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CHARACTERIZING UNCERTAINTY AVERSION THROUGH PREFERENCE FOR MIXTURES[1]

by

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Abstract

Uncertainty aversion is often modelled as (strict) quasi-concavity of preferences over uncertain acts. A theory of uncertainty aversion may be characterized by the pairs of acts for which strict preference for a mixture between them is permitted. This paper provides such a characterization for two leading representations of uncertainty averse preferences: those of Schmeidler [20] (Choquet expected utility) and of Gilboa and Schmeidler [13] (maxmin expected utility with a non-unique prior). This characterization clarifies the relation between the two theories. *Journal of Economic Literature* Classification Number: D81.

1 Introduction

A large body of work has recently emerged in economics and decision theory with the goal of representing behavior in the face of subjective uncertainty that may violate the independence axiom of subjective expected utility theory. One branch of this literature, and the one that will be the focus below, considers preferences that may violate independence by displaying a preference for facing risk (or "objective" probabilities) as opposed to uncertainty. This preference is known as *uncertainty aversion*. One motivation for examining these preferences are the well-known problems posed by Ellsberg [9] and the huge experimental literature that has followed, in which many individuals behave as if they were uncertainty averse.

There are several ways that one could imagine defining uncertainty aversion. The definition that I will use here, and the one that has dominated the literature so far, is due to Schmeidler [20]. It states that for any two acts which an individual is indifferent between, a mixture, or randomization[1], over these two acts is at least as preferred as either act. One may interpret this requirement as saying that the individual likes smoothing expected utility across states. This smoothing has the effect of making the outcome less subjective, and therefore such a mixing operation could be called "objectifying"[2]. Thus, in a natural sense, such an individual is displaying an aversion to uncertainty. An equivalent way of stating this characteristic is to say that preferences are quasi-concave ($f \succeq g$ and $\alpha \in (0, 1)$ implies $\alpha f + (1-\alpha)g \succeq g$). In particular, observe that if $f \sim g$ then quasi-concavity allows $\alpha f + (1-\alpha)g \succ g$ while independence requires $\alpha f + (1-\alpha)g \sim g$.

From this viewpoint, a theory of uncertainty averse preferences may be characterized by the set of violations of independence in the direction of strict quasi-concavity that it allows. The goal of this paper is to provide a characterization of this kind for two leading axiomatic theories of uncertainty aversion, the Choquet expected utility theory of Schmeidler [20] and the maxmin expected utility theory of Gilboa and Schmeidler [13]. Such a characterization is useful not only from the point of view of theoretical understanding, but also as a guide to the design of experiments testing one theory of

[1] There is some controversy in this literature as to whether or not a randomization should be considered equivalent to a mixture as defined in a formal sense. The correctness and (in large part) interpretation of the analysis below is independent of one's position in this debate. The objects of study are acts, while the issue of randomization concerns which acts are feasible in particular settings.

[2] I thank Mark Machina for suggesting this term.

uncertainty aversion against another. Furthermore, in the emerging literature applying these theories (e.g. Dow and Werlang [7], Klibanoff [14], Lo [15], Eichberger and Kelsey [8], Marinacci [17] on game theory; Wakker [21] on optimism and pessimism; Dow and Werlang [6], Chateauneuf, Kast, and Lapied [3], Epstein and Wang [10] on financial markets; Mukerji [18] and Ghirardato [11] on contracting; and others) too often one theory or the other is adopted without much recognition of the ways in which the theories differ.

The next section introduces the Choquet expected utility and maxmin expected utility theories and points out the known, yet frequently ignored, fact that under uncertainty aversion, maxmin expected utility is a strict generalization of Choquet expected utility. Section 3 presents the main theorems characterizing the acts for which no convex combination is ever strictly preferred to both acts themselves under maxmin expected utility and under Choquet expected utility respectively. Section 4 concludes.

