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THE LOSER'S CURSE AND INFORMATION AGGREGATION IN COMMON VALUE AUCTIONS

by

Wolfgang Pesendorfer
and
Jeroen M. Swinkels

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Abstract

We consider an auction in which k identical objects of unknown value are auctioned off to n bidders. The k highest bidders get an object and pay the $k+1^{st}$ bid. Bidders receive a signal that provides information about the value of the object. We characterize the unique symmetric equilibrium of this auction. We then consider a sequence of auctions A_r with n_r bidders and k_r objects. We show that price converges in probability to the true value of the object if and only if both $k_r \to \infty$ and $n_r - k_r \to \infty$, i.e., the number of objects and the number of bidders who do not receive an object in equilibrium go to infinity.

1 Introduction

In his paper "A bidding model of perfect competition," Wilson (1977) establishes a remarkable result about common value auctions. Consider an auction for a single object of unknown value. Each of n players receives a signal about the value, and then submits a bid. The highest bidder wins the object and pays his bid. Under appropriate conditions on the structure relating value to signals, price converges in probability to the true value of the object as the number of bidders goes to infinity. Thus the auction aggregates the information that is diffused through the economy. Milgrom (1979) provides a precise characterization of the signal structures that give convergence.

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Wilson interprets this result as providing insight into the general way in which price formation processes might aggregate diffuse information. The advantage of the auction setting is that it provides an explicit model of how individual actions and information translate into prices. This is in contrast to the price taking model of a competitive market that is characterized by equilibrium conditions on prices, but provides no explanation of how prices satisfying those conditions are arrived at. Milgrom (1981) carries this theme further, arguing that auction models address basic flaws in the rational expectations paradigm. He also extends the convergence result to uniform price k object auctions. Such auctions are a more appropriate model of many price setting environments.

It is tempting to interpret these results as holding rather positive news about the ability of markets to aggregate information. However, the conditions needed for aggregation are very strong. In equilibrium, winning the object conveys the information that $n-1$ other bidders received less favorable news about the value of the object. As $n-1$ grows, this winner's curse becomes arbitrarily strong and so for high bids to be optimal the bidder's own signal must be correspondingly powerful. In Wilson's setting, for any v', there must be a signal that is impossible for $v < v'$. Working in a setting with a finite set of values, Milgrom (1979) shows that this is essentially necessary: a necessary (and sufficient) condition for full information aggregation is that for every v', and any M, there is a signal that gives a likelihood ratio on v' vs. $v < v'$ of at least M. That is, there must be signals that come arbitrarily close to ruling out values below v'.[1] Both the intuition and the result generalize to uniform price auctions with any fixed number of objects (Milgrom (1981)).

These assumptions are very strong: for every value v' there is a signal s' such that *no matter what other information the bidder infers from the behavior of his opponents*, he still puts very small probability on v being less than v'. We thus view the Wilson and Milgrom results as essentially negative: *only under extreme informational assumptions is information fully aggregated.*

This brings us to our point of departure from the existing literature: for many

[1] To see why these conditions are sufficient note that while s' essentially rules out values below v', the strength of the winner's curse essentially rules out values above v' when a bidder with signal s' wins in equilibrium. Therefore, the player knows that if he wins with signal s' the value of the object is essentially v'. Since many other bidders will have received signals nearly as favorable as s', competition for the object rules out prices much below v'. The convergence result follows.

auctions where information aggregation is of interest, and certainly when using these auctions as models of other markets, it seems appropriate not only to let the number of buyers grow large, but to let *supply* grow large as well.

We consider the allocation of a set of k identical objects by an auction mechanism. The objects have unknown value v. Each of n bidders receives one of finitely many signals about v before submitting his bid. Signals satisfy the strict monotone likelihood ratio property. A single signal can only change the likelihood of any subset of values by a multiplicative factor uniformly bounded away from 0 and ∞. Thus the posterior of a bidder after having received any signal has every possible value v in its support and density bounded away from zero and infinity. An example satisfying our assumptions is the information structure with two signals, $s = 1, 2$, where the probability of receiving signal 1 is strictly decreasing in v.

Bidders chose bids as a function of their signal about value. The k highest bidders each get an object and pay a price equal to the $k+1^{st}$ highest bid, receiving a payoff equal to the difference between value and price.

We characterize the unique symmetric equilibrium of this auction. We then explore the limiting properties of the equilibrium as the number of bidders and objects grows large. In particular, we consider a sequence of auctions with n_r bidders and k_r objects and ask under what conditions the equilibrium price of the auction converges in probability to the true value of the object. We call this property full information aggregation.

We show that a necessary and sufficient condition for full information aggregation is that $k_r \to \infty$ and $n_r - k_r \to \infty$, i.e., both the number of objects sold and the number of bidders who do not receive an object goes to infinity. We call this property *double largeness.*

In an extension we ask what fraction of bidders must receive an informative signal for full information aggregation to hold. Consider the following environment. With probability γ_r a bidder receives a binary signal of the form discussed above. With probability $1 - \gamma_r$ the bidder receives no information. We show that full information aggregation will hold as long as double largeness holds and γ_r does not converge to zero too fast. More precisely, we require that $\gamma_r \sqrt{n_r} \to \infty$.

Our results stand in stark contrast to those in the fixed k setting. With fixed k, full information aggregation requires very strong signals. If supply is growing then there is full information aggregation even if individual signals are only minimally informative. This is true even if supply is growing very slowly relative to demand, and only a vanishing fraction of bidders is informed. Thus non-negligible supply can be a substitute for strong signals. This provides a much more encouraging foundation for a belief in the information aggregation properties of markets.

Winning with a bid b conveys the negative information that at most k of the other bidders chose to bid above b upon receiving their signals. This is the winner's curse. The driving force behind our results is that in our setting there is also a *loser's* curse: losing with a bid b conveys the positive information that at most $n-k$ of the other bidders chose to bid *below* b. If both $k_r \to \infty$ and $n_r - k_r \to \infty$ then the two effects together imply that the information that a bid is on the threshold between winning and losing conveys a very precise estimate of value. Since the precise value of the bid only matters when it is on this threshold, this will imply that the equilibrium price converges to the true value.

A feature of our result is that it is *not* necessary for supply to grow proportionately to demand: k_r/n_r can, for example, converge to 0. This is surprising since one might then expect the winner's curse to overwhelm the loser's curse. Some intuition comes from examining the properties of our equilibria: as k_r/n_r goes to zero a growing fraction of the bidders essentially "sit out" the auction by submitting bids very close to 0. The number of "active" bidders grows proportionally with k_r, and so the winner's and loser's curse effects balance out.

We devote considerable space to the question of uniqueness of equilibrium within the class of symmetric equilibria. While this result is of some independent interest, it is critical to the interpretation we place on our results. The equilibrium examined is surprisingly good at information aggregation. However, this equilibrium has a rather complex structure, and one might ask if there are more natural equilibria that do not satisfy full information aggregation. If this were true then full information aggregation would not be a fundamental property of this market, but rather a consequence of a special choice of equilibrium.

We think the restriction to symmetric equilibria is quite natural in our setting, since the symmetry assumption can be viewed as an *anonymity* assumption.

Anonymity seems a natural requirement when modeling large markets.[2] So, establishing that no other symmetric equilibrium exists goes some distance in strengthening our conviction that information aggregation is a feature of the market under consideration, and not just of a particular equilibrium.[3].

Section 2 characterizes the equilibrium of the model for fixed k and n and discusses the uniqueness result. Section 3 explores the behavior of the equilibrium price distribution as the auction grows large. It states and proves the main theorem and examines the cases in which k_r/n_r goes to zero and in which the fraction of informed bidders becomes small. Section 4 concludes.

1.1 Related Literature

Palfrey (1985) asks whether aggregate output in a Cournot oligopoly with decentralized information about marginal cost and demand converges to the output of the corresponding full information setting as the number of firms goes to infinity. Palfrey shows that a sufficient and essentially necessary condition for this result is that the likelihood matrix relating signals to states is "invertible". This requires in particular that there are at least as many signals as states. In contrast, our results hold even if there are only two signals although our state space is infinite. In Palfrey's model, *average* output must be equal to the full information output in each state. Thus, state by state, the overproduction of some firms must be exactly compensated by the underproduction of other firms. In our model, all that matters is that in each state v, the right number of bidders bids above the correct v. The magnitude by which bidders under or overbid is irrelevant.

Feddersen and Pesendorfer (1994) analyze two candidate elections with common values and give conditions under which full information aggregation is obtained in equilibrium. Full information aggregation in an election means that the candidate

[2] Anonymity also plays a central role in the literature on convergence of decentralized trading models to competitive equilibria: see Gale (1988), and Rubinstein and Wolinsky (1990) who show how a violation of anonymity can lead to a non-competitive limit.

[3] A result stating that the symmetric equilibrium is the only equilibrium (symmetric or otherwise) would be even nicer. This is false, because of the equilibrium in which players $1, \ldots, k$ bid more than the highest possible value regardless of their signal, while the others, knowing they have no chance of winning at a price less than the highest possible value, bid 0. If one rules out weakly dominated strategies then we suspect that the symmetric equilibrium is indeed unique, but we are unable to show this.

elected is the same as would be elected if all the private information were common knowledge. Similar to the $k+1^{st}$ price auction analyzed here, in a voting model the action of a player (the vote) only matters when he is pivotal and so a voter (like a bidder in an auction) has to condition on being pivotal. Thus the mechanism that guarantees full information aggregation in the present model is also at work in elections.

A natural question is the degree to which our results depend on the assumption of identical preferences. In Pesendorfer and Swinkels (1995) we explore information aggregation results in a model in which objects have a common component of value - a quality - about which players have some information, but in which players also have idiosyncratic differences in tastes. There, we show that in the limit, objects are allocated to those who value them most, and price reveals true quality. Thus, our information aggregation result is robust to the introduction of preference diversity.

We concentrate here on the pure common value for several reasons. First, the driving force behind information aggregation is clearest in this simple case. Second, this setting differs from that of Wilson and Milgrom only in having weaker signals and non-negligible supply. Thus it is the assumption of fixed supply that makes their strong information assumptions necessary. Third, in the pure common value case we are able to characterize equilibria explicitly and prove uniqueness. This allows us to show both necessity and sufficiency of double largeness for information aggregation. With differing tastes we give a (stronger) sufficient condition for full information aggregation but cannot show necessity. Finally, the forces driving information aggregation in the setting with idiosyncratic tastes are somewhat different than those at work here. In particular, the pure common values result cannot be thought of as a special case of the result with idiosyncratic tastes.

2 Model and Equilibrium

We begin by characterizing the equilibrium of the model with a fixed number of bidders and objects. There are n bidders, labelled $i = 1, \ldots, n$ and k identical objects. Each bidder puts value v on a single object, and 0 on further objects. The value v is common across players, but unknown; v is drawn according to a distribution $F(.)$, with support $[0,1]$.

