUCTURES TO ASSET REPACKAGING[1]

by
Nabil I. Al-Najjar[2]

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Abstract:

The paper provides a framework to study asset repackaging in a large asset economy, modeled as an atomless measure space of assets. The main theorem shows that, given an initial economy with strict factor space F, any economy obtained by repackaging these assets has a unique factor space $F' \subset F$. Thus, in contrast to earlier literature, factor structures are robust to the repackaging of asset.

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[1] Forthcoming in *Journal of Mathematical Economics.*

[2] Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208.
[Phone: 847-491-5426; Fax: 847-467-1220; e-mail: al-najjar@nwu.edu

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1. INTRODUCTION

Factor-based asset pricing models explain variations in asset returns in terms of factor-risks representing systematic aggregate effects. Since these models have little to say about the definition or boundaries of individual assets, their starting point is typically a representation of the economy in which the definition of individual assets is taken as given. However, the particular representation of the asset economy is likely to contain a degree of arbitrariness because the boundaries of financial assets can change with time and may be ambiguously defined. This suggests viewing assets as packages or portfolios of more elementary assets. The paper provides a formal model of asset repackagings and examines their effect on the factor structure of asset returns.

One motivation for examining this problem is the work of Shanken (1982), Gilles and LeRoy (1991) and Bray (1994a and 1994b). Consider the usual model of a large asset economy as an infinite sequence of random returns $E = \{\tilde{R}_1, \tilde{R}_2, \ldots\}$; let S be the linear space they span, representing the returns on all possible portfolios formed through linear combinations of assets in E. These authors argued that if returns in E are generated by a strict factor space F, and if F' is any other finite dimensional subspace of S, then there is a sequence economy E' with returns spanning S and having F' as its strict factor space. Since the choice of F' is arbitrary, the factor structures of the two economies can be chosen to have a trivial intersection and/or different dimension. This despite the fact that the sets of attainable returns of the two economies coincide and every asset in one economy can be obtained as a portfolio (a repackaging) of assets in the other.

Such a result can have serious consequences for factor-based pricing models, such as the APT (Ross (1976)). Asset prices in such models are explained in terms of their exposures to factor-risks reflecting the fundamentals of the economy. This explanation is severely undermined if the identity of factor-risks were to hinge on the particular representation of assets, or if the factors were to drastically change due to seemingly innocuous repackagings. The apparent vulnerability of factor-based pricing models to asset repackaging led some authors to conclude that: "Factor structure, being merely a feature of a particular arbitrary representation of the space of attainable returns .. cannot have anything essential to do with asset pricing" (Gilles and LeRoy (1991), p. 214).

The present paper provides a more careful examination of what is meant by repackaging assets. I introduce a model in which a large economy is modeled as an atomless measure space of assets. In this model, repackaging is a process of 'slicing' assets into shares, then shuffling and reconstituting these shares into new assets.[1] The key observation is that repackaging has to be consistent with the supply of assets: it can neither create new returns which did not exist in the original economy, nor destroy old ones. This is formally modeled by requiring the shuffling of shares to be done in a measure-preserving manner. The main theorem shows that, starting with an asset economy with strict factor space F, any repackaging yields a new asset economy with a unique strict factor space $F' \subset F$. While repackaging does not create new factors, a repackaged economy can have strictly fewer factors than the original economy: repackaging may eliminate a factor by combining negatively correlated risks to form a portfolio (not necessarily well diversified) that is free of that risk. See Section 4 for an example illustrating this point.

In contrast to earlier work, repackaged economies in this model do not have an arbitrary factor structure and the concept of factor-risk has a substantive content independent of the particular representation of assets. To understand the source of this contrast, it is useful to take a closer look at the concept of repackaging used in earlier papers. These papers viewed two sequence economies as 'equivalent' if their returns span the same space. Satisfying this *spanning criterion* implies that each asset can be expressed as a portfolio of assets in the other economy. But, doing so for all assets simultaneously may be inconsistent with the supply of assets in the original economy. To see this, consider the following example:

Example: Let θ, $\epsilon_1, \epsilon_2 \ldots$ be i.i.d. and define the economies:

E	E'
$\tilde{R}_1 = \theta$	$\tilde{R}'_1 = \theta$
$\tilde{R}_2 = \epsilon_1$	$\tilde{R}'_2 = \epsilon_1$
$\tilde{R}_3 = \theta + \epsilon_2$	$\tilde{R}'_3 = \epsilon_1 + \epsilon_2$
$\tilde{R}_4 = \theta + \epsilon_3$	$\tilde{R}'_4 = \epsilon_1 + \epsilon_3$

While E and E' span the same portfolio space, a unit of ϵ_1-risk is needed to form every new asset $\tilde{r}'_n$, $n > 2$, in E'. Although E contains only one unit of ϵ_1-risk, infinitely many units of this risk are used to build E' from E. In economy E there is a fundamental difference between ϵ_1- and θ-risks in terms of the *measure* of the set of assets that carry them: the first is carried by a 'negligible' subset of assets, while the latter is carried by a large subset. The spanning criterion does not take into account this difference.

The model with a continuum of assets resolves this problem because there is an explicit assignment of weights to subsets of assets (their measure). In particular, there is a clear distinction between risks carried by negligible sets, and those carried by a subset of assets of positive measure. Within this framework, the idea that no risk is used in portfolio formation in a way that exceeds its supply has a natural and simple expression.

Modeling large economies using the continuum follows a long tradition in Economics. In the context of large asset economies, the continuum framework was used in a companion paper (Al-Najjar (1994)) to study asset pricing and factor analysis. Motivating this approach is the observation that a description of an economy is not complete without a specification of the relative weight of agents' characteristics. A useful analogy is an exchange economy described as one with "N consumers, in which each consumer can be of one of two possible types". This description would not be sufficient for competitive analysis because it does not pin down the distribution of consumers' characteristics, without which aggregate supply and demand are not even well- defined. In our context, the relevant characteristics are assets' random returns, so a meaningful description requires an explicit statement about the distribution of these characteristics. The continuum model used here provides a framework in which individual assets are negligible, yet the distributions of asset characteristics is well defined. This is in contrast with the traditional approach of modeling a large asset economy as infinite sequence of assets in which there is no sensible way of specifying a distribution of characteristics where individual assets are negligible (have zero weight).

2. THE MODEL

An economy $E = \left(T, \{\tilde{R}_t : t \in T\}\right)$ consists of a collection of assets indexed by a set T, and a gross return $\tilde{R}_t$ for each $t \in T$. Each $\tilde{R}_t$ is a random variable with finite mean and variance defined on a probability space (Ω, Σ, P).[2] The asset economy is 'large' if each individual asset is negligible relative to the rest of the economy. To model this, assets are indexed by an atomless measure space $(T, \mathcal{T}, \tau)$, where τ is a probability measure assigning to each subset $A \subset T$ its weight $\tau(A)$ relative to the rest of the market. For concreteness, I assume that $T = [0, 1]$, $\mathcal{T}$ is the set of Borel measurable subsets of [0,1], and τ is the Lebesgue measure.

Given an economy E with returns $\tilde{R}_t$, its *covariance function* $Cov_R : [0, 1] \times [0, 1] \to I\!R$ is defined in a manner analogous to the covariance matrix for finite collections of assets. Specifically, imagine a 'matrix' with a continuum of rows and columns so the entry at row t and column s is $Cov_R(t, s) = cov(\tilde{R}_t, \tilde{R}_s)$. The covariance function provides a convenient way to represent correlation patterns and to introduce various conditions on asset returns. Throughout, it is assumed that Cov_R is *measurable* as a function on $[0, 1] \times [0, 1]$, and *bounded* in the sense that there is a real number A such that $|Cov_R(t, s)| \le A < \infty$ for all t and s.

If p_t denotes the price of asset t, then the rate of return on that asset is $\tilde{r}_t = \dfrac{\tilde{R}_t}{p_t}$, provided $p_t \neq 0$. A trivial asset is one that pays a gross return of zero with probability 1 (*i.e.*, $\tilde{R}_t = 0$ with probability 1). For such assets, $p_t = 0$ and we set $\tilde{r}_t = 0$ as a notational convention. To ensure that the boundedness and measurability properties hold for rates of returns, we also need to assume that asset prices p_t are measurable and uniformly bounded away from zero.