2 Two Models of Uncertainty Aversion

2.1 Notation and Set-up

Throughout the paper, preferences are a binary relation, $\succeq$, on functions (acts) $f : S \to Y$ where S is a finite set of states of the world, X a set of prizes, and Y the set of all probability measures with finite support (lotteries) on X. Mixtures of acts are taken pointwise, and thus the set of acts is closed under mixtures. Lotteries are evaluated according to an affine utility function $u : Y \to \Re$.

2.2 Two Models

A leading representation of uncertainty averse preferences is the Choquet expected utility representation axiomatized by Schmeidler [20]. Here preferences are represented by the Choquet integral of a utility function with respect to a capacity or non-additive measure. One of the properties which characterizes such preferences is *comonotonic independence*. Two acts, f and g, are said to be comonotonic if, for no pair of states of the world s and s',

$f(s) \succ f(s')$ and $g(s') \succ g(s)$.[3] Preferences satisfy comonotonic independence if, for any acts f and g, $f \succeq g$ if and only if $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$ for all $\alpha \in (0,1)$ and all h such that f, g, h are pairwise comonotonic. This is simply a restriction of the standard independence axiom (e.g. Anscombe and Aumann [1]) to pairwise comonotonic acts. From this axiom, the following is immediate:

Result 1 *Suppose that preferences satisfy comonotonic independence. Then for any comonotonic acts f and g, for each $\alpha \in (0,1)$, either $f \succeq \alpha f + (1-\alpha)g$ or $g \succeq \alpha f + (1-\alpha)g$ or both.*

Thus, strict preference for mixtures cannot occur with comonotonic acts. Notice that this observation derives from comonotonic independence *alone* and is in no way implied by uncertainty aversion per se.

Now consider the second main representation of preferences incorporating uncertainty aversion, namely the representation axiomatized in Gilboa and Schmeidler [13]. In this work, the axiom of comonotonic independence is replaced by an alternative axiom, denoted C-independence. C-independence requires the independence axiom to hold only when the act h used to form the mixtures gives the same expected utility in every state of the world.[4] Intuitively, acts which yield the same expected utility in every state leave no room for uncertainty about which state will occur to matter. C-independence is the assumption that mixing with such an act will not change either the way in which the decision maker perceives her uncertainty or the way in which she allows her attitude towards uncertainty to affect her preferences.

Gilboa and Schmeidler [13] showed that C-independence and the standard assumptions of weak order, continuity and monotonicity together with uncertainty aversion imply that preferences can be represented by the minimum expected utility of an act, where the minimum is taken over a closed, convex set of probability measures. Notice that an act which yields constant expected utility across states is comonotonic with any other

[3] $f(s)$ should be understood as an act which gives the outcome that act f gives in state s no matter which state occurs.

[4] Technically the axiom is more restrictive, requiring h to give the same lottery over outcomes in each state of the world, but together with the assumptions of weak order, continuity and monotonicity the axiom as described is implied. Note that the assumptions of weak order, continuity and monotonicity were also assumed in the Choquet expected utility theory of Schmeidler [20].

act. In fact, comonotonic independence, weak order, continuity, and monotonicity imply C-independence. This means that, under the assumption of uncertainty aversion, any preferences that can be represented by Choquet expected utility can also be represented in the sets of measures framework (Schmeidler [19], [20]). The converse is not true, however, as the following example makes clear.

	s_1	s_2	s_3
f	1.5	2	3.5
g	0	2.1	4

Set of measures: $B = \{(p_1, p_2, p_3) | p_1 = p_3, \sum_{i=1}^{3} p_i = 1, 0 \leq p_i \leq 1\}$

Example 1

In example 1, an individual must choose over two possible pure acts, f and g, which give the expected utilities indicated above in the three possible states of the world. Observe that f and g are comonotonic. Suppose that the individual's preferences can be represented by minimum expected utility over the set of measures B (i.e. the set of all probability measures which assign equal weight to s_1 and s_3). Straightforward calculation shows that f and g each give a minimum expected utility of 2, while, for example, a half-half mixture between f and g gives a minimum expected utility of 2.05. Therefore, this uncertainty averse individual will strictly prefer a mixture over f and g compared to either act alone. Since this violates comonotonic independence it shows that these preferences cannot be represented in the Choquet expected utility framework, and also demonstrates that comonotonicity is not enough, in general, to guarantee that an uncertainty averse individual will not strictly prefer to objectify by mixing over acts. What is the right condition to guarantee no strict preference for mixtures in the maxmin expected utility representation? Is the comonotonicity condition a necessary as well as sufficient condition for no preference for mixing under Choquet expected utility? The next section provides results to answer these questions.