Each bidder i receives a signal $s \in S = \{1, ..., M\}$. Conditional on v, signals are independent across players, with a probability $\pi(s, v)$ of signal s. The probability distribution of signals given v is

$$\Pi(s, v) \equiv \sum_{s'=1}^{s} \pi(s', v).$$

We make the following assumptions:

Assumption 1 *F has a differentiable density $f(.)$. There is $\eta > 0$ such that $1/\eta > f(v) > \eta$ and $|f'(v)| < 1/\eta$ $\forall v \in [0, 1]$.*

Assumption 2 *$\pi(s, v)$ is continuously differentiable in v. There is an $\eta > 0$ such that $\eta < \pi(s, v) < 1 - \eta$ and such that for $s > s'$, $\frac{d}{dv}\left(\frac{\pi(s,v)}{\pi(s',v)}\right) > \eta$.*

Assumption 1 implies that the prior over values has no mass points and that the support of the prior is all of $[0, 1]$. Assumption 2 implies that the signals satisfy the strict monotone likelihood ratio property (MLRP) and that the likelihood ratio $\frac{\pi(s,v)}{\pi(s',v)}$ is uniformly bounded for all pairs of signals s, s'. The differentiability and boundedness assumptions are made for technical convenience. Strict MLRP implies that by sampling a large number of signals v can be determined with arbitrary accuracy. As a consequence, a large population of bidders always has information that, if properly aggregated, would determine v with great precision. Clearly, without this property we could not expect the equilibrium price to converge in probability to v.[4]

Each bidder i submits a bid b_i as a function of his signal. We will consider symmetric Nash equilibria. The cumulative distribution for bids conditional on a signal s is denoted $H_s(\cdot)$ for $s \in S$. The support for $H_s(.)$ is denoted $[\underline{b}_s, \overline{b}_s]$ (supports will always turn out to be intervals).

Let d_k denote the k^{th} highest bid among all bidders except bidder 1. When $d_k = b$ we say that b is *pivotal*. We describe the equilibrium strategies from the perspective of bidder 1. Since strategies are symmetric this describes the whole equilibrium profile.

[4] Assume that the true value is $w = v + \sigma$, where as before, signals are IID conditional on v, and where both signals and v are independent of σ. Then, all our results would carry through in the sense that the true v would be revealed by price (which would converge to $v + \sigma$). So of course, the real point is that asymptotically the auction mechanism reveals all *available* information about quality.

Proposition 1 *There is a unique symmetric equilibrium $H_s(\cdot)$, $s = 1, \ldots, M$ of this auction. In the equilibrium, each $H_s(b)$ is continuous and for $s > s'$, $\underline{b}_s > \bar{b}_{s'}$. For each bid $b \in [\underline{b}_s, \bar{b}_s]$,*

$$b = E(v|d_k = b, s) = \int_0^1 v \frac{x(b,v)^{n-k-1}(1 - x(b,v))^{k-1}\pi(s,v)^2 f(v)}{\int_0^1 x(b,w)^{n-k-1}(1 - x(b,w))^{k-1}\pi(s,w)^2 f(w)dw} dv \qquad (1)$$

where $x(b,v) \equiv \Pi(s-1,v) + \pi(s,v)H_s(b)$.

Proof A formal proof is in the Appendix 1.

Proposition 1 says that types mix over disjoint intervals in such a way that for each bid b made by type s, the expected value of an object given that b is pivotal and bidder 1 received signal s is b. This makes sense given that a change in bid makes a difference only when the bid is on the threshold between winning and losing, i.e., when $d_k = b$. The equilibrium is the analog of the equilibrium described in Milgrom (1981, Theorem 1) for the case of a continuous signal space.

A technical point about $E(v|d_k = b, s)$ should be mentioned. Note that $x(b,v)$ is the probability of any given bidder bidding less than b given value v. So, the density reflects that for $d_k = b$, $n-k-1$ of players $2, \ldots, n$ must bid less than b, while $k-1$ must bid more. These are both positive probability events. However, the event $d_k = b$ is zero probability, since it involves a player bidding b. So, it is not clear that $E(v|d_k = b, s)$ need always be well defined, and indeed, it is possible to construct bid functions for which it is not (simply choose H_s functions in such a way that $\lim_{b' \to b} h_s(b')/h_{s'}(b')$ is not well defined). However, for bid functions with disjoint supports this is not a problem. The intuition is that once one knows that a player has a bid in $[\underline{b}_s, \bar{b}_s]$, further information about his bid is irrelevant to v. Lemma 5 in the appendix formalizes this.

2.1 Uniqueness

The contribution of proposition 1 is to demonstrate uniqueness within the set of symmetric equilibria. The two key steps are to establish that in equilibrium bids are weakly monotonic in signals and to establish that weakly monotonic equilibria cannot involve atoms.

At first blush weak monotonicity seems quite obvious. Assume a player with signal s finds b to be a better bid than $b < b'$. Then a player with signal $s' > s$ is more optimistic about the value of the object when b' wins and b loses. So, he also ought to find b' a better bid than b. The difficulty with this argument is that the player with signal s' may also have substantially more pessimistic views about what he will pay in circumstances when b loses and b' wins.[5]

Let Y be the event that b loses and b' wins. Now, $E(v|Y, s') - E(v|Y, s) > \epsilon$ for some $\epsilon > 0$. And, $E(p|Y, s') - E(p|Y, s)$ is at most $b' - b$. So, if b and b' differ by at most ϵ, then s' will also find b' a preferable bid to b. For any interval in the support of equilibrium bids, one can string together local comparisons like this to show that more favorable signals correspond to weakly higher bids.

Consider next the case where b and b' are the end points of an interval in which no one bids, and where b' is preferred to b by s. The difficult case is the one in which both b' and b are atoms. Since b' is preferred to b by s, $E(v|Y, s) - E(p|Y, s) \geq 0$. And as before, $E(v|Y, s') > E(v|Y, s)$. So, if $E(p|Y, s') \leq E(p|Y, s)$ then s' also prefers b' to b.

Assume $E(p|Y, s') > E(p|Y, s)$. Conditional on Y, price is either b or b'. So, it must be that the beliefs of s' shift weight away from the event $p = b$ and toward the event $p = b'$ compared to the beliefs of s. Since s could have bid just under b', it must be that $E(v|Y, p = b', s) \geq b'$. If also $E(v|Y, p = b, s) \geq b$, then since s' is better off than s in either case, s' also prefers b' to b. So consider the case that $E(v|Y, p = b, s) < b$. Then, compared to s, s' places less weight on the event where bidding b' instead of b is costly to s, and more on the event where bidding b' instead of b is profitable to s. Since s' does better than s in each event, s' again prefers b' to b.

It is in showing weak monotonicity that symmetry plays a role. If we were to allow asymmetric equilibria, the difficulty may arises that players other than i bid on the interval (b, b') but i does not. Then we cannot get from b to b' by local comparisons across types of i, but neither can we conclude that price is either b or b' given Y.

The second step is to show that weakly monotonic bid functions cannot involve atoms. If b is an atom, and b is pivotal then there may be several people tied at b. But then, winning or losing with a bid of b conveys information in addition to knowing that b is pivotal. This is so because when a bid of b is pivotal, the odds of getting

[5] Harstad and Levin (1985) claim to prove the uniqueness result at issue here in the context of a single object auction. Their proof does not address this difficulty with monotonicty or the possibility of atoms.

an object depend on how many other players submitted a bid at or above b. This in turn depends on the signals those players received.

To establish that atoms cannot be a part of a symmetric equilibrium we show that there is a winner's curse at atoms: when b is pivotal and a bid of b wins, value is lower on average than when b is pivotal but does not win. But then some bidder will have an incentive to either raise or lower b a bit.

3 Information Aggregation

Consider a sequence of auctions indexed by the positive integers, where the r^{th} auction A_r has n_r bidders and k_r objects for sale. The structure on values and signals is constant along the sequence, and satisfies all the assumptions already made. For each r, Proposition 1 establishes that there is a unique symmetric equilibrium. We will maintain the notation describing the equilibrium from the last section, with r subscripts where appropriate. Let the random variable p_r describe the price in this equilibrium. We want to understand the conditions under which p_r approximates v as r gets large.

Definition 1 *The sequence of auctions $\{A_r\}_{r=1}^{\infty}$ satisfies full information aggregation if $p_r - v$ converges to 0 in probability.*

3.1 A Necessary Condition for Full Information Aggregation

Consider the case $s = M$. Then

$$1 - x_r(b, v) = \pi(M, v)(1 - H_{Mr}(b))$$

and therefore Proposition 1 implies that

$$b_{Mr} = \int_0^1 v \frac{(1 - \pi(M, v)(1 - H_{Mr}(b)))^{n_r - k_r - 1}\, \pi(M, v)^{k_r+1} f(v)}{\int_0^1 (1 - \pi(M, w)(1 - H_{Mr}(b)))^{n_r - k_r - 1}\, \pi(M, w)^{k_r+1} f(w)dw} dv.$$

At $\bar{b}_{Mr}$ this reduces to

$$\bar{b}_{Mr} = \int_0^1 v \frac{\pi(M, v)^{k_r+1} f(v)}{\int_0^1 \pi(M, w)^{k_r+1} f(w)dw} dv \tag{2}$$

Similarly,

$$\underline{b}_{1r} = \int_0^1 v \frac{\pi(1, v)^{n_r - k_r + 1} f(v)}{\int_0^1 \pi(1, w)^{n_r - k_r + 1} f(w)dw} dv. \tag{3}$$

Equation (2) implies that *independent of n_r* the highest bid in A_r is the expected value of the object conditional on observing k_r signals M. The idea is that $d_{k_r} = \bar{b}_{Mr}$ conveys the information that k_r other players bid $\bar{b}_{Mr}$ and therefore received signal M. All other players bid less than $\bar{b}_{Mr}$ and since players bid less than $\bar{b}_{Mr}$ with probability 1 (regardless of their signal) this conveys no information about v. Thus, conditional on $\bar{b}_{Mr}$ being pivotal, the bidder's information is his own signal M plus the information that a set of k_r other bidders all received M. For any finite k_r this implies that $\bar{b}_{Mr}$ is strictly less than 1. Any time v is more than $\bar{b}_{Mr}$, price and value will diverge. Therefore, $k_r \to \infty$ is a necessary condition for full information aggregation. Similarly, Equation (3) reflects that when $\underline{b}_{1r}$ is pivotal, a bidder knows that, like himself, the $n_r - k_r$ bidders who bid $\underline{b}_{1r}$ received signal 1 while there is no information about signals received by the $k_r - 1$ other bidders. So, another necessary condition for full information aggregation is that $n_r - k_r \to \infty$. This suggests

Definition 2 *A sequence of auctions $\{A_r\}_{r=1}^{\infty}$ satisfies double largeness if $k_r \to \infty$ and $n_r - k_r \to \infty$.*

3.2 Sufficiency

We have seen that double largeness is a necessary condition for full information aggregation. We now show that it is also sufficient.