Let F be the finite dimensional linear space spanned by some zero-mean orthonormal set of random variables $\{\tilde{\delta}_1, \ldots, \tilde{\delta}_K\}$. The process of asset returns $\tilde{R}_t$ can be expressed as:

$$\tilde{R}_t = E\tilde{R}_t + \mathrm{Proj}_F \tilde{R}_t + \tilde{h}_t. \tag{1}$$

That is, $\tilde{h}_t$ is the residual obtained by 'regressing' $\tilde{R}_t$ on F. While the $\tilde{h}_t$'s may be correlated with each other, the definition of orthogonal projections ensures that they must be orthogonal to F.

A finite dimensional subspace F is a *factor space* for E if:

1- The residuals, $\tilde{h}_t$, are *idiosyncratic* in the sense that the covariance function Cov_h of the residuals is zero almost everywhere on $[0,1] \times [0,1]$; and

2- There is no proper subspace $F' \subset F$ with property (1).

An economy E has *strict factor structure* if it has a (finite dimensional) factor space.

Condition (1) says that, for almost every pair of assets (t,s), the residuals $\tilde{h}_t$ and $\tilde{h}_s$ are uncorrelated. In other words, the residuals left after removing all variations in asset returns that can be explained by F contain no significant systematic component. Condition (2) requires that F is minimal in the sense of containing no superfluous factors that do not contribute to explaining asset returns. Unlike condition (1), this condition is not restrictive: if a linear space F satisfying Condition (1) exists, then we can always extract a subspace F' that satisfies both conditions.[3]

The representation (1) can be written directly in terms of the factors $\{\tilde{\delta}_1, \ldots, \tilde{\delta}_K\}$ to produce the equivalent (and more familiar) representation:

$$\tilde{R}_t = E\tilde{R}_t + \beta_{1t}\tilde{\delta}_1 + \cdots + \beta_{Kt}\tilde{\delta}_K + \tilde{h}_t \tag{2}$$

The coefficients β_{kt} are the betas familiar from Finance textbooks, representing asset t's exposure to various factor risks.[4]

It is easy to see that an asset economy with factor space spanned by $\{\tilde{\delta}_1, \ldots, \tilde{\delta}_K\}$ has a measurable covariance function if the betas (*i.e.*, the functions $t \mapsto \beta_{kt}$, $k = 1, \ldots, K$) are measurable in t, and the residuals have measurable covariance function. A slightly more involved argument can be used to show the converse, namely that measurability of the covariance function implies the measurability of the betas.

3. FACTOR SPACES IN REPACKAGED ECONOMIES

3.1. Repackaging: A Definition

Repackaging can be intuitively thought of as a process where each asset return $\tilde{R}_t$ is 'sliced' into smaller pieces which are then shuffled and reconstituted into new assets. Formally, a *repackaging* $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ consists of two sequences of measurable functions, $\lambda_i : [0, 1] \to [0, 1]$, $\alpha_i : [0, 1] \to [-1, 1]$, $i = 1, 2, \ldots$ satisfying a number of conditions described below. Here, $\alpha_i(t)$ represents the relative weight of the ith piece of asset t which, after shuffling, is assigned to a new (repackaged) asset with index $\lambda_i(t)$. Negative values of $\alpha_i(t)$ reflect short-sale of asset t.

A repackaged economy E' relative to $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ is an economy with assets indexed by $s \in [0, 1]$ and returns $\tilde{R}'_s$ defined by viewing asset s as a portfolio in the original economy with support

$$\{t : t = \lambda_i^{-1}(s),\ i = 1, 2, \ldots\}$$

and weights

$$\{\alpha_i(\lambda_i^{-1}(s)),\ i = 1, 2, \ldots\}.$$

Thus, the gross return on asset s is given by:

$$\tilde{R}'_s = \sum_{i=1}^{\infty} \alpha_i(\lambda_i^{-1}(s))\ \tilde{R}_{\lambda_i^{-1}(s)}. \tag{3}$$

This equation says that the return on the repackaged asset s is the weighted average of the gross returns of the pieces used in forming it. The assumption of linearity of returns implicit in (3) is a substantive one and is discussed in Section 4 below.

