Discussion Paper No. 1137

THE CLASSIFICATION OF CONTINUATION PROBABILITIES

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July 1995

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ABSTRACT. It is known that the only subgame perfect equilibrium for finitely repeated Prisoner's Dilemma games consists of "defecting" in every round. Finitely repeated games are only representative of a class of indefinitely repeated games where the sole subgame perfect equilibrium is noncooperative. This broader class of repeated games with "quasifinite" continuation probabilities is defined.

A matrix inequality is recalled that when solved by a cooperation vector, induces a subgame perfect equilibrium. A condition for continuation probabilities indicates when this matrix inequality can be satisfied at equality by a cooperation vector. The associated strategy is a cooperative subgame perfect equilibrium.

1. INTRODUCTION

The types of equilibria for finitely and infinitely repeated generalized, Prisoner's Dilemma games are markedly different. Past work has concentrated on the graph of equilibrium payoffs in repeated Prisoner's Dilemma games, see Stahl [11]. In Jones [5] and [6], the geometry of subgame perfect equilibria for indefinitely repeated, generalized Prisoner's Dilemma games is examined. Also in Jones [6], the conditions for the existence of cooperative subgame perfect publicly correlated equilibria is given. The characteristics of these equilibria fit naturally between the equilibria of finitely and infinitely repeated games.

Carroll [4] examines indefinite terminating points of repeated Prisoner's Dilemma game. But as Becker and Cudd [3] promote in their response to Carroll, there are cooperative equilibria for some indefinitely repeated games. In this paper, continuation probabilities are classified as to whether there exists *any* generalized Prisoner's Dilemma games with cooperative equilibria. Continuation probabilities satisfying this condition are called "quasifinite" since they have the equilibrium properties of finitely repeated Prisoner's Dilemma games. The benefits are obvious: defecting in

Key words and phrases. Continuation Probabilities, Prisoner's Dilemma, Subgame Perfect Equilibrium, Indefinitely Repeated Games.

*This research stems from my dissertation work under D.G. Saari at Northwestern University; it was supported in part by NSF grant IST 9103180. This research has benefitted from comments and suggestions by E. Kalai, D. Saari, and S. Zabell.

every round is the only subgame perfect equilibrium for *any* generalized Prisoner's Dilemma game (regardless of the payoff matrix) repeated indefinitely by a quasifinite continuation probability. Quasifinite games encompass the class of games examined by Carroll, but also show that a positive probability of the game continuing to the next round is not sufficient for the existence of a cooperative equilibrium for some generalized Prisoner's Dilemma game.

In Jones [5] and [6], publicly correlated strategies have associated cooperation vectors. When a matrix inequality equivalent to the one-stage-deviation principle is satisfied, a cooperation vector represents a subgame perfect strategy. The matrix and its spectral radius depend on the discount parameter and the continuation probability. When a value associated with the stage game is less than the spectral radius, then there are cooperative subgame perfect equilibria. When this value is greater than the spectral radius, then there does not exist any cooperative subgame perfect equilibria. When this inequality is satisfied at equality, the cooperation vector may or may not define a viable strategy. The existence of a cooperative equilibria in these cases is equated with a limit converging. This limit classifies which continuation probabilities have cooperative strategies associated with them that will satisfy the matrix equality. Examples demonstrate some of the subtleties involved in this analysis. Some general properties (such as monotonicity) of continuation probabilities are considered.

Background material, the model, and pertinent previous results appear in Section 2. The Classification Theorem appears in Section 3. The existence of cooperative equilibria satisfying the matrix equality is examined in Section 4. Throughout the paper, examples illustrate the analysis of games which previously were difficult, if not impossible, to analyze.

2. Indefinitely Repeated, Generalized Prisoner's Dilemma Games

I proceed by introducing notation for a stage game with standard information, *i.e.*, where there is only one state of nature and the players know all the past moves of each player. Let $N = \{1, 2, \ldots, n\}$ be the set of all players. Assume that there is a finite action space and that a utility function (u_i for player i) maps this set into the real numbers. In the past, the discount parameter was considered as both a discount on future payoffs and as the probability that the game continues to the next round (*e.g.*, Murnighan and Roth [9]). In this paper, the discount parameter $\delta \in (0, 1]$ denotes how the players' utilities decrease over time. The uncertainty about the game continuing is captured by the continuation probability.