Theorem 1 *$p_r - v$ converges in probability to 0 if and only if $\{A_r\}_{r=1}^{\infty}$ satisfies double largeness.*

Proof We have already established necessity. So, let $D_r(v|Y)$ be the distribution of v conditional on any given event Y in A_r, and $d_r(v|Y)$ be the corresponding density ($d_r(v|Y)$ will exist for all the Y we consider). Let $X_r(b)$ be the event that of bidders $3, \ldots, n_r$ in A_r, $k_r - 1$ bid above b and the remaining $n_r - k_r - 1$ bid below b. Then,

$$\Pr(X_r(b)|v) = (x_r(v,b))^{n_r - k_r - 1}(1 - x_r(v,b))^{k_r - 1}.$$

For event $X_r(b)$ to occur, a fraction

$$q_r \equiv \frac{n_r - k_r - 1}{n_r - 2}$$

of bidders $3, \ldots, n_r$ bid below b. Thus, if we define $v_r^*(b)$ by

$$v_r^*(b) = \arg\min_v |x_r(v,b) - q_r|$$

then a simple calculation shows that $\Pr(X_r(b)|v)$ is maximized by $v = v_r^*(b)$ (see the proof of Lemma 1 below). Further, as n_r grows large $\Pr(X_r(b)|v)$ depends increasingly sensitively on $x_r(v,b)$. If $x_r(v,b)$ is in turn sufficiently responsive to v, then $\Pr(X_r(b)|v)$ will depend sensitively on v, and so the event $X_r(b)$ will contain a great deal of information about v. With double largeness this is in fact true. This is confirmed by the following lemma, which is the key to our results. Let $T_r = \cup_{s \in S} \operatorname{supp}(H_{rs}(.)) \setminus \{\bar{b}_{Mr}, \underline{b}_{1r}\}$. That is, T_r is the support of possible bids in A_r, less the highest and lowest bid.[6]

Lemma 1 *Let A_r satisfy double largeness and let $C > 0$ and $\epsilon > 0$. Then there is r' such that for all $r > r'$, for all $b \in T_r$, and for all w, v such that either $w + \epsilon < v \leq v_r^*(b)$ or $v_r^*(b) < v < w - \epsilon$,*

$$\frac{\Pr(X_r(b)|v)}{\Pr(X_r(b)|w)} > C.$$

That is, $X_r(b)$ becomes much less likely as v moves away from $v_r^*(b)$.

We defer the proof of Lemma 1 to the next subsection. Here we show how Lemma 1 implies Theorem 1.

Because we are in a symmetric equilibrium of a symmetric model, when considering the event $d_k = b$, we can without loss assume that player 2 is the one who bid b, so that $d_k = b$ can be replaced by $X_r(b) \cap \{b_2 = b\}$ (since ties are zero probability). Similarly, the event $p_r = b$ can be replaced by $X_r(b) \cap \{b_2 = b\} \cap \{b_1 > b\}$. Note further that, conditional on v, bids are independent. So, if $b \in [\underline{b}_s, \bar{b}_s]$ for some s, then

$$d_r(v|d_{k_r} = b, s) = \frac{\Pr(X_r(b)|v)\, \pi(s,v)^2 f(v)}{\int_0^1 \Pr(X_r(b)|w)\, \pi(s,w)^2 f(w) dw}.$$

Similarly,

$$d_r(v|p_r = b) = \frac{\Pr(X_r(b)|v)\, \pi(s,v)(1 - x(b,v)) f(v)}{\int_0^1 \Pr(X_r(b)|w)\, \pi(s,w)(1 - x(b,w)) f(w) dw}$$

[6] The key results that follow can be extended to $\{\bar{b}_{Mr}, \underline{b}_{1r}\}$ but only at the cost of more complicated notation. Since the key result is a probabalistic one, the difference is irrelevant.

Let η be such that $\eta < \pi(s,v) < 1$, and $\eta < f(v) < 1/\eta$ for all s and v. (Such a η exists by Assumptions 1 and 2).

Pick $\epsilon > 0$, and $C > 0$. Choose r' so that Lemma 1 is satisfied for ϵ and C. Let $r > r'$, and $b \in T_r \cap \text{supp} H_{sr}$. Then

$$\begin{aligned}
& \frac{\Pr\left(|v - v_r^*(b)| < \epsilon \mid d_{k_r} = b, s\right)}{\Pr\left(|v - v_r^*(b)| > \epsilon \mid d_{k_r} = b, s\right)} \\
= \; & \frac{\int_{|v - v_r^*(b)| < \epsilon} \Pr\left(X_r(b)|v\right) \pi(s,v)^2 f(v) dv}{\int_{|v - v_r^*(b)| > \epsilon} \Pr\left(X_r(b)|v\right) \pi(s,v)^2 f(v) dv} \\
> \; & \eta^4 \frac{\int_{|v - v_r^*(b)| < \epsilon} \Pr\left(X_r(b)|v\right) dv}{\int_{|v - v_r^*(b)| > \epsilon} \Pr\left(X_r(b)|v\right) dv} \\
> \; & C\eta^4\epsilon.
\end{aligned}$$

where the first inequality uses that $\eta < \pi(s,v) < 1$ and $\eta < f(v) < 1/\eta$ and the second is from Lemma 1.

Similarly, for any $b \in T_r$

$$\frac{1 - x_r(b,v)}{1 - x_r(b,w)} > \eta$$

because bids depend only on signals, and the likelihood of any given signal is bounded below by η and above by 1. So

$$\frac{\Pr\left(|v - v_r^*(b)| < \epsilon \mid p_r = b\right)}{\Pr\left(|v - v_r^*(b)| > \epsilon \mid p_r = b\right)} > C\eta^4\epsilon.$$

Thus, far enough along the sequence of auctions, almost all the weight in the distribution of v conditional on either event becomes concentrated around $v_r^*(b)$. And, since $v \in [0,1]$, it thus follows that the expectation of v conditional on either event converges to $v_r^*(b)$. That is, for all $\epsilon > 0$, there is r' such that for all $r > r'$, for all $s \in S$, and for all $b \in T_r$,

$$\Pr(|v - v_r^*(b)| > \epsilon \mid d_{k_r} = b, s) < \epsilon$$

and so $|E(v \mid d_{k_r} = b, s) - v_r^*(b)| < 2\epsilon$, and similarly for the event $p_r = b$.

From the description of the equilibrium, $E(v|d_{k_r} = b, s) = b$. Since $|E(v|d_{k_r} = b, s) - v_r^*(b)| < 2\epsilon$, it follows that $|b - v_r^*(b)| < 2\epsilon$. Also $\Pr(|v - v_r^*(b)| > \epsilon \mid p_r = b) < \epsilon$ and therefore $\Pr(|v - b| > 3\epsilon \mid p_r = b) < \epsilon$. As this holds for all $b \in T_r$ and $\Pr(p_r \in T_r) = 1$, we conclude that $\Pr(|v - p_r| > 3\epsilon) < \epsilon$. Since ϵ was arbitrary, this establishes that $v - p_r$ converges to 0 in probability, proving Theorem 1. ■

It remains to prove Lemma 1.

Proof of Lemma 1 Let $b \in T_r$. A straightforward calculation shows that

$$\frac{\partial}{\partial v} \ln \Pr(X_r(b)|v) = \frac{\frac{\partial}{\partial v} x_r(v,b)}{x_r(v,b)(1-x_r(v,b))}[n_r - k_r - 1 - (n_r - 2)x_r(v,b)] \quad (4)$$

Now note that

$$\frac{\frac{\partial}{\partial v} x_r(v,b)}{x_r(v,b)(1-x_r(v,b))}$$

is always negative since $\frac{\partial}{\partial v} x_r(v,b)$ is negative, while $n_r - k_r - 1 - (n_r-2)x_r(v,b)$ has the same sign as

$$\frac{n_r - k_r - 1}{n_r - 2} - x_r(v,b).$$

So, $\frac{\partial}{\partial v} \ln \Pr(X_r(b)|v)$ is positive for $v < v_r^*(b)$ and negative for $v > v_r^*(b)$. We will prove that for any $\delta > 0$, and $D > 0$, there is r' such that for $r > r'$, for any $b \in T_r$ and for v such that $|v - v_r^*(b)| > \delta$, $|\frac{\partial}{\partial v} \ln d(v|X_r(b))| > D$. This will establish the lemma. To see this, pick $\epsilon > 0$, and let r' be such that when $v < v_r^*(b) - \epsilon/2$, $|\frac{\partial}{\partial v} \ln d(v|X_r(b))| > D$. Then, for $w + \epsilon < y < v_r^*(b)$,

$$\ln \frac{\Pr(X_r(b)|y)}{\Pr(X_r(b)|w)} = \ln \Pr(X_r(b)|y) - \ln \Pr(X_r(b)|w) > \frac{\epsilon}{2} D$$

and so

$$\frac{\Pr(X_r(b)|y)}{\Pr(X_r(b)|w)} > e^{\frac{\epsilon}{2} D}$$

which can be made arbitrarily large by choice of D.

So, let

$$(b_r, v_r) = \arg \min_{v, b \in T_r} \left|\frac{\partial}{\partial v} \ln d(v|X_r(b))\right| \quad \text{subject to } |v - v_r^*(b)| \geq \delta$$

Then, it is sufficient to show that

$$\left|\frac{\partial}{\partial v} \ln d(v|X_r(b))\right| = \left|\frac{\frac{\partial}{\partial v} x_r(v_r,b_r)}{x_r(v_r,b_r)(1-x_r(v_r,b_r))}[n_r - k_r - 1 - (n_r-2)x_r(v_r,b_r)]\right| \to \infty$$

To do this we distinguish three cases:

Case 1 $b_r \in \text{supp} H_{rs}, 1 < s < M$. In this case

$$\frac{\partial}{\partial v} x_r(v,b_r) = \frac{\partial}{\partial v} \left(\Pi(s-1,v) + \pi(s,v) H_{rs}(b_r)\right). \quad (5)$$

If $\frac{\partial \pi(s,v)}{\partial v} < 0$, then

$$\frac{\partial}{\partial v}\left(\Pi(s-1,v) + \pi(s,v)H_{rs}(b_r)\right) < \frac{\partial}{\partial v}\Pi(s-1,v).$$

If $\frac{\partial \pi(s,v)}{\partial v} > 0$, then

$$\frac{\partial}{\partial v}\left(\Pi(s-1,v) + \pi(s,v)H_{rs}(b_r)\right) < \frac{\partial}{\partial v}\Pi(s,v).$$

Thus by Assumption 2 there is an $\eta > 0$ such that

$$\frac{\frac{\partial}{\partial v}x_r(v_r,b_r)}{x_r(v_r,b_r)(1-x_r(v,b_r))} < -\eta \tag{6}$$

and therefore it is sufficient to show that

$$|n_r - k_r - 1 - (n_r - 2)x_r(v_r,b_r)| \to \infty \tag{7}$$

Since $v_r^*(b_r)$ minimizes $|x_r(v,b_r) - \frac{n_r-k_r-1}{n_r-2}|$ the fact that $|v_r - v_r^*(b_r)| > \delta$ together with (6) imply that

$$\left|\frac{n_r-k_r-1}{n_r-2} - x_r(v_r,b_r)\right| > \delta\eta$$

and hence $|n_r - k_r - 1 - (n_r-2)x_r(v_r,b_r))| \geq (n_r-2)\delta\eta \to \infty$.