3 Characterizing Preference for Mixtures

In examining when strict preference for mixtures is possible (or impossible) under the two theories, it is helpful to consider a previous result characterizing preference for mixtures under maxmin expected utility for a *specified* set of probability measures. While such results are of more interest in a setting where certain beliefs are focal (e.g. equilibrium beliefs in game theory), they will be used in proving the theorems to follow that apply to the whole domain of the respective theories.

Theorem 1 *(Klibanoff [14]) For any acts f and g such that $f \succeq g$, no mixture over these acts will be strictly preferred to either alone if and only if there exists some measure q in the set of measures such that q minimizes the expected utility of f over the set and such that the expected utility of f with respect to q is at least the expected utility of g with respect to q.*

For acts which the decision maker is indifferent between this simplifies to:

Theorem 2 *(Klibanoff [14]) For any acts f and g such that $f \sim g$, no mixture over these acts will be strictly preferred to either alone if and only if there exists some measure q in the set of measures such that q minimizes the expected utility of both f and g over the set of measures.*

Now we characterize the set of acts for which *no* G-S decision maker would have a strict preference for a mixture. This result and the corresponding result for the Choquet expected utility case are provided in the next two theorems.

Theorem 3 *Fix acts f and g. No convex combination of f and g will ever be strictly preferred to either alone (given G-S maxmin expected utility preferences) if and only if (i) f weakly dominates g or vice-versa (i.e. $u(f(s)) \geq (\leq) u(g(s))$, for all $s \in S$.)* **or** *(ii) there exists an $a \geq 0, b \in \Re$ such that either $u(g(s)) = au(f(s)) + b$ for all $s \in S$ or $u(f(s)) = au(g(s)) + b$ for all $s \in S$.*

Proof: The difficult direction is to show that no strict preference implies (i) or (ii). The key step in the proof is to show that the conditions for a convex combination to reduce uncertainty are equivalent to the existence of a pair of probability vectors satisfying a set of linear inequalities. This is done in the lemma below. The only task remaining is then to characterize existence. To do this I apply a well known result from the theory of linear inequalities, Motzkin's Theorem of the alternative (see e.g. Mangasarian [16]). The existence of a solution to the resulting alternative system is then (after a bit of rearrangement) shown to be equivalent to the conditions of the theorem.

Let the vector of utility payoffs to the act f be denoted $\mathcal{F}(\equiv \{u(f(s))\})$ and similarly for $\mathcal{G}$. The following lemma reduces the conditions for a convex combination of f and g to possibly reduce uncertainty to a question of existence of probabilities satisfying certain linear inequalities.

Lemma 1 *Fix $\mathcal{F}$ and $\mathcal{G}$. There exists a non-empty, closed, convex set of measures B for which some mixture of $\mathcal{F}$ and $\mathcal{G}$ is strictly preferred to either alone if and only if there exist probability vectors $p1$ and $p2$ satisfying:*

$$(i)\mathcal{F} \cdot p2 - \mathcal{F} \cdot p1 > 0$$

$$(ii)\mathcal{F} \cdot p1 - \mathcal{G} \cdot p1 < 0$$

$$(iii)\mathcal{G} \cdot p1 - \mathcal{G} \cdot p2 > 0$$

and

$$(iv)\mathcal{G} \cdot p2 - \mathcal{F} \cdot p2 < 0$$

Proof:

($\Leftarrow$) Suppose such $p1$ and $p2$ exist. Let B be the set of all convex combinations of $p1$ and $p2$. Either $f \succeq g$ or $g \succeq f$ or both. If $f \succeq g$ then by (i) and (ii), $p1$ is the only minimizer in B of the expected utility of f and the expected utility of f under $p1$, $\mathcal{F} \cdot p1$ is less than the expected utility of g under $p1$, $\mathcal{G} \cdot p1$. Therefore, by Theorem 1, there exists a mixture which is strictly preferred. If $g \succeq f$ then by (iii) and (iv) and Theorem 1 the same conclusion holds.