Definition 2.1. The *time-dependent continuation probability* β is defined by $\beta = (\beta_1, \beta_2, \ldots)$ where $\beta_k \in [0, 1]$. The number β_k is the probability that the game will continue to the k^{th} round given that the $(k-1)^{th}$ round occurs.

Notice that both finitely and infinitely repeated games can be represented by continuation probabilities. The continuation probability for a finitely repeated game of

r rounds is represented by the sequence consisting of r 1's followed by a tail of zeroes. For an infinitely repeated game, it is the sequence of all 1's. The continuation probability is assumed to be common knowledge. The expected payoff for the indefinitely repeated game weights future payoffs by both the discount parameter and the continuation probability.

I use a characterization of stage games generalizing the Prisoner's Dilemma that is found in Bernheim and Dasgupta [3]; a comparison to their work appears at the pertinent junctures. Specifically, I consider stage games with a single inefficient Nash equilibrium and normalized payoffs so that the each player receives zero utiles under the Nash equilibrium. The normalization implies that there exists at least one efficient strategy whose payoffs are positive and, thereby, strictly Pareto dominate the payoffs of the Nash equilibrium. Let Σ be the set of mixed strategies for the stage game and Σ_i be the set of mixed strategies for player i. Let the single Nash equilibrium in the stage game be denoted by σ^*. The normalization implies that $u_i(\sigma^*) = 0$ for all players i.

Define a function $f_i : \Sigma \to \mathbb{R}$ that gives the maximum gain to player i if her opponents stick to a fixed strategy σ and if she is allowed to deviate from σ. Therefore, $f_i(\sigma) = \max_{\tau_i \in \Sigma_i}[u_i(\tau_i, \sigma_{-i}) - u_i(\sigma)]$. Notice that $f_i(\sigma^*) = 0$ for every player i by the definition of a Nash equilibrium. Also, notice that $f_i(\sigma) > 0$ for at least one player i for all mixed strategies $\sigma \neq \sigma^*$ of the stage game.

Since σ^* is inefficient by assumption then there exists a $\tau^* \in \Sigma$ such that $u_i(\tau^*) > u_i(\sigma^*) = 0$ for all i. Let τ^* be a fixed strategy such that $u_i(\tau^*) > u_i(\sigma^*)$ for all i and

$$\tau^* \in \arg\min_{\tau} \left\{ \max_i \frac{f_i(\tau)}{u_i(\tau)} \right\}.$$

The strategy τ^* is an improvement from σ^* for every player, yet also minimizes the benefits from deviating from τ^*.

Assume that players use publicly correlated strategies. So, players essentially base their collective strategy on the public observance of the flipping of a λ-coin. Let $\lambda[\tau^*] + (1-\lambda)[\sigma^*]$ represent the correlated strategy where, with probability λ, the players all play τ^* and with probability $(1-\lambda)$ the players all play σ^*. The parameter λ represents the level of cooperation where $\lambda = 0$ is noncooperative and $\lambda = 1$ is purely cooperative. The pre-discounted expected utility for player i using this correlated strategy in the stage game is

$$u_i(\lambda[\tau^*] + (1-\lambda)[\sigma^*]) = \lambda u_i(\tau^*) + (1-\lambda)u_i(\sigma^*) = \lambda u_i(\tau^*).$$

Therefore, cooperation increases the expected payoff, but also gives a more lucrative incentive for players to deviate from the strategy since $f_i(\tau^*) > 0$ for all i.

Let S be the strategy profile for the indefinitely repeated game where the players play the correlated strategy $\lambda_k[\tau^*] + (1-\lambda_k)[\sigma^*]$ at round k until a player deviates. Enforcement is by a grim trigger mechanism, where deviation from τ^* warrants punishment with σ^* in all subsequent rounds. Deviation from $\lambda_k[\tau^*] + (1 - \lambda_k)[\sigma^*]$ is immediately obvious to the other players due to complete information. There is an important temporal aspect in this definition. I am requiring that the deviation is decided before the observation of the probabilistic event. One can think of this restriction as adhering to a mechanism and making the decision to deviate in a particular round at the beginning of the game. Similar results, however, hold if the deviation occurs after the event. Relying on grim trigger strategies is sufficient since any subgame perfect outcome can be supported by a grim trigger mechanism [1].