Case 2 $b_r \in \operatorname{supp}H_1$. In this case $x_r(v,b_r) = \pi(1,v)H_{1r}(b_r)$ and therefore

$$\begin{aligned}\frac{\frac{\partial}{\partial v}x_r(v,b_r)}{x_r(v_r,b_r)(1-x_r(v,b_r))} &= \frac{\frac{\partial}{\partial v}\pi(1,v)H_{rs}(b_r)}{\pi(1,v)H_{rs}(b_r)\left(1-\pi(1,v)H_{rs}(b_r)\right)}\\ &\leq \frac{\partial}{\partial v}\pi(1,v) < -\eta\end{aligned}$$

for some $\eta > 0$ by Assumption 2. So it is again sufficient to show that

$$|n_r - k_r - 1 - (n_r-2)x_r(v_r,b_r))| \to \infty \tag{8}$$

Since $x_r(v,b_r) = \pi(1,v)H_{1r}(b_r)$ it follows that $v_r^*(b_r)$ minimizes

$$\left|\pi(1,v) - \frac{n_r-k_r-1}{H_{1r}(b_r)(n_r-2)}\right|$$

Therefore, the fact that $|v_r - v_r^*(b_r)| > \delta$ and $\frac{\partial}{\partial v}\pi(1,v) < -\eta$ for all v implies that

$$\left|\pi(1,v_r) - \frac{n_r-k_r-1}{H_{1r}(b_r)(n_r-2)}\right| \geq \delta\eta \tag{9}$$

Note that

$$n_r - k_r - 1 - (n_r - 2)x_r(v_r, b_r) = n_r - k_r - 1 - (n_r - 2)H_{1r}(b_r)\pi(1, v_r). \qquad (10)$$

So, if $(n_r - 2)H_{r1}(b_r)$ stays bounded then (8) follows since $n_r - k_r \to \infty$ by double largeness. Conversely, if $(n_r - 2)H_{r1}(b_r) \to \infty$ then (8) follows from (9) since then

$$n_r - k_r - 1 - (n_r - 2)H_{1r}(b_r)\pi(1, v_r)| \geq \delta\eta(n_r - 2)H_{r1}(b_r).$$

Case 3 $b_r \in \text{supp}H_M$. In this case $x_r(v, b_r) = 1 - \pi(M, v)(1 - H_{Mr}(b_r))$ and hence

$$\begin{aligned}\frac{\frac{\partial}{\partial v}x_r(v, b_r)}{x_r(v, b_r)(1 - x_r(v, b_r))} &= \frac{\frac{\partial}{\partial v}(1 - \pi(M, v)(1 - H_{rs}(b_r)))}{(1 - \pi(M, v)(1 - H_{rs}(b_r)))\pi(M, v)(1 - H_{rs}(b_r))} \\ &\leq -\frac{\partial}{\partial v}\pi(M, v) < -\eta\end{aligned}$$

for some $\eta > 0$ by Assumption 2. Therefore, it is sufficient to show that

$$\begin{aligned}& |n_r - k_r - 1 - (n_r - 2)x_r(v_r, b_r)| \qquad (11) \\ = \; & |(n_r - 2)(1 - x_r(v_r, b_r)) - k_r + 1| \to \infty\end{aligned}$$

Since $1 - x_r(v, b_r) = \pi(M, v)(1 - H_{Mr}(b_r))$ it follows that $v_r^*(b_r)$ minimizes

$$\left|\pi(M, v) - \frac{k_r - 1}{(n_r - 2)(1 - H_{Mr}(b_r))}\right| \qquad (12)$$

Therefore, the fact that $|v_r - v_r^*(b_r)| > \delta$ and $\frac{\partial}{\partial v}\pi(M, v) > \eta$ implies that

$$\left|\pi(M, v_r) - \frac{k_r - 1}{(n_r - 2)(1 - H_{Mr}(b_r))}\right| \geq \delta\eta. \qquad (13)$$

Note that

$$(n_r - 2)(1 - x_r(v_r, b_r)) - k_r - 1 = \pi(M, v_r)(n_r - 2)(1 - H_{Mr}(b_r)) - (k_r + 1)$$

So, if $(n_r - 2)(1 - H_{Mr}(b_r))$ stays bounded then (11) follows since $k_r \to \infty$ by double largeness. Conversely if $(n_r - 2)(1 - H_r(b_r G)) \to \infty$ then (13) implies (11).

This proves the Lemma ■

We have proven that p_r converges to v in probability. The next remark establishes that convergence is uniform over all v.

Remark 1 *For all $\epsilon > 0$, there is r' such that for $r > r'$,* $\Pr(|p_r - v| > \epsilon|v) < \epsilon$ *for all $v \in [0, 1]$.*

Proof see Appendix 2.

3.3 A Small Fraction of Objects

Consider the case that $k_r/n_r \to 0$. Then, Theorem 1 states that full information aggregation will hold if and only if $k_r \to \infty$. In this section, we explore the properties of the equilibrium with $k_r/n_r \to 0$ to gain insight into this result.[7]

The key is the following Corollary. Let $b \in (0, E(v))$. Then, the expected number of bids above b grows at the same rate as k_r:

Corollary 1 *Pick $b \in (0, E(v))$. Then, there is $\delta > 0$ such that*

$$\frac{1}{\delta} > \frac{n_r(1 - x_r(v, b))}{k_r} > \delta \tag{14}$$

for all v and r.

Proof see Appendix 2.

Pick a small $\epsilon > 0$, and define a bidder to be *active* if he bids more than ϵ. By Corollary 1, the expected number of active bidders grows at the same rate as k_r. Thus, full information aggregation is equivalent to the number of active bidders growing without bound. Inactive bidders can be thought of as bidding essentially 0. An easy implication is that any time $k_r/n_r \to 0$, the lower bound of the bids made with M converges to 0 (since most bidders with signal M are inactive).

We have already seen that when k_r stays bounded, the top end of the support of bids remains less that 1, implying a failure of full information aggregation whenever v is sufficiently close to 1. We can now see that in fact when k_r stays bounded full information aggregation fails for all $v > \epsilon$. In particular, since the expected number of active bidders is bounded, it can easily be shown that regardless of v, there is a probability bounded away from 0 that no one bids above ϵ. Intuitively, very little is learned from inactive bidders, and so when the number of active bidders stays bounded, the amount of information in the event $p_r = b$ stays bounded when $b > \epsilon$.

3.4 A Small Fraction of Informed Bidders

In the previous section we held the information structure fixed as we increased the number of bidders. In this section we assume that each bidder receives useful information with probability γ_r where $\gamma_r \to 0$.

[7] Entirely analogous arguments hold in the case where $\frac{n_r - k_r}{n_r} \to 0$.

There are three signals, $S = \{1, 2, 3\}$. We assume that $\pi_r(2, v) = 1 - \gamma_r$ for all v, so that signal 2 provides no information. For $s \in \{1, 3\}$, $\pi_r(s, v) = \gamma_r \pi(s, v)$ where $\pi(1, v)$ is strictly decreasing in v, $\pi(1, v) + \pi(3, v) = 1$ and $\pi(s, v)$ satisfies Assumption 2.

For fixed r Proposition 1 holds. For fixed γ Assumption 2 is satisfied and hence if γ_r stays bounded away from zero then a slight modification of Theorem 1 implies full information aggregation also for this case. Thus the interesting question is what happens when $\gamma_r \to 0$. First note that if $n_r \gamma_r$ stays bounded then we cannot expect full information aggregation since a typical population of bidders does not have enough information to determine the value with great confidence. Thus $n_r \gamma_r \to \infty$ is a necessary condition for price to converge to the true value in probability.

The following theorem shows that a sufficient condition full information aggregation in this case is that in addition to double largeness, $\sqrt{n_r}\gamma_r \to \infty$. To give an intuition for this condition note that if there are m bidders with the uninformative signal then the standard deviation of the number of these bidders who bid above any given b is $\sqrt{(1 - H_{2r}(b))H_{2r}(b)m} \equiv \sigma_b$. To guarantee full information aggregation the actions of the informed bidders must in the limit determine whether the price is above or below b. A necessary condition for this is that σ_b must be small relative to the number of informed bidders. Since the number of bidders with signal 2 grows as n_r, σ_b grows as $\sqrt{n_r}$. Hence our condition guarantees that $\frac{n_r \gamma_r}{\sigma_b} \to \infty$ for all b. This implies that the informed bidders will typically dominate the noise introduced by bidders with no information.

Theorem 2 *Suppose Assumption 1 holds and that $n_r \gamma_r^2 \to \infty$. Let p_r denote the random variable that describes the equilibrium price in the auction with n_r bidders, k_r objects, and value v. Then, p_r converges in probability to v if and only if double largeness holds.*

Proof Necessity is as before. In Appendix 2 we reprove Lemma 1 for the current information structure. The remainder of the proof of Theorem 2 is the same as the proof of Theorem 1. ■

4 Conclusion

In this paper we give conditions under which a common value auction fully aggregates dispersed private information in the sense that the equilibrium price converges in probability to the true value of the object.

Note that in the pure common value setting there are no welfare consequences from full information aggregation or the lack thereof. As long as all objects are transferred from the buyer to the seller a Pareto efficient allocation is achieved. Hence the question remains, why do we care about full information aggregation?

Consider the following extension of the model. In addition to v nature also draws a value $w < v - \epsilon$ which is the valuation of the seller. Suppose the seller observes w and v and can decide whether to put the objects for sale or not. If he decides to put the objects up for sale then an auction as described in this paper will be conducted. Full information aggregation implies that the seller will always put the objects up for sale. Thus the analysis of the auction is unchanged by the introduction of this additional stage and an efficient allocation is achieved for every v, w.

If, on the other hand, the seller sells only one object, and hence the equilibrium price is bounded away from 1, then (for small ϵ) it cannot be an equilibrium for the seller to put the object up for sale irrespective of the realization of v and w. Thus an inefficient allocation will be realized with positive probability.

More generally, full information aggregation is linked to efficiency whenever we introduce an investment problem on the seller's side. If the seller has control over some component of the object's value then full information aggregation is necessary to provide the right incentives for efficiency.

On the buyers side, learning the true value of the object may provide valuable information for related decisions. For example, information aggregated in the price of a financial asset may be relevant for deciding how to construct the rest of a portfolio.

In the formal setting, there is only one type of object. However, more generally, one should think of the auction as being part of a larger economy. In that case, price in this market is relevant for trade-offs made between other commodities, and price again plays an allocative role.

5 Appendix 1: Existence and Uniqueness

Lemma 2 *Fix any symmetric strategy profile H_s, $s \in S$, and let $\bar{b} > \underline{b}$ be such that $Pr(\bar{b}$ wins, $\underline{b}$ loses $|r) > 0$ for all r. Then there exists an $\epsilon > 0$ independent of $\bar{b} > \underline{b}$ such that for $s > s'$,*

$$E(r|\bar{b} \text{ wins}, \underline{b} \text{ loses }, s) \geq E(r|\bar{b} \text{ wins}, \underline{b} \text{ loses }, s') + \epsilon$$

Proof Let

$$d(r) \equiv \frac{Pr(\bar{b} \text{ wins}, \underline{b} \text{ loses } |r) f(r)}{\int_0^1 Pr(\bar{b} \text{ wins}, \underline{b} \text{ loses } |r) f(r) dr}.$$

Then

$$E(r|\bar{b} \text{ wins}, \underline{b} \text{ loses }, s) = \int_0^1 r \frac{d(r)\pi(s,r)f(r)}{\int_0^1 d(w)\pi(s,w)f(w)dw} dr.$$

Note that

$$\frac{d(r)}{d(r')} = \frac{Pr(\bar{b} \text{ wins}, \underline{b} \text{ loses } |r)}{Pr(\bar{b} \text{ wins}, \underline{b} \text{ loses } |r')}$$

and let $\vec{s} = (s_1, \ldots, s_n)$ be the vector of signals received by the n players. Then,

$$\begin{aligned} Pr(\bar{b} \text{ wins}, \underline{b} \text{ loses } |r) &= \sum_{\vec{s} \in S^n} Pr(\bar{b} \text{ wins}, \underline{b} \text{ loses } |\vec{s}, r) \cdot Pr(\vec{s}|r) \\ &= \sum_{\vec{s} \in S^n} Pr(\bar{b} \text{ wins}, \underline{b} \text{ loses } |\vec{s}) \cdot Pr(\vec{s}|r) \end{aligned}$$

Further note that $1 > Pr(\vec{s}|r) > \eta^n$ since $\pi(s,r)$ is bounded below by η for all s and r and therefore it follows that $1/\eta^n > d(r)/d(r') > \eta^n$ independent of $\bar{b}, \underline{b}$.