We impose the following conditions on the repackaging $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$

1- For every t, $\sum_{i=1}^{\infty} \alpha_i(t) = 1$;

2- For every i, λ_i is a measure-preserving bijection onto $[0,1]$ (that is, λ_i and λ_i^{-1} are measurable and $\tau(A) = \tau(\lambda_i(A))$ for every i and every measurable subset of assets A);

3- Asset returns in the repackaged economy have a bounded and measurable covariance function.

Condition (1) says that the sum of pieces add up to the original assets, so no part of an asset is either eliminated or replicated as a result of repackaging. Condition (2) imposes a technical restriction on the shuffling of shares by ensuring that the new economy satisfies the necessary measurability requirements. More substantively, the measure-preserving part of that condition ensures that no risk that is idiosyncratic to a negligible subset of assets in the original economy can be blown up to have aggregate effects in the new economy.[5] Finally, Condition (3) ensures that returns in the new repackaged economy have well-behaved covariance function.

This definition of repackaging covers a wide range of possibilities. For example, it covers measurable permutations of assets, implying that the particular way assets are indexed as points in [0,1] is not important, as well as general forms of consolidation of several assets into larger ones and fragmentation of existing assets into smaller pieces. Also, since the composition of repackagings is also a repackaging, the definition provides a systematic way to obtain more complex repackagings starting from more elementary ones.

Further discussion of the three conditions used in defining repackagings; ways to relax them; and the intuition and interpretation of the Theorem can be found in Section 4.

3.2. Main Result

The main result of the paper can now be stated:

THEOREM: *If the initial economy E has a factor space F, then any repackaged economy E' has a unique factor space $F' \subset F$.*

The intuition underlying the Theorem is that in a model with a continuum of assets, each set of assets has a well defined mass relative to the rest of the economy. In particular, there is a straightforward way to define what is meant by a subset of assets being negligible (*i.e.*, has measure zero) and negligibility is not affected when pieces of assets are shuffled in a measure-preserving manner. Thus, any risk specific to a negligible subset of assets in the original economy will remain so after assets have been repackaged. In other words, repackaging cannot create new factors because a factor is a type of risk that can explain a non-negligible fraction of the variation in a non-negligible subset of assets.

Proof of the Theorem: For any random variable $\tilde{x}$, define $cov_{\tilde{x}}(\tilde{R}'_s) = cov(\tilde{x}, \tilde{R}'_s)$. Then

$$\begin{aligned} cov_{\tilde{x}}(\tilde{R}'_s) &= cov_{\tilde{x}}\left(\sum_{i=1}^{\infty} \alpha_i(\lambda_i^{-1}(s))\ \tilde{R}_{\lambda_i^{-1}(s)}\right) \\ &= \sum_{i=1}^{\infty} \alpha_i(\lambda_i^{-1}(s)) cov_{\tilde{x}}\left(\tilde{R}_{\lambda_i^{-1}(s)}\right). \end{aligned}$$

For each i, $\tilde{R}_{\lambda_i^{-1}(s)}$ represents the returns of the original economy after applying the measure-preserving permutation λ_i. The function $cov_{\tilde{x}}\left(\tilde{R}_{\lambda_i^{-1}(s)}\right)$ is the composition $s \mapsto t = \lambda_i^{-1}(s) \mapsto cov_{\tilde{x}}(\tilde{R}_t)$. The function λ_i^{-1} is measurable by assumption, while the measurability of $t \mapsto cov_{\tilde{x}}(\tilde{R}_t)$ follows from that of the covariance function of asset returns. To see this, note that $cov_{\tilde{x}}(\tilde{R}_t) = 0$, $t -$ a.e. whenever $\tilde{x} \perp F$, so we only need to consider $\tilde{x} \in F$. For such $\tilde{x}$, the measurability of $cov_{\tilde{x}}(\tilde{R}_t)$ follows from the measurability of the linear projections of $\tilde{R}_t$ onto F (see the remark at the end of Section 2). We therefore conclude that $s \mapsto cov_{\tilde{x}}\left(\tilde{R}_{\lambda_i^{-1}(s)}\right)$ is measurable, being the composition of two measurable functions.