Define $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ to be the cooperation vector associated with the strategy profile S. The cooperation vector Λ measures cooperation for the indefinitely repeated game just as λ_k measures cooperation for round k. The strategy profile S, with an associated cooperation vector Λ, is a subgame perfect equilibrium if it satisfies the one-stage deviation principle. The following proposition appears as a corollary to a proposition in Jones [5] and [6].

Proposition 2.1. *(Jones [5] and [6]) The strategy profile S with the associated cooperation vector Λ is a subgame perfect equilibrium if and only if, for every round k and every player i,*

$$\frac{f_i(\tau^*)}{u_i(\tau^*)}\Lambda \leq M(\delta, \beta)\Lambda \text{ where} \tag{2.1}$$

$$M(\delta,\beta) = \begin{pmatrix} 0 & \delta\beta_2 & \delta^2\beta_2\beta_3 & \delta^3\beta_2\beta_3\beta_4 & \delta^4\beta_2\beta_3\beta_4\beta_5 & \dots \\ 0 & 0 & \delta\beta_3 & \delta^2\beta_3\beta_4 & \delta^3\beta_3\beta_4\beta_5 & \dots \\ 0 & 0 & 0 & \delta\beta_4 & \delta^2\beta_4\beta_5 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The following theorem dictates, in part, when there exists a cooperation vector (and hence a subgame perfect equilibrium) satisfying Equation 2.1. The theorem appears in Jones [6] and utilizes the spectral radius of the matrix defined in Proposition 2.1. Jones [7] examines the existence of positive eigenvalues with associated eigenvectors containing all positive entries. These eigenvectors ultimately define cooperative strategies. Surprisingly, the existence of such eigenvalues and eigenvectors does not rely on the Krein-Rutman Theorem, the infinite dimensional counterpart to Perron-Fröbenius Theory (both of which have been successfully applied in mathematical economics previously *e.g.*, Kohlberg [8] and Samuelson and Solow [10]).

Let μ^* represent the spectral radius of $M(\delta, \beta)$. From Jones [7], when the continuation probability consists of positive entries, the spectral radius is given by

$\mu^* = \dfrac{\mathcal{E}\delta}{1-\mathcal{E}\delta}$, where $\mathcal{E} = \lim_{n\to\infty}\left(\prod_{j=2}^{n+1}\beta_j\right)^{\frac{1}{n}}$. To simplify notation, let μ be the maximum of $\left\{\frac{f_i(\tau^*)}{u_i(\tau^*)}\right\}_{i\in N}$. Again, in the theorem below assume that $\beta_k > 0$ for all k. If $\beta_k = 0$ then the game is finite.

Theorem 2.2. *a) When $\mu > \mu^*$, there does not exist a cooperative equilibrium satisfying $\mu\Lambda \leq M\Lambda$. b) When $\mu < \mu^*$, there exist cooperative equilibria satisfying $\mu\Lambda \leq M\Lambda$.*

The obvious omission from Theorem 2.2 is when $\mu = \mu^*$. The existence of a vector satisfying Equation 2.1 and defining a strategy (*i.e.*, all entries between 0 and 1 inclusive) is considered in Section 4. Further, notice that μ is always positive by the definition of τ^*. If the spectral radius of $M(\delta, \beta)$ is zero then there does not exist a cooperative subgame perfect equilibrium regardless of the stage game (because μ is never less than zero). This idea is used to classify all the continuation probabilities which do not have a cooperative subgame perfect equilibrium for *any* generalized Prisoner's Dilemma game.

3. The Classification Theorem

For a finitely repeated generalized Prisoner's Dilemma game, it is well known that defecting in every round is the only subgame perfect equilibrium. However, the continuation probabilities of finitely repeated games are only representatives of a class of continuation probabilities with this property. The theorem below fully represents these continuation probabilities. The subsequent corollary, remarks, and examples display some of the subtleties and implications of the classification theorem. Recall that $\mathcal{E} = \lim_{n\to\infty}\left(\prod_{j=2}^{n+1}\beta_j\right)^{\frac{1}{n}}$.