Thus it must be the case that $\Pr(r > E(r) + \delta) > \delta$ for some $\delta > 0$ (expectations and probabilities are taken with respect to the density d), where δ depends only on η and f. Then, since there is at least δ mass δ above $E(r)$, and since $r \in [0,1]$, there must be at least δ^2 mass below $E(r)$. Let $W_1 = [0, E(r)]$, $W_2 = (E(r), E(r)+\delta)$, and $W_3 = [E(r)+\delta, 1]$. Then,

$$E(r|s) = \sum_{i=1}^{3} \Pr(W_i|s) E(r|W_i, s).$$

Clearly $E(r|W_i, s)$ is increasing in i, and $E(r|W_3, s) > E(r|W_1, s) + \delta$. Since signals have the MLRP, $\Pr(W_i|s)$ stochastically dominates $\Pr(W_i|s')$. To establish the existence of our ϵ, it is thus enough to find ν depending only on δ such that $\Pr(W_3|s) - \Pr(W_3|s') > \nu$, and $\Pr(W_1|s) - \Pr(W_1|s') > \nu$ (because then $d(.|s)$ can be thought of as obtained from $d(.|s')$ by a series of transformations that at least weakly increased the expectation of v, and by shifting a mass ν a distance at least δ to the right). By assumption there is $\mu > 1$ such that $\frac{\pi(s,v)}{\pi(s,w)} > \mu \frac{\pi(s',v)}{\pi(s',w)}$ for all $v \in W_3$, $w \in W_1$, and so, $\frac{\Pr(W_3, s)}{\Pr(W_1, s)} > \mu \frac{\Pr(W_3, s')}{\Pr(W_1, s')}$. Since each of $\Pr(W_3)$ and $\Pr(W_1)$ was at least δ^2, we are done. ■

The following Lemma says that bidders with higher signals bid more than bidders with lower signals.

Lemma 3 *Suppose $H_s(\cdot)_{s \in S}$ is a symmetric equilibrium. Then for $s > s'$, $b \in suppH_s(\cdot)$ and $b' \in suppH_{s'}(\cdot)$ we have that $b \geq b'$.*

Proof For the given equilibrium, let BR_s denote the set of best responses for the signal s and let $K(b|s)$ denote the probability that bid b wins given signal s. Note that for each v all signals are received with probability larger than η and hence each bidder assigns at least probability η that any other bidder received signal s. This in turn implies that if b is a jump point of $K(\cdot|s)$ then b is also a jump point of $K(\cdot|s') \forall s'$; conversely, if $K(\cdot|s)$ is constant on some interval then $K(\cdot|s')$ is constant on the same interval. We will use the notation $b \succeq_s b'$ to indicate that bid b is weakly preferred to bid b' by a bidder with signal s.

First we show that $1 \geq \sup(\mathrm{supp}H_s(\cdot))$, for all s. Suppose to the contrary that there exists a positive probability that a bid strictly above 1 is made for some signal s. This implies that there is a positive probability that $k+1$ bids, and thus the price, are strictly larger than 1. But any bid that wins with positive probability at a price above 1 is strictly worse than a bid of 1.

Second we show that if $b \in \cup_s \mathrm{supp}H_s(\cdot)$ then $b \in BR_s$ for some s (the issue here is that $\mathrm{supp}H_s$ is closed, while the set of best responses may not be). $K(b|s)$ is a non-decreasing function of b. Therefore K is either continuous at b or b is a jump point. If b is a jump point then $b \in BR_s$ for some s since in particular, bids of b must then be made with positive probability in equilibrium. If b is not a jump point then payoffs are continuous in bids at b for all s. And since $b \in \mathrm{supp}H_s(\cdot)$ for some s there is a sequence of bids in BR_s converging to b and hence continuity of payoffs in bids implies that $b \in BR_s$.

Now let $\bar{b} = \sup_{s \leq M-1} \mathrm{supp}H_s(\cdot)$. We will show that $\inf BR_M \geq \bar{b}$. Let $Z \equiv \cup_s \mathrm{supp}H_s(\cdot) \cap [0, \bar{b}]$. Chose a $\delta > 0$ such that for all b, b' with $K(b'|s) > K(b|s)$ we have $E(v|b'$ wins, b loses, $M) - E(v|b'$ wins, b loses, $M-1) > \delta$. (Such a δ exists by Lemma 2.) Define $\{b_1, \dots, b_j, \dots, b_J\}$ inductively as follows:

$$\begin{aligned} b_1 &= 0 \\ b_{j+1} &= \begin{cases} \max Z \cap (b_j, b_j + \delta] \text{ if } Z \cup (b_j, b_j + \delta] \neq \emptyset \\ \min Z \cap [b_j + \delta, 1] \text{ otherwise.} \end{cases} \qquad j = 1, \dots, J-1 \\ b_J &= \bar{b} \end{aligned}$$

Clearly, since $b_{j+2} - b_j \geq \delta > 0$ the induction ends in a finite number of steps, with $b_J = \bar{b}$. By construction, all bids are in Z, and successive bids are either at most δ apart or are endpoints of a gap in Z.

For every $j \in \{1, \dots, J-1\}$ we will show that $b_{j+1} \succeq_M b$ for all $b \in [b_j, b_{j+1})$ with strict preference if (1) $b_{j+1} \in \text{supp} H_s(\cdot)$, for some $s < M$ and (2) $K(b_{j+1}|s) > K(b|s)$. By construction (1) and (2) hold when $j = J$. So this will imply $\bar{b} \succ_M b$ for all $b \in [b_{J-1}, b_J)$. Since $b_{j+1} \succeq_M b$ for all $b \in [b_j, b_{j+1})$ it follows that $\bar{b} \succ_M b$ for all $b < \bar{b}$ establishing the result.

So, choose $j \in \{1, \dots, J-1\}$ and let $b \in [b_j, b_{j+1})$. If $b_{j+1} \in \text{supp} H_M(\cdot)$ then trivially $b_{j+1} \succeq b$. If $K(b_{j+1}|s) - K(b|s) = 0$ then b_{j+1} and b earn the same payoff and again $b_{j+1} \succeq_M b$. Then, since $b_{j+1} \in BR_s$ for some s, we are left with the case where $b_{j+1} \succeq_s b$ for some $s < M$ and $K(b_{j+1}|s) - K(b|s) > 0$.

Case 1: $b_{j+1} - b_j < \delta$. Since $b_{j+1} \succeq_s b$ for $s < M$

$$E(v|\ b_{j+1} \text{ wins, } b \text{ loses, } s) \geq E(d_k|\ b_{j+1} \text{ wins, } b \text{ loses, } s) \geq b_j$$

But

$$\begin{aligned} E(v|\ b_{j+1} \text{ wins, } b \text{ loses, } M) &> E(v|\ b_{j+1} \text{ wins, } b \text{ loses, } s) + \delta \geq \\ b_j + \delta &\geq E(d_k|\ b_{j+1} \text{ wins, } b \text{ loses, } M) \end{aligned}$$

So $b_{j+1} \succ_M b$.

Case 2: $b_{j+1} - b_j > \delta$. Then by construction $(b_j, b_{j+1}) \cap (\cup_s \text{supp} H_s(\cdot)) = \emptyset$ and $b_{j+1} \succeq_s b$ for some $s < M$. And, since $K(b_{j+1}|s) - K(b|s) > 0$, it must thus be that $E(v|\ b_{j+1}$ wins, b loses, $s) - E(d_k|\ b_{j+1}$ wins, b loses, $s) \geq 0$. Since the first term on the rhs of this inequality increases when s is replace by s we would be done if

$$E(d_k|\ b_{j+1} \text{ wins, } b \text{ loses, } s) \geq E(d_k|\ b_{j+1} \text{ wins, } b \text{ loses, } M). \tag{15}$$

Note that either $d_k = b$ or $d_k = b_{j+1}$ since $(b_j, b_{j+1}) \cap (\cup_s \text{supp} H_s(\cdot)) = \emptyset$ and hence it follows that for (15) to be violated, it must be that

$$Pr(d_k = b_{j+1} | b_{j+1} \text{ wins}, b \text{ loses}, M) > Pr(d_k = b_{j+1} | b_{j+1} \text{ wins}, b \text{ loses}, s) > 0.$$

But,

$$\begin{aligned} & E(r | b_{j+1} \text{ wins}, b \text{ loses}, s) - E(d_k | b_{j+1} \text{ wins}, b \text{ loses}, s) \; = \\ & \Pr(d_k = b | b_{j+1} \text{ wins}, b \text{ loses}, s)(E(r | d_k = b, b_{j+1} \text{ wins}, b \text{ loses}, s) - b) + \\ & \Pr(d_k = b_{j+1} | b_{j+1} \text{ wins}, b \text{ loses}, s)(E(r | d_k = b_{j+1}, b_{j+1} \text{ wins}, b \text{ loses}, s) - b_{j+1}) \geq 0. \end{aligned}$$

Now, $E(r | d_k = b_{j+1}, b_{j+1} \text{ wins}, b \text{ loses}, s) - b_{j+1} \geq 0$ because bids just below b_{j+1} are feasible and $Pr(d_k = b_{j+1} | b_{j+1} \text{ wins}, b \text{ loses}, s) > 0$. Hence

$$E(r | d_k = b_{j+1}, b_{j+1} \text{ wins}, b \text{ loses}, M) - b_{j+1} > 0.$$

And,

$$E(r | d_k = b, b_{j+1} \text{ wins}, b \text{ loses}, M) - b > E(r | d_k = b, b_{j+1} \text{ wins}, b \text{ loses}, s) - b.$$

Since

$$Pr(d_k = b_{j+1} | b_{j+1} \text{ wins}, b \text{ loses}, M) > Pr(d_k = b_{j+1} | b_{j+1} \text{ wins}, b \text{ loses}, s),$$

it follows that

$$E(r | b_{j+1} \text{ wins}, b \text{ loses}, M) - E(d_k | b_{j+1} \text{ wins}, b \text{ loses}, M) > 0$$

which proves case 2.

Now we can repeat the argument to show that bidders with signals $s = M - 1$ bid above bidders with signals $s < M - 1$. In particular, let $\bar{b}' = \cup_{s < M-2} \text{supp} H_s$. Then, M never bids in $[0, \bar{b}')$, and so all incentive comparisons in $[0, \bar{b}')$ involve $M - 1$ and types lower than $M - 1$. Proceeding in this way establishes the lemma. ■

Next we show that if b is an atom, then conditional on b being pivotal, there is a winners curse. That is, winning is indeed bad news about the object and losing is good news about the object.