($\Rightarrow$) Suppose such a B exists. Consider the set $A_f = \{p \mid p \in \text{argmin}_{p \in B} \mathcal{F} \cdot p\}$ and $A_g = \{p \mid p \in \text{argmin}_{p \in B} \mathcal{G} \cdot p\}$. Consider $p1 \in A_f$ and $p2 \in A_g$. By definition of these sets we must have

$$(a)\mathcal{F} \cdot p2 - \mathcal{F} \cdot p1 \geq 0$$

and

$$(b)\mathcal{G} \cdot p1 - \mathcal{G} \cdot p2 \geq 0.$$

Suppose that (a) holds with equality for some such $p1$ and $p2$. Then if $g \succeq f$, $\mathcal{G} \cdot p2 \geq \mathcal{F} \cdot p2$ which implies that the condition in Theorem 1 is satisfied and no mixture is strictly preferred. If $f \succ g$, then $\mathcal{F} \cdot p1 = \mathcal{F} \cdot p2 > \mathcal{G} \cdot p2$ and again appealing to Theorem 1 no mixture is strictly preferred. Similar arguments show that if (b) holds with equality for some such $p1$ and $p2$ then no mixture is strictly preferred. Thus for there to be a mixture that is strictly preferred it must be that for all $p1 \in A_f$ and $p2 \in A_g$,

$$(i)\mathcal{F} \cdot p2 - \mathcal{F} \cdot p1 > 0$$

and

$$(iii)\mathcal{G} \cdot p1 - \mathcal{G} \cdot p2 > 0.$$

Can it be that $\mathcal{F} \cdot p1 - \mathcal{G} \cdot p1 \geq 0$? This and (iii) would imply $\mathcal{F} \cdot p1 - \mathcal{G} \cdot p2 > 0$ which implies $f \succ g$ and thus by Theorem 1 and the hypothesis no mixture would be strictly preferred. Therefore,

$$(ii)\mathcal{F} \cdot p1 - \mathcal{G} \cdot p1 < 0$$

must hold. By an analogous argument,

$$(iv)\mathcal{G} \cdot p2 - \mathcal{F} \cdot p2 < 0$$

must hold as well and we are done. *QED*

Now that the lemma has been proved, the next step in proving the theorem is to combine conditions (i)-(iv) with the restrictions implied by the fact that $p1$ and $p2$ must be probability vectors. To this end, let n be the number of states in S. Then $\mathcal{F}, \mathcal{G}, p1$ and $p2$ are n-vectors. Let $\mathcal{F}$ and $\mathcal{G}$ be row vectors and $p1$ and $p2$ be column vectors. Let e be a row n-vector of 1's. Let

$$p = \begin{bmatrix} p1 \\ p2 \end{bmatrix}$$

and

$$A = \begin{bmatrix} -\mathcal{F} & \mathcal{F} \\ \mathcal{G} - \mathcal{F} & 0 \\ \mathcal{G} & -\mathcal{G} \\ 0 & \mathcal{F} - \mathcal{G} \end{bmatrix}.$$