Definition 3.1. The continuation probability β is *quasifinite* if and only if there does not exist a cooperative subgame perfect equilibrium for *any* generalized Prisoner's Dilemma game when indefinitely repeated under β.

Theorem 3.1. *A continuation probability β is quasifinite if and only if $\mathcal{E} = 0$.*

The proof of Theorem 3.1 appears in the Appendix. Bernheim and Dasgupta [3] examine indefinitely repeated, generalized Prisoner's Dilemma games with both finite and continuous action spaces where the continuation probability is asymptotically finite.

Definition 3.2. (Bernheim and Dasgupta [3]) The continuation probability β is *asymptotically finite* if and only if $\lim_{k\to\infty}\beta_k = 0$.

They prove that there does not exist a cooperative subgame perfect equilibrium for any generalized Prisoner's Dilemma game with finite action spaces when the continuation probability is asymptotically finite [3]. Their result follows from Theorem 3.1 and appears as the following corollary. The proof appears in the Appendix.

Corollary 3.2. *An asymptotically finite continuation probability is quasifinite.*

However, there is a difference between quasifinite and asymptotically finite continuation probabilities. The following examples and remarks make some of the distinctions more clear.

Remark 3.3. *There exist quasifinite continuation probabilities which are not asymptotically finite.*

Example 3.4. *Consider the continuation probability defined by*

$$\beta_k = \begin{cases} 1 & if \quad k \text{ is odd} \\ \left(\frac{1}{2}\right)^{\frac{k}{2}} & if \quad k \text{ is even.} \end{cases}$$

It follows that

$$\mathcal{E} = \lim_{n\to\infty}\left(\prod_{j=2}^{n+1}\beta_j\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{2}\right)^{\frac{n}{8}+\frac{1}{4}} = 0.$$

The continuation probability β is quasifinite, but not asymptotically finite.

The continuation probability in Example 3.4 does not converge to zero, but does have a subsequence converging to zero. This is necessary, but not sufficient, to guarantee that the continuation probability is quasifinite.

Proposition 3.5. *If β is quasifinite and $\beta_k > 0$ for all k, then β has a subsequence converging to zero.*

Proof. Assume that no subsequence converges to zero. Hence, there exists a lower bound, $\epsilon > 0$, such that $\beta_k \geq \epsilon$ for all k. It follows that

$$\mathcal{E} = \lim_{n\to\infty}\left(\prod_{j=2}^{n+1}\beta_j\right)^{\frac{1}{n}} \geq \lim_{n\to\infty}\left(\prod_{j=2}^{n+1}\epsilon\right)^{\frac{1}{n}} = \epsilon > 0.$$

This contradicts that $\mathcal{E} = 0$. Therefore, a subsequence of β converges to zero. □

Remark 3.6. *It is not sufficient for a subsequence to converge to zero for a continuation probability to be quasifinite.*

For some generalized Prisoner's Dilemma game, there exist cooperative equilibria when indefinitely repeated by continuation probabilities with subsequences that converge to zero, but are not quasifinite. Under such a stage game, although the subsequence indicates that there are shocks which may cause the game to end with higher and higher probability, the existence of a cooperative equilibrium indicates that the players are still optimistic about the future. The following example demonstrates a continuation probability with a subsequence converging to zero which is not quasifinite. This example (with a specific stage game and equilibrium strategy) is thoroughly examined in Jones [5] and [6].

Example 3.7. *Let β be the continuation probability defined by*

$$\beta_k = \begin{cases} \left(\frac{3}{4}\right)^{\sqrt{k}-1} & if \quad \sqrt{k} \in \mathbb{N} \\ 1 & otherwise. \end{cases}$$

By definition,

$$\mathcal{E} = \lim_{n\to\infty} \left(\prod_{j=2}^{n+1} \beta_j\right)^{\frac{1}{n}} = \lim_{n^2\to\infty} \left(\prod_{j=2}^{n^2+1} \beta_j\right)^{\frac{1}{n^2}} = \lim_{n^2\to\infty} \left(\prod_{j=2}^{n} \left(\frac{3}{4}\right)^j\right)^{\frac{1}{n^2}}$$

$$= \lim_{n^2\to\infty} \left[\left(\frac{3}{4}\right)^{\frac{n(n-1)}{2}}\right]^{\frac{1}{n^2}} = \left(\frac{3}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2}.$$

Although β has a subsequence converging to zero, it is not quasifinite. By Theorem 3.1, there exists a cooperative subgame perfect equilibrium for some generalized Prisoner's Dilemma game, indefinitely repeated by β.