Lemma 4 *Suppose $H_s(.)_{s\in S}$ satisfies that for $s > s'$ and $b \in suppH_s(\cdot)$, $b' \in suppH_{s'}(\cdot)$ we have that $b \geq b'$. Let b be a bid that is an atom under $H_s(\cdot)$, for at least one of $s \in S$. Suppose further that the bid b is made with probability strictly less than one by at least one type. Then, there is winner's and loser's curse at b. That is,*

$$E(v|b \text{ wins}, b = d_k, s) < E(v|b = d_k, s) < E(v|b \text{ loses}, b = d_k, s).$$

Proof Fix a bid b. We will show that $\Pr(b \text{ wins } |d_k = b, v, s)$ is strictly decreasing in v. This establishes the result, since it establishes that winning shifts beliefs towards lower values, and losing shifts beliefs towards higher values. By $\#A$, $\#E$ and $\#B$ we denote the number of bidders above, equal to, and below b. So, note that

$$\begin{aligned}\Pr(b \text{ wins } |d_k &= b, v, s) = \Pr(b \text{ wins } |d_k = b, v) \\ &= \sum_{a=0}^{k-1} \Pr(b \text{ wins}|\#A = a, d_k = b, v)\Pr(\#A = a|d_k = b, v).\end{aligned}$$

where the first equality holds since once v is known, s carries no new information about the other players' bidding behavior. The summation ends at $k-1$ because for $a > k-1$, $\Pr(b \text{ wins}|\#A = a) = 0$. This summation can be usefully thought about as taking the expectation of the function $\Pr(b \text{ wins}|\#A = a, d_k = b, v)$ with respect to the distribution $\Pr(\#A = a|d_k = b, v)$ on $\#A$.

We establish 3 claims:

(1) $\Pr(\#A = a|d_k = b, v)$ is stochastically non-decreasing in v, and stochastically strictly increasing if there is any type that bids above b with positive probability.

(2) $\Pr(b \text{ wins}|\#A = a, d_k = b, v)$ is strictly decreasing in a.

(3) $\Pr(b \text{ wins}|\#A = a, d_k = b, v)$ is non-increasing in v,and strictly decreasing in v if there is any type that bids below b with positive probability.

By (3), when we increase v, the function of which we are taking expectations decreases term by term (at least weakly). (1) establishes that changing v shifts the weight in the expectation towards higher a. (2) establishes that the function decreases in a. Together, this establishes that $\Pr(b \text{ wins } |d_k = b, v, s)$ is non-decreasing in v. And, since not all bidding is concentrated at b, at least one of (1) or (3) is strict, so that in fact $\Pr(b \text{ wins } |d_k = b, v, s)$ is decreasing in v.

We take the claims in order:

Proof of Claim 1 Note that

$$\Pr(\#A = a|d_k = b, v) = \frac{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e, \#A = a|v)}{\sum_{a'=0}^{k-1} \sum_{e'=k'-a'}^{n-1-a'} \Pr(\#E = e', \#A = a'|v)}.$$

So,

$$\begin{aligned}\frac{\Pr(\#A = a+1|d_k = b, v)}{\Pr(\#A = a|d_k = b, v)} &= \frac{\sum_{e=k-a-1}^{n-1-a-1} \Pr(\#E = e, \#A = a+1|v)}{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e, \#A = a|v)} \\ &= \frac{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e-1, \#A = a+1|v)}{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e, \#A = a|v)}\end{aligned}$$

Now,

$$\frac{\Pr(\#E = e-1, \#A = a+1|v)}{\Pr(\#E = e, \#A = a|v)} =$$

$$\frac{\binom{n-1}{a+1}\binom{n-1-a-1}{e-1}\Pr(A|v)^{a+1}\Pr(E|v)^{e-1}\Pr(B|v)^{n-1-a-e}}{\binom{n-1}{a}\binom{n-1-a}{e}\Pr(A|v)^{a}\Pr(E|v)^{e}\Pr(B|v)^{n-1-a-e}} = \frac{e}{(a+1)}\frac{\Pr(A|v)}{\Pr(E|v)}.$$

So,

$$\frac{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e-1, \#A = a+1|v)}{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e, \#A = a|v)} = \frac{\sum_{e=k-a}^{n-1-a} \frac{e}{(a+1)}\frac{\Pr(A|v)}{\Pr(E|v)}\Pr(\#E = e, \#A = a|v)}{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e, \#A = a|v)} =$$

$$\frac{1}{(a+1)}\frac{\Pr(A|v)}{\Pr(E|v)}\sum_{e=k-a}^{n-1-a} e\frac{\Pr(\#E = e, \#A = a|v)}{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e, \#A = a|v)}$$

Now, by weak monotonicity of bids in signals, if $\Pr(A|s) > 0$ for some type s, then $\Pr(A|s) = 1$ for all higher types, and if $\Pr(E|s) > 0$ for some type s, then $\Pr(A|s) = 0$ for all types below s. So, since signals satisfy the MLRP with respect to v, it follows that $\frac{\Pr(A|v)}{\Pr(E|v)}$ is non-decreasing in v, and strictly increasing if bids above b are in the support for any type. So, it is enough to show that

$$\sum_{e=k-a}^{n-1-a} e\frac{\Pr(\#E = e, \#A = a|v)}{\sum_{e=k-a}^{n-1-a} \Pr(\#E = e, \#A = a|v)}$$

is non-decreasing in v. Since e is an increasing function of e, it is enough to show that

$$\frac{\Pr(\#E = e+1, \#A = a|v)}{\Pr(\#E = e, \#A = a|v)}$$

is non-decreasing in v. But, this is equal to

$$\frac{\binom{n-1}{a}\binom{n-1-a}{e+1}\Pr(A|v)^{a}\Pr(E|v)^{e+1}\Pr(B|v)^{n-1-a-e-1}}{\binom{n-1}{a}\binom{n-1-a}{e}\Pr(A|v)^{a}\Pr(E|v)^{e}\Pr(B|v)^{n-1-a-e}} = \frac{n-1-a-e}{e+1}\frac{\Pr(E|v)}{\Pr(B|v)}.$$

An entirely analogous argument to before establishes that $\frac{\Pr(E|v)}{\Pr(B|v)}$ is non-decreasing in v. So, $\frac{\Pr(\#A=a+1, d_k=b, v)}{\Pr(\#A=a, d_k=b, v)}$ is non-decreasing in v, and strictly increasing if $\Pr(A|s) > 0$ for some type.

Proof of Claim 2

$$\Pr(b \,\text{wins}|d_k = b, \#A = a, v) = \sum_{e=k-a}^{n-a-1} \frac{\Pr(\#E = e|\#A = a, v)}{\sum_{e=k-a}^{n-a-1} \Pr(\#E = e|\#A = a, v)} \frac{k-a}{e+1}. \qquad (16)$$

which is best thought of the expectation of $\frac{k-a}{e+1}$ with respect to a measure on e. Similarly,

$$\begin{aligned} &\Pr(b \,\text{wins}|d_k = b, \#A = a+1, v) \\ =& \sum_{e=k-a-1}^{n-a-2} \frac{\Pr(\#E = e|\#A = a+1, v)}{\sum_{e=k-a-1}^{n-a-2} \Pr(\#E = e|\#A = a+1, v)} \frac{k-a-1}{e+1} \\ =& \sum_{e=k-a}^{n-a-1} \frac{\Pr(\#E = e-1|\#A = a+1)}{\sum_{e=k-a}^{n-a-1} \Pr(\#E = e-1|\#A = a+1)} \frac{k-a-1}{e}. \end{aligned}$$

Now, whenever $e \geq k-a$,

$$\frac{k-a-1}{e} < \frac{k-a}{e+1}.$$

So,

$$\begin{aligned} &\Pr(b \,\text{wins}|d_k = b, \#A = a+1) \\ <& \sum_{e=k-a}^{n-a-1} \frac{\Pr(\#E = e-1|\#A = a+1)}{\sum_{e=k-a}^{n-a-1} \Pr(\#E = e-1|\#A = a+1)} \frac{k-a}{e+1}. \end{aligned}$$

which can be thought of as an expectation of the function $\frac{k-a}{e+1}$ with respect to a different measure on e. Now, since $\frac{k-a}{e+1}$ is a strictly decreasing function of e, it is thus enough to show that

$$\frac{\Pr(\#E = e-1|\#A = a+1, v)}{\Pr(\#E = e|\#A = a, v)}$$

has the MLRP in e. Now,

$$\Pr(\#E = e|\#A = a, v) = \binom{n-1-a}{e} \Pr(E|v)^e \Pr(B|v)^{n-1-a-e}$$

while

$$\Pr(\#E = e-1|\#A = a+1, v) = \binom{n-1-a-1}{e-1} \Pr(E|v)^{e-1} \Pr(B|v)^{n-1-a-e}$$

and so.

$$\frac{\Pr(\#E = \epsilon - 1 | \#A = a + 1, v)}{\Pr(\#E = \epsilon | \#A = a, v)} = \frac{\binom{n-1-a}{\epsilon-1}\Pr(E|v)^{\epsilon-1}\Pr(B|v)^{n-1-a-\epsilon}}{\binom{n-1-a-1}{\epsilon}\Pr(E|v)^{\epsilon}\Pr(B|v)^{n-1-a-\epsilon}} = \frac{\epsilon}{(n-a-1)\Pr(E|v)}$$

which is clearly strictly increasing in ϵ. So.

$$\Pr(b \text{ wins} | d_k = b, \#A = a, v)$$

is strictly decreasing in a.

Proof of Claim 3

$$\begin{aligned} \Pr(b \,\text{wins} | \#A &= a, d_k = b, v) \\ &= \sum_{\epsilon = k-a}^{n-1-a} \Pr(\#E = \epsilon | \#A = a, d_k = b, v)\frac{k-a}{\epsilon+1}. \end{aligned}$$

Trivially. $\frac{k-a}{\epsilon+1}$ is decreasing in ϵ. So. it is enough to show that $\Pr(\#E = \epsilon | \#A = a, d_k = b, v)$ is stochastically non-decreasing in v. But.

$$\Pr(\#E = \epsilon | \#A = a, d_k = b, v) = \frac{\binom{n-1-a}{\epsilon}\Pr(E|v)^{\epsilon}\Pr(B|v)^{n-1-a-\epsilon}}{\sum_{\epsilon=k-a}^{n-1-a}\binom{n-1-a}{\epsilon}\Pr(E|v)^{\epsilon}\Pr(B|v)^{n-1-a-\epsilon}}\frac{k-a}{\epsilon+1}.$$

So.

$$\frac{\Pr(\#E = \epsilon + 1 | \#A = a, d_k = b, v)}{\Pr(\#E = \epsilon | \#A = a, d_k = b, v)} = \frac{\binom{n-1-a}{\epsilon+1}\Pr(E|v)^{\epsilon+1}\Pr(B|v)^{n-1-a-\epsilon-1}}{\binom{n-1-a}{\epsilon}\Pr(E|v)^{\epsilon}\Pr(B|v)^{n-1-a-\epsilon}} = \frac{n-1-a-\epsilon}{\epsilon+1}\frac{\Pr(E|v)}{\Pr(B|v)}.$$

which is non-decreasing in v. and strictly increasing if there is an s that bids below b with positive probability. ■

Corollary 2 *If $H_s(.)$. $s \in \{1, \ldots, M\}$ are bid distributions corresponding to a symmetric equilibrium. then $H_s(\cdot)$ is continuous for all s (i.e.. there are no atoms).*

Proof Let b be an atom. Consider first the case that $b \in (0, 1)$. and at least one type bids b with probability less than 1. By bidding an amount an arbitrarily small amount below b. player 1 decreases his payoff by

$$\Pr(b \text{ wins}, d_k = b, s)(E(v | b \text{ wins}, d_k = b, s) - b).$$

For this not to be a profitable deviation, it must be that $E(v|b \text{ wins}, d_k = b, s) \geq b$. Similarly, for it not to be a profitable deviation to bid a little over b, it must be that $E(v|b \text{ loses}, d_k = b, s) \leq b$. By lemma 4, this is impossible.