Observe that (i)-(iv) is equivalent to $Ap > 0$. Furthermore the requirement that $p1$ and $p2$ be probabilities is equivalent to $p \geq 0$ and

$$\begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{1}$$

Equivalently, we can replace the normalization (1) with the condition

$$\begin{bmatrix} \epsilon & -\epsilon \end{bmatrix} p = 0$$

and the condition $p \geq 0$ with the equivalent

$$Ip \geq 0$$

where I is a 2n x 2n identity matrix. To summarize, we would like to characterize when there exists a p such that

$$(a) Ap > 0$$

$$(b) Ip \geq 0$$

and

$$(c) \begin{bmatrix} \epsilon & -\epsilon \end{bmatrix} p = 0.$$

By Motzkin's Theorem of the alternative (Mangasarian [16]), either (a), (b) and (c) has a solution p or

$$(*) \left\langle \begin{array}{c} A'y_1 + I'y_3 + \begin{bmatrix} \epsilon & -\epsilon \end{bmatrix}' y_4 = 0 \\ y_1 > 0, y_3 \geq 0 \end{array} \right\rangle$$

has a solution y_1, y_3, y_4, but never both. (Note that $y_1 > 0$ means that each element of y_1 is greater than or equal to zero with at least one element strictly positive. $y_3 \geq 0$ means almost the same thing except that it allows all elements to be zero.)

All that remains is to rewrite system (*) to get an interpretable condition (namely the one in the theorem.) First notice that since the elements of y_3 are all non-negative, (*) has a solution if and only if

$$(**) \left\langle \begin{array}{c} A'y_1 + \begin{bmatrix} \epsilon & -\epsilon \end{bmatrix}' y_4 \leq 0 \\ y_1 > 0 \end{array} \right\rangle$$

has a solution y_1, y_4. Adding up the inequalities determined by the first line of (**) yields

$$(\mathcal{G} - \mathcal{F})'(y_{2:}^1 - y_{4:}^1) \leq 0.$$

where

$$y_1 = \begin{bmatrix} y_{11}^1 \\ y_{21}^1 \\ y_{31}^1 \\ y_{41}^1 \end{bmatrix}.$$

This implies that either $y_{21}^1 = y_{41}^1$ or one of f and g is weakly dominated by the other. So, a solution to (*) exists if and only if either weak dominance between f and g holds or (**) is satisfied with $y_{21}^1 = y_{41}^1$. Imposing the latter restriction and disaggregating the inequality in (**) we obtain the system

$$\left\langle \begin{array}{c} \mathcal{G}'(y_{21}^1 + y_{31}^1) - \mathcal{F}'(y_{11}^1 + y_{21}^1) + \epsilon y_4 \leq 0 \\ -\mathcal{G}'(y_{21}^1 + y_{31}^1) + \mathcal{F}'(y_{11}^1 + y_{21}^1) - \epsilon y_4 \leq 0 \\ y_1 > 0 \end{array} \right\rangle$$

which is equivalent to

$$(***) \left\langle \begin{array}{c} \mathcal{G}'(y_{21}^1 + y_{31}^1) - \mathcal{F}'(y_{11}^1 + y_{21}^1) + \epsilon y_4 = 0 \\ y_1 > 0 \end{array} \right\rangle$$

. Observe that without loss of generality y_{21}^1 can be set to zero since it can be incorporated into y_{11}^1 and y_{31}^1. Now, suppose that one of y_{11}^1 or y_{31}^1 is zero. Then a solution will exist if and only if either g or f or both are constant utility acts. Finally, consider the remaining case where both y_{11}^1 and y_{31}^1 are positive. Here a solution exists if and only if there exists an $\alpha > 0$, $\beta > 0$, and y_4 such that

$$\alpha\mathcal{G}' - \beta\mathcal{F}' + \epsilon y_4 = 0. \tag{2}$$

This last condition is equivalent to

$$\mathcal{G}' = a\mathcal{F}' + b\epsilon', \text{ for some } a > 0,\ b \in \Re. \tag{3}$$

Now note that the case where $a = 0$ corresponds to the cases where g is a constant act. If only f is a constant act, simply reverse the roles of f and g and again set $a = 0$.