4. Solutions to the Matrix Equality

Recall from Section 2 that μ is defined as the maximum of $\left\{\frac{f_i(\tau^*)}{u_i(\tau^*)}\right\}_{i\in X}$. Jones [6] proves that a cooperative equilibrium exists if $\mu^* > \mu$ and that only the noncooperative equilibrium exists if $\mu^* < \mu$.

In this section, I examine the condition necessary for a cooperative equilibrium exists and $\mu^* = \max\limits_{i\in X} \frac{f_i(\tau^*)}{u_i(\tau^*)}$. This is equivalent to answering the question, "When does a cooperation vector satisfy $M\Lambda = \mu^*\Lambda$?" As is proved in Jones [7], every value is an eigenvalue of M with an eigenvector consisting of positive entries. In fact, the eigenvector associated with any eigenvalue is unique up to multiples. Recall that a vector defines a strategy if all its entries are between 0 and 1. Any vector with all positive entries can be rescaled to represent probabilities as long as the entries are bounded, *i.e.*, if the vector is in ℓ^∞. However, the vector may be in $\mathbb{R}^\infty$, but not ℓ^∞. If the vector is unbounded, it cannot be rescaled so that all of its entries are probabilities. The following proposition indicates when a vector defines a strategy, and when it doesn't. But, it is the corollary and examples that indicate the utility of this proposition.

Proposition 4.1. *A nontrivial vector Λ^* satisfying $M\Lambda = \mu^*\Lambda$ can be rescaled to define a cooperative subgame perfect equilibrium if and only if*

$$\lim_{k \to \infty} \mathcal{E}^k \left(\prod_{j=2}^{k+1} \beta_j \right)^{-1} < \infty. \tag{4.1}$$

The following corollary and examples help characterize the necessary properties of the continuation probability for a strategy to satisfy the matrix equality. The proofs of Proposition 4.1 and Corollary 4.2 appear in the Appendix.

Corollary 4.2. *If $\{\beta_k\}_{k=1}^{\infty}$ is monotonically nonincreasing, then a nontrivial vector satisfying $M\Lambda = \mu^*\Lambda$ can define a cooperative subgame perfect equilibrium under suitable rescaling.*

Player 2

Player 1		defect	cooperate
	defect	0,0	3,-1
	cooperate	-1,3	2,2

Bimatrix form of the Prisoner's Dilemma
Figure 4.1

Example 4.3. *Consider the stage game given in Figure 4.1. Notice that it satisfies the conditions of a generalized PD game given in Section 2. Specifically, $\tau^* =$ (cooperate, cooperate), $u_i(\tau^*) = 2$, and $f_i(\tau^*) = 1$. Let $\delta = \frac{2}{3}$. Let the continuation probability be defined by $\beta_k = \frac{1}{2} - \left(\frac{1}{2}\right)^k$. It follows that $\mathcal{E} = \frac{1}{2}$. Therefore, $\mu^* = \frac{1}{2} = \mu$.*

Corollary 4.2 guarantees that there exists a nontrivial solution to $M\Lambda = \mu^*\Lambda = \mu\Lambda$ which will define a strategy upon rescaling. The only solutions to $M\Lambda \geq \mu\Lambda$ are multiples of an eigenvector satisfying the matrix equality.

When β is montonically increasing, Theorem 4.1 provides information about the rate of convergence to determine if there exists a cooperative subgame perfect equilibrium for the matrix equality. The following two examples demonstrate this rate of convergence.