The next step is to show that if $\tilde{x}$ is orthogonal to F, then $cov_{\tilde{x}}(\tilde{R}_t) = 0$, $t - a.e.$ Note that $cov_{\tilde{x}}(\tilde{R}_t) = cov_{\tilde{x}}(\mathrm{Proj}_F \tilde{R}_t) + cov_{\tilde{x}}(\tilde{h}_t) = cov_{\tilde{x}}(\tilde{h}_t)$. Let H denote the linear space spanned by $\{\tilde{h}_t : t \in T\}$. Since the projection of $\tilde{x}$ on the orthogonal complement of H has zero covariance with every $\tilde{h}_t$, we may assume, without loss of generality, that $\tilde{x} \in H$. From the definition of H, there is countable set of indices $A \subset T$ and real numbers $\{b_\alpha : \alpha \in A\}$ such that $\tilde{x} = \sum_{\alpha \in A} b_\alpha \tilde{h}_\alpha$. Then $cov_{\tilde{x}}(\tilde{h}_t) = \sum_{\alpha \in A} b_\alpha\ cov(\tilde{h}_\alpha, \tilde{h}_t)$. Since F is a strict factor space for E, the residuals $\tilde{h}_t$ are idiosyncratic, so for every α, $cov(\tilde{h}_\alpha, \tilde{h}_t) = 0$, $t -$ a.e. This implies that $cov_{\tilde{x}}(\tilde{h}_t)$ (hence $cov_{\tilde{x}}(\tilde{R}_t)$) is zero $t -$ a.e., being the countable sum of functions that are zero almost everywhere.

The assumption that each λ_i is a measure-preserving bijection implies, in particular, that they preserve sets of measure zero. Thus, for any random variable $\tilde{x}$, $cov_{\tilde{x}}(\tilde{R}_t) = 0$, $t -$ a.e. if and only if $cov_{\tilde{x}}(\tilde{R}_{\lambda_t^{-1}(s)}) = 0$, s – a.e. This implies that, for $\tilde{x}$ orthogonal to F, $cov_{\tilde{x}}\left(\tilde{R}_{\lambda_t^{-1}(s)}\right) = 0$, s – a.e. and, consequently, that $cov_{\tilde{x}}(\tilde{R}'_s) = 0$, s – a.e.

This shows that any $\tilde{x}$ orthogonal to F explains variations in only a negligible subset of assets in the repackaged economy. We now take F as a candidate factor space for E', with corresponding residuals $\tilde{h}''_s$. That is, we write returns in E' in the form:

$$\tilde{R}'_s = E\tilde{R}'_s + \text{Proj}_F \tilde{R}'_s + \tilde{h}''_s,$$

and let Cov'' denote the covariance function of the residuals $\tilde{h}''_s$. It is easy to check that Cov'' is bounded and measurable. We now use the fact that $cov_{\tilde{x}}(\tilde{R}'_s)$ is zero a.e. for every $\tilde{x} \perp F$ to show that $\tilde{h}''_t$ is idiosyncratic. By Fubini's theorem, we have

$$\int_{[0,1]\times[0,1]} Cov''(t,s)\ d\tau\ d\tau) = \int_{[0,1]} \int_{[0,1]} Cov''(t,s)|\, d(\tau \times \tau)$$

$$= \int_{[0,1]} \left[\int_{[0,1]} |cov_{\tilde{h}''_t}(\tilde{h}''_s)|\, d\tau \right] d\tau.$$

Since $\tilde{h}''_t$ is orthogonal to F for every t, the argument above implies that $cov_{\tilde{h}''_t}(\tilde{R}_s) = 0$, s – a.e., hence $cov_{\tilde{h}''_t}(\tilde{h}''_s) = 0$, s – a.e.. Thus, the integral in square brackets is zero for every t. This implies that Cov'' is zero almost everywhere on $[0,1] \times [0,1]$ so the residuals $\tilde{h}''_s$ are indeed idiosyncratic.