Pulling the different possibilities together, we have that a solution to (*) exists if and only if either f and g are ordered by weak dominance or

$$\mathcal{G}' = a\mathcal{F}' + b\epsilon', \text{ for some } a \geq 0,\ b \in \Re \tag{4}$$

or,

$$\mathcal{F}' = a\mathcal{G}' + be', \text{ for some } a \geq 0,\ b \in \Re. \tag{5}$$

Our application of Motzkin's Theorem now yields the desired conclusion. *QED*

The analogue for Choquet expected utility is given in the next theorem. Note that this result is related to that of Bassanezi and Greco [2] who show that the Choquet integral is additive for all capacities if and only if the functions being integrated are comonotonic.

Theorem 4 *Fix acts f and g. No convex combination of f and g will ever be strictly preferred to either alone (given Choquet expected utility preferences) if and only if (i) f weakly dominates g or vice-versa (i.e. $u(f(s)) \geq (\leq) u(g(s))$, for all $s \in S$.)* **or** *(ii) f and g are comonotonic.*

Proof:

($\Leftarrow$) It is straightforward that (i) implies the weakly dominant act will be at least as good as any mixture. Result 1 stated earlier says that (ii) implies no mixture strictly preferred.

($\Rightarrow$) We will show that Not ((i) or (ii)) implies there exists a mixture that may be strictly preferred to both f and g. Not ((i) or (ii)) implies f, g not comonotonic and no weak dominance between them. Since the two acts are not comonotonic, there exist states $s_f, s_g \in S$ such that $f(s_f) \succ f(s_g)$ and $g(s_g) \succ g(s_f)$. Consider the restriction of f and g to $\{s_f, s_g\}$. There are two possible cases:

Case I: Neither restricted act weakly dominates the other. In this case, without loss of generality assume that $f(s_f) \succeq g(s_g) \succ g(s_f) \succeq f(s_g)$. Consider a capacity v such that $v(\{s_f\}) = v(\{s_g\}) = 0$ and $v(\{s_f, s_g\}) = 1$. Relative to this capacity, we can calculate the Choquet expected utility (CEU) of f and g: $CEU(f) = u(f(s_g))$ and $CEU(g) = u(g(s_f))$. By continuity of preferences, there exists an $\alpha^* \in (0, 1)$ such that

$$\alpha^* g(s_g) + (1 - \alpha^*) f(s_g) \succ g(s_f).$$

Taking the CEU of this convex combination with respect to v yields,

$$\begin{aligned} CEU(\alpha^* g + (1-\alpha^*)f) &= \min[\alpha^* u(g(s_f)) + (1-\alpha^*)u(f(s_f)), \alpha^* u(g(s_g)) + (1-\alpha^*)u(f(s_g))] \\ &> u(g(s_f)) \geq u(f(s_g)). \end{aligned}$$

Therefore $\alpha^*g + (1-\alpha^*)f \succ f$ and $\alpha^*g + (1-\alpha^*)f \succ g$ for this v.

This proves the claim for the case where neither restricted act weakly dominates the other. Now we examine the remaining possibility:

Case II: One restricted act weakly dominates the other. Without loss of generality assume $f(s_f) \succ f(s_g) \succeq g(s_g) \succ g(s_f)$. Since, over the whole space, S, we assumed neither act weakly dominates the other, there must exist an $s' \in S$ such that $g(s') \succ f(s')$.

There are several possibilities. First, suppose that $g(s') \succeq g(s_g)$. Then $g(s') \succ f(s')$ implies $f(s_g) \succ f(s')$ so that f and g are not comonotonic on $\{s_g, s'\}$ and Case I applies to the restriction of f and g to $\{s_g, s'\}$.

Another possible ordering of the states by g is $g(s_g) \succ g(s') \succeq g(s_f)$. Here $g(s') \succ f(s')$ implies $f(s_f) \succ f(s')$ and Case I applies to the restriction of f and g to $\{s_f, s'\}$.