Example 4.4. *Again, consider the stage game given in Figure 4.1. Let the discount parameter $\delta = \frac{1}{3}$. Let the continuation probability be defined by $\beta_k = \frac{k}{k+1}$ for all k. It follows that*

$$\mathcal{E} = \lim_{n\to\infty}\left(\prod_{j=2}^{n+1}\beta_j\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)^{\frac{1}{n}} = 1.$$

since $\lim\limits_{n\to\infty}\frac{1}{n}\ln\left(\frac{2}{n+1}\right) = 0$. The spectral radius is $\frac{1}{2}$. Therefore, a nontrivial vector defines a cooperative subgame perfect equilibria if $\Lambda^* \in \ell^\infty$ where $\lambda_1^* = 1$ and

$$\lambda_{k+1}^* = \mathcal{E}^k\left(\prod_{j=2}^{k+1}\beta_j\right)^{-1} = \frac{k+1}{2}.$$

Obviously, $\lim\limits_{k\to\infty}\lambda_k^* = \infty$. By Theorem 4.1, there does not exist a cooperative subgame perfect equilibrium for this indefinitely repeated game.

The continuation probability of the preceding example was monotonically increasing and converged to 1 too slowly. Compare this to a monotonically increasing continuation probability that converges to 1 more quickly.

Example 4.5. *Let the stage game given in Figure 4.1 and $\delta = \frac{1}{3}$. Let the continuation probability be defined by $\beta_k = e^{-\left(\frac{1}{2}\right)^k}$. As is the last example, β is monotonically increasing and $\mathcal{E} = 1$. As before, $\mu^* = \frac{1}{2} = \mu$ which means that a cooperative subgame perfect equilibrium exists only if a nontrivial eigenvector defines a legitimate strategy. Theorem 4.1 holds since*

$$\lim_{k\to\infty}\mathcal{E}^k\left(\prod_{j=2}^{k+1}\beta_j\right)^{-1} = \lim_{k\to\infty}\left(\prod_{j=2}^{k+1}e^{-\left(\frac{1}{2}\right)^j}\right) = \lim_{k\to\infty}e^{-\sum_{j=2}^{k+1}\left(\frac{1}{2}\right)^j} = \sqrt{e} < \infty.$$

Hence, a cooperative solution exists for $M\Lambda = \mu\Lambda$.

5. Conclusion

Finitely repeated games only have a single noncooperative subgame perfect equilibrium for *any* generalized Prisoner's Dilemma game. This equilibrium property is not limited to finitely repeated games; quasifinite games generalize finitely repeated games. When a Prisoner's Dilemma game is indefinitely repeated by a continuation probability from the class of quasifinite continuation probabilities, there does not exist a cooperative equilibrium regardless of the discount parameter and the stage game.

Additionally, continuation probabilities have been classified as to whether nontrivial vectors, with associated strategies, exist which satisfy a matrix equality. The induced strategies are subgame perfect equilibria. Although there exists a nontrivial equilibrium for the matrix equality when the continuation probability is monotonically nonincreasing, in general, the existence of a cooperative equilibrium is equated with a limit converging to a finite value. These facts and the examples in the paper demonstrate that the analysis of indefinitely repeated, generalized Prisoner's Dilemma games contain subtleties that are not present in the finitely and infinitely repeated games.

6. Appendix

Proof. (of Theorem 3.1) ($\Rightarrow$) If β is quasifinite, then the matrix inequality $M\Lambda \geq \frac{f_i(\tau^*)}{u_i(\tau^*)}\Lambda$ is never satisfied for *any* stage game. However, for any value of $\frac{f_i(\tau^*)}{u_i(\tau^*)} > 0$, there exists a stage game. And, $M\Lambda \geq \frac{f_i(\tau^*)}{u_i(\tau^*)}\Lambda$ has a solution as long as $\mu^* > \frac{f_i(\tau^*)}{u_i(\tau^*)}$. Since $\mu^* \geq 0$, the only possibility is for $\mu^* = 0$. The spectral radius is zero only when $\mathcal{E} = 0$.