Note that an atom in $H_1(\cdot)$ at 0 implies a positive probability of the event that $d_k = 0$. However, $\Pr(v \leq 0) = 0$. So, it must be that $E(v|d_k = 0, 1) > 0$. But, then by Lemma 4, $E(v|0 \text{ loses}, d_k = 0, 1) > E(v|d_k = 0, 1) > 0$, contradicting that 0 is an optimal bid with signal 1. An analogous argument rules out an atom in $H_M(.)$ at 1. Consider finally the case that all types bid b with probability 1. Then, winning or losing with a bid of b conveys no information about value. But, then since $E(v|s) < E(v|s+1)$ at least one type will have a profitable deviation. ■

Our next lemma establishes that $E(v|d_k = b)$ is well defined as the limit of $E(v|d_k \in (b', b''))$ for a sequence of intervals of bids (b', b'') converging to b.

Lemma 5 *Let $(H_s(.))_{s \in S}$ be atomless and have disjoint supports. Then, $E(v|d_k = b, s)$ is well defined for all $b \in \cup_{s \in S}(suppH_s(\cdot))$ with*

$$E(v|d_k = b, s) = \int_0^1 v \frac{x(v,b)^{n-k-1}(1-x(v,b))^{k-1}\pi(s,v)^2 f(v)}{\int_0^1 x(v,b)^{n-k-1}(1-x(v,b))^{k-1}\pi(s,w)^2 f(w)dw} dv \qquad (17)$$

where $x(v,b) = \Pi(s-1,v) + \pi(s,v)H_s(b)$. For any sequence of intervals $\{[\underline{b}_j, \bar{b}_j]\}_{j=1}^{\infty}$ with non-empty intersection with $suppH_s(\cdot)$ and $\lim \bar{b}_j = \lim \underline{b}_j \to b$, $E(v|d_k = b, s) = \lim_{j \to \infty} E(v|d_k \in I_j, s)$.

Proof Consider $b \in \text{supp}H_s(\cdot)$. Then, let $I = [\underline{b}, \bar{b}]$ be an interval having non-empty intersection with $\text{supp}H_s(\cdot)$. Define $\#_I$ as the random variable denoting the number of bids in the interval I. Define $D(v|d_k \in I) \equiv \Pr(v' \leq v|d_k \in I)$ and note that

$$\Pr(v' \leq v|d_k \in I) = \frac{1}{\Pr(d_k \in I)} [\Pr(d_k \in I, \#_I = 1)\Pr(v' \leq v|d_k \in I, \#I = 1) + \Pr(v \leq \tilde{v}|d_k \in I, \#_I > 1)]$$

Now, because there are no atoms in the bid functions, as the length of the interval I denoted by $|I|$, goes to zero, $\frac{\Pr(d_k \in I, \#_I > 1)}{\Pr(d_k \in I, \#_I = 1)}$ goes to zero. So,

$$D(v|d_k \in I) \to \Pr(v' \leq v|d_k \in I, \#_I = 1) \qquad (18)$$

as $|I| \to 0$. Denote by I^- the interval $[0, \underline{b}]$ and denote by I^+ the interval $[0, \bar{b}]$ But,

$$\Pr(v' \leq v|d_k \in I, \#_I = 1) = \Pr(v' \leq v|\#_{I^-} = k-1, \#_I = 1, \#_{I^+} = n-k-1)$$

For $B \subset [0,1] \cap \mathrm{supp} H_s$, define $x(v,B) = \Pi(s-1,v) + \pi(s,v)H_s(B)$. By Bayes' rule and (18),

$$\begin{aligned} D(v|d_k \in I) &\cong \frac{\int_0^v (n-1)\binom{n-2}{k-1} x(v,I^-)^{n-k-1}(1-x(v,I^+))^{k-1}\pi(s,v)H_s(I)f(v)dv}{\int_0^1 (n-1)\binom{n-2}{k-1} x(w,I^-)^{n-k-1}(1-x(w,I^+))^{k-1}\pi(s,w)H_s(I)f(w)dw} \\ &= \frac{\int_0^v x(v,I^-)^{n-k-1}(1-x(v,I^+))^{k-1}\pi(s,v)f(v)dv}{\int_0^1 x(w,I^-)^{n-k-1}(1-x(w,I^+))^{k-1}\pi(s,w)f(w)dw} \end{aligned}$$

From this,

$$D(v|d_k \in I,s) \cong \int_0^v \frac{x(v,I^-)^{n-k-1}(1-x(v,I^+))^{k-1}\pi^2(s,v)f(v)}{\int_0^1 x(w,I^-)^{n-k-1}x(w,I^+)^{k-1}\pi^2(s,w)f(w)dw}dv.$$

Taking limits, we can define $E(v|d_k = b,s)$ as

$$\int_0^1 v\frac{x(v,b)^{n-k-1}(1-x(v,b))^{k-1}\pi^2(s,v)f(v)dv}{\int_0^1 x(w,b)^{n-k-1}(1-x(w,b))^{k-1}\pi^2(s,w)f(w)dw}dv \blacksquare$$

Finally, we establish an intuitive necessary condition on equilibrium bids.

Lemma 6 *For any equilibrium* $(H_s(.))_{s\in S}$ *and for any* $b \in \mathrm{supp}H_s(\cdot)$, $E(v|d_k = b,s) = b$.

Proof Given the equilibrium, let BR_s be the set of best responses when the signal is s. Then, $BR_s \cap \mathrm{Int}(\mathrm{supp}H_s(\cdot))$ is a dense subset of $\mathrm{supp}H_s(\cdot)$. By Corollary 2 the $H_s(.)$ are continuous. Examining (17), it is thus clear that $E(v|d_k = b,s)$ is also continuous in b, and so it is enough to establish that $E(v|d_k = b,s) = b$ for $b \in BR_s \cap \mathrm{Int}(\mathrm{supp}H_s(\cdot))$. But, for any such b, and any $b' > b$, the change in payoff from bidding b' instead of b is

$$\Pr(d_k \in (b,b')|s)E(v - d_k|d_k \in (b,b'),s)) \geq \Pr(d_k \in (b,b')|s)E(v - b'|d_k \in (b,b'),s)).$$

Since $b \in \mathrm{Int}(\mathrm{supp}H_s(\cdot))$, $\Pr(d_k = (b,b')|s) > 0$. So, since b is optimal, it must be that $E(v|d_k \in (b,b'),s) \leq b'$ for any $b' > b$, and thus that $E(v|d_k = b,s) \leq b$. An analogous argument shows that $E(v|d_k = b,s) \geq b$, establishing the result. $\blacksquare$

Proof of Proposition 1 If there is an equilibrium $(H_s(.))_{s\in S}$, then by the preceding lemmas, $H_s(b)$ must satisfy

$$b = \int_0^1 v\frac{x(v,b)^{n-k-1}(1-x(v,b))^{k-1}\pi(s,v)^2 f(v)dv}{\int_0^1 x(w,b)^{n-k-1}(1-x(w,b))^{k-1}\pi(s,w)^2 f(w)dw} \tag{19}$$

where $x(v, b) = \Pi(s-1, v) + \pi(s, v)H_s(b)$ for all $b \in \text{supp}H_s(\cdot)$.

We will establish that for each s there is a unique $H_s(.)$ satisfying these two equations, and then show that $H_s(.), s \in S$, is an equilibrium.

Equation (17) implies that $E(v|d_k = b, s)$ is continuous in $H_s(b)$. Furthermore the following argument shows that $E(v|d_k = b, s)$ is strictly increasing in $H_s(b)$ for all $b \in \text{supp}H_s(\cdot)$. Denote by $\tilde{S} = S \times [0, 1]$ the following continuous version of our signal space. $F(s, y; v) = \prod(s-1, v) + \pi(s, v)y$. By a standard property of MLRP (see Milgrom 1981) the k^{th} order statistic of the signal (s, x) satisfies *strict MLRP* in the signal (s, x). But this implies that if z_k denotes the k^{th} order statistic of (s, x) then $E(v|z_k = (s, x))$ is strictly increasing in x. But clearly this is equivalent to $E(v|d_k = b, s)$ being strictly increasing in H_s.

Thus, there is at most one $H_s(b)$ that satisfies (19) for any given b. By definition of $\underline{b}_s$, (19) is solved at $\underline{b}_s$ by setting $H_s(\underline{b}_s) = 0$ and at $\bar{b}_s$ by setting $H_s(\bar{b}_s) = 1$.and has no solution with $H_s(b) \in [0, 1]$ for b below $\underline{b}_s$ or above $\bar{b}_s$. Since $E(v|d_k = b, s)$ is strictly increasing in $H_s(b)$, there is a unique $H_s(b) \in [0, 1]$ satisfying (19) for each $b \in [\underline{b}_s, \bar{b}_s]$, and $H_s(b)$ defined in this way is strictly increasing in b. Since $H_s(b)$ is not allowed to jump, this thus implies that there is exactly one $H_s(.)$ that satisfies (19) at each point in its support and does not involve atoms.

All that remains is to show that the $H_s(.)$ we have defined form an equilibrium. By construction, it is clear that all bids in $[\underline{b}_s, \bar{b}_s]$ are equivalent from the point of view of i when his signal is s (because, in particular, we have constructed the $H_s(\cdot)$ so that the derivative of payoff with respect to bids is zero). It remains to be shown that no other bid is better. For every bid $b \in [\underline{b}_{s'}, \bar{b}_{s'}]$, $s' > s$, $E(v|d_k = b, s') = b$, from which $E(v|d_k = b, s) < b$, and so $\bar{b}_s$ is a better bid for s than is any bid in $[\underline{b}_{s'}, \bar{b}_{s'}]$ (i.e., derivative of payoff with respect to bids negative on $[\underline{b}_{s'}, \bar{b}_{s'}]$ for signals s). For $s' \geq s$, if $b \in [\bar{b}_{s'}, \underline{b}_{s'+1}]$ then the payoff for the bidder is the same as if he bid $\bar{b}_{s'}$ and hence it follows that the payoff of s when bidding $\bar{b}_s$ is greater than or equal to the payoff of any bid larger than $\bar{b}_s$. An analogous argument establishes that any bid lower than $\underline{b}_s$ does not increase the bidders payoff. ■

6 Appendix 2: Miscellaneous Proofs

Proof of Remark 1 Let G_r denote the joint probability distribution on v and p in auction r. Let

$$E = \{(v, p) : |v - p| > \epsilon/2\}$$

By Theorem 1 there is an r' such that for $r > r'$

$$G_r(E) < \eta \cdot \epsilon^2/2.$$

Let $r > r'$. Let $Z_r(p|v)$ be the probability distribution over prices in A_r given v. Then

$$\begin{aligned} Z_r(p|v) &= \Pr[\{\#bids \leq p\} \geq n_r - k_r - 1] = \\ &= \sum_{j=n_r-k_r-1}^{n_r} \binom{n_r}{j} x_r(p, v)^{n_r-j}(1 - x_r(p, v))^j. \end{aligned}$$

Since $x_r(p, v)$ is strictly decreasing in v it follows that $Z_r(p|v)$ is strictly decreasing in v. Suppose contrary to the claim that

$$1 - Z_r(v + \epsilon|v) > \epsilon$$

for some $v \in [0, 1]$. Clearly this implies that $v + \epsilon < 1$. By the monotonicity of $Z_r(p|v)$ in v this in turn implies that

$$1 - Z_r(v + \epsilon|v') > \epsilon, \forall v' \in [v, v + \epsilon/2]$$

But this implies that

$$G_r(E) \geq \epsilon \cdot \int_v^{v+\epsilon/2} f(v)dv \geq \eta\epsilon^2/2.$$

since $f(v) > \eta$. This contradicts the hypothesis that $G_r(E) < \eta\epsilon^2/2$. An analogous argument shows that $Z_r(v - \epsilon|v) \leq \epsilon$ for all $v \in [0, 1]$ and hence the claim follows. ■

Proof of Corollary 1 We previously argued that

$$\frac{1}{\eta} > \frac{(1 - x_r(b, v))}{(1 - x_r(b, v))} > \eta.$$

It is thus enough to establish (14) for some given v.