Finally, assume (the only remaining possibility) that $g(s_f) \succ g(s') \succ f(s')$. Consider a capacity v such that $v(\{s_f, s_g, s'\}) = 1$, $v(\{s_f, s_g\}) = k$, and v is zero on all other subsets of $\{s_f, s_g, s'\}$. Choose $k \in (0,1)$ to satisfy

$$ku(f(s_g)) + (1-k)u(f(s')) = ku(g(s_f)) + (1-k)u(g(s')).$$

Such a k exists under our ordering assumptions. Using the capacity v,

$$CEU(f) = ku(f(s_g)) + (1-k)u(f(s')),$$

and

$$CEU(g) = ku(g(s_f)) + (1-k)u(g(s')).$$

Thus for this capacity v and utility u, $f \sim g$. Now, using the fact that we can represent the CEU preferences under v as the maxmin expected utility over the set of probability measures that are in the core of v (i.e. $\{p \mid p(s_f) + p(s_g) \geq k, p(s_f) + p(s_g) + p(s') = 1\}$) (Schmeidler [19], [20]), we can apply Theorem 2 to show that some convex combination will be strictly preferred to both f and g.

To summarize, in each of the possible cases where Not ((i) or (ii)) holds the above has shown that there exists a convex combination that may be strictly preferred to both f and g. *QED*

To facilitate a comparison with theorem 3 the following corollary is provided:

Corollary 1 *Fix acts f and g. No convex combination of f and g will ever be strictly preferred to either alone (given Choquet expected utility preferences) if and only if (i) f weakly dominates g or vice-versa (i.e. $u(f(s)) \geq (\leq) u(g(s))$, for all $s \in S$.)* **or** *(ii) there exists an act h and weakly increasing functions w and x on $\Re$ such that, for all $s \in S$, $u(f(s)) = w(u(h(s))$ and $u(g(s)) = x(u(h(s))$.*

Proof: By Denneberg [5, Proposition 4.5], two functions $d, e : S \to \Re$ are comonotonic if and only if there exists a function $z : S \to \Re$ and weakly increasing functions w, x on $\Re$ such that $d = w(z)$ and $e = x(z)$. Let $d = u \circ f$, $e = u \circ g$, and $z = u \circ h$ and the result follows from theorem 4 and the fact that f and g are comonotonic if and only if $u \circ f$ and $u \circ g$ are. *QED*

To see how this result compares to theorem 3, observe that if we require w and x to be affine then condition (ii) of the corollary is equivalent to condition (ii) of theorem 3. While Choquet expected utility prevents strict preference for mixture for acts that are weakly increasing tranformations of the same utility payoffs, maxmin expected utility does so only if the transformations are affine. Intuitively, this says that maxmin EU decision makers care about the cardinal properties of the distribution of utilities across states when facing uncertainty, while Choquet EU individuals only consider the (roughly) ordinal properties.

Remark: As the results above concern strict preference for mixture, the reader may wonder whether this addresses all the relevant possibilities for strict quasi-concavity of the preferences. Specifically, can there exist acts f and g satisfying (i) and (ii) of the appropriate theorem above such that indifference curves over mixtures are strictly quasi-concave, yet no mixture is strictly preferred? It is easily seen that the answer may be yes only if (ii) is violated. To see this note that if (ii) is satisfied then for any G-S preferences the same probability measure will be used to evaluate all mixtures, generating linear indifference curves. Conversly, if (ii) is violated then $u(f)$ and $u(g)$ are not related by a positive affine transformation and therefore order probability measures distinctly. Given one minimizing measure for f and another for g, it follows that the measure used to evaluate $\alpha f + (1-\alpha)g$ must generate more than the minimum expected utility level for one of the two acts, producing strict quasi-concavity of preferences. Arguments similar to the ones above could be used to show this more formally and demonstrate it for the Choquet case as well.[5] There is then no essential loss in limiting our analysis, as we have,

[5]See also Ghirardato, Klibanoff, and Marinacci [12].

to preference for mixtures. Furthermore, by examining the preference for mixtures case, we see that only weak dominance limits the extent of the quasi-concavity permitted by a violation of (ii).[6]

Thus we have a characterization of the strict preference for mixtures (or strict quasi-concavity) that the two theories allow. It is hoped that this will lead to further exploration of both the empirical importance and the theoretical interest in distinguishing between the two representations.