Thus, purging the returns $\tilde{R}'_s$ from all variations that can be explained by F leaves residuals $\tilde{h}''_s$ that are idiosyncratic, so F satisfies Condition (1) of the definition of strict factor structure relative to E'. We now show that we can extract a unique linear subspace $F' \subset F$ that satisfies both conditions. The set of random variable $\tilde{x} \in F$ such that $cov_{\tilde{x}}(\tilde{R}'_s) = 0$, s – a.e. is a linear subspace F_0 of F. Let F' be the orthogonal complement of F_0 in F and write

$$\tilde{R}'_s = R'_s + \text{Proj}_{F'} \tilde{R}'_s + \tilde{h}'_s.$$

Clearly, the new residuals $\tilde{h}'_s$ differ from the old residuals $\tilde{h}''_s$ only on a set of measure zero of assets, and so must themselves be idiosyncratic. It is also clear by construction that no proper subspace of F' can be a factor space for the repackaged economy E', so F' is indeed a factor structure for E'. The uniqueness of F' follows from uniqueness of F_0 and orthogonal complements.

Q.E.D.

3.3. Technical Remarks

1- The definition of repackaging includes simplifying restrictions that are not essential for the main argument of this paper. The boundedness part of Condition (3) would have been superfluous if we chose to impose a uniform upper bound on the number of pieces an asset can be divided into. The measurability part of that condition is in fact redundant, even when countable divisions are allowed, but was maintained because the proof that E' has a measurable covariance function has little relevance to the main point made in this paper.

2- Allowing assets to be divided into countably many pieces is done purely for convenience and can be modified in several ways without significant consequences on the result. For example, we can impose a uniform bound on the number pieces an asset can be divided into by simply requiring that all but finitely many of the α_i functions are identically zero. The added flexibility of a countable number of divisions has the advantage of simplifying the description of repackagings in which some assets might be sliced more finely than others. In the other direction, nothing in the proof seems to hinge on the assumption that the number of possible divisions of an asset is countable. It is therefore likely that a similar result would hold if we allow some assets to be divided into a continuum of pieces, instead of countably many as in the present framework. Sums would then have to be appropriately replaced by integrals and some measurability issues may need to be dealt with.

3- The assumption that assets are indexed by an atomless measure space reflects the intuition (found in many pricing theories, such as the APT) that individual assets become negligible as the economy becomes large. However, the model may also be interpreted to include assets whose weights relative to the rest of the economy remains large regardless of how many other assets are added. To motivate this interpretation, note that in asset pricing applications, assets are identified by their random rate of returns. For instance, in a model with a finite number of assets, there is no relevant distinction, from asset pricing point of view, between one asset with return $\tilde{R}_t$, and N assets each with return $\tilde{R}_t/N$. Using this observation, an asset with return $\tilde{R}_t$ and weight $\alpha \in [0,1]$ can be represented by creating copies of this return and assigning them to a subset of assets of measure α. The definitions and analysis of this paper can then be applied without change.

4. DISCUSSION AND INTERPRETATION

Remark 1: The substantive restriction imposed by the definition of a repackaging is contained in Equation (3) which implies that the way assets are combined in the economy is neutral in the sense that repackaging leaves total wealth unaffected. This assumption might be objected to because a set of assets might be more productive when combined together in a single larger asset (or firm), so the return on the firm may be greater than the sum of the returns on its individual components. This suggests that returns in the repackaged economy should not be computed linearly, as in Equation (3).

Three points are worth mentioning in response. First, the proof of the Theorem is not too sensitive to the assumption that returns in the repackaged economy are calculated linearly. All that matters is that in calculating these returns, no risk specific to a set of measure zero of assets in the original economy can be blown into a factor because of repackaging. This is consistent with many alternative methods of calculating returns in the repackaged economy. Linearity does not seem to be crucial to the result, but is retained to simplify the comparison with earlier papers. Second, if the original economy had no arbitrage opportunities, then every potentially profitable combination of assets would have been formed through the appropriate consolidation assets in the original economy.[6] Third, our definition of repackaging is general enough to accommodate additional conditions asserting that there are, for example, geographical, fiscal, or informational-related reasons that give one representation of assets inherent advantages over others. The main result states that the factor structure is preserved even when assets are repackaged in an essentially arbitrary manner.