($\Leftarrow$) From Jones [6], there exists cooperative equilibria when

$$\mu^* = \frac{\mathcal{E}\delta}{1 - \mathcal{E}\delta} > \frac{f_i(\tau^*)}{u_i(\tau^*)} \text{for every player } i.$$

The ratio $\frac{f_i(\tau^*)}{u_i(\tau^*)}$ is always positive and depends solely on the stage game. If $\mathcal{E} = 0$ then $\mu^* = 0$. It follows that $0 > \frac{f_i(\tau^*)}{u_i(\tau^*)}$ is never satisfied for *any* generalized Prisoner's Dilemma game. Hence, β is quasifinite. $\square$

Proof. (of Corollary 3.2) Since $\lim_{k\to\infty} \beta_k = 0$, then for any $\epsilon > 0$, there exists some $n \in \mathbb{N}$ such that $\beta_k \leq \epsilon$ for all $k > n$. It follows that the spectral radius of the continuation probability

$$\eta_k = \begin{cases} \beta_k & if \quad k \leq n \\ \epsilon & if \quad k > n \end{cases}$$

is greater than the spectral radius associated with β. Consequently, $\mathcal{E}_\eta \geq \mathcal{E}_\beta$ for any ϵ. However, $\mathcal{E}_\eta = \epsilon$. Therefore, $\mathcal{E}_\beta \leq \epsilon$ for any ϵ; this implies that $\mathcal{E}_\beta = 0$. Applying Theorem 3.1, there are no cooperative equilibria for any indefinitely repeated, generalized Prisoner's Dilemma game when the continuation probability is asymptotically finite. $\square$

Proof. (of Proposition 4.1) By Jones [7], a solution to $M\Lambda = \mu^*\Lambda$ is a multiple of the vector Λ^* where $\lambda_1^* = 1$ and

$$\lambda_{k+1}^* = (\mathcal{E})^k \left(\prod_{j=2}^{k+1} \beta_j\right)^{-1} \quad \text{for } k \geq 1.$$

($\Leftarrow$) If Equation 4.1 holds then

$$\lim_{k\to\infty} \lambda_{k+1}^* = \lim_{k\to\infty} (\mathcal{E})^k \left(\prod_{j=2}^{k+1} \beta_j\right)^{-1}$$

and Λ^* is bounded and consists of positive entries. Upon rescaling, this vector defines a strategy and hence a subgame perfect equilibrium.

($\Rightarrow$) If Λ^* can be rescaled to define a cooperative subgame perfect equilibrium then Equation 4.1 holds. □

Proof. (of Corollary 4.2) Since $\{\beta_k\}_{k=1}^{\infty}$ is monotonically nonincreasing and bounded below by 0, then $\lim_{k\to\infty} \beta_k$ exists. Assume $\lim_{k\to\infty} \beta_k = \alpha$.

Suppose that $\alpha > \mathcal{E}$. It follows that

$$\mathcal{E} = \lim_{n\to\infty} \left(\prod_{j=2}^{n+1} \beta_j\right)^{\frac{1}{n}} \geq \lim_{n\to\infty} \left(\prod_{j=2}^{n+1} \alpha\right)^{\frac{1}{n}} = \alpha,$$

which is a contradiction.

Suppose that $\mathcal{E} > \alpha$. Since the sequence converges monotonically, there exists an N such that $n > N$ implies that $\beta_n < \frac{\alpha+\mathcal{E}}{2}$. Define a continuation probability η by

$$\eta_k = \begin{cases} \beta_k & if \quad k \leq N \\ \frac{\alpha+\mathcal{E}}{2} & if \quad k > N \end{cases}$$

Let $\mathcal{E}_\eta = \lim_{n\to\infty} \left(\prod_{j=2}^{n+1} \eta_j\right)^{\frac{1}{n}}$. It follows that $\mathcal{E}_\eta \geq \mathcal{E}$ since $\eta_k \geq \beta_k$ for all k. But $\mathcal{E}_\eta = \frac{\alpha+\mathcal{E}}{2} < \mathcal{E}$. Therefore, $\mathcal{E} = \alpha$.

Since $\{\beta_k\}_{k=1}^{\infty}$ is monotonically nonincreasing and converges to $\mathcal{E}$ then $\frac{\mathcal{E}}{\beta_k} < 1$ for all k. Consequently, $\lim_{k\to\infty} (\mathcal{E})^k \left(\prod_{j=2}^{k+1} \beta_j\right)^{-1} < \infty$. By Theorem 4.1, a nontrivial eigenvector associated with μ^* can be rescaled to define a cooperative subgame perfect equilibrium. □

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