4 Conclusion

Theories of uncertainty aversion may differ in the circumstances under which they allow violations of independence, and in particular strict preference for mixtures. This paper has provided a characterization of those acts which may never admit such a strict preference for the two leading representations of uncertainty aversion: maxmin expected utility and Choquet expected utility. The fact that these characterizations are substantially different has implications for empirical testing of the theories as well as for those trying to apply one or the other model and wondering what the consequences of the modelling choice are. Fundamentally, Choquet decision makers view variations of expected utilities across states in an (practically) ordinal way, while maxmin EU decision makers care about the full cardinal properties of such variation.

[6] An alternative reason for interest in preference for mixtures per se is that such preferences correspond precisely to violations of the analogue for uncertain acts of the *betweeness* property for preferences under risk (e.g. Dekel [4]).

References

[1] ANSCOMBE, F.J. AND R.J. AUMANN (1963) "A definition of subjective probability," *The Annals of Mathematical Statistics* **34**, 199–205.

[2] BASSANEZI, R. AND G. GRECO (1984) "Sull' additivita dell' integrale," *Rend. Sem. Mat. Univ. Padova* **72**, 249–275.

[3] CHATEAUNEUF, A., R. KAST, AND A. LAPIED (1992) "Choquet Pricing for Financial Markets with Frictions," mimeo, G.R.E.Q.E.

[4] DEKEL, E. (1986) "An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom," *Journal of Economic Theory* **40**, 304–318.

[5] DENNEBERG, D. (1994) *Non-additive Measure and Integral*, Kluwer Academic, Dordrecht.

[6] DOW, J. AND S. WERLANG (1992) "Uncertainty aversion, risk aversion and the optimal choice of portfolio," *Econometrica* **60**, 197–204.

[7] ——(1994) "Nash Equilibrium under Knightian Uncertainty: Breaking Down Backward Induction," *Journal of Economic Theory* **64**, 305–324.

[8] EICHBERGER, J. AND D. KELSEY (1995) "Non-additive Beliefs and Game Theory," Mimeo, University of Saarbrucken.

[9] ELLSBERG, D. (1961) "Risk, ambiguity, and the Savage axioms," *Quarterly Journal of Economics* **75**, 643–669.

[10] EPSTEIN, L. AND T. WANG (1994) "Intertemporal Asset Pricing Under Knightian Uncertainty," *Econometrica* **62**, 283–322.

[11] GHIRARDATO, P. (1994) "Agency Theory with Non-additive Uncertainty," mimeo, U.C. Berkeley.

[12] GHIRARDATO, P., P. KLIBANOFF, AND M. MARINACCI (1996) "Linearity with Multiple Priors," mimeo, Northwestern and CalTech.

[13] GILBOA, I. AND D. SCHMEIDLER (1989) "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics* **18**, 141–153.

[14] KLIBANOFF, P. (1994) "Uncertainty, Decision, and Normal Form Games," mimeo, Northwestern University.

[15] LO, K.C. (1995) "Equilibrium in Beliefs under Uncertainty," mimeo, University of Toronto.

[16] MANGASARIAN, O. (1969) *Nonlinear Programming*, McGraw-Hill.

[17] MARINACCI, M. (1996) "Ambiguous Games," mimeo, Northwestern University.

[18] MUKERJI, S. (1996) "Ambiguity Aversion and Incompleteness of Contractual Form," mimeo, University of Southampton.

[19] SCHMEIDLER, D. (1986) "Integral representation without additivity," *Proceedings of the American Mathematical Society* **97**, 255-261.

[20] ——(1989) "Subjective probability and expected utility without additivity," *Econometrica* **57**, 571–587.

[21] WAKKER, P. (1990) "Characterizing Optimism and Pessimism Directly through Comonotonicity," *Journal of Economic Theory* **52**, 453–463.