Remark 2: [7] Suppose that an asset provides a unique hedging opportunity not offered by any other asset. This insurance possibility would not have existed without this asset, suggesting that such asset might carry a greater weight in the economy than would be consistent with our assumption of an atomless measure space of assets (*i.e.*, the asset becomes an atom). To take an extreme example, consider an economy with asset returns $\tilde{r}_0 = -\tilde{\eta}$, and $\tilde{r}_t = \tilde{\eta}$, for $t > 0$. Asset $t = 0$ is a perfect hedge against the risk of any other asset, creating a new opportunity of a riskless return that would not have existed otherwise. However, since only one unit of this asset is available, one would expect that in a large economy this asset can be used to hedge against the risk in only a small subset of the remaining assets. In the framework of this paper, this intuition is captured by the requirement that the permutations λ_i's are measure preserving, so the hedge provided by asset $t = 0$ can be used only against risks in a set of measure zero of the remaining assets. Thus, although asset $t = 0$ provides a unique hedging opportunity against economy-wide risk, its role remains negligible because it is available in too small a quantity to make much difference in improving risk sharing in the economy.

Remark 3: Fix a random variable $\tilde{\eta}$ and define the asset economy E by setting $\tilde{R}_t = \tilde{\eta}$ if $t \in [0, 1/2)$, and $\tilde{R}_t = -\tilde{\eta}$ for $t \in [1/2, 1]$. Consider the following repackaging: $\lambda_1(t) = t$, $\lambda_2(t) = -t$, and $\alpha_1(t) = \alpha_2(t) = 0.5$ for $0 \leq t \leq 1$. The original economy has a strict factor structure with a single factor $\tilde{\eta}$ and corresponding exposure (beta) which is 1 on $[0, 1/2)$ and -1 on $[1/2, 1]$. The repackaged economy, on the other hand, has a zero-dimensional factor structure because, with the exception of asset $s = 1/2$, returns in the repackaged economy are given by

$$\tilde{R}'_s = (\tilde{R}_s + \tilde{R}_{1-s})2 = (\tilde{\eta} - \tilde{\eta})/2 = 0.$$

There is nothing special or pathological here; the same phenomenon also appears in richer, more complex examples. The intuition is that, even if all risk in the original economy is factor-risk, the distribution of the exposures of various assets to this risk may be such that there are readily available hedges against it. The key observation is that the exposures (the betas) change sign so a factor-risk may be eliminated by using portfolios with appropriate weights assigned to assets with positive and negative exposures to the factor.

Note that the elimination of risk in this example, is not achieved via diversification as in the APT. Diversification, which is effective in dealing with idiosyncratic risk, is irrelevant here since all risk is factor-risk and repackaging involves no more than two assets at a time. Rather, repackaging reduces the number of factors through the cancellation of risks in assets with negatively correlated exposures to factor risk. This phenomena is not special to the present model (consider, for example, a sequence economy in which assets with an even index have return $\tilde{\eta}$ and assets with an odd index have return $-\tilde{\eta}$).

Remark 4: The last Remark suggests that it might be useful to distinguish between factor risks according to whether they can be eliminated through a simple recombination of assets. Thus, the $\tilde{\eta}$ factor risk in the example in Remark 3 above may be viewed as corresponding to a *reducible* factor risk because a simple repackaging of assets is sufficient to diversify it away. On the other hand, if $\tilde{R}_t = \tilde{\eta}$ for all t, then no repackaging can eliminate the $\tilde{\eta}$ factor risk in this case.

An appropriate definition and characterization of the reducibility of factor risks may be a bit more elusive than suggested in the simple one-factor examples above. One complication that might arise in a multi-factor settings is that recombining assets in a particular way to hedge against one type of factor risk might introduce or increase exposure to other types of factor risks. This suggests that one is unlikely to be able to characterize the reducibility of one factor in isolation from the way it is combined with other factors in the economy.

ENDNOTES

1- As in earlier papers, returns on repackaged assets are obtained linearly from returns of their components. See the more detailed discussion in Section 4.

2- Detailed discussion of this construction and its use in the type of models considered here can be found in Al-Najjar (1994, 1995).

3- See the last step of the proof of the main theorem.

4- The betas are often defined in terms of the *rates* of returns (the $\bar{r}_t$'s). If the function p_t is well-behaved, then the betas defined here can be converted to betas in terms of rates of returns in a straightforward manner.

5- The result depends only on the requirement that the rearrangement is done in an absolutely continuous manner.

6- This is basically the argument made by Gilles and LeRoy (1991).

7- I thank a referee for raising this point.

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