First, assume $k_r \to \infty$ (along a subsequence). We know that in that situation, beliefs conditional on $X_r(b)$ converge to $v_r^*(b)$, and $v_r^*(b)$ converges to b. But, $v_r^*(b)$ was chosen to make $x_r(v, b)$ as close as possible to $\frac{n_r - k_r - 1}{n_r - 2}$. Since $b \in (0, 1)$, $v_r^*(b)$ is interior for r large enough, and so $x_r(v_r^*(b), b) = \frac{n_r - k_r - 1}{n_r - 2}$, from which (14) is immediate.

Assume that for all r along a subsequence, $k_r < C$. Since C is finite, but the expected number of players with signal M grows without bound, equilibrium profits with signal M must go to 0. If $\frac{n_r(1-x_r(v,b))}{k_r} \to 0$, then since $k_r < C$, $n_r(1-x_r(v,b)) \to 0$ for all v, and so the probability that no one bids at or above b goes to 1. Consider bidding b with signal M. This bid wins whenever it is made, and so $E(v|b \text{ wins}, M) = E(v|M) > E(v) > b$. Thus, the profits with signal M are bounded away from 0, a contradiction.

Next, assume that (along a subsequence), $\frac{n_r(1-x_r(v,b))}{k_r} \to \infty$. Then, by (12), $v_r^*(b) \to 0$. But, also $n_r(1 - x_r(v, b)) \to \infty$, and so examining the proof of case 3 in Lemma 1, (see in particular equation (11)), it is then the case that $E(v|X_r(b), s)$ converges to $v_r^*(b)$. Since $v_r^*(b) \to 0$, this contradicts that $E(v|X_r(b), s) = b > 0$. ■

Proof of Lemma 1 for the information structure of section 3.4 For all b,

$$\Pr(X_r(b)|v) = x_r(v, b)^{n_r - k_r - 1}(1 - x_r(v, b))^{k_r - 1}$$

where $x(v, b) = \begin{cases} \gamma_r \pi(1, v) H_{1r}(b) & \text{if } s = 1 \\ (1 - \gamma_r) H_{2r}(b) + \gamma_r \pi(1, v) & \text{if } s = 2 \\ 1 - \gamma_r \pi(3, v)(1 - H_{3r}(b)) & \text{if } s = 3 \end{cases}$.

As before we have that

$$\frac{\partial}{\partial v} \ln \Pr(X_r(b)|v)) = \frac{\frac{\partial}{\partial v} x_r(v, b)}{x_r(v, b)(1 - x_r(v, b))}[n_r - k_r - 1 - (n_r - 2)x_r(v, b))] \qquad (20)$$

As before, define $v_r^*(b)$ by

$$v_r^*(b) = \arg\max_v \left| x_r(v, b) - \frac{n_r - k_r - 1}{n_r - 2} \right|.$$

Let

$$(b_r, v_r) = \arg \min_{v, b \in T_r} \frac{\partial}{\partial v} |\ln d(v|X_r(b))| \quad \text{subject to } |v - v_r^*(b)| \geq \epsilon$$

It is again sufficient to show that

$$\left| \frac{\frac{\partial}{\partial v} x_r(v_r, b_r)}{x_r(v_r, b_r)(1 - x_r(v_r, b_r))}[n_r - k_r - 1 - (n_r - 2)x_r(v_r, b_r))] \right| \to \infty.$$

We again distinguish three cases:

Case 1: $b_r \in \text{supp}H_{2r}$. In this case

$$\frac{\frac{\partial}{\partial v}x_r(v_r,b_r)}{x_r(v_r,b_r)(1-x_r(v_r,b_r))}[n_r-k_r-1-(n_r-2)x_r(v_r,b_r))] =$$
$$\frac{\frac{\partial}{\partial v}\pi(1,v_r)}{x_r(v_r,b_r)(1-x_r(v_r,b_r))}\gamma_r[n_r-k_r-1-(n_r-2)x_r(v_r,b_r))].$$

Again note that $\frac{\partial}{\partial v}\pi(1,v) < -\eta$ for some $\eta > 0$ by Assumption 2. Thus it suffices to show that

$$\begin{aligned} & |\gamma_r[n_r-k_r-1-(n_r-2)x_r(v_r,b_r)]| \qquad (21) \\ = & \left|\gamma_r^2(n_r-2)\pi(1,v)-[\gamma_r(n_r-k_r-1)-\gamma_r(n_r-2)(1-\gamma_r)H_{r2}(b_r)]\right| \to \infty. \end{aligned}$$

Since $v_r^*(b_r)$ minimizes $\left|\pi(1,v)-\frac{1}{\gamma_r}\left[\frac{n_r-k_r-1}{n_r-2}-(1-\gamma_r)H_{2r}(b_r)\right]\right|$the fact that $|v_r - v_r^*(b_r)| > \epsilon$ together with $\frac{\partial}{\partial v}\pi(1,v) < -\eta$ imply that

$$\left|\pi(1,v)-\frac{1}{\gamma_r}\left[\frac{n_r-k_r-1}{n_r-2}-(1-\gamma_r)H_{2r}(b_r)\right]\right| > \epsilon\eta$$

and hence (21) follows from the fact that $\gamma_r^2 n_r \to \infty$ (and thus $\gamma_r^2 n_r \to \infty$).

Case 2: $b_r \in \text{supp}H_{1r}$. In this case $x_r(v,b_r) = \gamma_r\pi(1,v)H_{1r}(b_r))$ and hence

$$\begin{aligned} & \frac{\frac{\partial}{\partial v}x_r(v,b_r)}{x_r(v,b_r)(1-x_r(v,b_r))}[n_r-k_r-1-(n_r-2)x_r(v_r,b_r))] \\ = & \frac{\pi'(1,v)}{\pi(1,v)(1-\pi(1,v)H_{1r}(b_r))}\left(n_r-k_r-1-\gamma_r(n_r-2)H_{1r}(b_r)\pi(1,v)\right). \end{aligned}$$

Again $\frac{\partial}{\partial v}\pi(1,v) < -\eta < 0$, and hence it is sufficient to show that

$$(n_r-k_r-1-\gamma_r(n_r-2)H_{1r}(b_r)\pi(1,v)) \to \infty. \qquad (22)$$

Since $v_r^*(b_r)$ minimizes $\left|\pi(1,v)-\frac{n_r-k_r-1}{H_{1r}(b_r)\gamma_r(n_r-2)}\right|$it follows from $|v-v_r^*(b_r)| > \epsilon$ that

$$\left|\pi(1,v)-\frac{n_r-k_r-1}{\gamma_r(n_r-2)H_{1r}(b_r)}\right| \geq \epsilon\eta. \qquad (23)$$

If $\gamma_r(n_r-2)H_{1r}(b_r)$ stays bounded then (22) follows from the fact that $n_r - k_r \to \infty$. Conversely, if $\gamma_r(n_r-2)H_{1r}(b_r) \to \infty$ then (23) implies (22).

Case 3: $b_r \in \text{supp}H_{3r}$. In this case $x_r(v,b_r) = 1-\gamma_r\pi(3,v)(1-H_{3r}(b_r))$ and hence

$$\begin{aligned} & \frac{\frac{\partial}{\partial v}x_r(v,b_r)}{x_r(v,b_r)(1-x_r(v,b_r))}[n_r-k_r-1-(n_r-2)x_r(v_r,b_r))] \\ = & \frac{\frac{\partial}{\partial v}\pi(3,v)}{\pi(3,v)(1-\pi(3,v)(1-H_{3r}(b_r))}[(n_r-2)(1-H_{3r}(b_r))\gamma_r\pi(3,v)-(k_r-1)] \end{aligned}$$

Since $\frac{\partial}{\partial v}\pi(M, v) > \eta > 0$, it is again sufficient to show that

$$|\gamma_r(n_r - 2)\pi(3, v)(1 - H_{3r}(b_r))\gamma_r\pi(3, v) - (k_r - 1)| \tag{24}$$

Since $v_r^*(b_r)$ minimizes $\left|\pi(3, v) - \frac{1}{\gamma_r}\left[\frac{k_r - 1}{(n_r - 2)(1 - H_{3r}(b_r))}\right]\right|$ it follows from $|v - v_r^*(b_r)| > \epsilon$ that

$$\left|\pi(3, v) - \frac{1}{\gamma_r}\left[\frac{k_r + 1}{(n_r - 2)(1 - H_{3r}(b_r))}\right]\right| \geq \epsilon\eta \tag{25}$$

If $\gamma_r(n_r - 2)\pi(3, v)[1 - H_{3r}(b_r)]$ stays bounded then (24) follows from the assumption that $k_r \to \infty$. Conversely, if $\gamma_r(n_r - 2)\pi(3, v)[1 - H_{3r}(b_r)] \to \infty$ then (24) follows from (25). ■

References

[1] Feddersen, Timothy, and Wolfgang Pesendorfer. 1994. "Voting Behavior and Information Aggregation in Elections with Private Information." CMSEMS Discussion Paper No. 1117, December 1994.

[2] Gale, Douglas. 1986. "Bargaining and Competition Part I: Characterization". *Econometrica*, 54, 785-806.

[3] Harstad, Ronald M. and Dan Levin, 1986. "Symmetric Bidding in Second-Price, Common Value Auctions". *Economics Letters*, **20**, 315-319.

[4] Milgrom, Paul R.,1979. "A Convergence Theorem for Competitive Bidding with Differential Information." *Econometrica*, **47**, 670-688.

[5] Milgrom, Paul R., 1981. "Rational Expectations, Information Acquisition, and Competitive Bidding." *Econometrica*, **49**, 921-943.

[6] Milgrom, Paul R., and Robert J. Weber, 1982. "A Theory of Auctions and Competitive Bidding." *Econometrica*, **50**, 1089-1122.

[7] Palfrey, Thomas R. 1985. "Uncertainty Resolution, Private Information Aggregation and the Cournot Competitive Limit". *Review of Economic Studies*, 69-83.

[8] Pesendorfer, Wolfgang and Jeroen M. Swinkels, 1995. "Efficiency and Information Aggregation in Auctions with Private and Common Values". mimeo, Northwestern University.

[9] Rubinstein, Ariel and Asher Wolinsky. 1990. "Decentralized Trading, Strategic Behaviour and the Walrasian Outcome." *Review of Economic Studies*, **57**, 63-78.

[10] Wilson, Robert. 1977. "A Bidding Model of Perfect Competition." *Review of Economic Studies* **44** 511-518.